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Corrections and supplement to the paper "Reduction of models over a discrete valuation ring"

By

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1. In this short note we shall correct a proposition given in [1] and rewrite, in part, the proof of a theorem depending on the proposition. Moreover we shall add some results on calculus of generalized cycles on absolutely irreducible models over a discrete valuation ring, which were treated in [1].

We shall generalize the definitions of products and intersections of generalized cycles, which have a sense in [1], whenever all the components of generalized cycles dominate the same place of the ground ring. In other words we shall define these calculus without any restriction on generalized cycles. Then, making use of properties of the operation ρ defined in [1], we shall see that some important results on calculus of generalized cycles in [1] remain also true.

The notations and the terminologies are the same as those of [1]. In particular we shall fix a ground ring $\mathfrak{o}$ with the quotient field k, the maximal ideal $\mathfrak{p}=(\pi)$ and the residue class field κ.

2. At first place we shall generalize the definition of a product of generalized cycles on absolutely irreducible models. Let M and N be two absolutely irreducible affine models over $\mathfrak{o}$, whose affiine rings are $\mathfrak{o}[x]$ and $\mathfrak{o}[y]$ respectively. Then $A=\mathfrak{o}[x]\otimes_{\mathfrak{o}}\mathfrak{o}[y]$ is the affine ring of the affine model $M\times N$. Let P and Q be spots of M and N corresponding to the prime ideals $\mathfrak{m}$ and $\mathfrak{n}$ of $\mathfrak{o}[x]$ and $\mathfrak{o}[y]$ respectively. If P and Q dominate the same place of the ground ring $\mathfrak{o}$, $P\times Q$ is defined in the sense of [1] and it is easy to see that the components $R_1, \cdots, R_t$ of $P\times Q$ correspond to the minimal prime ideals $\mathfrak{l}_1, \cdots, \mathfrak{l}_t$ of $\mathfrak{a}=(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n})A$, and that the coefficient of R_i in $P\times Q$ is equal to $l(R_i/\mathfrak{a}R_i)$. Therefore it is natural to give the same definition of $P\times Q$ as above in the case where P and Q dominate diffierent places of $\mathfrak{o}$. Let P and Q dominate $\mathfrak{o}$ and k respectively. Then the minimal prime ideals $\mathfrak{n}'_1, \cdots, \mathfrak{n}'_s$ of $(\pi, \mathfrak{n})\mathfrak{o}[y]$ correspond to the induced spots of Q over $\mathfrak{p}$, and it is easily seen that a minimal prime ideal of $(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n})A$ is a minimal prime ideal of $(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n}'_i)A$ for some i and vice versa. This means that any component of $P\times Q$ is a component of $P\times Q'_i$ for some induced spot Q'_i, and that, conversely, any component of $P\times Q'_i$ is a component of $P\times Q$. Therefore it seems to be significant to define a product of cycles on any absolutely irreducible models over $\mathfrak{o}$ as follows. Let M and N be two absolutely irreducible models over $\mathfrak{o}$. Let $(P, \mathfrak{m})$ be a spot of the closed subset $M-M_k$ of M and $(Q, \mathfrak{n})$ a spot of the open subset N_k of N. Let $Q'_i, \cdots, Q'_s$ be all the induced spots of Q over $\mathfrak{p}$ and that $\mathfrak{n}_i$ is the prime ideal of Q'_i corresponding to Q. Then P and Q'_i dominate the same place $\mathfrak{o}$, and hence $P\times Q'_i$ is defind in the sense of [1]. If $(R, \mathfrak{l})$ is a component of $P\times Q'_i$, then it is easy to see that R is a quotient ring of $P\otimes_{\mathfrak{o}}Q'_i$ and that the length $l(R/(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n}_i)R)$ is finite. Then we put $P\times Q=\sum_{i=1}^{s}\sum_{R} l(R/(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n}_i)R)$, where R runs over all the components of $P\times Q'_i$, and call this generalized cycle the product of P and Q. Similary we can define $P\times Q$ in the case where P and Q are in M_k and $N-N_k$ respectively.

PROPOSITION 1. *Let M and N be two absolutely irreducible models over $\mathfrak{o}$. Let $(P, \mathfrak{m})$ be in $M-M_k$ (resp. in M_k) and $(Q, \mathfrak{n})$ in N_k (resp. in $N-N_k$). Then we have*

$$P\times Q=P\times\rho(Q) \quad (resp.\ P\times Q=\rho(P)\times Q)$$

on $M\times N$.

PROOF. By the definition, $P\times Q$ and $P\times\rho(Q)$ have the same components. Let $(R, \mathfrak{l})$ be one of these compontents. Then R dominates $\mathfrak{o}$. Let $(Q', \mathfrak{n}')$ be the induced spot of Q dominated by R and

$\mathfrak{n}_1$ the prime ideal of Q' corresponding to Q. Then we can easily see that

$$l(R/\mathfrak{m}\otimes 1,\ 1\otimes\mathfrak{n}_1)R) = l(R/(\mathfrak{m}\otimes 1,\ 1\otimes(\pi,\ \mathfrak{n}_1))R)$$
$$= l((P/\mathfrak{m}\otimes_\kappa Q'/(\pi,\ \mathfrak{n}_1))\bar{\mathfrak{l}}),$$

where $\bar{\mathfrak{l}}$ is the prime ideal of $P/\mathfrak{m}\otimes_\kappa Q'/(\pi, \mathfrak{n}_1)$ corresponding to the maximal ideal $\mathfrak{l}$ of R. By Lemma 2 in [1], the righ hand side of the above equality is equal to

$$\mathfrak{l}(R/(\mathfrak{m}\otimes 1,\ 1\otimes\mathfrak{n}')R)l(P/\mathfrak{m})l(Q'/(\pi,\ \mathfrak{n}))$$
$$= l(R\{(\mathfrak{m}\otimes 1,\ 1\otimes n')R)\mu(Q;\ Q').$$

This means that the coefficient of R in $P\times Q$ is equal to that of $P\times\rho(Q)$. q.e.d.

Remark: Let $(\mathfrak{o}^*, \mathfrak{p}^*)$ be a ground ring extension of $(\mathfrak{o}, \mathfrak{p})$, and let P and Q be as the above proposition. Then, using Propositions 1 and 5 in [1], we can easily see that $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P\times Q) = \sigma_{\mathfrak{o}^*/\mathfrak{o}}(P)\times\sigma_{\mathfrak{o}^*/\mathfrak{o}}(Q)$. In other words. we can remove the restriction that $X\times Y$ is well defined in Proposition 1 in [1].

3. Let M be an absolutely irreducible model over $\mathfrak{o}$. Let P be a spot in the closed subset $M-M_k$ of M and Q a spot in the open subset M_k of M. Let $(R, \mathfrak{l})$ be a component of $M(P)\cap M(Q)$. Let us put $\mathfrak{O}=R\otimes_{\mathfrak{o}}R$ and $\mathfrak{d}=\mathfrak{d}(\mathfrak{O})$. If $\mathfrak{m}$ and $\mathfrak{n}$ are the prime ideals of R corresponding to P and Q, then $\mathfrak{l}'=(\mathfrak{l}\otimes 1, \mathfrak{d})\mathfrak{O}$ is a minimal prime ideal of $(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n}, \mathfrak{d})\mathfrak{O}$. Let $\bar{\mathfrak{O}}$ be the quotient ring $\mathfrak{O}_{\mathfrak{l}'}$, of $\mathfrak{O}$ with respect to $\mathfrak{l}'$, and we denote the multiplicity $e(\mathfrak{d}\bar{\mathfrak{O}}/(\mathfrak{m}\otimes 1,\ 1\otimes\mathfrak{n})\bar{\mathfrak{O}})$ by $i_{\mathfrak{o}}(R;\ P\cdot Q)$. We shall define the intersection of P and Q by $P\cdot Q=\sum_{i=1}^{s} i_{\mathfrak{o}}(R_i;\ P\cdot Q)R_i$, where $R_1, \cdots, R_s$ are all the components of $M(P)\cap M(Q)$. It is evident to see that $P\cdot Q=Q\cdot P$. This definition is a natural generalization of the definition given in [1].

PROPOSITION 2. *Let M be an absolutely irreducible model over $\mathfrak{o}$. Let P and Q be spots in $M-M_k$ and M_k respectively. Then any component R of $P\cdot Q$ is that of $P\cdot\rho(Q)$ and the coefficient of R in* $P\cdot Q$ is equal to that of $P\cdot\rho(Q)$.

PROOF. The first assertion is easily seen from the definition of $P\cdot Q$. Let the notations be as the above. Then the coefficient of R in $P\cdot Q$ is equal to $e(\mathfrak{d}\overline{\mathfrak{O}}/(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n})\overline{\mathfrak{O}})$. Using Lemma 8 and Lemma 10 in [1], we see that this value is equal to $\sum_{h=1}^{t} e(\mathfrak{d}\overline{\mathfrak{O}}/\mathfrak{P}_h) l(\overline{\mathfrak{O}}_{\mathfrak{P}_h}/(\mathfrak{m}\otimes 1, 1\otimes(\pi, \mathfrak{n}))\overline{\mathfrak{O}}_{\mathfrak{P}_h})$, where $\mathfrak{P}_h$ runs over all the minimal prime divisors of $(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n})\overline{\mathfrak{O}}=(\mathfrak{m}\otimes 1, 1\otimes(\pi, \mathfrak{n}))\overline{\mathfrak{O}}$. In fact it is easily seen that these prime ideals have the same corank. Now put $\mathfrak{n}'_h=\mathfrak{P}_h \cap(1\otimes R)$ and $Q'_h=R_{\mathfrak{n}'_h}$. Then $\mathfrak{P}_h$ is a minimal prime divisor of $(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n}'_h)\overline{\mathfrak{O}}$ and we can see that

$$l(\overline{\mathfrak{O}}_{\mathfrak{P}_h}/(\mathfrak{m}\otimes 1, 1\otimes(\pi, \mathfrak{n}))\overline{\mathfrak{O}}_{\mathfrak{P}_h})$$
$$=l((\overline{\mathfrak{O}}_{\mathfrak{P}_h}/(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n}'_h)\overline{\mathfrak{O}}_{\mathfrak{P}_h})e(\pi Q'_h/\mathfrak{n}Q'_h)$$

(cf. the proof of Proposition 1). Lte $\mathfrak{n}'_{i_1}, \cdots, \mathfrak{n}'_{i_s}$ be all the different members among $\mathfrak{n}'_1, \cdots, \mathfrak{n}'_t$. Then we have

$$e(\mathfrak{d}\overline{\mathfrak{O}}/(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n})\overline{\mathfrak{O}})$$
$$=\sum_{h=1}^{t} e(\mathfrak{d}\mathfrak{O}/\mathfrak{P}_h) l(\overline{\mathfrak{O}}_{\mathfrak{P}_h}/(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n}'_h)\overline{\mathfrak{O}}_{\mathfrak{P}_h}) e(\pi Q'_h/\mathfrak{n}Q'_h)$$
$$=\sum_{j=1}^{s}\left\{\sum_{\mathfrak{n}'_h=\mathfrak{n}'_i} e(\mathfrak{d}\overline{\mathfrak{O}}/\mathfrak{P}_h) l(\overline{\mathfrak{O}}_{\mathfrak{P}_h}/(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n}'_h)\overline{\mathfrak{O}}_{\mathfrak{P}_h}) e(Q'_h/\mathfrak{n}Q'_h)\right\}$$
$$=\sum_{j=1}^{s} e(\mathfrak{d}\mathfrak{O}/(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n}'_{i_j})\mathfrak{O}) e(\pi Q'_{i_j}/\mathfrak{n}Q'_{i_j})$$
$$=\sum_{j=1}^{s} i_0(R;\ P\cdot Q'_{i_j})\cdot\mu(Q;\ Q'_{i_j}).$$

The right hand side of this equality is equal to the coefficient of R in $P\cdot\rho(Q)$. q.e.d.

COROLLARY. *Let M be an absolutely irreducible model over $\mathfrak{o}$ such that M has only one generating spot P_0 over $\mathfrak{p}$. Then we have the equality $P_0\cdot Q=P_0\cdot\rho(Q)$ for any spot in the open subset M_k of M.*

PROOF. By the definition, the components of $P_0\cdot Q$ are the induced spots of Q over $\mathfrak{p}$. On the other hand, since P_0 is the unique generating spot over $\mathfrak{p}$, all the induced spots of Q are specializations of P_0. Therefore the components of $P_0\cdot\rho(Q)$ are the induced spots of Q over

$\mathfrak{p}$. The corollary is a direct conseqtence of this fact and Proposition 2. q.e.d.

4. In this section we shall correct some points in [1]. First Proposition 2 in §2 of [1] should be read as follows:

PROPOSITION 2 in [1]. *Let M be an absolutely irreducible model over $\mathfrak{o}$. Let $\mathfrak{o}^*$ be a ground ring extension of $\mathfrak{o}$ and let X and Y be generalized cycles in M such that $X \cdot Y$ is well defined. Then any component of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(X \cdot Y)$ is a component of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(X) \cdot \sigma_{\mathfrak{o}^*/\mathfrak{o}}(Y)$ and the coefficient of R^* in $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(X \cdot Y)$ is equal to that of R^* in $\sigma_{\mathfrak{o}_*/\mathfrak{o}}(X) \cdot \sigma_{\mathfrak{o}^*/\mathfrak{o}}(Y)$.*

In fact a component of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(X) \cdot \sigma_{\mathfrak{o}^*/\mathfrak{o}}(Y)$ may not appear in $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(X \cdot Y)$. The proof need not be corrected.

Here we shall remark that the restriction on X and Y in this Proposition 2 can be removed, if we understand $X \cdot Y$ as in §3. The first assertion is verified withaut any modification. As to the second, let R be the component of $P \cdot Q$ such that R^* is a component of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(R)$. Then R is a component of $P \cdot \rho(Q)$ and the coefficient of R in $P \cdot Q$ is equal to that of $P \cdot \rho(Q)$ by Proposition 2 in §3. Therefore the coefficient of R^* in $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P \cdot Q)$ is equal to that of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P \cdot \rho(Q))$, which is the coefficient of R^* in $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P) \cdot \sigma_{\mathfrak{o}^*/\mathfrak{o}}(\rho(Q)) = \sigma_{\mathfrak{o}^*/\mathfrak{o}}(P) \cdot \rho(\sigma_{\mathfrak{o}^*/\mathfrak{o}}(Q))$ by Proposition 2 in [1]. Since we have already seen that R^* is a component of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P) \cdot \sigma_{\mathfrak{o}^*/\mathfrak{o}}(Q)$, we see that the coefficient of R^* in $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P \cdot Q)$ is equal to that of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P) \cdot \sigma_{\mathfrak{o}^*/\mathfrak{o}}(Q)$.

Using Proposition 2 in [1], we proved Theorem 2 in [1]. However the results of Theorem 2 need not be changed. For it is enough, in the proof of Therem 2, to correct the last part (the part from the 9-th line to the 16-th line in p. 146) as follows:

Let P and Q be components of X and Y respectively such that R is a component of $\rho(P) \cdot \rho(Q)$. If P^* and Q^* are components of P and Q over $\mathfrak{o}^*$ respectively such that $\rho(P^*) \cdot \rho(Q^*)$ has R^* as a component, the coefficient c of R^* in $\rho(P^*) \cdot \rho(Q^*)$ is equal to that of $\rho(P^* \cdot Q^*)$ by Proposition 8. On the other hand, from the fact that R is a proper component of $\rho(P) \cdot \rho(Q)$ and from Proposition 2, it is easy to see that the coefficient c_1 of R^* in $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\sigma(P)\cdot\rho(Q))$ is equal to that of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\rho(P))\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\rho(Q))=\rho(\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P))\cdot\rho(\sigma_{\mathfrak{o}^*/\mathfrak{o}}(Q))$. Therefore c is equal to c_1. Next we show that c is equal to the coefficient of R^* in $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\rho(P\cdot Q))=\rho(\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P\cdot Q))$. For this, it is enough to show that any component S^* of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P)\cdot\sigma_{\mathfrak{o}^*/\mathfrak{o}}(Q)$ having R^* as a specialization is a component of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P\cdot Q)$. If S^* is not a component $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P\cdot Q)$, there exists a component S_1^* of $\rho_{\mathfrak{o}^*/\mathfrak{o}}(P\cdot Q)$ which is a generalization of S^* but not equal to S^*. Then there exist spots S and S_1 in $M(P)\cap M(Q)$ such that R is a component of $\rho(S)$, and such that S^* and S_1^* are components of S and S_1 over $\mathfrak{o}^*$ respectively. From this, we easily see that there exists a component R_1 of $\rho(S_1)$ which is a generalization of R. This means that $R=R_1$ and hence $S^*=S_1^*$, a contradiction. Therefore the coefficient of R^* in $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P\cdot Q))$ is equal to that of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\rho(P)\cdot\rho(Q))$.

REFERENCES

[1] H. Yanagihara, "Reduction of models over a discrete valuation ring", J. Math. Kyoto Univ. vol. 2 (1963), pp. 123-156.

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