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Homotopy Groups of $SU(3)$, $SU(4)$ and $Sp(2)$

By

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§1. Introduction

Let $\pi_i(G)$ be the i-th homotopy group of a topological group G. For $i \leqq 23$ and for $G=SU(3)$, $SU(4)$, $Sp(2)$, the groups $\pi_i(G)$ are computed and the results are given by the following table:

$i=$	3	4	5	6	7	8	9	10	11	12
$\pi_i(SU(3)) \cong$	Z	0	Z	Z_6	0	Z_{12}	Z_3	Z_{30}	Z_4	Z_{60}
$\pi_i(SU(4)) \cong$	Z	0	Z	0	Z	Z_{24}	Z_2	$Z_{120}+Z_2$	Z_4	Z_{60}
$\pi_i(Sp(2)) \cong$	Z	Z_2	Z_2	0	Z	0	0	Z_{120}	Z_2	Z_2+Z_2

$i=$	13	14	15	16	17	18
$\pi_i(SU(3)) \cong$	Z_6	$Z_{84}+Z_2$	Z_{36}	$Z_{252}+Z_6$	$Z_{30}+Z_2$	$Z_{30}+Z_6$
$\pi_i(SU(4)) \cong$	Z_4	$Z_{1680}+Z_2$	$Z_{72}+Z_2$	$Z_{504}+Z_2+Z_2+Z_2+Z_2$	$Z_{40}+Z_2+Z_2+Z_2$	$Z_{2520}+Z_{12}+Z_2$
$\pi_i(Sp(2)) \cong$	Z_4+Z_2	Z_{1680}	Z_2	Z_2+Z_2	Z_{40}	$Z_{2520}+Z_2$

$i=$	19	20	21	22	23
$\pi_i(SU(3)) \cong$	$Z_{12}+Z_6$	$Z_{60}+Z_6$	Z_6	$Z_{66}+Z_2$	$Z_{12}+Z_2$
$\pi_i(SU(4)) \cong$	$Z_{12}+Z_2$	$Z_{60}+Z_2$	$Z_{16}+Z_2$	$Z_{2640}+Z_4+Z_2+Z_2$	$Z_{24}+Z_2+Z_2+Z_2+Z_2$
$\pi_i(Sp(2)) \cong$	Z_2+Z_2	$Z_2+Z_2+Z_2$	$Z_{32}+Z_2$	$Z_{5280}+Z_2+Z_2$	$Z_2+Z_2+Z_2$

These results are stated in Theorems 4.1, 5.1, 6.1, in which generators of the 2-primary components are given, The computations will be done by use of the homotopy exact sequences associated with the bundles $SU(3)/SU(2)=S^3$, $Sp(2)/Sp(1)=S^7$ and $SU(4)/SU(2)=S^5\times S^7$ and the results [7], [3] on the homotopy groups of spheres S^n. In §2, we shall discuss on properties of the composition and the secondary composition operators with respect to the homotopy exact sequences for fibre spaces, in particular, for S^3-bundles over S^n. The homotopy groups of $SU(3)$, $Sp(2)$ and $SU(4)$ will be computed in §4, §5 and §6, respectively, after auxiliary computations on boundary homomorphisms in §3.

§2. Homotopy exact sequences for fibre spaces

Let (X, p, B) be a fibre space. Then we have the following homotopy exact sequence (2.1) associated with the fibre space:

$$(2.1) \qquad \cdots \to \pi_i(F) \xrightarrow{i_*} \pi_i(X) \xrightarrow{p_*} \pi_i(B) \xrightarrow{\Delta} \pi_{i-1}(F) \to \cdots,$$

where F is the fibre $p^{-1}(x_0)$ on a base point x_0 of B, $i: F \to X$ the inclusion map and Δ is the boundary homomorphism defined by the commutativity of the diagram

$$\begin{array}{ccc} \pi_i(X, F) & & \\ \cong \big\downarrow p_* & \searrow \partial & \\ \pi_i(B) & \xrightarrow{\Delta} & \pi_{i-1}(F). \end{array}$$

Let E^{i+1} be the unit $(i+1)$-cube and $S^i = \partial E^{i+1}$ the unit i-sphere. The composition $\alpha \circ \beta = \beta^*(\alpha)$, $\beta \in \pi_j(S^i)$, defines a correspondence $\beta^*: \pi_i(Y) \to \pi_j(Y)$, $Y = F, X, B$, such that it commutes with the homomorphisms i_* and p_* of (2.1), that is,

$$(2.1)' \qquad i_*(\alpha \circ \beta) = (i_* \alpha) \circ \beta \quad \textit{and} \quad p_*(\alpha \circ \beta) = (p_* \alpha) \circ \beta.$$

For the boundary homomorphism Δ, we have the formula

$$(2.2) \qquad \Delta(\alpha \circ E\beta) = (\Delta(\alpha)) \circ \beta, \qquad \alpha \in \pi_{i+1}(B),$$

where $E: \pi_j(S^i) \to \pi_{j+1}(S^{i+1})$ is a suspension homomorphism given by the commutativity of the diagram

$$\begin{array}{ccc} \pi_j(S^i) & \xrightarrow{E} & \pi_{j+1}(S^{i+1}) \\ \cong \nwarrow \partial & & \nearrow p'_* \\ & \pi_{j+1}(E^{i+1}, S^i) & \end{array}$$

(p' pinches S^i and preserves the orientations).

Theorem 2.1. *Assume that* $\alpha \in \pi_{i+1}(B)$, $\beta \in \pi_j(S^i)$ *and* $\gamma \in \pi_k(S^j)$ *satisfy the conditions* $(\Delta\alpha)\circ\beta=0$ *and* $\beta\circ\gamma=0$. *For an arbitrary element* δ *of* $\{\Delta\alpha, \beta, \gamma\} \subset \pi_{k+1}(F)$, *there exists an element* $\varepsilon \in \pi_{j+1}(X)$ *such that*

$$p_*\varepsilon = \alpha\circ E\beta \quad \textit{and} \quad i_*\delta = \varepsilon\circ E\gamma .$$

Proof. Let E_+^{k+1} (resp. E_-^{k+1}) be the upper-(resp. lower-)hemisphere of S^{k+1}. As the definition of the secondary composition $\{\Delta\alpha, \beta, \gamma\}$, δ is represented by a mapping $H: S^{k+1}\to F$ such that $H|E_+^{k+1}=A\circ\bar{c}$, $A|S^j=a\circ b$, $\bar{c}|S^k=c$, $H|E_-^{k+1}=a\circ B$, $B|S^k=b\circ c$ for mappings $A: E_+^{j+1}\to F$, $B: E_-^{k+1}\to S^i$, $\bar{c}: (E_-^{k+1}, S^k)\to(E_+^{j+1}, S^j)$ and representatives a (resp. b, c) of $\Delta\alpha$ (resp. β, γ). We orient (E_-^{j+1}, S^j) and (E^{j+1}, S^j) coherently. By the definition of Δ, there exists an extension $\bar{a}: (E_-^{i+1}, S^i)\to(X, F)$ of $a=\bar{a}|S^i$ such that $p\circ\bar{a}$ represents α. Let $\bar{b}: (E_-^{j+1}, S^j)\to(E_-^{i+1}, S^i)$ be an extension of b. Then $p\circ\bar{a}\circ\bar{b}$ represents $\alpha\circ E\beta$. Define a mapping $G: S^{j+1}\to X$ by setting $G|E_-^{j+1}=\bar{a}\circ\bar{b}$ and $G|E_+^{j+1}=A$. Then G represents an element ε of $\pi_{j+1}(X)$ such that $p_*\varepsilon=\alpha\circ E\beta$. Let $E_-c: (E_-^{k+1}, S^k)\to(E_-^{j+1}, S^j)$ be an extension of c, then $E\gamma$ is respresented by a mapping Ec given by $Ec|E_+^{k+1}=\bar{c}$ and $Ec|E_-^{k+1}=E_-c$. The mapping B gives a null-homotopy of $b\circ c$. By use of the homotopy, we see that $\bar{b}\circ\bar{c}$ is homotopic to B rel. S^k. It follows that H is homotopic to $G\circ Ec$. Therefore we have $i_*\delta=\varepsilon\circ E\gamma$. q. e. d.

The following lemmas will be used in later.

Lemma 2.2. *Let G be a compact Lie group,* $\alpha\in\pi_i(G)$ *and* $\beta_1, \beta_2\in\pi_j(S^i)$. *If* $E\beta_1=E\beta_2$, *then*

$$\alpha\circ\beta_1 = \alpha\circ\beta_2 .$$

Proof. Let (E_G, p, B_G) be a universal G-bundle. Then $\Delta_G: \pi_{i+1}(B_G)\to\pi_i(G)$ is an isomorphism. Let $f: S^{i+1}\to B_G$ be a representative of $\Delta_G^{-1}(\alpha)$ and let $(X, p, S^{i+1}=X/G)$ be a principal G-bundle induced by f. Then $\Delta(\iota_{i+1})=\alpha$ for $\Delta: \pi_{i+1}(S^{i+1})\to\pi_i(G)$ and for the class ι_{i+1} of the identity of S^{i+1}. Then we have $\alpha\circ\beta_1=\Delta(\iota_{i+1})\circ\beta_1=\Delta(\iota_{i+1}\circ E\beta_1)=\Delta(E\beta_1)$. Similarly, $\alpha\circ\beta_2=\Delta(E\beta_2)$. Thus $\alpha\circ\beta_1=\alpha\circ\beta_2$ if $E\beta_1=E\beta_2$. q. e. d.

For a principal G-bundle $(X, p, S^{i+1}=X/G)$ over S^{i+1}, the class

$$\Delta(\iota_{i+1}) = \chi(X)$$

will be refered as *the characteristic class* of the bundle. $\chi(X)$ determines the bundle up to equivalence [6].

Lemma 2. 3. *Let $i \geqq 2$ and let $\mathcal{C}_p$ be the class of finite abelian groups without p-torsion (p: a prime). Assume that $q\chi(X) = q'\chi(X')$ for integers q, q' prime to p. Then $\pi_j(X)$ and $\pi_j(X')$ are $\mathcal{C}_p$-isomorphic to each other for all j. In particular, if the order of $\chi(X)$ is finite and prime to p, then $\pi_j(X)$ is $\mathcal{C}_p$-isomorphic to $\pi_j(S^{i+1}) \oplus \pi_j(G)$.*

Proof. It is sufficient to prove the case $q'=1$. Let $\alpha = \chi(X)$, then the bundle (X, p, S^{i+1}) is induced by a mapping f as in the previous proof. Let $g: S^{i+1} \to S^{i+1}$ be a mapping of degree q. Then the composition $f \circ g$ induces a bundle (X'', p, S^{i+1}) with $\chi(X'') = \Delta(q\iota_{i+1}) = q\chi(X) = \chi(X')$. Thus X'' is equivalent to X' and $\pi_j(X'') \cong \pi_j(X')$ for all j. Consider the homomorphism between the exact sequence (2. 1) and that of (X'', p, S^{i+1}), induced by g. The homomorphism is identical on $\pi_j(F) = \pi_j(G)$ and $\mathcal{C}_p$-isomorphic on $\pi_j(S^{i+1}) = \pi_j(B)$ by Serre's $\mathcal{C}$-theory [5]. Then it is $\mathcal{C}_p$-isomorphic between $\pi_j(X)$ and $\pi_j(X'') \approx \pi_j(X')$ by the five lemma.

If q is the order of $\chi(X)$ and it is prime to p. Then $\chi(X') = 0$ and hence X' is equivalent to the trivial bundle $S^{i+1} \times G$. The last assertion follows. q. e. d.

Now we consider the case $G = S^3$ $(= Sp(1) = SU(2))$. We may consider that the classifying space B_{S^3} is an infinite dimensional quaternion projective space $S^4 \cup e^8 \cup e^{12} \cup \cdots$. Let $i: S^4 \to B_{S^3}$ be the inclusion map. In the diagram

$$\begin{array}{ccc} \pi_{n+1}(B_{S^3}) & \xrightarrow[\approx]{\Delta_{S^3}} & \pi_n(S^3) \\ & {\nwarrow}^{i_*} \quad {\swarrow}^{E} & \\ & \pi_{n+1}(S^4) , & \end{array}$$

the relation

$$\Delta_{S^3}(i_*(E\alpha)) = \alpha , \qquad \alpha \in \pi_n(S^3) ,$$

holds for the case $\alpha = \iota_3$. Then it follows from (2. 2) that the relation holds for arbitrary element α of $\pi_n(S^3)$. Define a homomorphism

$$E^* : \pi_{n+1}(S^4) \longrightarrow \pi_n(S^3)$$

by setting

$$E^* = \Delta_{S^3} \circ i_* .$$

Let $h: S^7 \to S^4$ be Hopf's fibre map. h is the attaching map of e^8. Thus $h_* \pi_{n+1}(S^7)$ is contained in the kernel of E^*. By concerning the isomorphism

$E + h_* : \pi_n(S^3) \oplus \pi_{n+1}(S^7) \cong \pi_{n+1}(S^4)$, we have

(2. 3) $E^* \circ E =$ *the identity and* h_**-image* $=$ *the kernel of* E^*.

Denote by

$$\pi_j(X : p)$$

the p-primary component of $\pi_j(X)$.

The elements $\nu_4 \in \pi_7(S^4)$ and $\nu' \in \pi_6(S^3 : 2)$ of [7] are characterized by the properties:

$$H(\nu_4) = \iota_7 , \quad H(\nu') = \eta_5 \neq 0 \text{ and } 2E\nu_4 = 2\nu_5 = E^2\nu' .$$

It is known that $H(\{h\}) = \iota_7$ (for suitable choice of the orientation of S^7). Then $\{h\} = \nu_4 + aE\nu' + bE\alpha$ for some integers a, b and an element α of order 3. Here, we replace ν_4 by $\nu_4 + aE\nu'$ and ν' by $(2a+1)\nu'$. Then the above properties hold and hence the results in [7] still hold for the new choice of ν_4 and ν'.

Lemma 2. 4. *For the above choice of* ν_4 *and* ν', *we have that* h *represents* $\nu_4 + E\alpha$ (α *generates* $\pi_6(S^3 : 3) \cong Z_3$) *and*

$$\nu_4 \circ \pi_{n+1}(S^7 : 2) = \operatorname{Ker} E^* \cap \pi_{n+1}(S^4 : 2) .$$

Proof. Obviously $\{h\} = \nu_4 + E(b\alpha)$. $b \not\equiv 0$ (mod 3) since mod 3 reduced power operation $\mathcal{P}^1$ is not trivial in quaternion projective plane $S^4 \cup e^8$ [1]. Then, replacing $b\alpha$ by α, we have the first assertion and $3\nu_4 = \nu_4 \circ 3\iota_7 = \{h\} \circ 3\iota_7$. Since S^7 is an H-space, $3\iota_7 \circ \beta = 3\beta$ for all $\beta \in \pi_{n+1}(S^7)$. It follows that $3\iota_7 \circ \pi_{n+1}(S^7 : 2) = \pi_{n+1}(S^7 : 2)$. Thus we have

$$\begin{aligned} \nu_4 \circ \pi_{n+1}(S^7 : 2) &= \nu_4 \circ 3\iota_7 \circ \pi_{n+1}(S^7 : 2) = h_* \pi_{n+1}(S^7 : 2) \\ &= h_* \pi_{n+1}(S^7) \cap \pi_{n+1}(S^4 : 2) \\ &= \operatorname{Ker} E^* \cap \pi_{n+1}(S^4 : 2) . \end{aligned}$$

q. e. d.

Consider a principal S^3-bundle $(X, p, S^n=X/S^3)$ over S^n and its boundary homomorphism

$$\Delta : \pi_{j+1}(S^n) \longrightarrow \pi_j(S^3) .$$

Theorem 2.5. *For* $\alpha \in \pi_{i+1}(S^n)$ *and* $\beta \in \pi_{j+1}(S^{i+1})$, *we have*

$$\Delta(\alpha\circ\beta) = E^*(E(\Delta\alpha)\circ\beta) .$$

In particular,
$$\Delta(\alpha) = E^*(E(\Delta\iota_n)\circ\alpha) .$$

Proof. Let $f: S^n \to B_{S^3} = S^4 \cup e^8 \cup \cdots$ be a mapping which induces the bundle (X, p, S^n). Since $i_* : \pi_n(S^4) \to \pi_n(B_{S^3})$ is equivalent to E^* and it is an epimorphism, there exists a mapping $f' : S^n \to S^4$ homotopic to f in B_{S^3}. From the commutativity of the diagram

$$\begin{array}{ccccc} & & \pi_{j+1}(S^n) & \xrightarrow{\ \Delta\ } & \pi(S_j^3) \\ & \swarrow^{f'_*} & \downarrow{\scriptstyle f_*} & \nearrow_{\Delta_{S^3}} & \\ \pi_{j+1}(S^4) & \xrightarrow{\ i_*\ } & \pi_{j+1}(B_{S^3}) , & & \end{array}$$

it follows $\Delta(\alpha\circ\beta) = \Delta_{S^3} f_*(\alpha\circ\beta) = \Delta_{S^3}(f_*(\alpha)\circ\beta) = \Delta_{S^3}(i_* f'_*(\alpha)\circ\beta)$. In particular, $\Delta_{S^3} i_* f'_*(\alpha) = \Delta\alpha = E^* E(\Delta\alpha) = \Delta_{S^3} i_* E(\Delta\alpha)$ by (2.3). Thus $i_* f'_*(\alpha) = i_* E(\Delta\alpha)$ and $\Delta(\alpha\circ\beta) = \Delta_{S^3}(i_* E(\Delta\alpha)\circ\beta) = E^*(E(\Delta\alpha)\circ\beta)$.

q. e. d.

Theorem 2.6. *Assume that* $\alpha \in \pi_i(S^n)$, $\beta \in \pi_j(S^i)$ *and* $\gamma \in \pi_k(S^j)$ *satisfy the conditions*

$$E(\Delta\alpha) = E(\Delta\iota_n)\circ\alpha, \quad E(\Delta\alpha)\circ\beta = 0 \quad \textit{and} \quad \beta\circ\gamma = 0 .$$

For an arbitrary element δ *of* $\{E(\Delta\alpha), \beta, \gamma\}$, *there exists an element* ε *of* $\pi_j(X)$ *such that*

$$p_*(\varepsilon) = \alpha\circ\beta \quad \textit{and} \quad i_*(E^*\delta) = \varepsilon\circ\gamma ,$$

where $p_* : \pi_j(X) \to \pi_j(S^n)$ *and* $i_* : \pi_k(S^3) \to \pi_k(X)$ *are the homomorphisms in* (2.1).

In order to prove the above theorem, we consider, in general, a principal G-bundle $(X, p, B=X/G)$ which is induced by a mapping $f: B \to B_G$ from a universal G-bundle (E_G, p, B_G). Let $Z_f = B_G \bigcup_f B \times I$ be a mapping cylinder of f. Let $F: X \to E_G$ be the induced bundle map and $Z_F = E_G \bigcup_F X \times I$ a mapping cylinder of F. Then the projections of the bundles define a mapping $\bar{p} : (Z_F, X)$

$\to (Z_f, B)$. E_G and thus Z_F are contractible. It follows that

$$\partial : \pi_{i+1}(Z_F, X) \longrightarrow \pi_i(X)$$

are isomorphisms for all i. Define a homomorphism

$$P : \pi_i(X) \longrightarrow \pi_{i+1}(Z_f, B)$$

by the formula $P = p_* \circ \partial^{-1}$. Then it is verified directly that the following diagram is commutative:

$$\begin{array}{ccccccccc} \cdots \to & \pi_i(G) & \xrightarrow{i_*} & \pi_i(X) & \xrightarrow{p_*} & \pi_i(B) & \xrightarrow{\Delta} & \pi_{i-1}(G) & \to \cdots \\ & \uparrow \Delta_G & & \downarrow P & & \| & & \uparrow \Delta_G & \\ \cdots \to & \pi_{i+1}(B_G) & \xrightarrow{j_*} & \pi_{i+1}(Z_f, B) & \xrightarrow{\partial} & \pi_i(B) & \xrightarrow{f_*} & \pi_i(B_G) & \to \cdots . \end{array} \tag{2.4}$$

Since Δ_G are isomorphisms for all i, so are P, by the five lemma. That is,

(2.5) *the two sequences in* (2.4) *are equivalent.*

Proof of Theorem 2.6. Consider a mapping $f' : S^n \to S^4$ in the proof of Theorem 2.5 which represents $E(\Delta\iota_n)$, and let $Z_{f'} = S^4 \bigcup_{f'} S^n \times I$ be a mapping cylinder of f'. Natural maps induce the following homomorphism of two exact sequences:

$$\begin{array}{ccccccc} \cdots \to \pi_{i+1}(S^4) & \xrightarrow{j_*} & \pi_{i+1}(Z_{f'}, S^n) & \xrightarrow{\partial} & \pi_i(S^n) & \xrightarrow{f'_*} & \pi_i(S^4) \\ \downarrow i_{1*} & & \downarrow i_{2*} & & \| & & \downarrow i_{1*} \\ \cdots \to \pi_{i+1}(B_{S^3}) & \xrightarrow{j_*} & \pi_{i+1}(Z_f, S^n) & \xrightarrow{\partial} & \pi_i(S^n) & \xrightarrow{f_*} & \pi_i(B_{S^3}) . \end{array}$$

In the upper sequence, we apply a similar discussion as in the proof of Theorem 2.1. Then we have that for an arbitrary element δ of $\{f'_*(\alpha), \beta, \gamma\} = \{E(\Delta\alpha), \beta, \gamma\} \subset \pi_{k+1}(S^4)$, there exists an element $\mathcal{E}'$ of $\pi_{j+1}(Z_{f'}, S^n)$ such that

$$\partial \mathcal{E}' = \alpha \circ \beta \quad \text{and} \quad j_* \delta = \mathcal{E}' \circ \tilde{\gamma} ,$$

where $\tilde{\gamma}$ is the inverse image of γ under $\partial : \pi_{k+1}(E^{j+1}, S^j) \approx \pi_k(S^j)$. Define $\mathcal{E}$ by the formula $\mathcal{E} = P^{-1}(i_{2*}\mathcal{E}')$. Then we have

$$p_* \mathcal{E} = p_* P^{-1}(i_{2*}\mathcal{E}') = \partial \mathcal{E}' = \alpha \circ \beta$$

and

$$\begin{aligned} i_*(E^*\delta) &= i_*(\Delta_{S^3} i_{1*}\delta) = P^{-1} j_*(i_{1*}\delta) = P^{-1} i_{2*}(j_*\delta) = P^{-1} i_{2*}(\mathcal{E}' \circ \tilde{\gamma}) \\ &= (P^{-1}(i_{2*}\mathcal{E}')) \circ \gamma = \mathcal{E} \circ \gamma , \end{aligned}$$

by the commutativity of (2.4) and the above diagram, (2.3) and by the property $P^{-1}(\alpha\circ\tilde{\gamma})=(P^{-1}\alpha)\circ\gamma$ of P. q.e.d.

§3. The boundary homomorphisms for the fiberings $SU(3)/SU(2)$ and $Sp(2)/Sp(1)$

We shall apply the theory of the previous section to the bundles $(SU(3), p, S^5=SU(3)/SU(2))$ and $(Sp(2), p, S^7=Sp(2)/Sp(1))$, where $SU(2)\cong Sp(1)\cong S^3$.

The boundary homomorphisms

$$\Delta:\pi_5(S^5)\longrightarrow\pi_4(S^3)\quad\text{and}\quad\Delta:\pi_7(S^7)\longrightarrow\pi_6(S^3)$$

are epimorphisms, since $\pi_4(SU(3))=0$ [6] and $\pi_6(Sp(2))=0$ [1] (see also [2]). Therefore, the characteristic classes of the above two bundles are generators of $\pi_4(S^3)\cong Z_2$ and $\pi_6(S^3)\cong Z_{12}$ respectively. The elements η_3 and ν' of [7] are generators of $\pi_4(S^3:2)=\pi_4(S^3)$ and $\pi_6(S^3:2)\cong Z_4$ respectively. Then we have

Lemma 3.1. i). *For the homomorphisms* $\Delta:\pi_i(S^5)\to\pi_{i-1}(S^3)$, *we have* $\Delta(E\alpha')=\eta_3\circ\alpha'$ *for* $\alpha'\in\pi_{i-1}(S^4)$ *and* $\Delta\alpha=E^*(\eta_4\circ\alpha)$ *for* $\alpha\in\pi_i(S^5)$. $\eta_4\circ\alpha\in E\pi_{i-1}(S^3)$ *if and only if* $\eta_7^2\circ H(\alpha)=0$ *and whence* $E(\Delta\alpha)=\eta_4\circ\alpha$.

ii). *For the homomorphisms* $\Delta:\pi_i(S^7)\to\pi_{i-1}(S^3)$, *we have*

$$\Delta(E\alpha')=\pm\nu'\circ\alpha'\qquad \textit{for}\quad \alpha'\in\pi_{i-1}(S^6:2)$$

and $\Delta\alpha=E^*(\pm E\nu'\circ\alpha)$, $E(\Delta\alpha)=\pm E\nu'\circ\alpha$ *for* $\alpha\in\pi_i(S^7:2)$.

Proof. The first two assertions of i) follow immediately from (2.2) and Theorem 2.5, where $\Delta\iota_5=\eta_3$. By the exactness of the sequence

$$\pi_{i-1}(S^3)\xrightarrow{E}\pi_i(S^4)\xrightarrow{H}\pi_i(S^7),$$

$\eta_4\circ\alpha\in E\pi_{i-1}(S^3)$ if and only if $H(\eta_4\circ\alpha)=0$. $H(\eta_4\circ\alpha)=H(E\eta_3\circ\alpha)=E(\eta_3\times\eta_3)\circ H(\alpha)=\eta_7^2\circ H(\alpha)$ by Proposition 2.2 of [7]. If $\eta_4\circ\alpha=E\beta$, then $E(\Delta\alpha)=E(E^*E\beta)=E\beta=\eta_4\circ\alpha$ by (2.3).

In ii), replacing $\pm\nu'$ by $\Delta\iota_7$, the similar assertions are true. Then it is sufficient to prove the relations

$$E(\Delta\iota_7)\circ\alpha = \pm E\nu'\circ\alpha \quad \text{and} \quad H(E\nu'\circ\alpha) = 0 \quad \text{for} \quad \alpha\in\pi_i(S^7:2).$$

Since $E^2\nu'=2\nu_5$ and $2\nu_5^2=0$, we have $H(E\nu'\circ\alpha)=E(\nu'\times\nu')\circ H(\alpha)=4\nu_7^2\circ H(\alpha)=0$. There exists an odd integer t such that $3t\alpha=\alpha$, since the order of α is a power of 2. $3t\alpha=(3t\iota_7)\circ\alpha$ since S^7 is an H-space. Then we have $E(\Delta\iota_7)\circ\alpha=E(\Delta\iota_7)\circ(3t\iota_7)\circ\alpha=3tE(\Delta\iota_7)\circ\alpha=(\pm E\nu')\circ\alpha=\pm(E\nu'\circ\alpha)$. This completes the proof of the lemma.

We introduce necessary results on the homotopy groups of spheres. According to [7], we denote, for odd n,

$$\pi_i^n = \begin{cases} \pi_n(S^n) & \text{if } i = n, \\ \pi_i(S^n:2) & \text{if } i \neq n. \end{cases}$$

Then the results on π_i^n for $n=3, 5, 7$ are listed in the following table:

(3.1)

$i=$	1, 2	3	4	5	6	7	8	9	10	11	12
$\pi_{i+1}^7\cong$	0	0	0	0	Z	Z_2	Z_2	Z_8	0	0	Z_2
generators					ι_7	η_7	η_7^2	ν_7			ν_7^2
$\pi_{i+1}^5\cong$	0	0	Z	Z_2	Z_2	Z_8	Z_2	Z_2	Z_2	Z_2	Z_2
generators			ι_5	η_5	η_5^2	ν_5	$\nu_5\eta_8$	$\nu_5\eta_8^2$	ν_5^2	σ'''	ε_5
$\pi_i^3\cong$	0	Z	Z_2	Z_2	Z_4	Z_2	Z_2	0	0	Z_2	Z_2+Z_2
generators		ι_3	η_3	η_3^2	ν'	$\nu'\eta_6$	$\nu'\eta_6^2$			ε_3	μ_3, $\eta_3\varepsilon_4$

$i=$	13	14	15	16
$\pi_{i+1}^7\cong$	Z_8	$Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2+Z_2$	Z_8+Z_2
generators	σ'	$\sigma'\eta_{14}$, $\bar{\nu}_7$, ε_7	$\sigma'\eta_{14}^2$, ν_7^3, μ_7, $\eta_7\varepsilon_8$	$\nu_7\sigma_{10}$, $\eta_7\mu_8$
$\pi_{i+1}^5\cong$	$Z_2+Z_2+Z_2$	Z_8+Z_2	$Z_8+Z_2+Z_2$	$Z_2+Z_2+Z_2$
generators	ν_5^3, μ_5, $\eta_5\varepsilon_6$	$\nu_5\sigma_8$, $\eta_5\mu_6$	ζ_5, $\nu_5\bar{\nu}_8$, $\nu_5\varepsilon_8$	ν_5^4, $\nu_5\mu_8$, $\nu_5\eta_8\varepsilon_9$
$\pi_i^3\cong$	Z_4+Z_2	$Z_4+Z_2+Z_2$	Z_2+Z_2	Z_2
generators	ε', $\eta_3\mu_4$	μ', $\varepsilon_3\nu_{11}$, $\nu'\varepsilon_6$	$\nu'\mu_6$, $\nu'\eta_6\varepsilon_7$	$\nu'\eta_6\mu_7$

i	17	18	19	20
$\pi^7_{i+1}\cong$	Z_8+Z_2	0	Z_2	Z_8+Z_4
generators	$\zeta_7,\ \bar\nu_7\nu_{15}$		$\nu_7\sigma_{10}\nu_{17}$	$\sigma'\sigma_{14},\ \kappa_7$
$\pi^5_{i+1}\cong$	Z_2+Z_2	Z_2+Z_2	Z_2+Z_2	Z_2+Z_2
generators	$\nu_5\sigma_8\nu_{15},\ \nu_5\eta_8\mu_9$	$\nu_5\zeta_8,\ \nu_5\bar\nu_8\nu_{16}$	$\rho^{\mathrm{IV}},\ \bar\varepsilon_5$	$\mu_5\sigma_{14},\ \eta_5\varepsilon_6$
$\pi^3_i\ \cong$	Z_2	Z_2	Z_2+Z_2	$Z_4+Z_2+Z_2$
generators	$\varepsilon_3\nu_{11}^2$	$\bar\varepsilon_3$	$\mu_3\sigma_{12},\ \eta_3\bar\varepsilon_4$	$\bar\varepsilon',\ \bar\mu_3,\ \eta_3\mu_4\sigma_{13}$

$i=$	21	22	23
$\pi^7_{i+1}\cong$	$Z_8+Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2+Z_2$
generators	$\rho'',\ \sigma'\bar\nu_{14},\ \sigma'\varepsilon_{14},\ \bar\varepsilon_7$	$\sigma'\mu_{14},\ E\zeta',\ \mu_7\sigma_{10},\ \eta_7\bar\varepsilon_8$	$\sigma'\eta_{14}\mu_{15},\ \nu_7\kappa_{10},\ \bar\mu_7,\ \eta_7\mu_8\sigma_{17}$
$\pi^5_{i+1}\cong$	$Z_4+Z_2+Z_2$	$Z_8+Z_2+Z_2$	Z_8+Z_2
generators	$\nu_5\kappa_8,\ \bar\mu_5,\ \eta_5\mu_6\sigma_{15}$	$\zeta_5\sigma_{16},\ \nu_5\bar\varepsilon_8,\ \eta_5\bar\mu_6$	$\zeta_5,\ \nu_5\mu_8\sigma_{17}$
$\pi^3_i\ \cong$	$Z_4+Z_2+Z_2$	Z_4+Z_2	Z_2+Z_2
generators	$\mu'\sigma_{14},\ \nu'\bar\varepsilon_6,\ \eta_3\bar\mu_4$	$\bar\mu',\ \nu'\mu_6\sigma_{15}$	$\nu'\eta_6\mu_7\sigma_{16},\ \nu'\bar\mu_6$

Here, we use the notations of [7]. For the simplicity, we omit the symbol "$\circ$" of the composition operator. The results are given in [7] except the group π^3_{23} which is given in [3].

Proposition 3.2. i). *For the homomorphism* $\Delta:\pi^5_{i+1}\to\pi^3_i$, *we have the following table*:

$\alpha=$	η_5	η_5^2	ν_5	$\nu_5\eta_8$	$\nu_5\eta_8^2$	ν_5^2	σ'''	ε_5	ν_5^3	μ_5	$\eta_5\varepsilon_6$
$\Delta\alpha=$	η_3^2	$2\nu'$	$\nu'\eta_6$	$\nu'\eta_6^2$	0	0	0	$\eta_3\varepsilon_4$	0	$\eta_3\mu_4$	$2\varepsilon'$

$\alpha=$	$\nu_5\sigma_8$	$\eta_5\mu_6$	ζ_5	$\nu_5\bar\nu_8$	$\nu_5\varepsilon_8$	ν_5^4	$\nu_5\mu_8$
$\Delta\alpha=$	$\varepsilon_3\nu_{11}+\nu'\varepsilon_6$	$2\mu'$	$\nu'\mu_6$ mod $\nu'\eta_6\varepsilon_7$	0	$\nu'\eta_6\varepsilon_7$	0	$\nu'\eta_6\mu_7$

$\alpha=$	$\nu_5\eta_8\varepsilon_9$	$\nu_5\sigma_8\nu_{15}$	$\nu_5\eta_8\mu_9$	$\nu_5\zeta_8$	$\nu_5\bar\nu_8\nu_{16}$	ρ^{IV}	$\bar\varepsilon_5$	$\mu_5\sigma_{14}$
$\Delta\alpha=$	0	$\varepsilon_3\nu_{11}^2$	0	0	0	0	$\eta_3\bar\varepsilon_4$	$\eta_3\mu_4\sigma_{13}$

$\alpha=$	$\eta_5\bar\varepsilon_6$	$\nu_5\kappa_8$	$\bar\mu_5$	$\eta_5\mu_6\sigma_{15}$	$\zeta_5\sigma_{16}$	$\nu_5\bar\varepsilon_8$	$\eta_5\bar\mu_6$	$\bar\zeta_5$	$\nu_5\mu_8\sigma_{17}$
$\Delta\alpha=$	$2\bar\varepsilon'$	$\nu'\bar\varepsilon_6$	$\eta_3\bar\mu_4$	$2\mu'\sigma_{14}$	$\nu'\mu_6\sigma_{15}$	0	$2\bar\mu'$	$\nu'\bar\mu_6$ mod β	$\beta=\nu'\eta_6\mu_7\sigma_{16}$

ii). *For the homomorphisms* $\Delta : \pi_{i+1}^7 \to \pi_i^3$, $i \neq 6$, *we have the following table*:

$\alpha=$	η_7	η_7^2	ν_7	ν_7^2	σ'	$\sigma'\eta_{14}$	$\bar{\nu}_7$	ε_7	$\sigma'\eta_{14}^2$	ν_7^3
$\Delta\alpha=$	$\nu'\eta_6$	$\nu'\eta_6^2$	0	0	$2\varepsilon'$	0	$\varepsilon_3\nu_{11}$	$\nu'\varepsilon_6$	0	0

$\alpha=$	μ_7	$\eta_7\varepsilon_8$	$\nu_7\sigma_{10}$	$\eta_7\mu_8$	ζ_7	$\bar{\nu}_7\nu_{15}$	$\nu_7\sigma_{10}\nu_{17}$	$\sigma'\sigma_{14}$	κ_7	ρ''	$\sigma'\bar{\nu}_{14}$
$\Delta\alpha=$	$\nu'\mu_6$	$\nu'\eta_6\varepsilon_7$	0	$\nu'\eta_6\mu_7$	0	$\varepsilon_3\nu_{11}^2$	0	0	$\pm\bar{\varepsilon}'$	0	0

$\alpha=$	$\sigma'\varepsilon_{14}$	$\bar{\varepsilon}_7$	$\sigma'\mu_{14}$	$E\zeta'$	$\mu_7\sigma_{16}$	$\eta_7\bar{\varepsilon}_8$	$\sigma'\eta_{14}\mu_{15}$	$\nu_7\kappa_{10}$	$\bar{\mu}_7$	$\eta_7\mu_8\sigma_{17}$
$\Delta\alpha=$	0	$\nu'\bar{\varepsilon}_6$	0	0	$\nu'\mu_6\sigma_{15}$	0	0	0	$\nu'\bar{\mu}_6$	$\nu'\eta_6\mu_7\sigma_{16}$

Proof.

i). It follows directly from the formula $\Delta(E\alpha')=\eta_3\circ\alpha'$ of Lemma 3.1, i), that the table is true for $\alpha=\eta_5,\ \varepsilon_5,\ \mu_5,\ \bar{\varepsilon}_5,\ \mu_5\sigma_{14},\ \bar{\mu}_5$.

The relation $\eta_3\nu_4=\nu'\eta_6$ in (5.9) of [7] implies the formula

$$\Delta(\nu_5\circ E\beta) = \nu'\eta_6\beta \qquad \text{for} \quad \beta\in\pi_i^7.$$

Then the cases $\alpha=\nu_5,\ \nu_5\eta_8,\ \nu_5\varepsilon_8,\ \nu_5\mu_8,\ \nu_5\mu_8\sigma_{17}$ follow immediately. The cases $\alpha=\nu_5\eta_8^2,\ \nu_5^2$ are obvious, since $\pi_9^3=\pi_{10}^3=0$, hence the cases $\alpha=\nu_5^3,\ \nu_5^4$ follow.

We have also

$$\begin{aligned}
\Delta(\nu_5\bar{\nu}_8) &= \nu'\eta_6\bar{\nu}_7 = \nu'\nu_6^3\in\pi_9^3\circ\nu_9^2 = 0 && \text{by (5.9) of [7]},\\
\Delta(\nu_5\bar{\nu}_8\nu_{16}) &= \Delta(\nu_5\bar{\nu}_8)\circ\nu_{15} = 0 && \text{by (2.2)},\\
\Delta(\nu_5\zeta_8) &= \nu'\eta_6\zeta_7\in\nu'E^2\pi_{16}^4 = \nu'\nu_6\circ\pi_{18}^9 && \text{by Theorem 7.9 of [7]}\\
&\subset\pi_9^3\circ\pi_{18}^9 = 0,\\
\Delta(\nu_5\kappa_8) &= \nu'\eta_6\kappa_7 = \nu'\bar{\varepsilon}_6 && \text{by (10.23) of [7]},\\
\Delta(\nu_5\bar{\varepsilon}_8) &= \nu'\eta_6\bar{\varepsilon}_7 = \nu'\nu_6\sigma_9\nu_{16}^2 && \text{by Lemma 12.10 of [7]}\\
&\in\pi_9^3\circ\sigma_9\nu_{16}^2 = 0.
\end{aligned}$$

The relations $\eta_3^3=2\nu'$, $\eta_3^2\varepsilon_5=2\varepsilon'$, $\eta_3^2\mu_5=2\mu'$, $\eta_3^2\bar{\varepsilon}_5=2\bar{\varepsilon}'$ and $\eta_3^2\bar{\mu}_5=2\bar{\mu}'$ are obtained in (5.3), Lemma 6.6, (7.7), Lemma 12.3 and Lemma 12.4 of [7] respectively. Then the cases $\alpha=\eta_5^2,\ \eta_5\varepsilon_6,\ \eta_5\mu_6,\ \eta_5\mu_6\sigma_{15},\ \eta_5\bar{\varepsilon}_6$ and $\eta_5\bar{\mu}_6$ follow.

We have also

$$\Delta(\nu_5\eta_8\varepsilon_9) = \nu'\eta_6^2\varepsilon_8 = \nu'(2E^3\varepsilon') \in 2\pi_{16}^3 = 0$$

and $$\Delta(\nu_5\eta_8\mu_9) = \nu'\eta_6^2\mu_8 = \nu'(2E^3\mu') \in 2\pi_{17}^3 = 0,$$

by Theorems 7.7 and 10.3 of [7].

The remaining cases: $\alpha=\sigma'''$, $\nu_5\sigma_8$, ζ_5, $\nu_5\sigma_8\nu_{15}$, ρ^{IV}, $\zeta_5\sigma_{16}$, $\bar{\zeta}_5$, $\Delta(\alpha)$ are verified directly from i) of the following Lemma 3.3, in virtue of Lemma 3.1, i) and (2.3).

ii). It follows directly from the formula $\Delta(E\alpha')=\pm\nu'\circ\alpha'$ of Lemma 3.1, ii) that the table is true for $\alpha=\eta_7$, η_7^2, ε_7, μ_7, $\eta_7\varepsilon_8$, $\eta_7\mu_8$, $\bar{\varepsilon}_7$, $\mu_7\sigma_{16}$, $\bar{\mu}_7$ and $\eta_7\mu_8\sigma_{17}$.

Obviously $\nu'\nu_6\in\pi_9^3=0$. Thus we have

$$\Delta(\nu_7\circ E\beta) = \pm\nu'\nu_6\circ\beta = 0 \qquad \text{for} \quad \beta\in\pi_i^9.$$

Then the cases $\alpha=\nu_7$, ν_7^2, ν_7^3, $\nu_7\sigma_{10}$, $\nu_7\sigma_{10}\nu_{17}$ and $\nu_7\kappa_{10}$ follow.

The relations $\nu'\bar{\nu}_6=\varepsilon_3\nu_{11}$ and $\eta_6\bar{\varepsilon}_7=\nu_6\sigma_9\nu_{16}^2$ are obtained in (7.12) and Lemma 12.10 of [7]. Then the cases $\alpha=\bar{\nu}_7$, $\bar{\nu}_7\nu_{15}$, and $\eta_7\bar{\varepsilon}_8$ follow.

Next we prove

$$(3.2) \qquad E\nu'\circ\sigma' = 2E\varepsilon' = \eta_4^2\varepsilon_6.$$

By Lemma 5.4, Lemma 5.14 and (7.10) of [7], we have

$$E^2(E\nu'\circ\sigma') = 4\nu_6\sigma_9 = E^2(\eta_4^2\varepsilon_6) = E^2(2E\varepsilon').$$

By Lemma 3.1, ii), $E\nu'\circ\sigma'=E\Delta(\sigma')\in E\pi_{13}^3$. It is seen in Theorem 7.3 of [7] that $E^2|E\pi_{13}^3: E\pi_{13}^3\to\pi_{16}^6$ is a monomorphism. It follows the relation (3.2).

Applying the homomorphism E^* to (3.2), we have

$$\Delta(\sigma') = E^*E(\Delta\sigma') = E^*(E\nu'\circ\sigma') = E^*E(2\varepsilon') = 2\varepsilon' = \eta_3^2\varepsilon_5.$$

If $2\beta=0$, then $\Delta(\sigma'\circ E^2\beta)=\varepsilon'\circ 2E\beta=0$, and the cases $\alpha=\sigma'\eta_{14}$, $\sigma'\eta_{14}^2$, $\sigma'\bar{\nu}_{14}$, $\sigma'\mu_{14}$, $\sigma'\varepsilon_{14}$, $\sigma'\eta_{14}\mu_{15}$ follow immediately.

We have $E\zeta'=\sigma'\eta_{14}\varepsilon_{15}$ in (12.4) of [7]. Then the case $\alpha=E\zeta'$ follows. We have $\varepsilon_5\sigma_{13}=0$ in Lemma 10.7 of [7]. Then

$$\Delta(\sigma'\sigma_{14}) = \Delta(\sigma')\sigma_{13} = \eta_3^2\varepsilon_5\sigma_{13} = 0.$$

The remaining cases: $\alpha=\zeta_7$, κ_7, ρ'' follow from ii) of the following Lemma 3.3 in virtue of Lemma 3.1, ii) and (2.3).

Lemma 3. 3. i).

$$\eta_4\sigma''' = 0\,, \qquad \eta_4\nu_5\sigma_8 = E(\varepsilon_3\nu_{11}+\nu'\varepsilon_6)\,,$$
$$\eta_4\nu_5\sigma_8\nu_{15} = E(\varepsilon_3\nu_{11}^2)\,, \qquad \eta_4\zeta_5 \equiv E(\nu'\mu_6) \mod E(\nu'\eta_6\varepsilon_7)\,,$$
$$\eta_4\rho^{\mathrm{IV}} = 0\,, \qquad \eta_4\zeta_5\sigma_{16} = E(\nu'\mu_6\sigma_{15})\,,$$
$$\eta_4\bar{\zeta}_5 \equiv E(\nu'\bar{\mu}_6) \mod E(\nu'\eta_6\mu_7\sigma_{16})\,.$$

ii). $\nu'\zeta_6=0$, $(E\nu')\kappa_7=E\bar{\varepsilon}'$ *and* $E\nu'\rho''=0$.

Proof. i). The relation $\eta_4\sigma'''=0$ is already obtained in (7.4) of [7].

We have
$$\begin{aligned} \eta_4\nu_5\sigma_8 &= E\nu'\eta_7\sigma_8 \\ &= E\nu'(\sigma'\eta_{14}+\bar{\nu}_7+\varepsilon_7) && \text{by (7.9) of [7]} \\ &= E\Delta(\sigma'\eta_{14}+\bar{\nu}_7+\varepsilon_7) && \text{for } \Delta \text{ of ii)} \\ &= E(\varepsilon_3\nu_{11}+\nu'\varepsilon_6)\,, \end{aligned}$$
and
$$\eta_4\nu_5\sigma_8\nu_{15} = E(\varepsilon_3\nu_{11}^2+\nu'\varepsilon_6\nu_{14}) = E(\varepsilon_3\nu_{11}^2)$$
since $\varepsilon_6\nu_{14}=0$ by (7.14) of [7].

Next, we have
$$\begin{aligned} \eta_4\zeta_5 &\in \eta_4\circ\{\nu_5, 8\iota_8, E\sigma'\}_1 && \text{by the definition of } \zeta_5 \\ &\subset \{\eta_4\nu_5, 8\iota_8, E\sigma'\}_1 && \text{by Proposition 1.2 of [7]} \\ &= \{E\nu'\eta_7, 8\iota_8, E\sigma'\}_1 \end{aligned}$$
and
$$\begin{aligned} E\nu'\mu_7 &\in E\nu'\circ\{\eta_7, 2\iota_8, E^3\sigma'''\}_1 && \text{by Lemma 6.5 of [7]} \\ &\subset \{E\nu'\eta_7, 2\iota_8, 4E\sigma'\}_1 && \text{by Lemma 5.14 of [7]} \\ &\subset \{E\nu'\eta_7, 8\iota_8, E\sigma'\}_1 && \text{by Proposition 1.2 of [7].} \end{aligned}$$

It follows
$$\eta_4\zeta_5 \equiv E\nu'\mu_7 \qquad \mod E\nu'\eta_7\circ E\pi_{15}^7+\pi_9^4\circ E^2\sigma'\,.$$

$\eta_6\circ\pi_{15}^7$ is generated by $\eta_6\sigma'\eta_{14}$, $\eta_6\bar{\nu}_7$ and $\eta_6\varepsilon_7$. $\eta_6\sigma'\eta_{14}=(4\bar{\nu}_6)\eta_{14}=0$ by (7.4) of [7]. $\nu'\eta_6\bar{\nu}_7=\nu'\nu_6^3=\Delta(\nu_7^3)=0$ by (7.3) of [7]. $\pi_9^4\circ E^2\sigma'=\pi_9^4\circ(2\sigma_9)=2\pi_9^4\circ\sigma_9=0$ by Lemma 5.14 of [7]. Thus we have that $E\nu'\eta_7\circ E\pi_{15}^7+\pi_9^4\circ E^2\sigma'$ is generated by $E(\nu'\eta_6\varepsilon_7)$, and
$$\eta_4\zeta_5 \equiv E(\nu'\mu_6) \qquad \mod E(\nu'\eta_6\varepsilon_7)\,.$$

It follows from the relation $\varepsilon_7\sigma_{15}=0$ in Lemma 10.7 of [7]
$$\eta_4\zeta_5\sigma_{16} = E(\nu'\mu_6\sigma_{15})\,.$$

By (10.12) of [7], $\eta_7^2\circ H(\rho^{IV})=\eta_7^2\circ 4\zeta_9=0$. It follows from Lemma 3.1, i) $\eta_4\circ\rho^{IV}\in E\pi_{19}^3$. $E^2(\eta_4\circ\rho^{IV})=\eta_6(2E\rho''')=2\eta_6\circ E\rho'''=0$ by (10.15) of [7]. It is seen in Theorem 12.6 of [7] that $E^2|E\pi_{19}^3$ is a monomorphism. Thus we have

$$\eta_4\circ\rho^{IV}=0\,.$$

For $\bar{\zeta}_5$, we have, similarly to ζ_5,

$$\begin{aligned}
&\eta_4\bar{\zeta}_5\in\eta_4\circ\{\zeta_5,\,8\iota_{16},\,2\sigma_{16}\}_1\\
&\qquad\subset\{\eta_4\zeta_5,\,8\iota_{16},\,2\sigma_{16}\}_1=\{E\nu'\mu_7+xE\nu'\eta_7\varepsilon_8,\,8\iota_{16},\,2\sigma_{16}\}_1\\
&\qquad\subset\{E\nu'\mu_7,\,8\iota_{16},\,2\sigma_{16}\}_1+x\{E\nu'\eta_7\varepsilon_8,\,8\iota_{16},\,2\sigma_{16}\}_1\,,\quad x=0\text{ or }1\,,\\
&E\nu'\bar{\mu}_7\in E\nu'\circ E^4\{\mu_3,\,2\iota_{12},\,8\sigma_{12}\}_1\subset E\nu'\circ\{\mu_7,\,2\iota_{16},\,8\sigma_{16}\}_1\\
&\qquad\subset\{E\nu'\mu_7,\,8\iota_{16},\,2\sigma_{16}\}_1\,,\\
&E\nu'\varepsilon_7\mu_{15}\in E\nu'\varepsilon_7\circ\{\eta_{15},\,2\iota_{16},\,8\sigma_{16}\}_1\\
&\qquad\subset\{E\nu'\varepsilon_7\eta_{15},\,2\iota_{16},\,8\sigma_{16}\}_1=\{E\nu'\eta_7\varepsilon_8,\,2\iota_{16},\,8\sigma_{16}\}_1\,.
\end{aligned}$$

Thus $$\eta_4\bar{\zeta}_5\equiv E\nu'\bar{\mu}_7+xE\nu'\varepsilon_7\mu_{15}\quad\bmod G\,,$$

where $G=E\nu'\mu_7\circ E\pi_{23}^{15}+E\nu'\eta_7\varepsilon_8\circ E\pi_{23}^{15}+\pi_{17}^4\circ 2\sigma_{17}$ is generated by the following elements: $E\nu'\mu_7\bar{\nu}_{16}$, $E\nu'\mu_7\varepsilon_{16}$, $E\nu'\eta_7\varepsilon_8\bar{\nu}_{16}$, $E\nu'\eta_7\varepsilon_8\varepsilon_{16}$, $\pi_{17}^4\circ 2\sigma_{17}\subset 2\pi_{24}^4=0$ (*cf.* Theorem 7.1 of [7] and [3]). $\mu_5\nu_{14}\in E^2\pi_{15}^3=0$ by Theorem 7.6 of [7]. Then

$$\begin{aligned}
\mu_7\bar{\nu}_{16}&=E^2\mu_5\circ E^2\{\nu_{14},\,\eta_{17},\,\nu_{18}\} &&\text{by Lemma 6.2 of [7]}\\
&=E^2\{\mu_5,\,\nu_{14},\,\eta_{17}\}\circ\nu_{21} &&\text{by Proposition 1.4 of [7]}\\
&\subset E^2\pi_{19}^5\circ\nu_{21}\subset 4\pi_{21}^7\circ\nu_{21} &&\text{by Theorem 10.3 of [7]}\\
&\subset 4\pi_{24}^7=0 &&\text{by Theorem 12.7 of [7]}.
\end{aligned}$$

By use of the anti-commutativity of the composition operator and the relation $\bar{\nu}_{16}+\varepsilon_{16}=\eta_{16}\sigma_{17}$ in Lemma 6.4 of [7],

$$\varepsilon_7\mu_{15}=\mu_7\varepsilon_{16}=\mu_7\eta_{16}\sigma_{17}+\mu_7\bar{\nu}_{16}=\eta_7\mu_8\sigma_{17}\,.$$

We have also $\eta_7\varepsilon_8\varepsilon_{16}=\eta_7\varepsilon_8\bar{\nu}_{16}=\eta_7\nu_8\sigma_{11}\nu_{18}^2=0$ by Lemma 12.10 and (5.9) of [7]. Consequently, we have obtained the relation

$$\eta_4\bar{\zeta}_5\equiv E(\nu'\bar{\mu}_6)\qquad\bmod E(\nu'\eta_6\mu_7\sigma_{16})\,.$$

ii). We have

$$\begin{aligned} \nu'\zeta_6 &\in \nu'\circ\{\nu_6, 8\iota_9, E^2\sigma'\} && \text{by the definition of } \zeta_6 \\ &= -\{\nu', \nu_6, 8\iota_9\}\circ E^3\sigma' && \text{by Proposition 1.4 of [7]} \\ &\subset \pi_{10}^3\circ E^3\sigma' = 0 . \end{aligned}$$

It follows $\nu'\zeta_6=0$.

The relation $E\bar{\varepsilon}'=E\nu'\circ\kappa_7$ is given in Lemma 12.3 of [7].

Next we have

$$\begin{aligned} E\nu'\circ\rho'' &\in E\nu'\circ\{\sigma', 8\iota_{14}, 2\sigma_{14}\}_1 && \text{by the definition of } \rho'' \\ &\subset \{E\nu'\circ\sigma', 8\iota_{14}, 2\sigma_{14}\}_1 \\ &= \{\eta_4^2\varepsilon_6, 8\iota_{14}, 2\sigma_{14}\}_1 && \text{by (3.2)} \\ &\subset \{\eta_4^2, 8\varepsilon_6, 2\sigma_{14}\}_1 = \{\eta_4^2, 0, 2\sigma_{14}\}_1 \equiv 0 . \end{aligned}$$

Thus $E\nu'\circ\rho''\in\eta_4^2\circ E\pi_{21}^5+\pi_{15}^4\circ 2\sigma_{15}$. From table (3.1) and Proposition 3.2, i), we have that $\eta_4^2\circ E\pi_{21}^5$ is generated by $\eta_4^2\mu_6\sigma_{15}$ and $\eta_4^2\eta_6\bar{\varepsilon}_7=2\eta_4E\bar{\varepsilon}'=0$. We see in Theorem 7.3 of [7] that $2\pi_{15}^4$ is generated by $2E\mu'=\eta_4^2\mu_6$. It follows that

$$E\nu'\circ\rho'' = x(\eta_4^2\mu_6\sigma_{15}) \qquad \text{for} \quad x=0 \text{ or } 1 .$$

Now, consider the composition $\eta_8\circ\rho'\in\pi_{24}^8$. We have $H(\eta_8\circ\rho')=\eta_{15}^2\circ 8\sigma_{17}=0$. It follows that $\eta_8\circ\rho'\in E\pi_{23}^7$. By Lemma 10.9 of [7], $E^5(\eta_8\circ\rho')=\eta_{13}\circ E(2\rho_{13})=2\eta_{13}\circ\rho_{14}=0$. The kernel of $E^6:\pi_{23}^7\to\pi_{29}^{13}$ is generated by $\sigma'\mu_{14}$, $E\zeta'$ and $\eta_7\bar{\varepsilon}_8$, since Theorem 12.6 and Theorem 12.10 of [7]. It follows that $\eta_6^3\circ\rho'=\eta_6^2(\eta_8\circ\rho')$ is a linear combination of the following three elements:

$$\begin{aligned} \eta_6^2E\sigma'\mu_{15} &= \eta_6(4\bar{\nu}_7)\mu_{15} = 0 && \text{by (7.4) of [7]}, \\ \eta_6^2E^2\zeta' &= \eta_6^2E\sigma'\eta_{15}\varepsilon_{17} = 0 && \text{by (12.4) of [7]}, \\ \eta_6^2\eta_8\bar{\varepsilon}_9 &= 4\nu\bar{\varepsilon}_9 = 0 && \text{by (5.7) of [7]}. \end{aligned}$$

Therefore, we have obtained

$$\eta_6^3\circ\rho' = 0 .$$

We have $E^3(E\nu'\circ\rho'')=E^4\nu'\circ E(2\rho')=E^4(2\nu')\circ E\rho'=E(\eta_6^3\circ\rho')=0$ by (5.5) and the page 107 of [7]. It follows that

$$\begin{aligned} 4x(\zeta_7\sigma_{18}) &= E^3(x\eta_4^2\mu_6\sigma_{15}) && \text{by (7.14) of [7]} \\ &= E^3(E\nu'\circ\rho'') = 0 . \end{aligned}$$

This implies $x=0$ and

$$E\nu'\circ\rho''=0,$$

since the element $\zeta_7\sigma_{18}$ is of order 8 by Theorem 12.8 of [7].

Consequently the proof of Lemma 3.3 and hence Proposition 3.2 is established.

§ 4. The homotopy groups $\pi_i(SU(3))$ for $i\leqq 23$

In this section, we shall prove the following

Theorem 4.1. *The homotopy groups $\pi_i(SU(3))$ for $i\leqq 23$ and generators of their 2-primary components are listed in the following table:*

$i=$	1, 2	3	4	5	6	7	8	9
$\pi_i(SU(3))\cong$	0	Z	0	Z	Z_2+Z_3	0	Z_4+Z_3	Z_3
gen. of 2-comp.		$i_*\iota_3$		$[2\iota_5]$	$i_*\nu'$		$[2\iota_5]\circ\nu_5$	

$i=$	10	11	12	13	14
$\pi_i(SU(3))\cong$	Z_2+Z_{15}	Z_4	Z_4+Z_{15}	Z_2+Z_3	$Z_4+Z_2+Z_{21}$
gen. of 2-comp.	$[\nu_5\eta_8^2]$	$[\nu_5^2]$	$[\sigma''']$	$i_*\varepsilon'$	$[\nu_5^2]\circ\nu_{11}, i_*\mu'$

$i=$	15	16	17
$\pi_i(SU(3))\cong$	Z_4+Z_9	$Z_4+Z_2+Z_{63}+Z_3$	$Z_2+Z_2+Z_{15}$
gen. of 2-comp.	$[2\iota_5]\circ\nu_5\sigma_8$	$[2\iota_5]\circ\zeta_5, [\nu_5\bar{\nu}_8]$	$[\nu_5^2]\circ\nu_{11}^2, [\nu_5\eta_8\varepsilon_9]$

$i=$	18	19	20
$\pi_i(SU(3))\cong$	$Z_2+Z_2+Z_{15}+Z_3$	$Z_4+Z_2+Z_3+Z_3$	$Z_4+Z_2+Z_{15}+Z_3$
gen. of 2-comp.	$i_*\bar{\varepsilon}_3, [\nu_5\eta_8\mu_9]$	$[\sigma''']\circ\sigma_{12}, [\nu_5\bar{\nu}_8]\circ\nu_{16}$	$[\rho^{IV}], i_*\bar{\varepsilon}'$

$i=$	21	22	23
$\pi_i(SU(3))\cong$	Z_2+Z_3	$Z_2+Z_2+Z_{33}$	$Z_4+Z_2+Z_3$
gen. of 2-comp.	$i_*\mu'\sigma_{14}$	$i_*\bar{\mu}', [2\iota_5]\circ\nu_5\kappa_8$	$[2\iota_5]\circ\zeta_5\sigma_{16}, [\nu_5\bar{\varepsilon}_8]$

Here, we denote by $[\alpha]$ an element of $\pi_i(SU(3))$ such that $p_*[\alpha]=\alpha\in\pi_i(S^5)$ and $[\alpha]\in\pi_i(SU(3):2)$ for $i>5$. The following relations hold:

(4. 1) $$2[\nu_5^2] = i_*\varepsilon_3\,,\ \ 2[\sigma'''] = i_*\mu_3\,,\ \ 2([\nu_5^2]\circ\nu_{11}) = i_*\varepsilon_3\nu_{11}\,,$$
$$2([\sigma''']\circ\sigma_{12}) = i_*\mu_3\sigma_{12}\,,\ \ 2[\rho^{IV}] \equiv i_*\bar{\mu}_3 \quad \mod i_*\bar{\varepsilon}'\,.$$

Since $\chi(SU(3))=\Delta\iota_5=\eta_3$ is an element of order 2, we have, by Lemma 2. 3, isomorphisms

$$\pi_i(SU(3):p) \cong \pi_i(S^5\times S^3:p) \cong \pi_i(S^3:p)\oplus\pi_i(S^5:p)$$

for odd prime p and all i. Then the above results on the odd components follow immediately from the following table:

(4. 2)

$i=$	1, 2, 3, 4, 5	6	7	8	9	10	11	12	13	14
odd comp. of $\pi_i(S^3)\cong$	0	Z_3	0	0	Z_3	Z_{15}	0	0	Z_3	Z_{21}
odd comp. of $\pi_i(S^5)\cong$	0	0	0	Z_3	0	0	0	Z_{15}	0	0

$i=$	15	16	17	18	19	20	21	22	23
odd comp. of $\pi_i(S^3)\cong$	0	Z_3	Z_{15}	Z_{15}	Z_3	Z_3	Z_3	Z_{33}	0
odd comp. of $\pi_i(S^5)\cong$	Z_9	Z_{63}	0	Z_3	Z_3	Z_{15}	0	0	Z_3

.

The table is given by Chapter XIII of [7] and [3].

Consider the exact sequence (2. 1) for the bundle $(SU(3), p, S^5 = SU(3)/S^3)$. Then it induces an exact sequence

(4. 3)
$$0\to \text{Coker}(\Delta:\pi_{i+1}^5\to\pi_i^3)\xrightarrow{i_*}\pi_i(SU(3):2)\xrightarrow{p_*}\text{Ker}(\Delta:\pi_i^5\to\pi_{i+1}^3)\to 0\,,$$

for $i>5$. We see also the exactness of (4. 3) holds for $i\leqq 5$ if we replace $\pi_i(SU(3):2)$ by $\pi_i(SU(3))$. Then we easily have the results in Theorem 4. 1 for $i\leqq 3$.

By concerning the table (3. 1) and Proposition 3. 2, the following lemma is directly verified.

Lemma 4. 2. **i).** *The homomorphisms* $\Delta:\pi_{i+1}^5\to\pi_i^3$ *are epimorphisms for* $i=4, 5, 7, 8, 9, 10, 15, 16, 17, 23$. *For the other values of* i, $3<i<24$, *we have the following table of the cokernel of* Δ:

$i=$	6	11	12	13	14	18	19	20	21	22
Coker $\Delta\cong$	Z_2	Z_2	Z_2	Z_2	Z_2+Z_2	Z_2	Z_2	Z_2+Z_2	Z_2	Z_2
repr. of gene.	ν'	ε_3	μ_3	ε'	μ', $\varepsilon_3\nu_{11}$	$\bar{\varepsilon}_3$	$\mu_3\sigma_{12}$	$\bar{\varepsilon}'$, $\bar{\mu}_3$	$\mu'\sigma_{14}$	$\bar{\mu}'$

.

ii). *The homomorphisms* $\Delta : \pi_i^5 \to \pi_{i-1}^3$ *are monomorphisms for* $i=4, 6, 7, 9, 13, 21$. *For the other values of* i, $3<i<24$, *we have the following table of the kernel of* Δ:

$i=$	5	8	10	11	12	14	15	16	17
Ker. $\Delta\cong$	Z	Z_4	Z_2	Z_2	Z_2	Z_2	Z_4	Z_4+Z_2	Z_2+Z_2
generators	$2\iota_5$	$2\nu_5$	$\nu_5\eta_8^2$	ν_5^2	σ'''	ν_5^3	$2\nu_5\sigma_8$	$2\zeta_5, \nu_5\bar{\nu}_8$	$\nu_5^4, \nu_5\eta_8\varepsilon_9$

$i=$	18	19	20	22	23
Ker. $\Delta\cong$	Z_2	Z_2+Z_2	Z_2	Z_2	Z_4+Z_2
generators	$\nu_5\eta_8\mu_9$	$\nu_5\zeta_8, \nu_5\bar{\nu}_8\nu_{16}$	ρ^{IV}	$2\nu_5\kappa_8$	$2\zeta_5\sigma_{16}, \nu_5\bar{\varepsilon}_8$

Now we compute $\pi_i(SU(3):2)$ by dividing into three cases of i.

Case 1: $i=4, 6, 7, 9, 13, 21$. For these values of i, it follows from the exactness of (4.3) and ii) of Lemma 4.2 that $\pi_i(SU(3):2)$ is isomorphic to the cokernel of $\Delta : \pi_{i+1}^5 \to \pi_i^3$ under the injection homomorphism i_*. Thus Theorem 4.1 is established for these values of i, by i) of Lemma 4.2.

Case 2: $i=5, 8, 10, 15, 16, 17, 23$. For these values of i, it follows from the exactness of (4.3) and i) of Lemma 4.2 that $\pi_i(SU(3):2)$ ($\pi_5(SU(3))$ if $i=5$) is isomorphic to the the kernel of $\Delta : \pi_i^5 \to \pi_{i-1}^3$ under the projection homomorphism p_*. Thus Theorem 4.1 is established for these values of i, by ii) of Lemma 4.2, the naturality (2.1)′ and by the following relations:

$$(4.4) \qquad 2\iota_5\circ\nu_5 = 2\nu_5, \quad 2\iota_5\circ\nu_5\sigma_8 = 2\nu_5\sigma_8, \quad 2\iota_5\circ\zeta_5 = 2\zeta_5$$
$$\text{and} \quad 2\iota_5\circ\zeta_5\sigma_{16} = 2\zeta_5\sigma_{16}.$$

In general, $E(2\iota_5\circ\alpha)=2E\alpha=E(\alpha)$ for $\alpha\in\pi_i(S^5)$. For $i=8$ and $i=15$, the homomorphisms $E : \pi_i(S^5)\to\pi_{i+1}(S^6)$ are monomorphisms [7]. It follows the first two relations of (4.4). For $i=16$, the kernel of E is generated by $\nu_5\bar{\nu}_8+\nu_5\varepsilon_8$ (by (7.7) of [7]). Thus $2\iota_5\circ\zeta_5=2\zeta_5+x(\nu_5\bar{\nu}_8+\nu_5\varepsilon_8)$ for $x=0$ or 1. By the exactness of (2.1) and by i) of Proposition 3.2,

$$0 = \Delta(p_*([2\iota_5]\circ\zeta_5)) = \Delta(2\iota_5\circ\zeta_5)$$
$$= \Delta(2\zeta_5+x(\nu_5\bar{\nu}_6+\nu_5\varepsilon_8)) = x\nu'\eta_6\varepsilon_7\,.$$

It follows that $x=0$ and hence $2\zeta_5=2\iota_5\circ\zeta_5$. We have also $2\iota_5\circ\zeta_5\sigma_{16} =(2\zeta_5)\sigma_{16}=2\zeta_5\sigma_{16}$.

Case 3: $i=11, 12, 14, 18, 19, 20, 22$. In this case, we have to determine the extension (4.3).

First consider the case $i=11$. By Lemma 4.2, we have an exact sequence:

$$0 \to Z_2 \xrightarrow{i_*} \pi_{11}(SU(3):2) \xrightarrow{p_*} Z_2 \to 0 .$$

The first Z_2 is generated by ε_3 and the second by ν_5^2. By (6.1) of [7],

$$E\{\eta_4, \nu_5^2, 2\iota_{11}\} \subset \{\eta_5, \nu_6^2, 2\iota_{12}\}_1 = \varepsilon_5+\eta_5\circ E\pi_{12}^5+2\pi_{13}(S^5).$$

$2\pi_{13}(S^5)=0$ since $\pi_{13}(S^5)\cong Z_2$. $\eta_5\circ E\pi_{12}^5$ is generated by $E(\eta_4\circ\sigma''')$. $\eta_4\circ\sigma'''=0$ by (7.4) of [7]. Thus $\{\eta_5, \nu_6^2, 2\iota_{12}\}_1$ consists of a single element $\varepsilon_5=E\varepsilon_4$. Then $\varepsilon_4\in\{\eta_4, \nu_5^2, 2\iota_{11}\}$ since $E:\pi_{22}(S^4)\to\pi_{13}(S^5)$ is a monomorphism. By Theorem 2.6, there exists an element $\alpha\in\pi_{11}(SU(3))$ such that $p_*(\alpha)=\nu_5^2$ and $i_*\varepsilon_3=i_*E^*\varepsilon_4=\alpha\circ 2\iota_{11}=2\alpha$. $\alpha=[\nu_5^2]$ or $\alpha=[\nu_5^2]+i_*\varepsilon_3=3[\nu_5^2]$. Therefore we have proved

$$\pi_{11}(SU(3):2) = \{[\nu_5^2]\} \cong Z_4 \quad\text{and}\quad 2[\nu_5^2] = i_*\varepsilon_3 .$$

Consider the case $i=12$. Similarly as above, it is sufficient to prove

$$i_*\mu_3 = i_*E^*\{\eta_4, \sigma''', 2\iota_{12}\} .$$

We have

$$\begin{aligned} E^2\{\eta_4, \sigma''', 2\iota_{12}\} &\subset \{\eta_6, E^2\sigma''', 2\iota_{14}\}_1 \\ &= \{\eta_6, 2E\sigma'', 2\iota_{14}\}_1 \quad \text{by Lemma 5.15 of [7]} \\ &= \{\eta_6, 2\iota_6\circ E\sigma'', 2\iota_{14}\}_1 . \end{aligned}$$

By use of relation $\eta_5\bar{\nu}_6=\nu_5^3$ of [7 ; (7.3)] and Lemma 6.5 of [7], we have

$$\begin{aligned} \mu_6\in\{\eta_6, 2\iota_7, E^2\sigma'''\}_1 &= \{\eta_6, 2\iota_7, E\sigma''\circ 2\iota_{14}\}_1 \\ &\subset \{\eta_6, 2\iota_7\circ E\sigma'', 2\iota_{14}\}_1 . \end{aligned}$$

The secondary composition $\{\eta_6, 2\iota_7\circ E\sigma'', 2\iota_{14}\}_1$ is a coset of $\eta_6\circ E\pi_{14}^6+2\pi_{15}(S^6)=\eta_6\circ E\pi_{14}^6$ which is generated by $\eta_6\bar{\nu}_7=\nu_6^3=E^2\nu_4^3$ and $\eta_6\varepsilon_7=E^2(\eta_4\varepsilon_5)$. Since $E^2:\pi_{13}(S^4)\to\pi_{15}(S^6)$ is a monomorphism, we have $E^*\{\eta_4, \sigma''', 2\iota_{12}\}=\mu_3+\{\eta_3\varepsilon_4\}$ by (2.3) and Lemma 2.4. $i_*(\eta_3\varepsilon_4)=i^*(\Delta\varepsilon_3)=0$. Thus $i_*E^*\{\eta_4, \sigma''', 2\iota_{12}\}=i_*\mu_3$, and we have proved

$\pi_{12}(SU(3):2)=\{[\sigma''']\}\cong Z_4$ and $2[\sigma''']=i_*\mu_3$.

For the case $i=14$, we have an exact sequence:

$$0\to Z_2\oplus Z_2 \longrightarrow \pi_{14}(SU(3):2)\xrightarrow{p_*} Z_2\to 0\,,$$

where $Z_2\oplus Z_2$ is generated by $i_*\mu'$ and $i_*(\varepsilon_3\nu_{11})$ and Z_2 by ν_5^3. The first relation of (4.1) implies the third one: $2([\nu_5^2]\circ\nu_{11})=(2[\nu_5^2])\circ\nu_{11}=i_*(\varepsilon_3)\circ\nu_{11}=i_*(\varepsilon_3\nu_{11})$. Then we have the result $\pi_{14}(SU(3):2)=\{[\nu_5^2]\circ\nu_{11},\ i_*\mu'\}\cong Z_4\oplus Z_2$.

Consider the case $i=18$. We have to prove that the sequence

$$0\to \{i_*\bar{\varepsilon}_3\} \longrightarrow \pi_{18}(SU(3):2)\xrightarrow{p_*}\{\nu_5\eta_8\mu_9\}\to 0$$

splits. By Lemma 6.5 and of [7],

$$\begin{aligned}\nu_5\eta_8\mu_9 &\in \nu_5\eta_8\circ\{\eta_9,\ 2\iota_{10},\ 8\sigma_{10}\}_5\\ &\subset \{\nu_5\eta_8^2,\ 2\iota_{10},\ 8\sigma_{10}\}_5\,.\end{aligned}$$

The last secondary composition is a coset of $\nu_5\eta_8^2\circ E^5\pi_{13}^5+\pi_{11}^5\circ 8\sigma_{11}=\{\nu_5\eta_8^2\varepsilon_{10},\ 8(\nu_5^2\sigma_{11})\}=\{4\nu_5^2\sigma_{11}\}=0$. Thus $\{\nu_5\eta_8^2,\ 2\iota_{10},\ 8\sigma_{10}\}_5$ consists of $\nu_5\eta_8\mu_9$, Since $2[\nu_5\eta_8^2]=0$, the secondary composition $\{[\nu_5\eta_8^2],\ 2\iota_{10},\ 8\sigma_{10}\}_5$ is defined and

$$p_*\{[\nu_5\eta_8^2],\ 2\iota_{10},\ 8\sigma_{10}\}_5\subset\{p_*[\nu_5\eta_8^2],\ 2\iota_{10},\ 8\sigma_{10}\}_5=\nu_5\eta_8\mu_9\,.$$

Then we may choose $[\nu_5\eta_8\mu_9]$ as an element of $\{[\nu_5\eta_8^2],\ 2\iota_{10},\ 8\sigma_{10}\}_5$. We have

$$\begin{aligned}2[\nu_5\eta_8\mu_9]&\in\{[\nu_5\eta_8^2],\ 2\iota_{10},\ 8\sigma_{10}\}_5\circ 2\iota_{18}\\ &\subset[\nu_5\eta_8^2]\circ E\{2\iota_9,\ 8\sigma_9,\ 2\iota_{16}\}\\ &\subset[\nu_5\eta_8^2]\circ\{2\iota_{10},\ 8\sigma_{10},\ 2\iota_{17}\}_1\end{aligned}$$

and $0=8\sigma_{10}\circ\eta_{17}\in\{2\iota_{10},\ 8\sigma_{10},\ 2\iota_{17}\}_1$ by Corollary 3.7 of [7]. It follows that $2[\nu_5\eta_8\mu_9]\equiv 0 \mod [\nu_5\eta_8^2]\circ\pi_{18}(S^{10})\circ 2\iota_{18}$.

Since $\pi_{18}(S^{10})\circ 2\iota_{18}=2\pi_{18}(S^{10})=0$, it follows that $2[\nu_5\eta_8\mu_9]=0$ and therefore the above sequence splits.

For the case $i=19$, we have an exact sequence:

$$0\to Z_2\longrightarrow \pi_{19}(SU(3):2)\xrightarrow{p_*} Z_2\oplus Z_2\to 0\,,$$

where Z_2 is generated by $i_*(\mu_3\sigma_{12})$ and $Z_2\oplus Z_2$ is generated by $\nu_5\zeta_8$ and $\nu_5\bar{\nu}_8\nu_{16}$. The relation $\nu_6\zeta_9=2\sigma''\sigma_{13}=E(\sigma'''\sigma_{12})$ in (10.7) of [7] implies that $\nu_5\zeta_8\equiv\sigma'''\sigma_{12} \mod \nu_5\bar{\nu}_8\nu_{16}$, since the kernel of $E:\pi_{19}^5$

$\rightarrow \pi_{20}^6$ is generated by $\nu_5\bar{\nu}_8\nu_{16}$. So, we may replace $\nu_5\zeta_8$ by $\sigma'''\sigma_{12}$. Then it is sufficient to prove the relations:

$$p_*([\sigma''']\sigma_{12}) = \sigma'''\sigma_{12}, \quad 2([\sigma''']\sigma_{12}) = i_*(\mu_3\sigma_{12}),$$
$$p_*([\nu_5\bar{\nu}_8]\nu_{16}) = \nu_5\bar{\nu}_8\nu_{16} \quad \text{and} \quad 2([\nu_5\bar{\nu}_8]\circ\nu_{16}) = 0.$$

But these relations follow from (2.1)$'$, $2[\sigma''']=i_*\mu_3$ and $2[\nu_5\bar{\nu}_8]=0$.

Consider the case $i=20$. By Lemma 4.2, we have an exact sequence:

$$0 \rightarrow Z_2 \oplus Z_2 \rightarrow \pi_{20}(SU(3):2) \rightarrow Z_2 \rightarrow 0.$$

For the results $\pi_{20}(SU(3):2)=\{[\rho^{\mathrm{IV}}], i_*\bar{\varepsilon}'\}$, it is sufficient to prove the last relation $2[\rho^{\mathrm{IV}}] \equiv i_*\bar{\mu}_3 \bmod i_*\bar{\varepsilon}'$ of (4.1).

By the definition of $\bar{\mu}_3$,

$$\bar{\mu} = E^\infty\bar{\mu}_3 \in E^\infty\{\mu_3, 2\iota_{12}, 8\sigma_{12}\} \subset \langle \mu, 2\iota, 8\sigma \rangle.$$

By (3.9) of [7],

$$\langle \mu, 2\iota, 8\sigma \rangle + \langle 8\sigma, \mu, 2\iota \rangle + \langle 2\iota, 8\sigma, \mu \rangle$$
$$\equiv \langle \mu, 2\iota, 8\sigma \rangle + 4\sigma\circ\langle 2\iota, \mu, 2\iota \rangle + \langle \mu, 8\sigma, 2\iota \rangle \equiv 0$$

mod $\mu\circ G_8 + 8\sigma\circ G_{10} + 2G_{17}$. $2G_{17}=0$ since $G_{17} \approx Z_2\oplus Z_2\oplus Z_2\oplus Z_2$. $8\sigma\circ G_{10} + 4\sigma\circ\langle 2\iota, \mu, 2\iota \rangle \subset 2(G_{10}:2)=0$ since $G_{10} \approx Z_6$. $\mu\circ G_8$ is generated by $\mu\varepsilon = \eta^2\rho = \eta\mu\sigma$, by Theorem 14.1 of [7]. Thus we have

$$\bar{\mu} \equiv \langle \mu, 2\iota, 8\sigma \rangle = \langle \mu, 8\sigma, 2\iota \rangle \quad \bmod \eta\mu\sigma.$$

Similarly, we have

$$\mu \in \langle \eta, 2\iota, 8\sigma \rangle = \langle \eta, 8\sigma, 2\iota \rangle.$$

By (3.7) of [7],

$$\bar{\mu} \in \langle \eta, \langle 8\sigma, 2\iota, 8\sigma \rangle, 2\iota \rangle + \langle \eta, 8\sigma, \langle 2\iota, 8\sigma, 2\iota \rangle \rangle.$$

By (3.10) of [7],

$$\langle 2\iota, 8\sigma, 2\iota \rangle \subset 8\sigma\circ G_1 + 2G_8 = 0.$$

By the definition of ρ^{IV},

$$E^\infty\rho^{\mathrm{IV}} \in E^\infty\{\sigma''', 2\iota_{12}, 8\sigma_{12}\} \subset \langle 8\sigma, 2\iota, 8\sigma \rangle.$$

Thus we have that $\langle \eta, E^\infty\rho^{\mathrm{IV}}, 2\iota \rangle$ consists of $\bar{\mu}$ and $\bar{\mu}+\eta\mu\sigma$. It follows from the relation $E^\infty\{\eta_4, \rho^{\mathrm{IV}}, 2\iota_{12}\} \subset -\langle \eta, E^\infty\rho^{\mathrm{IV}}, 2\iota \rangle$

$$\pi_{21}^4 \cap \{\eta_4, \rho^{\mathrm{IV}}, 2\iota_{12}\} \equiv \bar{\mu}_4 \quad \bmod \{\eta_4\mu_5\sigma_{14}\} + \mathrm{Ker}(E^\infty : \pi_{21}^4 \rightarrow G_{17}).$$

In Theorem 12.7 and Theorem 12.17 of [7] we see that $\mathrm{Ker}(E^{\infty}: \pi_{21}^{4} \to G_{17})$ is generated by $\nu_4\sigma'\sigma_{14}$ and $E\bar{\varepsilon}'$. It follows that

$$\pi_{20}^{3} \cap E^{*}\{\eta_4, \rho^{\mathrm{IV}}, 2\iota_{12}\} \equiv \bar{\mu}_3 \mod \{\eta_3\mu_4\sigma_{13}, \bar{\varepsilon}'\},$$

and $$E^{*}\{\eta_4, \rho^{\mathrm{IV}}, 2\iota_{12}\} \equiv \bar{\mu}_3 \mod \{\eta_3\mu_4\sigma_{13}, \bar{\varepsilon}'\} + 2\pi_{20}(S^3).$$

Apply Theorem 2.6, then we have that there exists an element ρ^{IV} such that

$$2[\rho^{\mathrm{IV}}] \equiv i_*\bar{\mu}_3 \mod i_*\bar{\varepsilon}',$$

where $i_*\eta_3\mu_4\sigma_{13} = 0$.

For the case $i = 22$, we have an exact sequence:

$$0 \to Z_2 \to \pi_{22}(SU(4):2) \to Z_2 \to 0.$$

By Lemma 2.2,

$$2([2\iota_5]\circ\nu_5\kappa_8) = [2\iota_5]\circ 2(\nu_5\kappa_8) = [2\iota_5]\circ 0 = 0,$$

since $E(2\nu_5\kappa_8) = 2\nu_6\kappa_9 = 0$ by Theorem 12.7 of [7]. Thus the above sequence splits.

§5. The homotopy groups $\pi_i(Sp(2))$ for $i \leq 23$

In this section we compute the groups $\pi_i(Sp(2))$ and the results are stated in the following

Theorem 5.1. *The homotopy groups $\pi_i(Sp(2))$ for $i \leq 23$ and generators of their 2-primary components are listed in the following table:*

$i=$	1, 2	3	4	5	6	7	8, 9	10	11
$\pi_i(Sp(2)) \cong$	0	Z	Z_2	Z_2	0	Z	0	$Z_8 + Z_{15}$	Z_2
gen. of 2-comp.		$i_*\iota_3$	$i_*\eta_3$	$i_*\eta_3^2$		$[12\iota_7]$		$[\nu_7]$	$i_*\varepsilon_3$

$i=$	12	13	14	15	16
$\pi_i(Sp(2)) \cong$	$Z_2 + Z_2$	$Z_4 + Z_2$	$Z_{16} + Z_{105}$	Z_2	$Z_2 + Z_2$
gen. of 2-comp.	$i_*\mu_3, i_*\eta_3\varepsilon_4$	$[\nu_7]\circ\nu_{10}, i_*\eta_3\mu_4$	$[2\sigma']$	$[\sigma'\eta_{14}]$	$[\sigma'\eta_{14}]\circ\eta_{15}, [\nu_7]\circ\nu_{10}^2$

$i=$	17	18	19
$\pi_i(Sp(2)) \cong$	$Z_8 + Z_5$	$Z_8 + Z_2 + Z_{315}$	$Z_2 + Z_2$
gen. of 2-comp.	$[\nu_7]\circ\sigma_{10}$	$[\zeta_7], i_*\bar{\varepsilon}_3$	$i_*\mu_3\sigma_{12}, i_*\eta_3\bar{\varepsilon}_4$

$i=$	20	21
$\pi_i(Sp(2))\cong$	$Z_2+Z_2+Z_2$	$Z_{32}+Z_2$
gen. of 2-*comp.*	$[\nu_7]\circ\sigma_{10}\nu_{17},\ i_*\eta_3\mu_4\sigma_{13}$	$[\sigma'\sigma_{14}],\ i_*\eta_3\bar{\mu}_4$

$i=$	22	23
$\pi_i(Sp(2))\cong$	$Z_{32}+Z_2+Z_2+Z_{165}$	$Z_2+Z_2+Z_2$
gen. of 2-*comp.*	$[\rho''],\ [\sigma'\bar{\nu}_{14}],\ [\sigma'\varepsilon_{14}]$	$[\sigma'\mu_{14}],\ [E\zeta'],\ [\eta_7\bar{\varepsilon}_8]$

We denote by $[\alpha]$ *an element of* $\pi_i(Sp(2))$ *such that* $p_*[\alpha]=\alpha\in\pi_i(S^7)$ *and, for* $i\neq 7$, $[\alpha]\in\pi_i(Sp(2):2)$.

The following relations hold:

(5.1) $\quad 2[\nu_7]\circ\nu_{10}=i_*\varepsilon',\ 4[2\sigma']=\pm i_*\mu',\ 8[\sigma'\sigma_{14}]=\pm i_*\mu'\sigma_{14}$

and $8[\rho'']=\pm i_*\bar{\mu}'$.

Since $\chi(Sp(2))=\Delta\iota_7$ is an element of order 12, we have from Lemma 2.3 isomorphisms

$$\pi_i(Sp(2):p)\cong\pi_i(S^7\times S^3:p)\cong\pi_i(S^7:p)\oplus\pi_i(S^3:p)$$

for odd prime $p\geq 5$ and all i.

For 3-primary components, we quote from [8] the following isomorphisms:

$$\pi_i(Sp(2):3)\cong\pi_i(B(3):3)\qquad\text{for all } i,$$

Then the results in Theorem 5.1 on the odd components follow immediately from the following table:

(5.2)

$i=$	1, 2, 3, 4, 5, 6, 7, 8, 9	10	11, 12, 13	14	15, 16	17	18	19, 20, 21	22	23
Σ *p-comp. of* $\pi_i(S^3)$, $p\geq 5$	0	Z_5	0	Z_7	0	Z_5	Z_5	0	Z_{11}	0
Σ *p-comp. of* $\pi_i(S^7)$, $p\geq 5$	0	0	0	Z_5	0	0	Z_7	0	Z_5	0
3-*comp. of* $\pi_i(B(3))$	0	Z_3	0	Z_3	0	0	Z_9	0	Z_3	0

The table is given by Chapter XIII of [7], [3] and Theorem 3 of [8].

The exact sequence (2.1) associated with the bundle $(Sp(2), p, S^7=Sp(2)/S^3)$ induces the following exact sequence:

(5.3) $$0 \to \mathrm{Coker}\,(\Delta : \pi_{i+1}^7 \to \pi_i^3) \xrightarrow{i_*} \pi_i(Sp(2):2) \to \mathrm{Ker}\,(\Delta : \pi_i^7 \to \pi_{i-1}^3) \to 0, \quad \textit{for } i>7.$$

By concerning the table (3.1) and ii) of Proposition 3.2, we have

Lemma 5.2. i). *The homomorphisms $\Delta : \pi_{i+1}^7 \to \pi_i^3$ are epimorphisms for $i=7, 8, 9, 10, 15, 16, 17$ and 23. For the other values of i, $6<i<23$, we have the following table*:

$i=$	11	12	13	14	18	19
Coker. Δ	Z_2	Z_2+Z_2	Z_2+Z_2	Z_4	Z_2	Z_2+Z_2
rep. of gene.	ε_3	$\mu_3, \eta_3\varepsilon_4$	$\varepsilon', \eta_3\mu_4$	μ'	$\bar{\varepsilon}_3$	$\mu_3\sigma_{12}, \eta_3\bar{\varepsilon}_4$

$i=$	20	21	22
Coker. Δ	Z_2+Z_2	Z_4+Z_2	Z_4
rep. of gene.	$\bar{\mu}_3, \eta_3\mu_4\sigma_{13}$	$\mu'\sigma_{14}, \eta_3\bar{\mu}_4$	$\bar{\mu}'$

ii). *The homomorphisms $\Delta : \pi_i^7 \to \pi_{i-1}^3$ are monomorphisms for $i=8, 9, 11, 12$ and 19. For the other values of i, $7<i\leq 23$, we have the following table*:

$i=$	10	13	14	15	16	17	18
Ker. Δ	Z_2	Z_2	Z_4	Z_2	Z_2+Z_2	Z_8	Z_8
generators	ν_7	ν_7^2	$2\sigma'$	$\sigma'\eta_{14}$	$\sigma'\eta_{14}^2, \nu_7^3$	$\nu_7\sigma_{10}$	ζ_7

$i=$	20	21	22	23
Ker. Δ	Z_2	Z_8	$Z_8+Z_2+Z_2$	$Z_2+Z_2+Z_2$
generators	$\nu_7\sigma_{10}\nu_{17}$	$\sigma'\sigma_{14}$	$\rho'', \sigma'\bar{\nu}_{14}, \sigma'\varepsilon_{14}$	$\sigma'\mu_4, E\zeta', \eta_7\bar{\varepsilon}_8$

We consider $\pi_i(Sp(2):2)$ by dividing into three cases.

Case 1: $i=8, 9, 11, 12$ and 19. For these values of i, it follows from the exactness of (5.3) and ii) of Lemma 5.2 that $\pi_i(Sp(2):2)$ is isomorphic to the cokernel of $\Delta: \pi^7_{i+1} \to \pi^3_i$ under the injection homomorphisms i_*. Then Theorem 5.1 is obtained by i) of Lemma 5.2.

Case 2: $i=10$, 15, 16, 17 and 23. For these values of i, it follows from the exactness of (5.3) and i) of Lemma 5.2 that $\pi_i(Sp(2):2)$ is isomorphic to the kernel of $\Delta: \pi^7_i \to \pi^3_{i-1}$ under the projection homomorphisms p_*. So, Theorem 5.1 is established for these values of i, by ii) of Lemma 5.2 and (2.1)′.

Case 3: $i=13$, 14, 18, 20, 21 and 22. We have to determine the extension (5.3). We remark that, by Lemma 2.3, we may consider that the sequence (5.3) is induced from the homotopy exact sequence associated with an S^3-bundle over S^7 having the characteristic class $\Delta\iota_7=\nu'$.

First consider the case $i=13$. By Lemma 5.2, we have an exact sequence

$$0 \to Z_2 \oplus Z_2 \to \pi_{13}(Sp(2):2) \to Z_2 \to 0 .$$

For the result $\pi_{13}(Sp(2):2)=\{[\nu_7]\circ\nu_{10}, i_*\eta_3\mu_4\}$, it is sufficient to prove the first relation of (5.1): $2[\nu_7]\circ\nu_{10}=i_*\mathcal{E}'$. $\mathcal{E}' \in \{\nu', 2\nu_6, \nu_9\}$ by the definition of $\mathcal{E}'$. Then it follows from Theorem 2.1

$$2[\nu_7]\circ\nu_{10} = [2\nu_7]\circ\nu_{10} = i_*\mathcal{E}' .$$

For the case $i=14$. We have an exact sequence

$$0 \to Z_4 \longrightarrow \pi_{14}(Sp(2):2) \xrightarrow{p_*} Z_4 \to 0 ,$$

where the first Z_4 is generated by $i_*\mu'$ and the second by $2\sigma'$. We have $p_*(2[2\sigma'])=4\sigma'=12\sigma'=p^*([12\iota_7]\circ\sigma')$. It follows

$$2[2\sigma'] \equiv [12\iota_7]\circ\sigma' \bmod i_*\mu' \quad\text{and}\quad 4[2\sigma'] \equiv [12\iota_7]\circ E\sigma'' \bmod 2i_*\mu' .$$

By the definition of ζ_5 and by (7.14) of [7],

$$\pm E^2\mu' = 2\zeta_5 \in 2\{\nu_5, 8\iota_8, E\sigma'\}_1 .$$

We have also

$$E^2\{\nu', 4\iota_6, \sigma''\} \subset \{2\nu_5, 4\iota_8, 2E\sigma'\}_1 \subset 2\{\nu_5, 8\iota_8, E\sigma'\}_1 .$$

It follows $\pm E^2\mu'\equiv E^2\{\nu', 4\iota_6, \sigma''\}$ mod $\{\nu_5\varepsilon_8, \nu_5\bar{\nu}_8\}$. Since $\{\nu_5\varepsilon_8, \nu_5\bar{\nu}_8\}$ is complementary to the image of $E^2: \pi_{14}(S^3)\to\pi_{16}(S^5)$ and since the kernel of E^2 is generated by $\varepsilon_3\nu_{11}$ and $\nu'\varepsilon_6$, we have

$$\pm\mu'\equiv\{\nu', 4\iota_6, \sigma''\} \quad \text{mod } \{\varepsilon_3\nu_{11}, \nu'\varepsilon_6\}\,.$$

Applying Theorem 2.1, we have

$$4[2\sigma']\equiv[12\iota_7]\circ E\sigma''\equiv i_*\mu' \quad \text{mod } 2i_*\mu'\,.$$

This proves $\pi_{14}(Sp(2):2)=\{[2\sigma']\}\cong Z_{16}$ and $4[2\sigma']=\pm i_*\mu'$.

For the case $i=18$, we have an exact sequence:

$$0\to Z_2\to\pi_{18}(Sp(2):2)\to Z_8\to 0\,.$$

where Z_2 is generated by $i_*\bar{\varepsilon}_3$ and Z_8 by ζ_7. By Theorem 2.1, we have

$$8[\zeta_7]=[\zeta_7]\circ 8\iota_{18}\in i_*\{\nu', \zeta_6, 8\iota_{17}\}_1$$

We have, by (7.4) and (5.5) of [7],

$$\begin{aligned}\{\eta_5, \zeta_6, 8\iota_{17}\}_1 &= \{\eta_5, 4\zeta_6, 2\iota_{17}\}_1 = \{\eta_5, \eta_6^2\mu_8, 2\iota_{17}\}_1\\ &= \{\eta_5^3, \mu_8, 2\iota_{17}\}_1 = \{4\nu_5, \mu_8, 2\iota_{17}\}_1\\ &= \{2\nu_5, 2\mu_8, 2\iota_{17}\}_1 = \{2\nu_5, 0, 2\iota_{17}\}_1\\ &\ni 0\,.\end{aligned}$$

Note that the equality holds, since these secondary compositions have the same indeterminacy $2\pi_{18}(S^5)\cong Z_3$. Then it follows that $8[\zeta_7]=0$, and the above sequence splits.

Consider the case $i=20$. We have an exact sequence:

$$0\to Z_2\oplus Z_2\longrightarrow\pi_{20}(Sp(2):2)\xrightarrow{p_*}Z_2\to 0\,,$$

where $Z_2\oplus Z_2$ is generated by $i_*\bar{\mu}_3$ and $i_*\eta_3\mu_4\sigma_{13}$ and Z_2 by $\nu_7\sigma_{10}\nu_{17}$. Obviously, $p_*([\nu_7\sigma_{10}]\circ\nu_{17})=\nu_7\sigma_{10}\nu_{17}$. $E(2\sigma_{10}\nu_{17})=2\sigma_{11}\nu_{18}=0$ by (7.20) of [7]. Then it follows from Lemma 2.2

$$2([\nu_7\sigma_{10}]\circ\nu_{17})=[\nu_7]\circ(2\sigma_{10}\nu_{17})=[\nu_7]\circ 0=0\,.$$

This shows that the above sequence splits.

Consider the case $i=21$. We have an exact sequence:

$$0\to Z_4\oplus Z_2\longrightarrow\pi_{21}(Sp(2):2)\xrightarrow{p_*}Z_8\to 0\,,$$

where $Z_4 \oplus Z_2$ is generated by $i_* \mu' \sigma_{14}$ and $i_* \eta_3 \bar{\mu}_4$ and Z_8 is generated by $\sigma' \sigma_{14}$. In the proof of the case $i=14$ we have an element $[2\sigma']$ such that $p_*[2\sigma'] = 2\sigma'$ and $i_* \mu' = \pm 4[2\sigma']$. Thus

$$p_*([2\sigma'] \circ \sigma_{14}) = 2\sigma' \sigma_{14} = p_*(2[\sigma' \sigma_{14}])$$

and $\pm i_* \mu' \sigma_{14} = 4[2\sigma'] \circ \sigma_{14}$.
It follows that

$$8[\sigma' \sigma_{14}] \equiv \pm i_* \mu' \sigma_{14} \mod 4 i_* \pi_{21}^3, \quad 4 i_* \pi_{21}^3 = 0.$$

Therefore we have $8[\sigma' \sigma_{14}] = \pm i_* \mu' \sigma_{14}$, and

$$\pi_{21}(Sp(2):2) = \{[\sigma' \sigma_{14}], i_* \eta_3 \bar{\mu}_4\} \approx Z_{32} \oplus Z_2.$$

Consider the case $i=22$. We have an exact sequence, by Lemma 5.2,

$$0 \to Z_4 \longrightarrow \pi_{22}(Sp(2):2) \xrightarrow{p_*} Z_8 \oplus Z_2 \oplus Z_2 \to 0,$$

where Z_4 is generated by $i_* \bar{\mu}'$ and $Z_8 \oplus Z_2 \oplus Z_2$ by ρ'', $\sigma' \bar{\nu}_{14}$ and $\sigma' \varepsilon_{14}$.

First we prove that the relation

$$2[\sigma' \bar{\nu}_{14}] = 2[\sigma' \varepsilon_{14}] = 0$$

holds for suitable choice of $[\sigma' \bar{\nu}_{14}]$ and $[\sigma' \varepsilon_{14}]$. $p_*([\sigma' \eta_{14}] \circ \sigma_{15}) = \sigma' \eta_{14} \sigma_{15} = \sigma' \bar{\nu}_{14} + \sigma' \varepsilon_{14}$ by Lemma 6.4 of [7]. Thus we may choose $[\sigma' \bar{\nu}_{14}]$ such that if $[\sigma' \varepsilon_{14}]$ is given then

$$[\sigma' \bar{\nu}_{14}] = [\sigma' \varepsilon_{14}] + [\sigma' \eta_{14}] \circ \sigma_{15}.$$

Since $2[\sigma' \eta_{14}] = 0$, we have $2[\sigma' \bar{\nu}_{14}] = 2[\sigma' \varepsilon_{14}]$. Let α be an element of the secondary composition $\{[\sigma' \eta_{14}], 2\iota_{15}, \nu_{15}^2\}_1$. We have

$$\begin{aligned} \sigma' \varepsilon_{14} &\in \sigma' \circ \{\eta_{14}, 2\iota_{15}, \nu_{15}^2\}_1 \\ &\subset \{\sigma' \eta_{14}, 2\iota_{15}, \nu_{15}^2\}_1 \supset p^* \{[\sigma' \eta_{14}], 2\iota_{15}, \nu_{15}^2\}_1. \end{aligned}$$

$\{\sigma' \eta_{14}, 2\iota_{15}, \nu_{15}^2\}_1$ is a coset of $\sigma' \eta_{14} \circ E\pi_{21}^{14} + \pi_{16}^7 \circ \nu_{16}^2$ which is generated by $\sigma' \eta_{14} \sigma_{15} = \sigma' \bar{\nu}_{14} + \sigma' \varepsilon_{14}$, $\sigma' \eta_{14}^2 \nu_{16}^2 = 0$, $\mu_7 \nu_{16}^2 \in \pi_{19}^7 \circ \nu_{19} = 0$, $\nu_7^3 \nu_{16}^2 = \bar{\nu}_7 \zeta_{15} \nu_{16}^2 = 0$ and $\eta_7 \varepsilon_8 \nu_{16}^2 = \varepsilon_7 \eta_{15} \nu_{16}^2 = 0$. Thus

$$p_* \alpha = \sigma' \varepsilon_{14} + x(\sigma' \bar{\nu}_{14} + \sigma' \varepsilon_{14}), \qquad x = 0 \text{ or } 1.$$

Set $[\sigma' \varepsilon_{14}] = \alpha + x([\sigma' \eta_{14}] \circ \sigma_{15})$, then $p_*[\sigma' \varepsilon_{14}] = \sigma' \varepsilon_{14}$. We have

$$2[\sigma'\varepsilon_{14}] \in \{[\sigma'\eta_{14}], 2\iota_{15}, \nu_{15}^2\}_1 \circ 2\iota_{22}$$
$$= -[\sigma'\eta_{14}]\circ E\{2\iota_{14}, \nu_{14}^2, 2\iota_{20}\} \quad \text{by Proposition 1.4 of [7]}$$
$$0 = -[\sigma'\eta_{14}]\circ \eta_{14}\nu_{15}^2$$
$$\in -[\sigma'\eta_{14}]\circ E\{2\iota_{14}, \nu_{14}^2, 2\iota_{20}\} \quad \text{by Corollary 3.7 of [7]}.$$

Thus $2[\sigma'\varepsilon_{14}]\equiv 0 \mod G$, where $G=[\sigma'\eta_{14}]\circ E\pi_{21}(S^{14})\circ 2\iota_{22}=2[\sigma'\eta_{14}]\circ E\pi_{21}(S^{14})=0$. We have proved the required relation:

$$2[\sigma'\bar{\nu}_{14}] = 2[\sigma'\varepsilon_{14}] = 0\,.$$

Next, by the definition of ρ'',

$$p_*\{[2\sigma'], 16\iota_{14}, \sigma_{14}\}_1 \subset \{p_*[2\sigma'], 16\iota_{14}, \sigma_{14}\}_1$$
$$= \{2\sigma', 16\iota_{14}, \sigma_{14}\}_1$$
$$\subset 2\{\sigma', 8\iota_{14}, 2\sigma_{14}\}_1 \ni 2\rho'' = 2p_*[\rho'']\,.$$

Thus $\{[2\sigma'], 16\iota_{14}, \sigma_{14}\}_1 \equiv 2[\rho''] \mod G$, where G is generated by $i_*\bar{\mu}'$, $[2\sigma']\circ\bar{\nu}_{14}$, $[2\sigma']\circ\varepsilon_{14}$ and $[\sigma'\eta_{14}]\circ\sigma_{15}$. It is easy to see that $4G=0$. Then we have

$$8[\rho''] = 4\{[2\sigma'], 16\iota_{14}, \sigma_{14}\}_1\,.$$

By the definition of $\bar{\mu}'$,

$$i_*\bar{\mu}' \in i_*\{\mu', 4\iota_{14}, 4\sigma_{14}\}_1$$
$$\subset \{i_*\mu', 4\iota_{14}, 4\sigma_{14}\}_1$$
$$= \pm\{4[2\sigma'], 4\iota_{14}, 4\sigma_{14}\}_1$$
$$\subset \pm\{[2\sigma'], 16\iota_{14}, 4\sigma_{14}\}_1$$
$$\supset \pm\{[2\sigma'], 16\iota_{14}, \sigma_{14}\}_1\circ 4\iota_{22}$$
$$= 8[\rho'']\,.$$

Thus $8[\rho'']\equiv \pm i_*\bar{\mu}' \mod [2\sigma']\circ\pi_{22}^{14}+\pi_{15}(Sp(2))\circ 4\sigma_{15}$. $\pi_{15}(Sp(2))\circ 4\sigma_{15} =4\pi_{15}(Sp(2):2)\circ\sigma_{15}=0$. $[2\sigma']\circ\pi_{22}^{14}$ is generated by $[2\sigma']\circ\bar{\nu}_{14}$ and $[2\sigma']\circ\varepsilon_{14}$, which are in $i_*\pi_{22}^3$ and of order at most 2. Then we have

$$8[\rho''] \equiv i_*\bar{\mu}' \mod 2i_*\pi_{22}^3\,, \qquad \text{i.e., } 8[\rho''] = \pm i_*\bar{\mu}'\,.$$

By these relations and the exactness of the last sequence, we have

$$\pi_{22}(Sp(2):2) = \{[\rho''], [\sigma'\bar{\nu}_{14}], [\sigma'\varepsilon_{14}]\} \cong Z_{32}\oplus Z_2\oplus Z_2\,.$$

§ 6. The homotopy groups $\pi_i(SU(4))$ for $i \leq 23$

We shall prove the following theorem mainly by use of the known results given in the previous sections.

From the fibering $SU(4)/SU(2)=SU(4)/S^3=S^5 \times S^7$, we have the following exact sequence:

$$(6.1) \quad \cdots \to \pi_i(S^3) \xrightarrow{i_*} \pi_i(SU(4)) \xrightarrow{p_*} \pi_i(S^5) \oplus \pi_i(S^7) \to \pi_{i-1}(S_3) \to \cdots .$$

We denote by $[\alpha \oplus \beta]$ an element of $\pi_i(SU(4))$ such that $p_*[\alpha \oplus \beta]=\alpha \oplus \beta \in \pi_i(S^5) \oplus \pi_i(S^7)$ and if $i>7$ and $\alpha \oplus \beta \in \pi_i^5 \oplus \pi_i^7$ then $[\alpha \oplus \beta] \in \pi_i(SU(4):2)$.

Theorem 6.1. *The homotopy groups $\pi_i(SU(4))$ for $i \leq 23$ and generators of th3ir 2-primary components are listed in the following table.*

$i=$	1, 2	3	4	5	6	7	8	9
$\pi_i(SU(4)) \cong$	0	Z	0	Z	0	Z	Z_8+Z_3	Z_2
gen. of 2-comp.		$i_*\varepsilon_3$		$[2\iota_5]$		$[\eta_5^2 \oplus 6\iota_7]$	$[\nu_5 \oplus \eta_7]$	$[\nu_5 \oplus \eta_7] \circ \eta_8$

$i=$	10	11	12	13	14
$\pi_i(SU(4)) \cong$	$Z_8+Z_2+Z_{15}$	Z_4	Z_4+Z_{15}	Z_4	$Z_{16}+Z_2+Z_{105}$
gen. of 2-comp.	$[\nu_7]$, $[\nu_5\eta_8^2]$	$[\nu_5^2]$	$[\sigma''']$	$[\nu_7] \circ \nu_{10}$	$[\eta_5\varepsilon_6 \oplus \sigma']$, $[\nu_5^2] \circ \nu_{11}$

$i=$	15	16
$\pi_i(SU(4)) \cong$	$Z_8+Z_2+Z_9$	$Z_8+Z_2+Z_2+Z_2+Z_2+Z_{63}$
gen. of 2-comp.	$[\nu_5 \oplus \eta_7] \circ \sigma_8$, $[\sigma'\eta_{14}]$	$[\zeta_5 \oplus \mu_7]$, $[\nu_5\bar{\nu}_8]$, $[\sigma'\eta_{14}] \circ \eta_{15}$, $[\nu_7] \circ \nu_{10}^2$, $[\nu_5 \oplus \eta_7] \circ \varepsilon_8$

$i=$	17	18
$\pi_i(SU(4)) \cong$	$Z_8+Z_2+Z_2+Z_2+Z_5$	$Z_8+Z_4+Z_2+Z_{315}+Z_3$
gen. of 2-comp.	$[\nu_7] \circ \sigma_{10}$, $[\nu_5^2] \circ \nu_{11}^2$, $[\nu_5\eta_8\varepsilon_9]$, $[\nu_5 \oplus \eta_7] \circ \mu_8$	$[\zeta_7]$, $[\nu_5 \oplus \eta_7] \circ \sigma_8\nu_{15}$, $[\nu_5\eta_8\mu_9]$

$i=$	19	20	21
$\pi_i(SU(4)) \cong$	$Z_4+Z_2+Z_3$	$Z_4+Z_2+Z_{15}$	$Z_{16}+Z_2$
gen. of 2-comp.	$[\sigma'''] \circ \sigma_{12}$, $[\nu_5\bar{\nu}_8] \circ \nu_{16}$	$[\rho^{IV}]$, $[\nu_7] \circ \sigma_{10}\nu_{17}$	$[\sigma'\sigma_{14}]$, $[\eta_5\bar{\varepsilon}_6 \oplus 2\kappa_7]$

$i=$	22	23
$\pi_i(SU(4))\cong$	$Z_{16}+Z_4+Z_2+Z_2+Z_{165}$	$Z_8+Z_2+Z_2+Z_2+Z_2+Z_3$
gen. of 2-comp.	$[\rho'']$, $[\nu_5\kappa_8\oplus\bar{\varepsilon}_7]$, $[\sigma'\bar{\nu}_{14}]$, $[\sigma'\varepsilon_{14}]$	$[\zeta_5\oplus\mu_7]\circ\sigma_{16}$, $[\nu_5\bar{\varepsilon}_8]$, $[\sigma'\mu_{14}]$, $[E\zeta']$, $[\eta_7\bar{\varepsilon}_8]$

.

We have the following relations:

(6.2) $$2[\nu_5^2] = i_*\mathcal{E}_3, \quad 2[\sigma'''] = i_*\mu_3, \quad 2[\nu_7]\circ\nu_{10} = i_*\mathcal{E}',$$
$$8[\eta_5\mathcal{E}_6\oplus\sigma'] = i_*\mu', \quad 2[\nu_5\oplus\eta_7]\circ\sigma_8\nu_{15} = i_*\mathcal{E}_3,$$
$$2[\sigma''']\circ\sigma_{12} = 2[\nu_5\zeta_8] = i_*\mu_3\sigma_{12}, \quad 2[\rho^{\mathrm{IV}}] = i_*\bar{\mu}_3,$$
$$8[\sigma'\sigma_{14}] = i_*\mu'\sigma_{14} \quad \textit{and} \quad 8[\rho''] = i_*\bar{\mu}'.$$

Consider the bundle $SU(4)/Sp(2)=S^5$. Since the order of its characteristic class $\Delta\iota_5=i_*\eta_3$ is 2, we have, by Lemma 2.3, isomorphisms

$$\pi_i(SU(4):p) \cong \pi_i(S^5\times Sp(2):p) \cong \pi_i(S^5:p)\oplus\pi_i(Sp(2):p)$$

for odd prime p. Then the results for odd components follow immediately from the tables (4.2) and (5.2).

From (6.1) we have the exactness, for $i>7$, of the following sequence:

(6.3) $$0 \to \operatorname{Coker}(\Delta : \pi_{i+1}^5\oplus\pi_{i+1}^7 \to \pi_i^3)$$
$$\xrightarrow{i_*} \pi_i(SU(4):2) \xrightarrow{p^*} \operatorname{Ker}(\Delta : \pi_i^5\oplus\pi_i^7 \to \pi_{i-1}^3) \to 0.$$

Obviously, the above Δ is the sum of the Δ's of (4.3) and (5.3). Then the following lemma follows from Proposition 3.2.

Lemma 6.2. i). *For the case* $i=8, 9, 10, 15, 16, 17$ *and* 23, *the homomorphisms* $\Delta : \pi_{i+1}^5\oplus\pi_{i+1}^7\to\pi_i^3$ *are epimorphisms. For the other values of* i, $7<i<23$, *we have the following table of the cokernel and representatives of their generators.*

$i=$	11	12	13	14	18	19	20	21	22
Coker. $\Delta\cong$	Z_2	Z_2	Z_2	Z_2	Z_2	Z_2	Z_2	Z_2	Z_2
rep. of gene.	ε_3	μ_3	ε'	μ'	$\bar{\varepsilon}_3$	$\mu_3\sigma_{12}$	$\bar{\mu}_3$	$\mu'\sigma_{14}$	$\bar{\mu}'$

.

ii). *The kernels of the homomorphisms* $\Delta : \pi_i^5\oplus\pi_i^7 \to \pi_{i-1}^3$, $7<i\leq 23$, *and their generators are listed in the following table*:

$i=$	8	9	10	11	12	13	14
Ker. $\Delta\cong$	Z_8	Z_2	Z_2+Z_8	Z_2	Z_2	Z_2	Z_8+Z_2
generators	$\nu_5\oplus\eta_7$	$(\nu_5\oplus\eta_7)\circ\eta_8$	$\nu_5\eta_8^2$, ν_7	ν_5^2	σ'''	ν_7^2	$\eta_5\varepsilon_6\oplus\sigma'$, ν_5^3

$i=$	15	16
Ker. $\Delta\cong$	Z_8+Z_2	$Z_8+Z_2+Z_2+Z_2+Z_2$
generators	$(\nu_5\oplus\eta_7)\circ\sigma_8$, $\sigma'\eta_{14}$	$\zeta_5\oplus\mu_7$, $\nu_5\bar{\nu}_8$, $\sigma'\eta_{14}^2$, ν_7^3, $(\nu_5\oplus\eta_7)\circ\varepsilon_8$

$i=$	17	18
Ker. $\Delta\cong$	$Z_8+Z_2+Z_2+Z_2$	$Z_8+Z_2+Z_2$
generators	$\nu_7\sigma_{10}$, ν_5^4, $\nu_5\eta_8\varepsilon_9$, $(\nu_5\oplus\eta_7)\circ\mu_8$	ζ_7, $\nu_5\eta_8\mu_9$, $(\nu_5\sigma_8\oplus\bar{\nu}_7)\circ\nu_{15}$

$i=$	19	20	21
Ker. $\Delta\cong$	Z_2+Z_2	Z_2+Z_2	Z_8+Z_2
generators	$\nu_5\zeta_8$, $\nu_5\bar{\nu}_8\nu_{16}$	ρ^{IV}, $\nu_7\sigma_{10}\nu_{17}$	$\sigma'\sigma_{14}$, $\eta_5\bar{\varepsilon}_6\oplus 2\kappa_7$

$i=$	22	23
Ker. $\Delta\cong$	$Z_8+Z_4+Z_2+Z_2$	$Z_8+Z_2+Z_2+Z_2+Z_2$
generators	ρ'', $\nu_5\kappa_8\oplus\bar{\varepsilon}_7$, $\sigma'\nu_{14}$, $\sigma'\varepsilon_{14}$	$(\eta_5\oplus\mu_7)\circ\sigma_{16}$, $\nu_5\bar{\varepsilon}_8$, $\sigma'\mu_{14}$, $E\zeta'$, $\eta_7\bar{\varepsilon}_8$

The results for $i\leq 7$ in Theorem 6.1 are verified without difficulties from the exactness of (6.1), so we omit the proof.

We shall compute the 2-primary components. We see that the above lemma, the exactness of (6.3) and the relations (6.2) imply the results for the 2-primary components in Theorem 6.1. So, it is sufficient to prove the relation (6.2).

The first, second, sixth and seventh relations in (6.1) follow immediately from the corresponding relations in (4.1). The third, eighth and ninth relations in (6.1) follow from (5.1). From the second relation of (5.1), we have

$$4[2\sigma'] = i_*\mu' \qquad (\text{in } \pi_{14}(SU(4))).$$

Since $p_*[2\sigma']=p_*(2[\eta_5\varepsilon_6\oplus\sigma'])$, we have $2[\eta_5\varepsilon_6\oplus\sigma']\equiv[2\sigma']$ mod $i_*\mu'$. It follows the fourth relation $8[\eta_5\varepsilon_6\oplus\sigma']=i_*\mu'$.

It remains to prove fifth relation

$$2[\nu_5\oplus\eta_7]\circ\sigma_6\nu_{15} = i_*\bar{\varepsilon}_3 .$$

We have

$$2[\nu_5\oplus\eta_7]\circ\sigma_8\nu_{15} = [\nu_5\oplus\eta_7]\circ 2\sigma_8\nu_{15} = [\nu_5\oplus\eta_7]\circ\nu_8\sigma_{11}$$

and $$[\nu_5\oplus\eta_7]\circ\nu_8 \equiv [\nu_5^2] \mod 2[\nu_5^2],$$

since $p_*([\nu_5\oplus\eta_7]\circ\nu_8)=\nu_5^2\oplus\eta_7\nu_8=\nu_5^2$. It follows

$$2[\nu_5\oplus\eta_7]\circ\sigma_8\nu_{15} = [\nu_5^2]\circ\sigma_{11} .$$

By Theorem 2.6, we have

$$[\nu_5^2]\circ\sigma_{11}\in i_*E^*\{\eta_4, \nu_5^2, \sigma_{11}\} .$$

By Proposition 1.2 ard (7.19) of [7], we have

$$E\{\eta_4, \nu_5^2, \sigma_{11}\}\subset\{\eta_5, \nu_6^2, \sigma_{12}\}_1\supset\{\eta_5, \nu_6, \nu_9\sigma_{12}\}_1 = \{\eta_5, \nu_6, 2\sigma_9\nu_{16}\}_1$$

and $$\{\eta_5, \nu_6, 2\sigma_9\nu_{16}\}_2\subset\{\eta_5, 2\nu_6, \sigma_9\nu_{16}\}_1\supset\{\eta_5, 2\iota_6, \nu_6\sigma_9\nu_{16}\}_1 .$$

The indeterminacy is $\eta_5\circ E\pi_{19}^5+\pi_{13}^5\circ\sigma_{13}+\pi_9^5\circ\sigma_9\nu_{16}$. $\eta_5\circ E\pi_{19}^5=\eta_5\circ 2\pi_{20}^6$ $=\eta_5\circ 2\iota_6\circ\pi_{20}^6=2\eta_5\circ\pi_{20}^6=0$, by Theorem 10.3 of [7]. $\pi_{13}^5\circ\sigma_{13}=\{\varepsilon_5\sigma_{13}\}$ $=0$ by Theorem 7.1 and Lemma 10.7 of [7]. $\pi_9^5\circ\sigma_9\nu_{16}=\{\nu_5\eta_8\sigma_9\nu_{16}\}$ $=\{\nu_5(E\sigma'\eta_{15}+\bar{\nu}_8+\varepsilon_8)\nu_{16}\}=0$ by (7.4), (5.9), (7.17) and (7.18) of [7]. It follows that

$$E\{\eta_4, \nu_5^2, \sigma_{12}\} = \{\eta_5, 2\iota_6, \nu_6\sigma_9\nu_{16}\}_1$$

and this consists of a single element. We have

$$\nu_5\sigma_8\nu_{15}\equiv\{\nu_5^2, 2\iota_{11}, \nu_{11}^2\}_1 \mod \nu_5\eta_8\mu_9 ,$$

since $H(\nu_5\sigma_8\nu_{15})=H\{\nu_5^2, 2\iota_{11}, \nu_{11}^2\}_1=\nu_9^3$ and the kernel of E is generated by $\nu_5\eta_8\mu_9$ (cf. Theorem 7.7 of [7]). Then

$$\nu_6\sigma_9\nu_{16}\in\{\nu_6^2, 2\iota_{12}, \nu_{12}^2\}$$

since $E(\nu_5\eta_8\mu_9)=0$. By Proposition 1.5 of [7],

$$\{\eta_5, 2\iota_6, \nu_6\sigma_9\nu_{16}\}_1(\in\{\eta_5, 2\iota_6, \{\nu_6^2, 2\iota_{12}, \nu_{12}^2\}\})$$
$$\in\{\{\eta_5, 2\iota_6, \nu_6^2\}, 2\iota_{13}, \nu_{13}^2\}+\{\eta_5, \{2\iota_6, \nu_6^2, 2\iota_{12}\}, \nu_{13}^2\} .$$

Here we have that $\{\eta_5, 2\iota_6, \nu_6^2\}$ consists of the element ε_5 by (6.1) and Theorem 7.1 of [7]. By Corollary 3.7 of [7], we have $0=\eta_6\nu_7^2\in\{2\iota_6, \nu_6^2, 2\iota_{12}\}$ and hence $\{2\iota_6, \nu_6^2, 2\iota_{12}\}=2\pi_{12}(S^6)$. So, we have

$$\{\eta_5, 2\iota_6, \nu_6\sigma_9\nu_{16}\}_1\in\{\varepsilon_5, 2\iota_{13}, \nu_{13}^2\}+\{\eta_5, 2\alpha, \nu_{13}^2\} .$$

By the definition of $\bar{\varepsilon}_5$, $\bar{\varepsilon}_5 \in \{\varepsilon_5, 2\iota_{13}, \nu_{13}^2\}$. We have also $0 \in \{\eta_5, \alpha, 0\} \subset \{\eta_5, 2\alpha, \nu_{13}^2\}$. It follows

$$\bar{\varepsilon}_5 \equiv \{\eta_5, 2\iota_6, \nu_6\sigma_9\nu_{16}\} \quad \mathrm{mod} \quad G,$$

where $G = \eta_5 \circ \pi_{20}(S^6) + \pi_{14}^5 \circ \nu_{14}^2 = \eta_5 \circ \pi_{20}^6 = 0$. By concerning the behavior of E in Theorem 10.5 of [7], we have

$$\bar{\varepsilon}_3 = E^*\{\eta_4, 2\iota_5, \nu_5\sigma_8\nu_{15}\}.$$

Therefore we have proved that

$$[\nu_5^2]\circ\sigma_{11} = i_*E^*\{\eta_4, 2\iota_5, \nu_5\sigma_8\nu_{15}\} = i_*\bar{\varepsilon}_3.$$

This completes the proof of Theorem 6.1.

§ 7. Problems

In the previous computations it seems that the following two problems are true.

Problem 7.1. *Is the following diagram commutative?*

$$\begin{array}{ccc}
\pi_i(S^5) & \xrightarrow{E^2} & \pi_{i+2}(S^7) \\
\downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} \\
\pi_{i-1}(S^3) & & \pi_{i+1}(S^3) \\
& {\searrow}^{E^2} \quad {\swarrow}_{H} & \\
& \pi_{i+1}(S^5). &
\end{array}$$

This is surely true for the suspension elements in $\pi_i(S^5)$.

Problem 7.2. *Let an element* α *of* $\pi_i(S^5)$ *satisfy the relations*

$$2\alpha = 0 \quad \textit{and} \quad \Delta\alpha = 0,$$

where Δ *is the boundary homomorphism for the bundle* $SU(3)/S^3 = S^5$. *Does there exist an element* $[\alpha]$ *of* $\pi_i(SU(3))$ *and an element* β *of* $\pi_i(S^3)$ *such that*

$$p_*[\alpha] = \alpha, \quad i_*\beta = 2[\alpha] \quad \textit{and} \quad H(\beta) = \alpha?$$

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