J. Math. Kyoto Univ.
3-3 (1964) 379~382.

Note on semi-reductive groups

By

Masayoshi NAGATA and Takehiko MIYATA

(Received June 15, 1964)

The notion of a semi-reductive group was introduced in the preceding paper. The present note contains two results on semi-reductive groups.

The first result is as follows: Let G be a semi-reductive group contained in $GL(n, K)$, K being a field of characteristic p (which may be zero). Let ρ be a rational representation of G of type $\begin{pmatrix} 1 & \tau \\ 0 & \rho' \end{pmatrix}$, ρ' being a representation of degree one less than the degree m of ρ. We consider the action of G defined by ρ on the polynomial ring P_m in indeterminates $X_1, \cdots, X_m$ over K. Let $\mathfrak{a}$ be a G-stable ideal in P_m such that $\Sigma X_i K \cap \mathfrak{a} = 0$ and let x_i be the class of X_i modulo $\mathfrak{a}$. The semi-reductivity of G implies the existence of a G-invariant f in $K[x_1, \cdots, x_m]$ such that f is monic and of positive degree in x_1. Now the result is:

Theorem 1. *If there is such an f of degree d in x_1 so that d is not a multiple of p and if x_1 is transcendental over $K[x_2, \cdots, x_m]$, then ρ is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & \rho' \end{pmatrix}$.*

The other result concerns with the case of algebraic linear group, and can be stated as follows:

Theorem 2. *If an algebraic linear group G is semi-reductive, then the radical of G is a torus.*

We shall note in this article that any one of these theorems implies the following fact:

Proposition. *Let K be a field of characteristic zero and let G be a subgroup of $GL(n, K)$. Then G is reductive if and only if G is semi-reductive.*

1. The proof of Theorem 1.

We set $y(\sigma)=x_1^\sigma-x_1$ for $\sigma\in G$. Then $y(\sigma)\in\Sigma_{i\geq 2}x_iK$. We express f as a polynomial in x_1 with coefficients in $K[x_2,\cdots,x_m]$ so that

$$f=x_1^d+c_1x_1^{d-1}+\cdots+c_d\,.$$

Then $f=f^\sigma=x_1^d+(c_1^\sigma+dy(\sigma))x_1^{d-1}+\cdots$. Thus we have $c_1^d+dy(\sigma)=c_1$ for any $\sigma\in G$. Set $x^*=x_1+(1/d)c_1$. Then $x^{*\sigma}=x^*$, and therefore the representation module Σx_iK is the direct sum of representation modules x^*K and $\Sigma_{i\geq 2}x_iK$, and the assertion is proved.

2. The proof of Theorem 2.

We shall make use of the following result of Bialynicki-Birula:[1)]

Lemma 1. *Let G' be a connected linear algebraic group and let H be a closed subgroup of G'. Let H_u be the unipotent part of the radical of H. If G'/H is affine, then G'/H_u is affine and furthermore the following condition is satisfied:*

Any finite-dimensional rational H_u-module M is an H_u-submodule of a finite-dimensional rational G'-module N such that the set of H_u-invariants in M coincides with the set of G'-invariants in N.

We shall modify the last condition in Lemma 1 as follows:

If $\rho=\begin{pmatrix}1 & \tau\\ 0 & \rho'\end{pmatrix}$ is a rational representation of H_u, ρ' being a rational representation of degree one less than the degree m of ρ, then there is a rational representation ρ^* of G' such that

$$\rho^*=\begin{pmatrix}1 & \tau^* & \tau_1^*\\ 0 & \rho'^* & \tau_2^*\\ 0 & \tau'^* & \rho_2^*\end{pmatrix}$$

and such that the restriction of ρ^* on H_u is of the form

$$\begin{pmatrix}1 & \tau & *\\ 0 & \rho' & *\\ 0 & 0 & *\end{pmatrix}$$

1) A. Bialynicki-Birula, On homogeneous affine space of linear algebraic groups, Amer. J. Math. 85 (1963), pp. 577–582.

The proof of this modification is done as follows: Let ρ_1 be the contragradient represntation to ρ, and let M be a representation module for ρ_1. Apply Lemma 1 to this M, and get N as stated. Then ρ^* is the contragradient representation to the representation of G defined by N.

Now we shall prove Theorem 2. Let G' be a connected linear algebraic group which contains G and let H_u be the unipotent part of the radical of G. Since G is semi-reductive, G'/G is affine.[2] Therefore the condition stated above holds for this pair of G' and H_u. Observe the situation as in the modification above. Consider the restriction ρ_G of ρ^* on G. Then the semi-reductivity of G implies the existence of a G-invariant f in $P_s=K[X_1, \cdots, X_s]$ under the action of G defined by ρ_G (s=degree of ρ^*), such that f is monic and of positive degree in X_1. Let f' be the polynomial obtained form f removing out all terms in which some of $X_{m+1}, \cdots, X_s$ appear. Then f' in an H_u-invariant in P_m under the action defined by ρ, and we see that H_u is semi-reductive. Thus Theorem 2 follows from the following lemma:

Lemma 2. *If U is a connected unipotent linear algebraic group of positive dimension, then U cannot be semi-reductive.*

Proof. Since the image of a semi-reductive group under a rational homomorphism is semi-reductive we may assume that U is of dimension 1. Then U has a faithful rational representation ρ of degree 2:

$$\rho=\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Consider the action of U on $P_2=K[X_1, X_2]$. Then X_2 is an invariant. Since U is not a finite group, the set of invariants cannot be larger than $K[X_2]$, and the assertion is proved.

3. Proofs of Proposition.

(1) First we give a proof of Proposition making use of Theorem 2. We note that, letting K be a universal domain,

Lemma 3. *Let G be a group contained in $GL(n, K)$, and let*

2) See §8 of the preceding paper.

$\bar{G}$ be the closure of G in Zariski topology of $GL(n, K)$. Then (i) *G is reductive if and only if $\bar{G}$ is reductive and* (ii) *G is semi-reductive if and only if so is $\bar{G}$.*

The proof of Lemma 3 is easy and we omit it.

Now, by Lemma 3, in order to prove Proposition, we may assume that G is algebraic. Then Theorem 2 shows that the radical of G is a torus if G is semi-reductive, whence G is reductive as is well known. The converse is obvious.

(2) Next we give a proof of Proposition making use of Theorem 1. We know that reductivity of a matric group G is equivalent to the condition that every rational representation of type $\begin{pmatrix} 1 & \tau \\ 0 & \rho' \end{pmatrix}$ is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & \rho' \end{pmatrix}$.[3] Therefore, in the zero characteristic case, the semi-reductivity of G implies the reductivity of G by virtue of Theorem 1. Thus Proposition is proved.

Errata to some of my papers

Masayoshi NAGATA

1. "Complete reducibility of rational representations of a matric group", this journal, 1-1 (1961):
   In foot-note 6) in p. 99, "an affine variety" should be read "a quasi-affine variety".
2. "Note on coefficient fields of complete local rings", Mem. Coll. Sci. Univ. Kyoto, 32 (1959):
   In the remark in p. 92, the statement (2) is under the condition stated in (1).
3. "On the closedness of singular loci," Publ. Math. Inst. Haut. Etudes Sci., 2 (1959):
   Proposition 4 is wrong, hence Theorems 4, 5 are not proved yet.

---

3) M. Nagata, Complete reducibility of rational representations of a matric group, this Journal, vol. 1, No. 1 (1961), pp. 87-99.