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On the branching process for Brownian particles with an absorbing boundary

By

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Introduction

Recentry A. V. Skorohod [4] gave a general treatment of the branching process from a standpoint of the theory of Markov processes. In this paper we shall apply this to discuss some problems of a branching process which was studied by B. A. Sevast'yanov [3]. We shall discuss in particular the problem of the extinction and some limiting property of the number of particles. As for the latter our result corresponds to that of T. E. Harris [1] in the case of age dependent branching processes. In a recent book of Harris [2] this result was strengthened to the almost sure convergence but in our case it seems difficult to apply his arguments and we could not succeed in this point.

§ 1 Preliminaries

In general a branching process with particles of one type on a locally compact separable Hausdorff space S is determined if we are given a Markov process $x_t(P_x, x \in S)$ on S and a system of branching measures $(p_n(x), \Pi_n(x,dy))_{n=0}^{\infty}$ where $p_n(x)$, $x \in S$ satisfies

$$0 \leq p_n(x) \leq 1, \qquad \sum_{n=0}^{\infty} p_n(x) = 1$$

and $\Pi_n(x, dy)$, $x \in S$, $y=(y_1, y_2, \cdots y_n) \in S^n$ is a probability measure

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1) Independently a quite similar idea was given by K. Ito at the seminar of probability theory at Kyoto University.

on S^n which is symmetric (i. e. invariant under any permutation of $(y_1,\cdots, y_n)$) for each $x \in S$. Usually x_t is a α_t-subprocess of a Hunt process $\tilde{x}_t$ $(\tilde{P}_x)$ on $\bar{S}=S\cup\{\triangle\}$ [2] with the property

$$\tilde{P}_x(\tilde{x}_{\tilde{\zeta}-}\in S,\ \tilde{\zeta}<+\infty)=0 \quad \text{for every } x\in S^{(3)}$$

and α_t is a continuous non-increasing multiplicative functional of $\tilde{x}_t$. Intuitively a particle of our branching process starts $a\in S$ according the law P_a and when $t=\zeta$ [4] and $x_{\zeta-}=x\in S$, then it branches into n particles $y_1, y_2,\cdots, y_n$ with probability $p_n(x)$ and the position of these particles is determined by the law $\Pi_n(x, dy)$. Each of these particles starts afresh and continues independently the same motion. When $x_{\zeta-}=\triangle$ then it remains $\triangle$ forever.

Now let $S^{(0)}=\{\triangle\}$, $S^{(1)}=S$ and $S^{(n)}$ be the symmetrization of S^n. If at time t the branching process consists of n particles then they define a point in $S^{(n)}$ and so it defines a stochastic process $X_t(\boldsymbol{P}_X;\ X\in \boldsymbol{S}=\bigcup_{n=0}^{\infty} S^{(n)})$ which is clearly a strong Markov process on $\boldsymbol{S}$ with right continuous trajectories. *In the sequel we shall give our arguments in terms of this 'large' Markov process* X_t.

We set

$$Z_t=n \quad if \quad X_t\in S^{(n)} \tag{1.1}$$

$$e_\infty=\sup\{t\,;\sup_{u\in[0,t)} Z_u<+\infty\}^{(5)} \tag{1.2}$$

$$e_\triangle=\inf\{t<e_\infty\,;Z_t=0\} \tag{1.3}$$

Z_t is nothing but the number of particles at time t and e_∞ and $e_\triangle$ are called *the explosion time* and *the extinction time* respectively. Set also

$$T=\inf\{t;\ Z_t\neq Z_0\} \tag{1.4}$$

$$T_k=T_{k-1}+\theta_{T_{k-1}}T,\ k=1,\ 2,\cdots \tag{1.5}$$

where $T_0=0$ and θ is the usual shift operator.

2) $\triangle$ is the point at infinity when S is not compact and an isolated point otherwise.

3) $\tilde{\zeta}$ is the terminal time of $\tilde{x}_t$-process; $\tilde{\zeta}=\inf\{t;\ \tilde{x}_t=\triangle\}$ $(\inf\phi=+\infty)$

4) ζ is the terminal time of x_t-process.

5) For $t\geq e_\infty$, we shall set $x_t=\triangle$.

§ 2 The extinction problem of the Sevast'yanov model

The branching process discussed in Sevast'yanov [3] is the following;

(2.1) $S=G\subset R^N$: a bounded domain with a sufficiently smooth boundary ∂G,

(2.2) $\tilde{x}_t(\tilde{P}_x,\ x\in G)$: the Brownian motion on G with ∂G as an absorbing barrier (so we identify ∂G with $\triangle$) determined by the diffusion equation

$$\frac{\partial u}{\partial t}=D\Delta u,\ (D>0: const.),$$

(2.3) $x_t: e^{-ct}$—subprocess of $\tilde{x}_t$ $(c>0: const)$,

(2.4) $p_n(x)=p_n,\quad p_0=0,\quad p_1<1,$

(2.5) $\pi_n(x,\ E)=\chi_E(x,x,\cdots,x)$ (6)

Set

(2.6) $$F[\xi]=\sum_{p=1}^{\infty} p_n\xi^n,\quad \text{for } 0\leq\xi\leq 1$$

then it is clear that $F[\xi]$ is strictly increasing and strictly convex and

$$F[0]=0,\ F[1]=1.$$

We shall assume $F'[1]<+\infty$, then we have

(2.7) $$\boldsymbol{P}_x(e_\infty=+\infty)=1\ ^{(7)}.$$

By a fundamental result of Skorohod [4], for $\hat{f}(x)\in C(G)$, $\|f\|\leq 1$, if we define $\hat{f}(X)$, $X\in S$ by

$$\begin{aligned}\hat{f}(X)&=1 && \text{if } X=\triangle,\\ &=f(x_1)f(x_2)\cdots f(x_n) && \text{if } X=(x_1,\ x_2,\cdots x_n)\in S^{(n)}\end{aligned}$$

and if we set

6) $\chi_E\ (x,x,\cdots,x)=1$ if $(x,\cdots,x)\ \in E$
$=0$ otherwise

7) Without this assumption the explosion happens in general: if we set $P_x(e_\infty=+\infty)\equiv u_\infty(x)$, $x\in G$ then $u_\infty(x)<1 \leftrightarrows \int^1 \frac{d\xi}{\xi-F[\xi]}<+\infty$ *Cf.* Harris [2] pp. 106-107. N. Ikeda gave another interesting proof of this fact. By the result given below we see also $u_\infty(x)=u_1(x)\equiv P_x(e_\triangle<+\infty)$ when $u_\infty<1$.

$$u(t,\ x)=\boldsymbol{T}_t\hat{f}(x)=\boldsymbol{E}_x\ [\hat{f}(X_t)],\quad x\in G$$

then u satisfies the following non-linear differential equation,

(2. 8) $\dfrac{\partial u}{\partial t}=D\varDelta u+c(F[u]-u),\ u(0+,\ x)=f(x),\ u(t,\ x)|_{x\to\partial G}=1.$

Set

(2. 9) $Z_t^E=\sum_{i=1}^{Z_t}\chi_E(X_t^{(i)}),\qquad E\in\boldsymbol{B}(G), X_t=(X_t^{(1)},\cdots X_t^{(Z_t)})\in S^{(Z_t)}$

(2.10) $$M(t,\ x,\ E)=\boldsymbol{E}_x(Z_t^E).$$

Then

$$u(t,\ x)=M_tf(x)\equiv\int_G M(t,x,dy)f(y)=\boldsymbol{E}_x(\sum_{i=1}^{Z_t}f(X_t^{(i)}))$$

satisfies the following parabolic differential equation

(2.11) $$\frac{\partial u}{\partial t}=D\varDelta u+a\cdot u,\ u(0+,x)=f(x),\ u(t,\ x)|_{x\to\partial G}=0$$

where

(2.12) $$a=c(F'[1]-1)$$

In fact (2.11) follows from (2. 8) at once by putting $u(t,\ x)=\boldsymbol{T}_t\hat{g}(x)$ where $g(x)=\lambda^{f(x)}$, $0<\lambda<1$ and differentiating with respect to λ and then putting $\lambda=1$.

Hence

$$M(t,x,E)=\int_E m(t,\ x,\ y)dy$$ [8]

with

(2.13) $$m(t,\ x,\ y)=e^{at}\ p(t,\ x,\ y)$$

where $p(t,\ x,\ y)$ is the transition probability density of $\tilde{x}_t$.

Now consider the eigenvalue problem:

$$(\varDelta+\lambda)\varphi=0,\qquad \varphi|_{x\to\partial G}=0$$

and let

8) dy=the Lebesgue measure.

$$0<\lambda_1<\lambda_2\leq\lambda_3\leq\cdots \text{ and } \varphi_1(x),\ \varphi_2(x),\cdots$$

be its eigenvalues and the corresponding normalized eigenfunctions. As is well-known $\varphi_1(x)>0$, $x\in G$ and

$$p(t, x, y)=\sum_{i=1}^{\infty} e^{-D\lambda_i t}\varphi_i(x)\varphi_i(y) \tag{2.14}$$

and so from (2.13)

$$m(t, x, y)=\sum_{i=1}^{\infty} e^{(a-D\lambda_i)t}\varphi_i(x)\varphi_i(y) \tag{2.15}$$

LEMMA 2.1
$\boldsymbol{P}_x(Z_t\to 0$ *or* $Z_t\to+\infty$ *when* $t\to+\infty)=1$ *for all* $x\in G$.

PROOF Take a positive integer $k\geq 1$. It is enough to show that the probability that Z_t takes the value of k infinitely often is zero. Set

$$R=R_1=\inf\{t\,;Z_t=k\}\quad(\inf\phi=+\infty)$$
$$S_1=R_1+\theta_{R_1}T^{(9)}$$
$$R_2=S_1+\theta_{S_1}R$$
$$S_2=R_2+\theta_{R_2}T$$
$$\cdots\cdots\cdots$$

Then for $x\in G$

$$\boldsymbol{P}_x(Z_t \textit{ takes } k \textit{ infinitely often})$$
$$=\boldsymbol{P}_x(\bigcap_n\{R_n<+\infty\})=\lim_{n\to\infty}\boldsymbol{P}_x(R_n<+\infty)$$

Noting that for every $x\in G$

$$\boldsymbol{P}_x(X_{T-}\in\partial G)=1-c.\int_0^{\infty}e^{-ct}\,dt\int_G p(t, x, y)dy\geq\alpha>0$$

we have

$$\boldsymbol{P}_x(R_1<\infty)\leq 1-\boldsymbol{P}_x(X_{T-}\in\partial G)\leq 1-\alpha$$
$$\boldsymbol{P}_x(R_2<\infty)=\boldsymbol{E}_x(\boldsymbol{P}_{X_{R_1}}(T+\theta_T R<\infty)\,;\ R_1<\infty)$$
$$\leq\boldsymbol{E}_x[(1-\prod_{i=1}^{k}\boldsymbol{P}_{X_{R_1}^{(i)}}(X_{T-}\in\partial G))\,;\ R_1<+\infty]$$
$$\leq(1-\alpha)(1-\alpha^k)$$

9) T is defined by (1.4).

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$$P_x(R_n<+\infty)\leq(1-\alpha)(1-\alpha^k)^{n-1} \longrightarrow 0 \ (n\to\infty).$$

Set

(2.16) $$u_1(x)=P_x(Z_t\to 0)=P_x(e_\Delta<+\infty) \quad x\in G$$

and call it the *extinction probability*.

THEOREM 2.1 (*Sevast'yanov*) *$u_1(x)$ is the smallest solution of*

(2.17) $$v(x)=h(x)+\int_G F[v(y)]K(x, dy), \quad 0\leq v\leq 1$$

where

(2.18) $$h_x=P_x(X_{T-}\in\partial G)=1-c.\int_0^\infty e^{-ct}dt\int_G p(t, x, y)dy$$

(2.19) $$K(x,E)=P_x(X_{T-}\in E)=c.\int_0^\infty e^{-ct}dt\int_E p(t, x, y)dy, \ E\in B(G).$$

PROOF [10] Since

$$u(t,x)\equiv P_x(e_\Delta<t)=P_x(T<t, X_{T-}\in\partial G)+\int_0^t\int_G F[u(t-s, y)] \times P_x(T\in ds, X_{T-}\in dy)$$

by letting $t\to\infty$ we obtain (2.17)

Now let v be any solution of (2.17). Set

(2.20) $$u^{(k)}(x)=P_x(Z_{T_k}=0)^{(11)}$$

then

(2.21) $$\begin{aligned} u^{(k)}(x)&=h(x)+E_x(P_{X_T}(Z_{T_{k-1}}=0)\,;\ X_{T-}\in G)\\ &\leq h(x)+E_x([P_{X_{T-}}(Z_{k-1}=0)]^{Z_T}\,;\ X_{T-}\in G)\\ &=h(x)+\int_G K(x, dy)F[u^{(k-1)}(y)]\end{aligned}$$

Now $u^{(o)}(x)=0\leq v(x)$ and if $u^{(k-1)}(x)\leq v(x)$ then

$$u^{(k)}(x)\leq h(x)+\int_G K(x, dy)F[u^{(k-1)}(y)]\leq h(x)+\int_G K(x,dy)F[v(y)] =v(x).$$

Thus for every k, $u^{(k)}(x)\leq v(x)$ and letting $k\to\infty$ we have

$$u_1(x)\leq v(x).$$

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10) The proof is essentially the same as that of [3].

11) T_k is defined by (1.5).

COROLLARY $u_1(x)$ *is the smallest solution of*

(2.22) $$D\Delta u=c.(u-F[u]),\ 0\leq u\leq 1,\ u(x)|_{x\to\partial G}=1$$

THEOREM 2.2 (*Sevast'yanov*) *If we set*

(2.23) $$\alpha=a-D\lambda_1=c\,(F'[1]-1)-D\lambda_1$$

then if $\alpha\leq 0$, $u_1(x)\equiv 1$ *while if* $\alpha>0$, $u_1(x)<1$ *for all* $x\in G$.

PROOF We shall give here a proof somewhat different from that of [3].

Suppose $\alpha\leq 0$ and we shall prove any solution u of (2.22) is $u\equiv 1$. Setting $v=1-u$ we have

$$D\Delta v=c\cdot f(1-v),\quad v|_{\partial G}=0,\quad 0\leq v\leq 1$$

where

$$f(\xi)=F[\xi]-\xi.$$

Since $v(x)\geq 0$,

$$D\Delta v=cf(1\text{-}v)=-c(f(1)-\mathrm{f}(1\text{-}v))\geq -cf'(1)v$$

and so

$$D\Delta v+av\geq 0.$$

Note also that, since $\mathrm{f}(\xi)$ is strictly convex, if $v(x)>0$ then

$$D\Delta v(x)+a\cdot v(x)>0.$$

Now

$$\int_G\varphi_1(x)[D\Delta v(x)+a\cdot v(x)]dx=-D\lambda_1\int_G\varphi_1(x)\,v(x)dx+a\int_G\varphi_1(x)v(x)dx$$
$$=\alpha\int_G\varphi_1(x)\,v(x)\,dx$$

and so if $\int_G\varphi_1(x)\,v(x)\,dx>0$, then $\alpha\int_G\varphi_1(x)\,v(x)\,dx>0$. But this is impossible since $\alpha\leq 0$. So $\int\varphi_1(x)\,v(x)\,dx=0$ and therefore $v(x)\equiv 0$.

Suppose $\alpha>0$. Take β, $0<\beta<1$ such that

$\beta.\dfrac{c+D\lambda_1+\alpha}{c+D\lambda_1}\leq 1$ and take ε, $0<\varepsilon<1$ such that $F'[1\text{-}\varepsilon]\geq\beta F'[1]$
Next take $\delta>0$ such that $\delta\ \max_{x\epsilon G}\varphi_1(x)\leq\varepsilon$. Set $w(x)=1-\delta\varphi_1(x)$ then

$$\begin{aligned}
h(x)+\int_G K(x, dy)F[x(y)] &= 1-\int_G K(x, dy)[1-F[w(y)]]\\
&=1-\int_G K(x, dy)[F[1]-F[w(y)]]\\
&=1-\int_G K(x, dy)F'[p_y](1-w(y))\\
&\leqq 1-\beta F'[1]\delta\int_G K(x, dy)\varphi_1(y) \quad (\because 1>p_y>w(y)\geq 1-\varepsilon\\
&=1-\beta\frac{c.F'[1]}{c+D\lambda_1}\delta.\varphi_1(x) \quad\quad \therefore F'[p_y]\geq F'[1\text{-}\varepsilon]\geq\beta F'[1])\\
&=1-\beta\frac{c+D\lambda_1+\alpha}{c+D\lambda_1}\delta.\varphi_1(x)\leq 1-\delta\varphi_1(x)=w(x).
\end{aligned}$$

Let $u^{(k)}(x)$ be defined by (2.20) then $u^{(0)}(x)=0\leq w(x)$ and by (2.21) if $u^{(k-1)}(x)\leq w(x)$ then $u^{(k)}(x)\leq w(x)$. So $u^{(k)}(x)\leq w(x)$ for every k and $u_1(x)=\lim_{k\to\infty}u^{(k)}(x)\leq w(x)<1$.

Thus if $\alpha>0$ (2.22) has at least two solutions; $u\equiv 1$ and $u_1(x)$ but we can show there is no other solution. This fact will be needed in § 3

LEMMA 2.2 *Let* $\alpha>0$ *then the equation* (2.22) *has just two solutions*; $u\equiv 1$ and $u_1(x)$.

PROOF Setting $v=1-u$, (2.22) is equivalent to

$$D\Delta v=c.f(1-v),\quad 0\leq v\leq 1,\quad v|=0 \tag{2.24}$$

where

$$f(\xi)=F[\xi]-\xi. \tag{2.25}$$

Since $f(\xi)\leq 0$ any solution v of (2.24) is superharmonic and so if $v(x)=0$ for some $x\epsilon G$ then $v\equiv 0$.

Now let v be a solution of (2.24) such that $v(x)>0$ for all $x\epsilon G$. Set $v_1=1-u_1$, then v_1 and v satisfy

$$D\Delta v_1-cf(1-v_1)=0$$
$$D\Delta v-cf(1-v)=0$$

and so

$$\int_G \{D(\Delta v_1 . v - \triangle v . v_1) - c[f(1-v_1). v - f(1-v). v_1]\} \, dx = 0$$

Noting

$$\int_G (\Delta v_1 \cdot v - \Delta v . v_1) dx = 0$$

we have

$$\int_G [f(1-v_1)v - f(1-v)v_1] \, dx$$

$$= \int_G [\frac{f(1) - f(1-v_1(x))}{v(x)} - \frac{f(1) - f(1-v_1(x))}{v_1(x)}] \, v_1(x) v(x) dx$$

$=0.$

But since $v(x) \leq v_1(x)$, $v(x).v_1(x) > 0$ and $f(\xi)$ is strictly convex we must have

$$\frac{f(1) - f(1-v(x))}{v(x)} = \frac{f(1) - f(1-v_1(x))}{v_1(x)} \qquad \text{a.e.}$$

and so $v(x) = v_1(x)$ a. e.. Therefore $v(x) \equiv v_1(x)$.

§ 3 Limiting properties of Z_t and Z_t^E.

In this section we shall assume

$$\alpha > 0 \quad and \quad F''[1] < +\infty.$$

Set

$$A(E) = \frac{\int_E \varphi_1(x) dx}{\int_G \varphi_1(x) dx} \qquad E \in B(G). \tag{3. 1}$$

It is clear that

$$A \cdot M_t = \int_G A(dx) M(t, x, .) = e^{\alpha t} A. \tag{3. 2}$$

In the sequel we shall prove that $\frac{Z_t^E}{Z_t}$ converges in a certain sense to the non random distribution $A(E)$.

THEOREM 3.1 *For* $E, F \in B(G)$

(3. 3) $\boldsymbol{E}_A(Z_t^E Z_{t+s}^E)$

$$=e^{\alpha t}\left\{cF''[1]\int_0^t e^{-\alpha u}du\int_G \boldsymbol{E}_x(Z_u^E)\boldsymbol{E}_x(Z_{u+s}^F)A(dx)+\int_E \boldsymbol{E}_x(Z_s^F)A(dx)\right\}^{(12)}$$

COROLLARY

(3.4) $$\boldsymbol{E}_A(Z_t^E Z_{t+s}^F)=e^{2\alpha t+\alpha s}\frac{cF''[1]}{\alpha}\int_E\varphi_1(x)dx\int_F\varphi_1(x)dx$$
$$\times\int_G\varphi_1^3(x)dx\left(\int_G\varphi_1(x)dx\right)^{-1}(1+O(e^{-\delta t}))$$

where

(3. 5) $$\delta=\alpha\wedge D(\lambda_2-\lambda_1)>0.$$

$O(e^{-\delta t})$ *is independent of s.*

PROOF Since there is no essential difference we shall prove for the simplicity the case $E=F=G$. First fix $s\geq 0$ and set

$$u(t,x\,;\lambda,\mu)=\boldsymbol{E}_x(\lambda^{Z_t}\mu^{Z_{t+s}})\quad 0<\lambda<1,\ 0<\mu<1.$$

Since

$$\boldsymbol{E}_x(\lambda^{Z_t}\mu^{Z_{t+s}})=\boldsymbol{E}_x(\lambda^{Z_t}\boldsymbol{E}_{X_t}(\mu^{Z_s}))=\boldsymbol{E}_x(\lambda^{Z_t}\prod_{i=1}^{Z_t}\boldsymbol{E}_{X_t^{(i)}}(\mu^{Z_s}))=\boldsymbol{T}_t\hat{f}(x)\quad x\in G$$

where

$$f(x)=\lambda\boldsymbol{E}_x(\mu^{Z_s})\quad x\in G,$$

u satisfies the Skorohod equation:

(3. 6) $$\frac{\partial u}{\partial t}=D\Delta u+c(F[u]-u),\quad u(0+,x)=\lambda\boldsymbol{E}_x(\mu^{Z_s}),\quad u(t,x)\big|_{x\to\partial G}=1.$$

Set

$$u_1(t,x\,;\mu)=\boldsymbol{E}_x(Z_t\mu^{Z_{t+s}})=\frac{\partial}{\partial\lambda}u(t,x\,;\lambda,)\Big|_{\lambda=1}$$

Differentiating both sides of (3. 6) with respect to λ and then putting $\lambda=1$ we obtain

(3. 7) $$\frac{\partial u_1}{\partial t}=D\Delta u_1+c(F'[v]-1)u_1,\ u_1(0+,x)=\boldsymbol{E}_x(\mu^{Z_s}),\ u_1(t,x)\big|_{x\to\partial G}=0$$

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12) $\boldsymbol{E}_A(\cdot)=\int_G\boldsymbol{E}_x(\cdot)A(dx)$

where

$$v(x)=\boldsymbol{E}_x(\mu^{Z_{t+s}})$$

Set

$$u_2(t,x)=\boldsymbol{E}_x(Z_tZ_{t+s})=\frac{\partial}{\partial\mu}u_1(t, x\,;\mu)\Big|_{\mu=1}$$

Differentiating both sides of (3.7) by μ and then putting $\mu=1$ we obtain

$$\frac{\partial u_2}{\partial t}=D\Delta u_2+c(F'[1]-1)u_2+cF''[1]\boldsymbol{E}_x(Z_t)\boldsymbol{E}_x(Z_{t+s}), \tag{3.8}$$

$$u_2(0+, x)=\boldsymbol{E}_x(Z_s),\quad u_2(t,x)\big|_{x\to\partial G}=0.$$

Now we expand $u_2(t, x)$ and $\boldsymbol{E}_x(Z_t)\boldsymbol{E}_x(Z_{t+s})$ in terms of eigenfunctions;

$$u_2(t, x)=\sum_{i=1}^{\infty}f_i(x)\varphi_i(x) \tag{3.9}$$

where

$$f_i(t)=\int_G u_2(t, x)\,\varphi_i(x)dx \tag{3.10}$$

and

$$\boldsymbol{E}_x(Z_t)\boldsymbol{E}_x(Z_{t+s})=\sum_{i=1}^{\infty}g_i(t)\varphi_i(x) \tag{3.11}$$

where

$$g_i(t)=\int_G \boldsymbol{E}_x(Z_t)\boldsymbol{E}_x(Z_{t+s})\,\varphi_i(x)dx. \tag{3.12}$$

Substituting (3.9) and (3.11) into (3.8) we have for $i=1,2,\cdots$

$$\begin{aligned} f_i'(t)&=-D\lambda_i f_i(t)+c(F'[1]-1)f_i(t)+cF''[1]g_i(t) \\ &=(a-D\lambda_i)f_i(t)+cF''[1]g_i(t),\quad f_i(0+)=e^{(a-D\lambda_i)s}\int_G\varphi_i(x)dx. \end{aligned} \tag{3.13}$$

We can easily solve (3.13) and obtain

$$\begin{aligned} f_i(t)=&e^{(a-D\lambda_i)t} \\ &\times\left\{\int_0^t e^{(a-D\lambda_i)u}\,cF''[1]g_i(u)du+e^{(a-D\lambda_i)s}\int_G\varphi_i(x)\,dx\right\}. \end{aligned} \tag{3.14}$$

So we have calculated $u_2(t, x)$ in the form (3.9) with $f_i(t)$ given by (3.14). Integrating both sides of (3.9) by $A(dx)$ we obtain (3.3) for the case $E=F=G$.

Now Corollary follows easilly from the formula

$$\boldsymbol{E}_x(Z_t^E)=\sum_{i=1}^{\infty} e^{(\alpha-D\lambda_i)}\int_E \varphi_i(x)dx\,\varphi_i(x),\quad t>0,\ E\in \boldsymbol{B}(G).$$

Set

$$W_t=\frac{Z_t}{e^{\alpha t}}$$

and

$$W_t^E=\frac{Z_t^E}{e^{\alpha t}A(E)},\qquad E\in \boldsymbol{B}(G).$$

THEOREM 3.2 (*Mean convergence of W_t and W_t^E*) *There exists a random variable $W\geq 0$ such that for every $x\in G$*

$$\boldsymbol{E}_x[(W_t-W)^2]=O(e^{-\delta t}) \tag{3.15}$$

and further for every $E\in \boldsymbol{B}(G)$

$$\boldsymbol{E}_x[(W_t^E-W)^2]=O(e^{-\delta t}). \tag{3.16}$$

PROOF From (3.4) we have

$$\boldsymbol{E}_A(W_t^E\,W_{t+s}^F)=\frac{cF''[1]}{\alpha}\int_G \varphi_1{}^3(x)dx\int_G \varphi_1(x)dx(1+O(e(^{-\delta t}))$$

where $O(e^{-\delta t})$ is independent of s. Hence

$$\boldsymbol{E}_A[(W_t^F-W_{t+s}^F)^2]=O(e^{-\delta t}).$$

In this formula taking $E=F=G$ we see that $W=\underset{t\to\infty}{\text{l.i.m.}}\ W_t$ exists and letting $F=G$ and $s\to\infty$ we see

$$\boldsymbol{E}_A[(W_t^E-W)^2]=O(e^{-\delta t}).$$

Now take $u>0$ and fix it. Then

$$\boldsymbol{E}_x[(W_{u+t}^E-W)^2]=\frac{1}{e^{2\alpha u}}\boldsymbol{E}_x\{\boldsymbol{E}x_u[(W_t^E-W)^2]\}$$

$$=\frac{1}{e^{2\alpha u}}\boldsymbol{E}_x(\sum_{i=1}^{Z_u}\boldsymbol{E}x_u^{(i)}[(W_t^E-W)^2])$$

$$+\frac{1}{e^{2\alpha u}}E_x\left(\sum_{\substack{i,j=1\\ i\neq j}}^{Z_u} E_{X_u^{(i)}}[W_t^E - W]\, E_{X_u^{(j)}}[W_t^E - W]\right)$$

$$=I_1+I_2, \quad X_u=(X_n^{(1)}, X_u^{(2)},\cdots,X_u^{(Z_u)}).$$

It is easy to see that $E_x(W_t^E - W)=O(e^{-D(\lambda_2-\lambda_1)t})$ where $O(\cdot)$ is independent of x. So we have

$$|I_2| \leq O(e^{-2D(\lambda_2-\lambda_1)t})E_x(Z_u^2)$$

Since there exists $C>0$ such that $m(u, x, y)\leq C\varphi_1(y)$ we have

$$I_1=\frac{1}{e^{2\alpha u}}\int_G m(u, x, y)E_y[(W_t^E - W)^2]dy$$
$$\leq C'E_A[(W_t^E - W)^2]=O(e^{-\delta t})$$

and the proof was complete.

THEOREM 3.3 *For every* $x \in G$

$$(W>0)=(e_\Delta=+\infty)$$

modulo P_x*—null set.*

PROOF It is clear $(W>0)\subseteq(e_\Delta=+\infty)$ and so it is enough to prove

$$P_x(W>0)=P_x(e_\Delta=+\infty)$$

equivalently

$$P_x(W=0)=P_x(e_\Delta<+\infty)\equiv u_1(x)$$

First it is easy to see that $u(x)=P_x(W=0)$ satisfies the equation (2.17) and so the equation (2.22). On the other hand since $W=\underset{t\to\infty}{\text{l.i.m.}}\, W_t$

$$E_x(W)=\lim_{t\to\infty}E_x(W_t)=\varphi_1(x)\int_G\varphi_1(x)dx>0$$

and so $u(x)<1$. Then by Lemma 2.2 $u(x)=u_1(x)$.

COROLLARY *Let* $\{t_n\}$ *be any sequence such that* $\sum_{n=1}^{\infty} e^{-\delta t_n} <+\infty$ *then*

$$\boldsymbol{P}_x\left(\frac{Z_{t_n}^E}{Z_{t_n}} \rightarrow A(E) \text{ when } n \rightarrow \infty \,|\, e_\Delta = +\infty\right) = 1$$

for every $x \in G$ *and* $E \in \boldsymbol{B}(G)$.

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