J. Math. Kyoto Univ.
3-1 (1963) 111-113.

A note on transitive permutation groups of degree $p=2q+1$, p and q being prime numbers

To Professor Y. Akizuki on the occasion of his 60th birthday

By

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(Communicated by Prof. Nagata, Aug. 19, 1963)

1. Let $p\geqq 5$ be a prime number and let Ω be the set of symbols $1, \cdots, p$. Let $\mathfrak{G}$ be a nonsolvable transitive permutation group on Ω. Let $p_0(\mathfrak{G})$ be the number of irreducible characters of $\mathfrak{G}$ whose degrees are divisible by p. It seems to be little known about the number $p_0(\mathfrak{G})$. In (**9**) it is shown that $p_0(\mathfrak{G})>0$. There exist a few groups with $p_0(\mathfrak{G})=1$; namely, $LF_2(l)$ as a permutation group of degree l ($l=5, 7, 11$), where $LF_2(l)$ denotes the linear fractional group over the field of 1 elements ((**2**), p. 286). In the present note, under the special condition that $\frac{1}{2}(p-1)=q$ is also a prime, we show that the converse of this fact holds; namely, we prove the following

Theorem. *Let $q=\frac{1}{2}(p-1)$ be also a prime. If $p_0(\mathfrak{G})=1$, then $p=5, 7, 11$ and $\mathfrak{G}$ is isomorphic to $LF_2(p)$.*

2. Throughout this section we assume that $q=\frac{1}{2}(p-1)$ is a prime. Then in (**6**), (**7**) and (**8**) we studied the structure of $\mathfrak{G}$ to some extent. In particular, we proved that such a group $\mathfrak{G}$ is triply transitive on Ω with the exception of $LF_2(7)$ and $LF_2(11)$. Now let us consider two irreducible characters $X_0(X)=\frac{1}{2}(\alpha(X)-1)(\alpha(X)-2)-\beta(X)$ and $X_{00}(X)=\frac{1}{2}\alpha(X)(\alpha(X)-3)+\beta(X)$ of the symmetric group on Ω, where $\alpha(X)$ and $\beta(X)$ respectively denote the the numbers of fixed symbols and the transpositions in the cycle structure of X ((**3**)). Then using the above mentioned triple transitivity of $\mathfrak{G}$ for $p>11$ we obtain the following

Lemma. *Let us assnme that $p>11$. Then X_0^0 restricted on $\mathfrak{G}$ is irreducible, and the decomposition of X_{00} restricted on $\mathfrak{G}$ into its irreducible parts has the following form:*

$$X_{00} = \sum_{i=1}^{s} (D, C)_i ,$$

where $(D, C)_i$ $(i=1,\cdots,s)$ has degree rp with $rs=q-1$.

The proof is similar to those of Lemmas 5–10 in (7). A detailed proof will appear elsewhere (Transitive permutation groups of degree $p=2q+1$, p and q being prime numbers, III).
Now by a theorem of Frobenius ((**4**)) $\mathfrak{G}$ is quadruply transitive on Ω if and only if $s=1$.

Proof of Theorem. Surely we can assume that $p>11$. Because of $p_0(\mathfrak{G})=1$, we can assume, by Lemma, that $\mathfrak{G}$ is quadruply transitive on Ω. Therefore X_{00} restricted on $\mathfrak{G}$ is irreducible.

If the order of $\mathfrak{G}$ is divisible by q^2, then $\mathfrak{G}$ contains a q-cycle. Thus by a theorem of Jordan ((**10**), 13.9) $\mathfrak{G}$ contains the alternating group on Ω. Since $p>11$, $\mathfrak{G}$ is sextuply transitive on Ω. Then using a theorem of Frobenius ((**4**)) we obtain that $p_0(\mathfrak{G})\geqq 3$, which contradicts our assumption $p_0(\mathfrak{G})=1$. Hence q divides the order of $\mathfrak{G}$ only to the first power. Let $\mathfrak{Q}$ be a Sylow q-subgroup of $\mathfrak{G}$ and let Q be a generator of $\mathfrak{Q}$. Then we have that $\alpha(Q)=1$. Let NsQ denote the normalizer of $\mathfrak{Q}$ in $\mathfrak{G}$.

If $\mathfrak{G}$ contains a class $\mathfrak{C}$ of conjugate involutions J such that $Ns\mathfrak{Q}\cap\mathfrak{C}$ is empty, then we obtain the equation

$$\text{(B)} \qquad 0 = \sum_{X} X(J)^2 X(Q)/X(1) ,$$

where X runs over all the irreducible characters of $\mathfrak{G}$ ((**1**), (21)). For $X=X_{00}$ we have that $X(J)^2X(Q)/X(1) = -\{\frac{1}{2}\alpha(J)(\alpha(J)-3)+\beta(J)\}^2/\frac{1}{2}p(p-3) = -\{\alpha(J)(\alpha(J)-4)+p\}^2/2p(p-3)$, because of $\alpha(J)+2\beta(J)=p$. Since $\alpha(J)$ is odd and smaller than p, $\alpha(J)(\alpha(J)-4)$ is not divisible by p. On the other hand, by our assumption $X(1)$ for $X\neq X_{00}$ is prime to p. Then (B) shows a contradiction. Hence for each class $\mathfrak{C}$ of conjugate involutions of $\mathfrak{G}$ we have that $Ns\mathfrak{Q}\cap\mathfrak{C}$ is not empty. In particular, we have that $\alpha(J)=3$ for every involution J of $\mathfrak{G}$.

Let $\mathfrak{M}$ be the maximal subgroup of $\mathfrak{G}$, which leaves the symbols 1, 2, 3 and 4 individually fixed. Then the order of $\mathfrak{M}$ is odd. Hence by a theorem of M. Hall ((5)) p is smaller than or equal to 11. This is a contradiction.

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