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Paths in a Finsler space

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The purpose of this paper is to introduce paths in a Finsler space from a standpoint of a connection in a principal bundle. In a Riemannian space, a geodesic is, of course, defined as an extremal of the length integral, and it is well known that a geodesic coincides with a path defined with respect to the Riemannian connection given by the Christoffel's symbols. On the other hand, a geodesic in a Finsler space is defined in like manner, but the explicit equation of a geodesic is obtained in various forms by several authors, according to the choice of a connection [1], [6].

In a previous paper [2] was presented the theory of a Finsler connection in a certain principal bundle Q. According to this definition of a Finsler connection, various paths may be obtained in a Finsler space. In the case of an ordinary connection it is known that the projection of any integral curve of every basic vector field in a bundle space is a path in the base manifold, and conversely, every path in the manifold is obtaind in this way [4, p. 63]. In the present paper, this theorem is taken as the standpoint of the definition of paths in a Finsler space.

The terminologies and signs of papers [2] and [3] will be used in the following without too much comment.

§1. Basic vector fields

We denote by $P(M, \pi, G)$ the bundle of frames of a differentiable n-manifold M, and by $B(M, \tau, F, G)$ the tangent vector bundle of M, where G is the full linear real group $GL(n, R)$ and F is the real vector n-space. In order to define a Finsler connection and parallelism, let us consider the induced bundle $\tau^{-1}P=Q(B, \bar{\pi}, G)$ and further the induced bundle $\tau^{-1}B=D(B, \bar{\tau}, F, G)$. Total spaces Q and D of these induced bundles are as follows:

$$Q=\{(b, p) \mid b \in B,\ p \in P,\ \tau(b)=\pi(p)\},$$
$$D=\{(b, \bar{b}) \mid b, \bar{b} \in B,\ \tau(b)=\tau(\bar{b})\}.$$

Then we have induced mappings $\eta: Q \rightarrow P$ and $\rho: D \rightarrow B$ which are given by $\eta(b, p)=p$ and $\rho(b, \bar{b})=\bar{b}$.

A *Finsler connection* (Γ^v, Γ^h) in Q is by definition [2, §1] a pair of distributions which satisfies the well known conditions for a connection, together with the further condition $\bar{\pi}\Gamma_q^v=B_b^v$, where B_b^v indicates the vertical subspace of the tangent vector space B_b to B at $b=\bar{\pi}(q)$. As is easily seen, the direct sum $\Gamma=\Gamma^v+\Gamma^h$ gives an ordinary connection in Q, which is called the *linear connection associated with the Finsler connection.*

If we put $\bar{\pi}\Gamma_q^h=H_b$, $\bar{\pi}(q)=b$, we have a distribution $H: b \in B \rightarrow H_b$ which is independent of the choice of $q \in \bar{\pi}^{-1}(b)$, and the tangent vector space B_b is the direct sum $B_b^v+H_b$. H is called the *non-linear connection in B induced from the Finsler connection.*

The induced bundle D over B is associated with the principal bundle Q in which the Finsler connection is defined, and we therefore obtain naturally a connection K in D corresponding to the Finsler cconnection [4, p. 43]. In order to obtain K, we consider a mapping $r_f: Q \rightarrow D$, $q=(b, p) \rightarrow (b, pf)$, where f is a fixed element of F, and then we have subspaces $K_d^v=r_f\Gamma_q^v$ and $K_d^h=r_f\Gamma_q^h$ of the tangent vector space D_d at d, where $r_f(q)=d$. The distribution $K: d \in D \rightarrow K_d= K_d^v+K_d^h$ is called the *connection associated with the Finsler connection.*

A concept of a lift arises from a connection [4, p. 26]. First, with respect to the associated linear connection Γ, we obtain the lift l_qX of a given tangent vector $X \in B_b$ to $q \in \bar{\pi}^{-1}(b)$, which is a unique horizontal vector at $q \in Q$ and covers X. Especially, l_qX belongs to Γ_q^h or Γ_q^v, according whether X is horizontal or vertical. Moreover, given a (piece-wise differentiable) curve $C=\{b_t\}$ in B, the lift $l(q_0)C$,

$q_0 \in \bar{\pi}^{-1}(b_0)$, to Q is by definition a horizontal curve $\{q_t\}$ in Q such that $\bar{\pi} q_t = b_t$. (Here, and in the following, t indicates always a parameter: $0 \leqq t \leqq 1$.) The lift $l(q_0)C$ is uniquely determined by its starting point q_0, and if the starting point is taken as $q_0 g$, $g \in G$, then a lift $l(q_0 g)C$ is easily verified to be given by $R_g l(q_0)C$ (R_g is a right translation of Q by $g \in G$).

Secondly, with respect to the non-linear connection H in B, we have also a lift $l_b X$ of a given tangent vector $X \in M_x$ to $b \in \tau^{-1}(x)$, and a lift $l(b_0)C$ of a given curve $C = \{x_t\}$ in M to B. Finally, with respect to the associated connection K in D, we have a lift $l_d X$ of $X \in B_b$ to $d \in \bar{\tau}^{-1}(b)$ and a lift $l(d_0)C$ of a given curve C in B to D.

We are now in a position to give the definition of basic vector fields $B^v(f)$ and $B^h(f)$, which will play an important rôle in all our subsequent considerations. First, the *v-basic vector field* $B^v(f)$ corresponding to a fixed element $f \in F$ is defined by the rule $B^v(f)_q = l_q(dp j_\gamma f) \in \Gamma^v_q$, $q = (b, p)$, where dp expresses the differential of an admissible mapping $p: F \to \tau^{-1}\pi(p)$, γ denotes the characteristic field: $Q \to F$ [2, p. 3], and j_f, $f \in F$, is the identification $F \to F_f$ [2, p. 3]. On the other hand, the *h-basic vector field* $B^h(f)$ is defined by the rule $B^h(f)_q = l_q l_b(pf) \in \Gamma^h_q$, $q = (b, p)$. If $e_1, \cdots, e_n$ is a fixed base of F, then we obtain $B^v(e_a) = B^v_a$ and $B^h(e_a) = B^h_a$, $a = 1, \cdots, n$, which are linearly independent from each other and span Γ^v and Γ^h respectively. In terms of a canonical coordinate (x^i, b^i, p^i_a) of a point q, those basic vector fields are expressed as

$$B^v_a = p^i_a\left(\frac{\partial}{\partial b^i} - p^j_b C^k_{ji} \frac{\partial}{\partial p^k_b}\right),$$

$$B^h_a = p^i_a\left(\frac{\partial}{\partial x^i} - F^j_i \frac{\partial}{\partial b^j} - p^j_b F^k_{ji} \frac{\partial}{\partial p^k_b}\right),$$

in which C^k_{ji}, F^j_i and F^k_{ji} are functions of arguments x^i and b^i only, and called *coefficients of the Finsler connection.*

§2. Parallel displacement

Let us consider a curve $C = \{b_t\}$ in B and take the lift $l(d_0)C =$

$\{d_t\}=\{(b_t, \bar{b}_t)\}$ of C to D, with respect to the associated connection K. Then we say as usual that d_t is obtained from d_0 by *parallel displacement along* C in B. Here $\rho d_t=\bar{b}_t$ is thought of as discribing another curve in B. Then we say that $\bar{b}_t$ is obtained from $\bar{b}_0$ by parallel displacement along C in B. Moreover, in the case of the Finsler connection, the curve C in B is looked upon as the curve $\tau C=\underline{C}=\{x_t\}$ in the base manifold M, together with the vector field b_t defined along $\underline{C}$. Then $\bar{b}_t$ is interpreted as another vector field defined along $\underline{C}$. We therefore express the above situation by saying that $\bar{b}_t$ is obtained from $\bar{b}_0$ by parallel displacement along $\underline{C}$ in M *with respect to the element of support* b_t [1, p. 4].

If a given curve C in B is vertical, the projection $\tau C=\underline{C}$ is obviously reduced to a single point x_0. In this special case, b_t and $\bar{b}_t$ are tangent vectors to M at the fixed x_0, which are rotating about x_0 as t varies.

From the definition of the associated connection K in D, it follows incidently that

Proposition 1. *Let $l(q_0)C=\{q_t\}=\{(b_t, p_t)\}$ be a lift of a given curve $C=\{b_t\}$ in B to Q, then any lift $l(d_0)C=\{d_t\}=\{(b_t, \bar{b}_t)\}$ of C to D is constructed by the rule $\bar{b}_t=p_t f$, where f is a fixed element $p_0^{-1}\bar{b}_0$ of F.*

As a consequence of the proposition, the parallel displacement as above defined is expressed in terms of a canonical coordinate as follows:

$$\frac{d\bar{b}^i}{dt}+\bar{b}^j\left(\Gamma^i_{jk}(x, b)\frac{dx^k}{dt}+C^i_{jk}(x, b)\frac{db^k}{dt}\right)=0, \tag{2. 1}$$

where $b_t=(x^i_t, b^i_t)$, $\bar{b}_t=(x^i_t, \bar{b}^i_t)$ and we put $\Gamma^i_{jk}=F^i_{jk}+C^i_{jl}F^l_k$. Finally, let us consider a curve $\underline{C}=\{x_t\}$ in M and take a lift $l(b_0)\underline{C}=\{b_t\}$ to B with respect to the non-linear connection H in B. The curve $\underline{C}$ together with its lift $l(b_0)\underline{C}$ is thought of as a special vecter field b_t defined along $\underline{C}$. We say that b_t is obtained from b_0 by *parallel displacement along* $\underline{C}$. In terms of a canonical coordinate, the parallel displacement of b_t is expressible by the equation

(2.2) $$\frac{db^i}{dt}+F^i_j(x,b)\frac{dx^j}{dt}=0,$$

where $x_t=(x^i_t)$ and $b_t=(x^i_t, b^i_t)$.

We see that equations (2.1) and (2.2) expressing parallel displacements coincide formally with that derived by several authors, see [6], in particular, p. 55 (3.18), p. 67 (1.3) and p. 82 (4.4).

§3. Horizontal paths

Definition. *The horizontal path C in B is the projection $\pi\bar{C}$ of an integral curve $\bar{C}$ of every h-basic vector field $B^h(f)$ on Q.*

If we put $C=\{b_t\}$ and take the projection $\tau C=\underline{C}=\{x_t\}$ on M, the tangent vector x_0' to $\underline{C}$ at x_0 is called the *initial direction* of the horizontal path C. By virtue of the definition of the non-linear connection H, it is obvious that C is horizontal, and hence the tangent vector b_0' to C at the starting point b_0 is obtained by $l_{b_0}x_0'$.

Proposition 2. *There exists uniquely a horizontal path by giving its starting point and initial direction.*

Proof. We first observe that, if the horizontal curve $C=\{b_t\}$ is the projection of an integral curve $\bar{C}=\{q_t\}=\{(b_t, p_t)\}$ of the h-basic vector field $B^h(f)$, the tangent vector b_t' to C at b_t is equal to $l_{b_t}p_tf$, as is easily seen from the definition of $B^h(f)$. Therefore, if $\underline{C}=\{x_t\}$ is the projection of $\underline{C}$ to M, the tangent vector x_t' to $\underline{C}$ at x_t is equal to p_tf.

Now, let any point b_0 of B and any direction $x_0'\in M_{x_0}$, $x_0=\tau(b_0)$, be given. If we take an arbitrary frame $p_0\in\pi^{-1}\tau(b_0)$, then the direction x_0' is expressed as p_0f, $f\in F$. The pair $(b_0, p_0)=q_0$ may be regarded as a point of Q, and then there exists a unique integral curve $\bar{C}=\{q_t\}=\{(b_t, p_t)\}$ through q_0 of the h-basic vector field $B^h(f)$, corresponding to the above $f\in F$. The projection $\pi\bar{C}=C=\{b_t\}$ is the desired horizontal path.

In order to complete the proof it is enough to show that the horizontal path C as above obtained is independent of the expression p_0f of the initial direction x_0'. If we take an another expression $p_0'f'$, there is an element $g\in G$ such that $p_0'=p_0g$, and hence $f'=g^{-1}f$. By virtue of the relation $B^h(g^{-1}f)=R_gB^h(f)$, we see that the integral curve $\overline{C'}$ of $B^h(f')$ through $q_0'=q_0g$ is given by $\overline{C'}=R_g\overline{C}$, and consequently we see $\pi\overline{C'}=\pi R_g\overline{C}=\pi\overline{C}$, which coincides with the above C.

Theorem 1. *Let $C=\{b_t\}$ be a horizontal curve in B and let $\tau C=\underline{C}=\{x_t\}$ be the image of C under the projection $\tau: B\rightarrow M$. The necessary and sufficient condition for C to be a horizontal path in B is that the tangent vector x_t' to $\underline{C}$ is obtained from x_0' by parallel displacement along $\underline{C}$ with respect to the element of support b_t.*

Proof. Suppose that C is a horizontal path and hence C is the projection of an integral curve $\overline{C}=\{q_t\}=\{(b_t, p_t)\}$ of the h-basic vector field $B^h(f)$. Since the tangent vector q_t' is equal to $l_{q_t}l_{b_t}(p_tf)$, we see that the tangent vector b_t' is $l_{b_t}p_tf$ and so the tangent vector x_t' is given by p_tf. Let us consider a curve $C^*=\{d_t\}=\{(b_t, x_t')\}$ in D, and it follows from Proposition 1 that C^* is a lift of C to D, because $\overline{C}=\{(b_t, p_t)\}$ is a lift of C to Q and that $x_t'=p_tf$. Thus we show the necessity of the condition in the theorem.

Conversely, if the condition holds for a horizontal curve C, then $C^*=\{d_t\}=\{(b_t, x_t')\}$ is a lift of C to D, by means of the definition of the parallelism, and hence Proposition 1 shows that $x_t'=p_tf$, where f is a fixed element of F and $\overline{C}=\{q_t\}=\{(b_t, p_t)\}$ is a lift of C to Q. Since C is assumed to be horizontal, the tangent vector q_t' to $\overline{C}$ is given by $l_{q_t}l_{b_t}(x_t')$, that is, $l_{q_t}l_{b_t}p_tf$, which is equal to $B^h(f)_{q_t}$. Thus $\overline{C}$ is an integral curve of $B^h(f)$, and we complete the proof.

In terms of a canonical coordinate, the expression of a horizontal path C is easily obtained by means of Theorem 1. Firstly, since C is horizontal, the equation (2. 2) is satisfied. Next, x_t' is parallel along $\{x_t\}$ with respect to b_t, and hence $x_t'=\overline{b}_t$ has to satisfy (2. 1). Accordingly the differential equation of a horizontal path is given as follows:

$$(3.1)\qquad \begin{aligned} &\frac{d^2x^i}{dt^2}+F^i_{jk}(x, b)\frac{dx^j}{dt}\frac{dx^k}{dt}=0,\\ &\frac{db^i}{dt}+F^i_j(x, b)\frac{dx^j}{dt}=0. \end{aligned}$$

The h-basic vector field $B^h(f)$ as above used is determined, of course, by choosing an element $f \in F$. We, however, can define an *intrinsic* h-horizontal vector field by making use of the characteristic field $\gamma : Q \to F$, $q=(b, p)$ $p^{-1}b \in F$. Namely, we denote by B^h the h-horizontal vector field which is defined by the rule $B^h_q = B^h(\gamma(q))_q$. Such a vector field B^h will be called the *h-characteristic vector field* on Q. We see at once that $B^h_q = l_q l_b(b)$, $q=(b, p)$, where the point $b \in B$ is to be thought of as the tangent vector at $x=\tau(b)$. Since $R_g B^h_q = B^h_{qg}$, we know that a projection through $b_0 \in B$ of an integral curve of B^h does not depend upon the choice of the starting point $q_0=(b_0, p_0)$ of the integral curve.

Definition. *The path in M is the projection $\tau\pi\overline{C}$ of an integral curve $\overline{C}$ of the h-characteristic vector field B^h.*

Corresponding to Proposition 2 for the case of a horizontal path, we shall show

Proposition 3. *The path $\underline{C}=\{x_t\}$ in M is uniquely determined by giving the starting point x_0 and the initial direction x_0'.*

Proof. We observe first that, if $\underline{C}=\{x_t\}$ is the projection of an integral curve $\overline{C}=\{q_t\}=\{(b_t, p_t)\}$ of B^h, then the tangent vector x_t' to $\underline{C}$ at x_t is equal to b_t, as is easily seen from the definition of B^h.

Now, let any point $x_0 \in M$ and any direction x_0' at x_0 be given. The direction x_0' is looked upon as the point $b_0=x_0'$ of B over x_0, and hence we have a projection $C=\{b_t\}$ through b_0 of an integral curve $\overline{C}=\{q_t\}=\{(b_t, p_t)\}$ of B^h. As have above shown, the curve C is uniquely detemined by its starting point $b_0=x_0'$. The projection $\underline{C}=\{x_t\}$ of C to M is the desired path, because the tangent vector q_t' to $\overline{C}$ is $l_{q_t} l_{b_t}(b_t)$, and so the tangent vector x_t' to $\underline{C}$ is equal to b_t, especially $x_0'=b_0$. This completes the proof.

It is to be remarked here that the propeety stated in Proposition 3 is analogous to that of a geodesic in a Riemannian manifold.

Theorem 2. *A curve $\underline{C}=\{x_t\}$ in M is a path in M if and only if the tangent vector x_t' to $\underline{C}$ is obtained from x_0' by parallel displacement along $\underline{C}$.*

Proof. Assume that $\underline{C}$ is a path in M, and then $\underline{C}$ is the projection of the horizontal curve $C=\{b_t\}$ in B, the latter being the projection of an integral curve $\overline{C}=\{q_t\}=\{(b_t, p_t)\}$ of B^h. The tangent vector q_t' to $\overline{C}$ is $l_{q_t}l_{b_t}(b_t)$, and hence the tangent vector x_t' to $\underline{C}$ is equal to b_t. Since $C=\{b_t\}=\{x_t'\}$ is horizontal, x_t' is parallel along $\underline{C}$. Consequently the necessity of the condition is shown. The sufficiency will be seen easily, observing that $C=\{b_t\}=\{x_t'\}$ is horizontal, and that the tangent vector to the lift $\overline{C}=\{q_t\}=\{(b_t, p_t)\}$ of C is equal to $l_{q_t}l_{b_t}(x_t')=l_{q_t}l_{b_t}(b_t)$.

Theorem 2 and the equation (2.2) gives at once the differential equation of a path in M in terms of a canonical coordinate as follows:

$$\frac{d^2x^i}{dt^i}+F^i_j\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt}=0. \tag{3.2}$$

§4. Vertical paths

The process by means of which we define horizontal paths in the last section is applied equally well when we use v-basic vector fields, instead of h-basic ones.

Definition. *The vertical path C is the projection $C=\pi\overline{C}$ of an integral curve $\overline{C}$ of every v-basic vector field $B^v(f)$ on Q.*

Let us consider an integral curve $\overline{C}=\{q_t\}=\{(b_t, p_t)\}$ of $B^v(f)$, and the projection $C=\pi\overline{C}=\{b_t\}$ on B. The tangent vector q_t' to $\overline{C}$ is equal to $B^v(f)_{q_t}=l_{q_t}(dp_t j_{\gamma_t} f)$, where $\gamma_t=\gamma(q_t)=p_t^{-1}b_t$, and hence the tangent vector b_t' to C is $dp_t j_{\gamma_t} f$. It is obvious that the vertical path C is vertical in B, and its projection τC is a single point x_0 in M. Therefore C is thought of as the tangent vector b_t rotating around the fixed point x_0.

Proposition 4. *There exists a unique vertical path by giving its starting point b_0 and the initial direction b_0'.*

Proof. We take an arbitrary frame $p_0\in\pi^{-1}\tau(b_0)$, and an element $f\in F$ such that $dp_0 b_0'=j_{\gamma_0}f$, where $\gamma_0=p_0^{-1}b_0$. Then we have the integral curve $\overline{C}$ through $q_0=(b_0, p_0)$ of $B^v(f)$ corresponding to the above $f\in F$. Put $\pi\overline{C}=C$, and then C is the desired vertical path, as will be easily verified. Moreover, in the similar way to the case of a horizontal path, it will be seen that C is well determined, independent of the choice of a frame p_0.

In order to examine the relation between a vertical path and parallel displacement, we consider a mapping

$$\sigma : B_b^v \rightarrow \tau^{-1}\tau(b), \quad X \rightarrow pj_f dp^{-1}X, \quad p \in \pi^{-1}\tau(b),$$

where $f=p^{-1}b$. As is easily verified, the mapping σ is well defined, independent of the choice of a frame p used. Thus, corresponding to a tangent vertical vector X at b, we have a point σX on the fibre through b. The point σX is called the *B-expression* of X. If $X= X^i(\partial/\partial b^i)_b$, the B-expression of X is the point having the canonical coordinate (x^i, X^i), where $b=(x^i, b^i)$.

Now, as has above shown, the tangent vector b_t' to a vertical path C is $dp_t j_{\gamma_t} f$, and hence we have the B-expression $\sigma b_t'=p_t f$. Therefore Proposition 1 shows that the curve $C^*=\{d_t\}=\{(b_t, \sigma b_t')\}$ is a lift of C to D. Conversely, if $C=\{b_t\}$ is a vertical curve in B such that the curve $C^*=\{d_t\}=\{(b_t, \sigma b_t')$ is a lift of C to D, it will be at once seen that C is a vertical path in B. Thus we have

Theorem 3. *The necessary and sufficient condition for a vertical curve C in B to be a vertical path is that the B-expression of the tangent vector to C is parallel along the curve C.*

In terms of a canonical coordinate, the B-expression of the tangent vector b_t' is the point $(x_t^i, db^i/dt)$, and then (2.1) gives the differential equation of a vertical path as follows:

$$\text{(4.1)} \qquad \frac{dx^i}{dt}=0, \quad \frac{d^2b^i}{dt^2}+C^i_{jk}(x_0, b)\frac{db^j}{dt}\frac{dx^k}{dt}=0,$$

where $x_t^i=x_0^i$ (=constants).

Similar to the definition of the h-characteristic vector field B^h, we have the *v-characteristic vector field* B^v, which is given by the rule $B_q^v=B^v(\gamma(q))_q=l_q(dpj_\gamma p^{-1}b)$, $q=(b, p)$. Since the projection of an integral curve $\overline{C}=\{q_t\}=\{(b_t, p_t)\}$ of B^v on the base M is a single point, we then are concerned with the projection $\pi\overline{C}=C=\{b_t\}$ on B.

The tangent vector b_t' to C is equal to $dp_t j_{\gamma_t}(p_t^{-1} b_t)$, and hence the B-expression $\sigma b_t'$ is equal to b_t. Conversely, if a vertical curve $C=\{b_t\}$ in B is such that $\sigma b_t'=b_t$, C is a projection of an integral curve of B^v, as will be easily verified. Thus we have

Proposition 5. *A vertical curve C in B is the projection of an integral curve of the v-characteristic vector field B^v on Q if and only if the B-expression of the tangent vector to C coincides with C itself.*

The definition of the B-expression does not depend upon a Finsler connection, and hence Proposition 5 shows that the curve C as above defined is out of all relation to the Finsler connection, and the equation is given by

$$(4.2) \qquad \frac{dx^i}{dt}=0, \quad \frac{db^i}{dt}=b^i.$$

The curve as just now considered will be of interest only in connection with a geometric interpretation of the equations (4.2), in particular the second one.

§5. Quasi-paths

There are three kinds of coefficients of a Finsler connection, that is, F^i_j, F^i_{jk} and C^i_{jk}. The first F^i_j take place in the equation (3.2) of a path in M, while the third C^i_{jk} appear in the equation (4.1) of a vertical path. However, as for the second F^i_{jk}, we have not yet an equation of the form

$$(5.1) \qquad \frac{d^2x^i}{dt^2}+F^i_{jk}\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}=0,$$

though we have already derived the equation (3.1) of a horizontal path, in which F^i_{jk} and further F^i_j have appeared. In order to consider a geometrical meaning of the above (5.1), we have to recall here a quasi-connection in P derived from a Finsler connection in Q [2, §2].

A quasi-connection $\Gamma_{(f)}$ with respect to a fixed element $f\in F$, more briefly, *quasi-f-connection* is by definition the distribution $\Gamma_{(f)}$; $p\in P$

$\to\Gamma_{(f)p}$ on P such that $\Gamma_{(f)p}=\eta\Gamma^h_q$, where $q=(pf, p)\in Q$. In a previous paper [2] we found the quasi-f-connection form $\omega^*_{(f)}$. The force of the prefix 'quasi-f' is that the form $\omega^*_{(f)}$ does not subject to the ordinary equation: $\omega^*_{(f)}R_g=ad(g^{-1})\omega^*_{(f)}$, but satisfies the equation (2.6) of [2]. This fact is also seen by the equation

$$R_g\Gamma_{(f)p}=\Gamma_{(g^{-1}f)pg}\,. \tag{5.2}$$

In fact, we see, according to the definition of $\Gamma_{(f)}$,

$$R_g\Gamma_{(f)p}=R_g\eta\Gamma^h_q=\eta R_g\Gamma^h_q=\eta\Gamma^h_{qg}\,,\quad q=(pf, p).$$

Since $qg=(pf, pg)=(pg\cdot g^{-1}f, pg)$, the equation (5.2) is proved.

Corresponding to $f_1\in F$, the *basic vector field* $B_{(f)}(f_1)$ of the quasi-f-connection is naturally obtaind by the rule $B_{(f)}(f_1)_p=\eta B^h(f_1)_q$, $q=(pf, p)$.

If an ordinary connection is given in P, then we have naturally the associated connection in B [4, p. 43]. We analogously obtain the connection H^* in B, corresponding to the quasi-f-connection in P as follows. That is, if we take a mapping $K_f: P\to B$, $p\to pf$, the distribution $H^*: b\in B\to H^*_b$ is defined by the rule $H^*_b=K_f\Gamma_{(f)p}$, $pf=b$. As above remarked, the quasi-f-connection in P depends upon the choice of $f\in F$ used, while we shall show that H^* does not so. To do this, if we take $f, f'\in F$ such that $b=pf=p'f'$, there exists an element $g\in G$ such that $p'=pg$, and so $f'=g^{-1}f$. By means of (5.2), we have

$$K_{f'}\Gamma_{(f')p'}=K_{f'}R_g\Gamma_{(f)p}=K_{gf'}\Gamma_{(f)p}=K_f\Gamma_{(f)p}\,,$$

as we wished to show. The distribution H^* determined in this way is called the *non-linear quasi-connection* in B.

With respect to the quasi-f-connection $\Gamma_{(f)}$ in P and the non-linear quasi-connection H^* in B, we can define, of course, the concepts of lifts and parallel displacements. Similar to Proposition 1, we can show immediately

Proposition 6. *Let* $C^*=\{p_t\}$ *be a lift of a given curve* $\underline{C}=\{x_t\}$

in M to P, with respect to the quasi-f-connection $\Gamma_{(f)}$. Then, a lift $C=\{b_t\}$ of $\underline{C}$ to B with respect to the non-linear quasi-connection H^* is constructed by $b_t=p_t f$.

We consider a particular basic vector field $B_{(f)}(f)$ of the quasi-f-connection, corresponding to the same $f\in F$. In the following, we shall denote this vector field by $B_{(f)}$ simply and call it the *self-basic vector field*. Further, as the image of $B_{(f)}$ under the mapping K_f, we have the quasi-horizontal vector field F^2 on B. This vector field on B is called the *F^2-vector field*. In terms of a canonical coordinate, F^2 is expressed by

$$F_b^2=b^i\left(\frac{\partial}{\partial x^i}-b^k F_{ki}^j(x,b)\frac{\partial}{\partial b^j}\right), \tag{5.3}$$

where $b=(x^i,b^i)$. It is to be noticed here that we have an another special vector field F^1 on B such that $F_b^1=\pi B^h(f)_q$, $q=(pf,p)$, $b=pf$. It is easy to see that F_b^1 is well determined independent of the choice of the expression $b=pf$. The vector field F^1 is obviously horizontal with respect to the non-linear connection H in B and is expressed by

$$F_b^1=b^i\left(\frac{\partial}{\partial x^i}-F_i^j(x,b)\frac{\partial}{\partial b^j}\right). \tag{5.4}$$

In a previous paper [3], we derived the equation [3, (1.1)], which gave the differential of the characteristic field γ. By virtue of that equation, we obtain the relation between above vector fields F^1 and F^2 as follows:

$$dp\gamma B^h(f)_q=F_b^1-F_b^2,\quad q=(pf,p),\ b=pf. \tag{5.5}$$

Since $F_b^1=\pi B^h(\gamma)_q$, $q=(pf,p)$, $b=pf$, we obtain

Proposition 7. *The path in M is the projection of an integral curve of the vector field F^1 on B.*

Corresponding to this characterization of a path, we now lay down the following definition.

Definition. *The quasi-path in M is the image of an integral curve of F^2 vector field on B under the projection $\tau: B\to M$.*

As a consequence of (5. 3), we now can recognize that the equation (5. 1) just is the differential equation satisfied by a quasi-path in M.

Let us consider a quasi-path $\underline{C}=\{x_t\}$ in M which is the projection τC of an integral curve $C=\{b_t\}$ of the F^2 vector field. By means of the definition of F^2, the curve C is the image of the integral curve $C^*=\{p_t\}$ of the self-basic vector field $B_{(f)}$ on P under the mapping K_f. From the relation $\tau K_f=\pi$ it follows that a quasi-path $\underline{C}$ just is the projection πC^*. The tangent vector p_t' to C^* is, by definition, equal to $B_{(f)p_t}=\eta B^h(f)_{q_t}$, $q_t=(p_tf, p_t)$, and hence the tangent vector is x_t' to $\underline{C}$ is expressed as $\pi\eta B^h(f)_{q_t}$, which is equal to $\tau\bar{\pi}B^h(f)_{q_t}=p_tf$ $=b_t$. Thus we have $x_t'=b_t$. From the viewpoint of the non-linear quasi-connection H^*, this fact permits us to state that the tangent vector field x_t' to the quasi-path $\underline{C}$ is parallel along $\underline{C}$ with respect to H^*.

Conversely, if this fact is true for a curve $\underline{C}=\{x_t\}$ in M, we have a quasi-horizontal curve $C=\{b_t\}=\{x_t'\}$ in B, the locus of the tangent vector x_t' to $\underline{C}$, and then Proposition 6 shows that there exists a lift $C^*=\{p_t\}$ of $\underline{C}$ to P with respect to the quasi-f-connection $\Gamma_{(f)}$ such that $x_t'=p_tf$, $f\in F$. Since C^* is horizontal, the tangent vector p_t' to C^* is written by $B_{(f)}(f_1)_{p_t}=\eta B^h(f_1)_{q_t}$, $q_t=(p_tf, p_t)$, where f_1 is some element of F. Since C^* is a lift of $\underline{C}$, we see that $x_t'=$ $\pi\eta B^h(f_1)_{q_t}=\tau\bar{\pi}B^h(f_1)_{q_t}=p_tf_1$, while $x_t'=p_tf$ as above shown. It follows that $f_1=f$ and $p_t'=\eta B^h(f)_{q_t}=B_{(f)p_t}$. Therefore C^* is an integral curve of the self-basic vector field $B_{(f)}$ and so $\underline{C}$ is a quasi-path certainly. Consequently, we give an alternative characterization of a quasi-path in

Theorem 4. *A curve $\underline{C}=\{x_t\}$ in M is a quasi-path if and only if the tangent vector x_t' to $\underline{C}$ is parallel along $\underline{C}$ with respect to the non-linear quasi-connection H^*.*

If the Finsler connection under consideration satisfies the condition F [3, §6], the concept of a quasi-path coincides with that of a path, which will be easily seen from (5. 5) and Proposition 7, or equations (3. 2) and (5. 1) concretely.

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