J. Math. Kyoto Univ.
5-1 (1965) 87-142

On iterated suspensions I.

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(Received October 21, 1965)

Introduction

The $(n+t)$-th homotopy groups $\pi_{n+t}(S^n)$ of n-spheres S^n are stable if $n>t+1$ with respect to Freudenthal's suspension homomorphism $S:\pi_{n+t}(S^n)\to\pi_{n+t+1}(S^{n+1})$, and the t-stem group π_t^S is the limit of $\{\pi_{n+t}(S^n)\}$.

Throughout this paper p will denote an odd prime which is fixed. $\pi_i(X, A:p)$ indicates the p-primary component of $\pi_i(X, A)$. Serre [10] obtained the following direct sum decomposition:

$$\pi_{i+1}(S^{2m}:p)\approx\pi_i(S^{2m-1}:p)\oplus\pi_{i+1}(S^{4m-1}:p).$$

So, we shall devote to consider the groups $\pi_{2m-1+t}(S^{2m-1}:p)$ and $2k$-fold iterated suspensions

$$S^{2k}:\pi_{2m-1+t}(S^{2m-1}:p)\to\pi_{2(m+k)-1+t}(S^{2(m+k)-1}:p).$$

Moore [8] and Serre [10] proved that the above homomorphism S^{2k} is an isomorphism if $t<2m(p-1)-2$, that is, the group $\pi_{2m-1+t}(S^{2m-1}:p)$ is stable if $m>(t+2)/2(p-1)$ and denoted by $(\pi_t^S:p)$.

The homomorphism S^2 is related with groups $\pi_i(\Omega^2S^{2m+1}, S^{2m-1})$ by the following exact sequence:

$$\cdots\to\pi_{i+1}(\Omega^2S^{2m+1}, S^{2m-1})\xrightarrow{\partial}\pi_i(S^{2m-1})\xrightarrow{S^2}\pi_{i+2}(S^{2m+1})$$
$$\xrightarrow{H^{(2)}}\pi_i(\Omega^2S^{2m+1}, S^{2m-1})\xrightarrow{\partial}\cdots.$$

In [13, Th. (8.3)] the author gave an exact sequence

$$\cdots\to\pi_{i+3}(S^{2mp+1}:p)\xrightarrow{\Delta}\pi_{i+1}(S^{2mp-1}:p)\to\pi_i(\Omega^2S^{2m+1},S^{2m-1}:p)$$
$$\to\pi_{i+2}(S^{2mp+1}:p)\to\cdots,\quad (i>2mp-2),$$

from which we shall have a direct sum decomposition

$$\pi_i(\Omega^2S^{2m+1},S^{2m-1}:p)\approx\pi^S_{i-2mp+2}\otimes Z_p+\mathrm{Tor}(\pi^S_{i-2mp+1},Z_p)$$

for $i<2mp^2-4$. This means that if we know the stable groups up to π^S_t, then we can estimate the unstable groups $\pi_{2m-1+t}(S^{2m-1}:p)$ for $2m-1+t<2mp^2-5$, such a case will be referred as a *meta-stable case.*

In the present paper, the iterated suspensions, in particular, four fold iterated suspensions S^4 are discussed. There are many numbers of unstable elements.

The first type of unstable elements is an elements γ of $\pi_{2m-1+t}(S^{2m-1}:p)$ such that $\gamma\notin\mathrm{Im}\,S^2$ and $S^2\gamma=0$. For example, if $t=2r(p-1)-2$, $r\not\equiv 0 \pmod p$ then such an element γ exists for $\mathrm{Max}(1,r/(p+1))<m\leq r$ and γ generates a direct factor isomorphic to Z_p (see Theorem 5.2.).

The second type of unstable elements is a pair of elements $\gamma\in\pi_{2m+1+t}(S^{2m+1}:p)$ and $\gamma'\in\pi_{2m-1+t}(S^{2m-1}:p)$ such that $\gamma\notin\mathrm{Im}\,S^2$, $p\cdot\gamma=S^2\gamma'\neq 0$ and $\gamma'\notin\mathrm{Im}\,S^2$. The possibility of the existence of such a pair will be proved for $t=2rp(p-1)-2$, $=2rp(p-1)-1$. For example, such a pair exists for $1<m<p$, $t=2p(p-1)-2$ and for $m=2$, $t=2rp(p-1)-1$. These examples reprove recent results of Gershenson [4] and Hardie [5]: $\pi_{2m-1+t}(S^{2m-1}:p)\approx Z_{p^2}$ for $2<m\leq p$ and $t=2p(p-1)-2$, $2p(p-1)-1$.

The third type of unstable elements exists: $\gamma\notin\mathrm{Im}\,S^2$, $S^{2p-4}\gamma\neq 0$ and $S^{2p-2}\gamma=0$. We may announce the existence of the fourth type of unstable elements: $\gamma\notin\mathrm{Im}\,S^2$, $S^{2p}\gamma\neq 0$ and $S^{2p+2}\gamma=0$.

In section 1 we shall prepare some notions of homotopy theory. Section 2 will be an introduction of the results of [13]. In section 3 we shall compute the cohomology of a space Q^{2m-1}_{2k} such that $\pi_i(Q^{2m-1}_{2k})\approx\pi_{i+1}(\Omega^2S^{2m+1},S^{2m-1})$. For dimensions less than $p(2mp-2)-2$

the cohomology ring $H^*(Q_{2k}^{2m-1}; Z_p)$ has a form $\Lambda(a_0, a_1, \cdots, a_{k-1}) \otimes Z_p[\Delta a_0, \Delta a_1, \cdots, \Delta a_{k-1}]$ with relations $\mathcal{P}^1 a_i = (m+i+1) a_{i+1}$, $\mathcal{P}^1 \Delta a_i = (m+i) \Delta a_{i+1}$, $i=0, 1, \cdots, k-2$. Section 4 will be a discussion on homotopy groups of Moore spaces $Y_p^n = S^{n-1} \cup_p e^n$, and the results will be applied for the existence of unstable elements of the first and the second types in section 5. Some results on meta-stable groups will be obtained in section 6. In section 7 we shall determine the groups $\pi_{2m-1+t}(S^{2m-1} : p)$ for $t<2(2p+3)(p-1)-3$.

In the forthcoming paper II, we shall discuss more delicate problems. One problem is how to compute unstable but not meta-stable groups which may be solved by clarifying the properties of $\Delta : \pi_{i+3}(S^{2mp+1} : p) \rightarrow \pi_{i+1}(S^{2mp-1} : p)$. The second problem is the structure of Q_{2k}^{2m-1} which can not be determined by cohomological operations $\mathcal{P}^t$. A relative J-homomorphism of mod p type will be introduced in order to solve the second problem. The other problems are the existence of unstable elements of the third and the fourth types and further computations of $\pi_{2m-1+t}(S^{2m-1} : p)$.

1. Preliminaries.

In this paper, all topological spaces will have the base points $*$, all maps and homotopies will preserve the base points. The set of the homotopy classes of maps $f : (X, A) \rightarrow (Y, B)$ will be denoted by

$$\pi(X, A; Y, B).$$

We shall use the following notations:

$\alpha = \{f\} \in \pi(X, A; Y, B)$: the homotopy class of a map $f : (X, A) \rightarrow (Y, B)$,

$\beta_* = g_* : \pi(X, A; Y, B) \rightarrow \pi(X, A; Y', B')$: the covariant map induced by a map $g : (Y, B) \rightarrow (Y', B')$, $\beta = \{g\}$, $\beta_*\{f\} = g_*\{f\} = \{g \circ f\}$,

$\gamma^* = h^* : \pi(X, A; Y, B) \rightarrow \pi(X', A'; Y, B)$: the contravariant map induced by a map $h : (X', A') \rightarrow (X, A)$, $\gamma = \{h\}$, $\gamma^*\{f\} = h^*\{f\} = \{f \circ h\}$,

$\beta\circ\alpha$ or simply $\beta\alpha$: the composition of homotopy classes α and β, $\beta\circ\alpha=\beta_*\alpha=\alpha^*\beta$,

$\pi(X;\ Y)=\pi(X, *;\ Y, *)$,

1_X: the identity of X,

$\iota_X\in\pi(X;\ X)$: the class of 1_X,

$X\wedge Y$: the reduced join of spaces X and Y, $X\wedge Y=(X\times Y)/(X\times *\cup *\times Y)$,

$f\wedge g: X\wedge Y\to X'\wedge Y'$: the reduced join of maps $f: X\to X'$ and $g: Y\to Y'$,

$\alpha\wedge\beta$: the reduced join of homotopy classes α and β, $\alpha\wedge\beta=\{f\wedge\beta\}$ for $\alpha=\{f\}$ and $\beta=\{g\}$,

$I=[0, 1]$: the unit interval with $*=(0)$,

$CX=X\wedge I$: cone over X,

S^n: the unit n-sphere $\{(t_1, \cdots, t_{n+1})\,|\,\sum t_i^2=1\}$,

E^{n+1}: the unit n-cube $\{(t_1, \cdots, t_{n+1})\,|\,\sum t_i^2\leq 1\}$ considered as CS^n,

$S^nX=X\wedge S^n$: n-fold suspension of a space X, $SX=S^1X$ and $S^{m+n}X=S^nS^mX$ by suitable identification $S^{m+n}=S^m\wedge S^n$,

$S^nf=f\wedge 1_{S^n}$: n-fold suspension of a map f, $Sf=S^1f$,

$\pi_n(X, A)=\pi(E^n, S^{n-1};\ X, A)$: the n-th homotopy group of (X, A),

$\pi_n(X)=\pi(S^n;\ X)$: the n-th homotopy group of X,

$\iota_n\in\pi_n(S^n)$: the class of the identity $1_n=1_{S^n}$ of S^n,

$S^n\alpha=\alpha\wedge\iota_n$: n-fold suspension of a homotopy class α, $S\alpha=S^1\alpha$,

$C_f=Y\cup_f CX$: mapping-cone of a map $f: X\to Y$, where we identify SC_f with C_{Sf} by the natural way,

$Y_q^n=S^{n-1}\cup e^n$: mapping-cone of a map $f: S^{n-1}\to S^{n-1}$ of degree q, $CY_p^n=Y_q^{n+1}$, $n\geqq 2$,

$\Omega(X;\ A, B)$: space of the paths $l: (I, 0, 1)\to(X, A, B)$ with the compact open topology,

$\Omega(X, A)=\Omega(X;\ *, A)$,

$\Omega X=\Omega(X;\ *, *)$: space of the loops in X,

$\Omega f: \Omega X\to\Omega Y(: \Omega(X, A)\to\Omega(Y, B))$: the map defined by a map $f: X\to Y(: (X, A)\to(Y, B))$, $\Omega f(l)(t)=f(l(t))$ for $l\in\Omega X$

$(\in \Omega(X, A))$, $t \in I$.

Throughout the paper all pairs (X, A) will have the same homotopy types of pairs of CW-complexes. We have Puppe's exact sequence [9] for a map $f: X \to Y$:

$$\cdots \xrightarrow{Sf^*} \pi(SX;\ W) \xrightarrow{\pi^*} \pi(C_f;\ W) \xrightarrow{i^*} \pi(Y;\ W) \xrightarrow{f^*} \pi(X;\ W),$$

where

$$i: Y \to C_f = Y \cup_f CX \quad \text{and} \quad \pi: C_f = Y \cup_f CX \to SX$$

are the inclusion and the pinching map of Y respectively. The pinching map π is defined as follows. Let

$$\varphi: (I, \dot{I}) \to (S^1, *)$$

be an orientation preserving map pinching the boundary $\dot{I}$ of I and

$$\chi: (CX, Y) \to (Y \cup_f CX, Y),\ \chi | X = f,$$

be the characteristic map, then

$$\pi \circ \chi = 1_X \wedge \varphi\ :\ CX = X \wedge I \to SX = X \wedge S^1.$$

Let $p: E \to B$ be a fibering (in the sense of Serre) and $F = p^{-1}(*)$ be the fibre over $*$, then we have an exact sequence

$$\cdots \xrightarrow{p_*} \pi(SX;\ B) \xrightarrow{\partial} \pi(X;\ F) \xrightarrow{i_*} \pi(X;\ E) \xrightarrow{p_*} \pi(X;\ B),$$

where $i: F \to E$ is the inclusion and ∂ is defined by

$$\partial = \partial' \circ p_*^{-1} \circ \pi^* : \pi(SX;\ B) \to \pi(CX, X;\ B, *) \leftarrow \pi(CX, X;\ E, F) \\ \to \pi(X;\ F),$$

$\partial'\{f\} = \{f | X\}$. Here p_* and π^* are one-to-one onto, and we may identify

$$\pi(SX;\ Y) = \pi(CX;\ Y, *)$$

by π^*. For example, $p: E = \Omega(Y, B) \to B$, $p(l) = l(1)$, is a fibering with the fibre $F = \Omega Y$, then the above sequence is equivalent to the following usual exact sequence:

$$\cdots \xrightarrow{\partial} \pi(SX;\ B) \xrightarrow{i_*} \pi(SX;\ Y) \xrightarrow{j_*} \pi(CX, X;\ Y, B) \xrightarrow{\partial} \pi(X;\ B).$$

Let $d'_X : X\times I\to CX=X\wedge I$ be a map defining CX. Put $i'(x)(t)=d'_X(x,t)$, $x\in X$, $t\in I$, then

$$i' : X\to\Omega(CX, X)$$

is an imbedding and a homotopy equivalence. For each element $\{f\}\in\pi(CX, X;\ Y, B)$ we put $\Omega\{f\}=\{\Omega f\circ i'\}\in\pi(X;\ \Omega(Y, B))$, then we have

(1. 1). $\Omega : \pi(CX, X;\ Y, B)\to\pi(X;\ \Omega(Y, B))$ *is one-to-one onto.*

For the pinching map $\pi : (CX, X)\to(SX, *)$, we have an imbedding

$$i=\Omega\pi\circ i' : X\to\Omega(SX)$$

which we call a canonical imbedding. From (1. 1) it follows

(1. 1)$'$. $\Omega\{f\}=\{\Omega f\circ i\}$, $\{f\}\in\pi(SX;\ Y)$, *defines a one-to-one onto map*

$$\Omega : \pi(SX;\ Y)\to\pi(X;\ \Omega Y).$$

Note that the sets $\pi(SX;\ Y)$ and $\pi(X;\ \Omega Y)$ form groups by canonical methods, and the map Ω of (1. 1)$'$ is an isomorphism. The group $\pi(S^2X;\ Y)$ is abelian. These (1. 1) and (1. 1)$'$ are generalizations of well-known isomorphisms:

$$\Omega : \pi_{n+1}(Y, B)\approx\pi_n(\Omega(Y, B)),\quad \pi_{n+1}(Y)\approx\pi_n(\Omega Y).$$

It is verified directly that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ \Omega(SX) & \xrightarrow{(\Omega Sf)} & \Omega(SY) \end{array}$$

is commutative. Thus the commutativity of the following diagram (1. 2) holds.

$$\text{(1. 2)}\qquad \begin{tikzcd} & \pi(SX;\ SY) \arrow[d, "\Omega", "\approx"'] \\ \pi(X;\ Y) \arrow[ur, "S"] \arrow[r, "i_*"'] & \pi(X;\ \Omega(SY)). \end{tikzcd}$$

The following relations hold.

(1.3). (i) $S(\alpha\circ\beta)=S\alpha\circ S\beta,\quad \alpha\in\pi(X;\ Y),\quad \beta\in\pi(W;\ X).$

(ii) $\Omega(\alpha\circ S\beta)=\Omega\alpha\circ\beta,\quad \alpha\in\pi(SX;\ Y),\ \beta\in\pi(W;\ X).$

(iii) $\alpha\circ(\beta_1+\beta_2)=\alpha\circ\beta_1+\alpha\circ\beta_2,$

$\alpha\in\pi(X;\ Y),\quad \beta_1,\beta_2\in\pi(SW;\ X).$

(iv) $(\alpha_1+\alpha_2)\circ S\beta=\alpha_1\circ S\beta+\alpha_2\circ S\beta,$

$\alpha_1,\alpha_2\in\pi(SX;\ Y),\ \beta\in\pi(W;\ X).$

For the suspension S we have

(1.4). *Assume that Y is r-connected space and K is a CW-complex. Then $S:\pi(K;\ Y)\to\pi(SK;\ SY)$ is one-to-one onto if dim $K<2r+1$ and onto if dim $K=2r+1$.*

This follows from the fact that $(\Omega(SY),\ Y)$ is $(2r+1)$-connected.

We denote the suspension limit of $\pi(X;\ Y)$ by

$$\pi^S(X;\ Y)=\lim\pi(S^nX;\ S^nY)$$

which is an abelian group. The notation

$$S^\infty:\pi(S^nX;\ S^nY)\to\pi^S(X;\ Y)$$

indicates the projection to the limit. (1.4) shows that if X is a finite dimensional CW-complex then the above S^∞ is an isomorphism for sufficiently large n. We shall also use the notation:

$$\pi_k^S=\pi^S(S^k;\ S^0)=\lim\pi_{n+k}(S^n).$$

It follows from (1.4) that

$$S^\infty:\pi_{n+k}(S^n)\to\pi_k^S$$

is an isomorphism for $n>k+1$. We have

(1.5). *Let p be an odd prime and $n\geq 4$. Then $\pi(Y_p^n;\ Y_p^n)$ is a cyclic group of order p generated by ι_Y, $Y=Y_p^n$. Thus the groups $\pi(Y_p^n;\ W)$ and $S\pi(SX;\ Y_p^n)$ are Z_p-modules.*

The following (1.6) will be used in later.

(1.6). *Let $X\supset A\supset B$ be a triple of spaces. Then the sequence*

$$\Omega(A, B) \xrightarrow{i'} \Omega(X, B) \xrightarrow{i} \Omega(X, A)$$

of the inclusions is a fibering up to homotopy equivalence, i.e., $\Omega(A, B)$ *and* $\Omega(X, B)$ *are deformation retracts of* $F=\Omega(\Omega(X, A), \Omega(X, B))$ *and* $E=\Omega(\Omega(X, A);\ \Omega(X, B), \Omega(X, A))$ *respectively, and the sequence is equivalent to the fibering* $F \rightarrow E \xrightarrow{p} \Omega(X, A)$. *Furthermore, the inclusion* i' *is equivalent to a fibering with a fibre* $\Omega(\Omega(X, A))$.

For the sake of simplicity, we shall use the following notations:

$$\Omega X=\Omega^1 X,\ \Omega^k X=\Omega(\Omega^{k-1}X),\ \Omega^k(X, A)=\Omega^{k-1}(\Omega(X, A)),\ k=2, 3, \cdots,$$
$$\Omega^k=\Omega^{k-1}\circ\Omega : \pi(S^k X;\ Y) \xrightarrow{\approx} \pi(X;\ \Omega^k Y),\ \Omega^1=\Omega,\quad k=2, 3, \cdots,$$
$$\Omega^{k+1}=\Omega^k\circ\Omega : \pi(CS^k X, S^k X;\ Y, B) \xrightarrow{\approx} \pi(X;\ \Omega^{k+1}(Y, B)),\ k=1, 2, \cdots.$$

The canonical imbeddings $S^{n+k}\subset\Omega S^{n+k+1}$, $k=0, 1, \cdots$, define a sequence of inclusions

$$S^n\subset\Omega S^{n+1}\subset\cdots\subset\Omega^k S^{n+k}\subset\Omega^{k+1}S^{n+k+1}\subset\cdots$$

such that the diagram

$$\begin{tikzcd} & \pi_{i+k}(S^{n+k}) \arrow[r, "S^h"] \arrow[d, "\approx"', "\Omega^k"] & \pi_{i+k+h}(S^{n+k+h}) \arrow[d, "\approx"', "\Omega^{k+h}"] \\ \pi_i(S^n) \arrow[ur, "S^k"] \arrow[r, "i_*"'] & \pi_i(\Omega^k S^{n+k}) \arrow[r, "i_*"] & \pi_i(\Omega^{k+h}S^{n+k+h}) \end{tikzcd}$$

is commutative. We put

$$(1.7)' \qquad Q_k^n=\Omega(\Omega^k S^{n+k}, S^n).$$

From the homotopy exact sequence associated with the fibering $p : Q_k^n \rightarrow S^n$ we have the following exact one:

$$(1.7) \qquad \cdots\rightarrow\pi_i(Q_k^n) \xrightarrow{p_*} \pi_i(S^n) \xrightarrow{S^k} \pi_{i+k}(S^{n+k}) \xrightarrow{H^{(k)}} \pi_{i-1}(Q_k^n)\rightarrow\cdots,$$

where

$$H^{(k)}=j_*\circ\Omega^{k+1} : \pi_{i+k}(S^{n+k})\rightarrow\pi_{i-1}(\Omega^{k+1}S^{n+k})\rightarrow\pi_{i-1}(Q_k^n)$$

for the inclusion $j : \Omega^{k+1}S^{n+k}\subset\Omega(\Omega^k S^{n+k}, S^n)=Q_k^n$.

For an abelian group G, we denote by

$$(G : p)$$

the p-primary component of G. Also we shall use the following notations:

$$\pi_i(X:p)=(\pi_i(X):p), \quad \pi_i(X, A:p)=(\pi_i(X, A):p).$$

We denote by C_p a class (in the sense of Serre [10]) of finite abelian gronps with vanishing p-primary components. If G and H are finitely generated abelian groups then a C_p-isomorphism $f:G\to H$ induces an isomorphism $f:(G:p)\to(H:p)$. For the convenience we introduce the following theorems of C-theory from [10].

(1.8) (i) *Assume that a pair (X, A) is 2-connected, A is simply connected and the homology groups $H_i(X)$, $H_i(A)$ are finitely generated for all i. If $H_i(X, A)\in C_p$ for $i<n$, then $\pi_i(X, A)\in C_p$ for $i<n$ and Hurewicz homomorphism $\tau:\pi_n(X, A)\to H_n(X, A)$ is a C_p-isomorphism.*

(ii) *Assume that spaces X and Y are 2-connected and $H_i(X)$, $H_i(Y)$ are finitely generated for all i. Let $f:X\to Y$ be a map. Then the following two conditions are equivalent*:

*) *$f^*:H^i(Y; Z_p)\to H^i(X; Z_p)$ is an epimorphism for $i<n$ and a monomorphism for $i\leq n$;*

**) *$f_*:\pi_i(X)\to\pi_i(Y)$ is a C_p-isomorphism for $i<n$ and a C_p-epimorphism for $i=n$.*

Serre also obtained the following (1.9) in [10].

(1.9) *The correspondence $(\alpha, \beta)\to S\alpha+[\iota_{2m}, \iota_{2m}]\circ\beta$ gives a C_p-isomorphism $\pi_i(S^{2m-1})+\pi_{i+1}(S^{4m-1})\to\pi_{i+1}(S^{2m})$, where $[\,,\,]$ indicates Whitehead product. Thus we have a direct sum decomposition*:

$$\pi_{i+1}(S^{2m}:p)\approx\pi_i(S^{2m-1}:p)+\pi_{i+1}(S^{4m-1}:p).$$

As a corollary we have

(1.10). (i) *$S:\pi_i(S^{2m-1}:p)\to\pi_{i+1}(S^{2m}:p)$ is a monomorphism and*

$$(k\iota_{2m-1})\circ\alpha=k\alpha$$

for arbitary integer k and $\alpha\in\pi_i(S^{2m-1}:p)$.

(ii) *For any element γ of $\pi_{i+1}(S^{2m}:p)$ there exists an element*

α *of* $\pi_i(S^{2m-1}:p)$ *such that* $S\gamma=S^2\alpha$.

(i) follows from (1. 9) and (1. 3), (iv). (ii) follows from (1. 9) and $S[\ ,\ ]=0$.

Theorem 1. 1. *Let* X *be a 2-connected space having finitely generated homology groups* $H_i(X)$ *for all* i. *Let* $u_i\in H^{n_i-1}(X;\ Z_p)$, $i=1,2,\cdots,r$, $4\leq n_1\leq n_2\leq\cdots\leq n_r$, *be elements on which Bockstein operators* δ/q_i, $q_i=p^{t_i}$, *are defined. Let* $Z_p\{u_i,\ (\delta/q_i)u_i\}$ *be a free* Z_p*-module with a base* $\{u_i,\ (\delta/q_i)u_i,\ i=1,\cdots,r\}$. *Assume that the natural homomorphism* $Z_p\{u_i,\ (\delta/q_i)u_i\}\to H^*(X;\ Z_p)$ *is an isomorphism for dimension less than* n_r *and a monomorphism for dimension* n_r. *Then there exist a CW-complex* K *and a map* $f:K\to X$ *satisfying the following conditions. There is a sequence* $*=K_0\subset K_1\subset\cdots\subset K_r=K$ *of subcomplexes of* K *such that* K_{i+1} *is a mapping cone of a map* $g_i:Y_{q_i}^{n_i-1}\to K_i$, $i=0,1,\cdots,r-1$. $f_*:\pi_j(K)\to\pi_j(X)$ *is a* C_p*-isomorphism for* $j<n_r$ *and* C_p*-epimorphism for* $j=n_r$.

Proof. The case $r=0$ is trivial. Assume that $n_s<n_{s+1}=\cdots=n_r$ and a complex $K'=K_s$ and a map $f_s=f':K_s\to X$ are constructed such that the conditions of the theorem are satisfied. By use of a mapping cylinder of f', we may assume that $K_s\subset X$. Then we have $H^j(X,\ K_s;\ Z_p)\approx H^{j-1}(\Omega(X,K_s);\ Z_p)$ for $j\leq n_r$. Applying (1. 8), (i) we see that $\pi_{n_r-2}(\Omega(X,K_s))$ is finite and the p-primary component of it is isomorphic to $Z_{q_{s+1}}+\cdots+Z_{q_r}$. Then there exist maps $g'_i:Y_{q_i}^{n_i-1}\to\Omega(X,K_s)$ the restrictions of which on S^{n_i-1} represent generators of Z_{q_i}. Put $g_i=p\circ g'_i$ for the fibering $p:\Omega(X,K_s)\to K_s$ and construct K_r. Then f_s is naturally extended over $f:K_r\to X$ and the required conditions are verified without difficulties. By induction on r the theorem is proved. q. e. d.

Theorem 1. 2. *Let* K *be a complex having a structure as in Theorem* 1. 1. *Assume that a map* $f:X\to Y$ *induces* C_p*-isomorphisms* $f_*:\pi_j(X)\to\pi_j(Y)$ *for* $j<N$ *and a* C_p*-epimorphism* f_* *for* $j=N$. *Then* $f_*:\pi(K;\ X)\to\pi(K;\ Y)$ *is one-to-one onto if* $\dim K<N$ *and onto if* $\dim K=N$.

Proof. Without loss of generality, we may assume that f is a fibering. Let F be a fibre. Then the assumption is equivalent to $\pi_j(F)\in\mathcal{C}_p$ for $j<N$. First consider the case $r=1$, $K=Y^n_q$, $q=p^t$. In Puppe's sequence

$$\pi_n(F)\xrightarrow{(q\iota)^*}\pi_n(F)\longrightarrow\pi(Y^n_q;\ F)\xrightarrow{i^*}\pi_{n-1}(F)\xrightarrow{(q\iota)^*}\pi_{n-1}(F)$$

$(q\iota)^*\alpha=q\alpha=p^t\alpha$. If $\pi_n(F)$, $\pi_{n-1}(F)\in\mathcal{C}_p$, then $(q\iota)^*$ are isomorphisms. Thus $\pi(Y^n_q;\ F)=0$ for $n<N$. Then the theorem is proved for $r=1$ by the exactness of the sequence

$$\pi(Y^n_q;\ F)\rightarrow\pi(Y^n_q;\ X)\xrightarrow{f^*}\pi(Y^n_q;\ Y)\rightarrow\pi(Y^{n-1}_q;\ F).$$

Assume that the theorem is true for K_{r-1}. Let $K=K_r=K_{r-1}\cup_g CY^n_q$, and consider the following diagram

$$\begin{array}{ccccccccc}
\pi(SK_{r-1};\ X) & \xrightarrow{Sg^*} & \pi(Y^{n+1}_q;\ X) & \xrightarrow{\pi^*} & \pi(K;\ X) & \xrightarrow{i^*} & \pi(K_{r-1};\ X) & \xrightarrow{g^*} & \pi(Y^n_q;\ X)\\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*}\\
\pi(SK_{r-1};\ Y) & \xrightarrow{Sg^*} & \pi(Y^{n+1}_q;\ Y) & \xrightarrow{\pi^*} & \pi(K;\ Y) & \xrightarrow{i^*} & \pi(K_{r-1};\ Y) & \xrightarrow{g^*} & \pi(Y^n_q;\ Y).
\end{array}$$

Remark that we can apply the five lemma to the diagram even if some of these homotopy sets do not form groups. Then we have that the theorem is true for $K=K_r$ and proved by induction on r.

Corollary 1.3. *In Theorem* 1.1. *assume further that the natural homomorphism into $H^*(X;\ Z_p)$ is an isomorphism for dimension n_r and a monomorphism for dimension n_r+1. Then complexes K satisfying the conditions of Theorem* 1.1 *are homotopy equivalent to each other and the map f is unique up to homotopy equivalences.*

Proof. Let $f:K\rightarrow X$ and $f':K'\rightarrow X$ satisfy the conditions of Theorem 1.1. Apply (1.8), (ii) and Theorem 1.2 to these maps, then we see that $f_*:\pi(K'';\ K)\rightarrow\pi(K'';\ X)$ and $f'_*:\pi(K'';\ K')\rightarrow\pi(K'';\ X)$ are one-to-one onto for $K''=K$ or $=K'$. Then $f_*^{-1}\{f'\}$ gives a homotopy equivalence $g:K'\rightarrow K$. If $K=K'$, then this shows that f is unique up to homotopy equivalences of K in itself.

2. Double suspensions.

We shall recall the results of [13] on $S^2: \pi_i(S^{2m-1}:p) \to \pi_{i+2}(S^{2m+1}:p)$ and add some necessary properties.

Let $S^n_\infty = S^n \cup e^{2n} \cup \cdots \cup e^{kn} \cup \cdots$ be the reduced product complex [6] of S^n and $S^n_k = S^n \cup \cdots \cup e^{kn}$ the kn-skeleton of S^n_∞. The canonical inclusion of S^n into ΩS^{n+1} is extended over S^n_∞ one-to-one way and continuously, and the resultant is a (singular) homotopy equivalence: $S^n_\infty \to \Omega S^{n+1}$. Thus we may consider that

$$S^n_\infty = \Omega S^{n+1},\ H^*(\Omega S^{n+1}) = H^*(S^n_\infty),\ \pi_i(S^n_\infty, S^n_k) = \pi_i(\Omega S^{n+1}, S^n_k).$$

The following (2.1), (2.2) and (2.3) are main results of [13].

(2.1) *There is a map* $h_p: (S^{2m}_\infty, S^{2m}_{p-1}) \to (\Omega S^{2mp+1}, *)$ *which induces* C_p*-isomorphisms* $h_{p*}: \pi_i(S^{2m}_\infty, S^{2m}_{p-1}) \to \pi_i(\Omega S^{2mp+1})$ *for all* i [13: Theorem (2.11)]. h_{p*} *is an isomorphism if* $i < 2mp + 2m - 1$.

(2.2) *There exist a space* Y *and maps* $h: (\Omega S^{2m}_{p-1}, S^{2m-1}) \to (Y, *)$ *and* $i: \Omega S^{2mp-1} \to Y$ *such that the maps induce* C_p*-isomorphisms*

$$h_*: \pi_i(\Omega S^{2m}_{p-1}, S^{2m-1}) \to \pi_i(Y)$$

and $\quad i_*: \pi_i(\Omega S^{2mp-1}) \to \pi_i(Y)$ (*isomorphic if* $i < 4mp - 4$)

for all i [13: Proposition (7.5), (7.3)'].

(2.3) *Let* $f: S^{2mp-1} \to S^{2mp-1}$ *be a map of degree* p. *Then there exists a homomorphism* ν *such that the diagram*

$$\begin{array}{ccccccc}
\pi_{i+3}(S^{2mp+1}) & \xleftarrow{S^2} & \pi_{i+1}(S^{2mp-1}) & \xrightarrow[\approx]{\Omega} & \pi_i(\Omega S^{2mp-1}) & \xrightarrow{\Omega f_*} & \pi_i(\Omega S^{2mp-1}) \\
\approx \downarrow \Omega^2 & & \downarrow & \nu & & & \downarrow i_* \\
\pi_{i+1}(\Omega^2 S^{2mp+1}) & \xleftarrow{h_{p*}} & \pi_{i+1}(\Omega S^{2m}_\infty, \Omega S^{2m}_{p-1}) & \xrightarrow{\partial} & \pi_i(\Omega S^{2m}_{p-1}, S^{2m-1}) & \xrightarrow{h_*} & \pi_i(Y)
\end{array}$$

is commutative (*see the proof of* [13: Theorem (8.3)]).

Put

$$Q'_m = \Omega(\Omega S^{2m}_{p-1},\ S^{2m-1}) \text{ and } Q_m = \Omega(\Omega^2 S^{2m+1}, \Omega S^{2m}_{p-1}).$$

By use of (1.8), we have the following (2.1)' and (2.2)' from (2.1) and (2.2).

(2. 1)′ *$H^*(Q_m: Z_p)$ and $H^*(\Omega^3S^{2mp+1}; Z_p)$ are naturally isomorphic. There is a C_p-isomorphism $I_0=\Omega^{-1}\circ h_{p*}\circ\Omega^{-2}: \pi_i(Q_m)\to\pi_{i+3}(S^{2mp+1})$ for all i (an isomorphism for $i<2mp+2m-3$), hence an isomorphism $I_0:\pi_i(Q_m:p)\approx\pi_{i+3}(S^{2mp+1}:p)$ for all i.*

(2. 2)′ *$H^*(Q'_m; Z_p)$ and $H^*(\Omega^2S^{2mp-1}; Z_p)$ are naturally isomorphic. $i_*^{-1}\circ h_*$ gives a C_p-isomorphism $I'_0:\pi_i(Q'_m)\to\pi_{i+2}(S^{2mp-1})$ for $i<4mp-5$ and an isomorphism $I'_0:\pi_i(Q'_m:p)\approx\pi_{i+2}(S^{2mp-1}:p)$ for all i.*

By use of (1. 3), (ii), we have the following (2. 1)″ and (2. 2)″.

$$\begin{aligned}&(2.\,1)'' \quad I_0(\alpha\circ\beta)=I_0\alpha\circ S^3\beta,\\ &(2.\,1)'' \quad I'_0(\alpha\circ\beta)=I'_0\alpha\circ S^2\beta,\end{aligned}\qquad \beta\in\pi_j(S^i:p).$$

Apply (1. 6) to the triple $(\Omega^2S^{2m+1}, \Omega S^{2m}_{p-1}, S^{2m-1})$, then we have a sequence

$$(2.\,4)\qquad Q'_m\xrightarrow{i'}Q_2^{2m-1}\xrightarrow{i}Q_m$$

of inclusions equivalent to a fibering, and we have an exact sequence

$$(2.\,4)'\qquad \cdots\to\pi_{i+1}(Q_m)\xrightarrow{\partial}\pi_i(Q'_m)\xrightarrow{i'_*}\pi_i(Q_2^{2m-1})\xrightarrow{i_*}\pi_i(Q_m)\to\cdots.$$

Put

$$\Delta=I'_0\partial I_0^{-1},\ I'=i'_*I_0'^{-1} \text{ and } I=I_0i_*,$$

then we have the following exact sequence.

$$(2.\,5)\qquad \cdots\to\pi_{i+4}(S^{2mp+1}:p)\xrightarrow{\Delta}\pi_{i+2}(S^{2mp-1}:p)\xrightarrow{I'}\pi_i(Q_2^{2m-1}:p)$$
$$\xrightarrow{I}\pi_{i+3}(S^{2mp+1}:p)\to\cdots,$$

where the first two groups are considered to be Z if $i=2mp-3$.

It follows form (2. 1)″ and (2. 2)″

(2. 6). *$I(\alpha\circ\beta)=I\alpha\circ S^3\beta$ and $I'(\alpha'\circ S^2\beta)=I'\alpha'\circ\beta$ for $\beta\in\pi_j(S^i:p)$.*

From (2. 3) and (1. 10), (i) we have

(2. 7) *The homomorphism Δ of (2. 5) satisfies the relation*

$$\Delta S^2\alpha=(p\iota_{2mp-1})\circ\alpha=p\alpha$$

for $\alpha\in\pi_{i+2}(S^{2mp-1}:p)\ (\in\pi_{2mp-1}(S^{2mp-1})$ *if* $i=2mp-3)$.

As a consequence of (2. 5), (2. 7) and (1. 7), we have

(2. 8). $\pi_i(Q_2^{2m-1}:p)=0$ *for* $i<2mp-3$, $\approx Z_p$ *for* $i=2mp-3$,
$S^2:\pi_i(S^{2m-1}:p)\rightarrow\pi_{i+2}(S^{2m+1}:p)$

is an isomorphism for $i<2mp-3$, *an epimorphism for* $i=2mp-3$ *and* $\quad S^\infty:\pi_{2m-1+t}(S^{2m-1}:p)\approx(\pi_t^S:p)\quad$ *for* $t<2m(p-1)-2$.

Furthermore Corollary (8. 7)$'$ of [13] states that for a mapping cylinder S_f^{2mp-1} of a map f of degree p we have an isomorphism

(2. 9). $\pi_i(Q_2^{2m-1}:p)\approx\pi_{i+2}(S_f^{2mp-1},S^{2mp-1})\quad$ *for* $i<2mp^2-4$.

Since $\pi_i(Q_2^{2m-1})\approx\pi_{i+1}(\Omega^2S^{2m+1},\ S^{2m-1})\approx\pi_{i-k+1}(\Omega^{k+2}S^{2m+1},\Omega^kS^{2m-1})\in C_p$ for $i<2mp-3$, we have by (1. 8), (i)

(2. 8)$'$ $\quad i^*:H^{i+1}(\Omega^2S^{2m+1};\ Z_p)\approx H^{i+1}(S^{2m-1};\ Z_p)$
and $\quad(\Omega^ki)^*:H^{i-k+1}(\Omega^{k+2}S^{2m+1};\ Z_p)\approx H^{i-k+1}(\Omega^kS^{2m-1};\ Z_p)$
for $i<2mp-3$.

Lemma 2. 1. $S^2:\pi_{2m-3+t}(S^{2m-3}:p)\rightarrow\pi_{2m-1+t}(S^{2m-1}:p)$ *is an epimorphism if* $m>2$ *and* $t<2m(p-1)-2$.

Proof. It is sufficient to prove the triviality of the homomorphism

$$H^{(2)}:\pi_{2m-1+t}(S^{2m-1}:p)\rightarrow\pi_{2m-4+t}(Q_2^{2m-3}:p)$$

in the exact sequence (1. 7). By use of (2. 8), (2. 9) and the fact $(\pi_t^S:p)=0$ for $0<t<2p-3$, we see that $\pi_{2m-4+t}(Q_2^{2m-3}:p)=\pi_{2m-2+t}(S_f^{2(m-1)p-1},S^{2(m-1)p-1})=0$ if $2m-2+t\neq2(m-1)p-1$ and $2m-2+t<2(m-1)p-1+2p-3$. Thus the lemma is proved except the case $t=2(m-1)p-(2m-1)$, in which case the above homomorphism is equivalent to mod p Hopf homomorphism [14] and is trivial if $m-1>1$ by [7], [11]. q. e. d.

Theorem 2. 2. *Assume that* $i<2p^2m-4$, *then there exists an exact sequence*

$$0 \to \pi_{i+2}(S^{2mp-1}) \otimes Z_p \xrightarrow{I} \pi_i(Q_2^{2m-1} : p) \xrightarrow{I'} \mathrm{Tor}(\pi_{i+3}(S^{2mp+1}), Z_p) \to 0$$

which splits, i.e.,

$$\pi_i(Q_2^{2m-1} : p) \approx \pi^S_{i-2mp+3} \otimes Z_p + \mathrm{Tor}(\pi^S_{i-2mp+2}, Z_p)$$

if $i < 2mp^2 - 5$.

Proof. Let $j = i+1$, $i+2$ and consider the homomorphisms Δ: $\pi_{j+2}(S^{2mp+1} : p) \to \pi_j(S^{2mp-1} : p)$ in the exact sequence (2.5), then we have an exact sequence

$$0 \to \mathrm{Coker}\,\Delta(j = i+2) \to \pi_i(Q_2^{2m-1} : p) \to \mathrm{Ker}\,\Delta(j = i+1) \to 0.$$

By the assumption and (2.8), $S^2 : \pi_j(S^{2mp-1} : p) \to \pi_{j+2}(S^{2mp+1} : p)$ is an isomorphism for $j = i+1$. Then it follows from the relation (2.7) that $\mathrm{Ker}\,\Delta(j = i+1) = S^2 \mathrm{Ker} f_*$, where $f_*(\alpha) = p\alpha$ for $\alpha \in \pi_{k+1}(S^{2mp-1} : p)$. Thus $\mathrm{Ker}\,\Delta(j = i+1) = S^2 \mathrm{Ker} f_* = \mathrm{Tor}(\pi_{i+3}(S^{2mp+1}), Z_p)$. Similarly we have Coker $\Delta(j = i+2) = \pi_{i+2}(S^{2mp-1}) \otimes Z_p$ by Lemma 2.1. Then the exactness of the sequence of the theorem is proved.

For the splitting of the sequence, by (2.9), it is sufficient to prove that $\pi_{i+2}(S_f^{2mp-1}, S^{2mp-1})$ is a Z_p-module. Consider the following commutative and exact diagram:

$$\begin{array}{ccccccc}
\cdots \to \pi_{i+2}(S^{2mp-1} : p) & \xrightarrow{f_*} & \pi_{i+2}(S^{2mp-1} : p) & \xrightarrow{j_*} & \pi_{i+2}(S_f^{2mp-1}, S^{2mp-1}) & \xrightarrow{\partial} & \cdots \\
\downarrow S^h & & \downarrow S^h & & \downarrow S^h & & \\
\cdots \to \pi_{i+2+h}(S^n : p) & \xrightarrow{F_*} & \pi_{i+2+h}(S^n : p) & \xrightarrow{j_*} & \pi_{i+2+h}(S_F^n, S^n : p) & \xrightarrow{\partial} & \cdots,
\end{array}$$

where $n = 2pm - 1 + h$ and $F = S^h f$. Let h be sufficiently large, then S^h is an isomorphism of $\pi_{i+2}(S^{2mp-1} : p)$ for $i < 2mp^2 - 5$. By the five lemma, $\pi_{i+2}(S_f^{2mp-1}, S^{2mp-1})$ is isomorphic to $\pi_{i+2+h}(S_F^n, S^n) \approx \pi_{i+2+h}(S_F^n / S^n)$ $= \pi_{i+2+h}(Y_p^{n+1})$ which is a Z_p-module by (1.5). This completes the proof of the theorem.

Lemma 2.3. *Let* a_0 *be a non-zero element of* $H^{2mp-3}(Q_2^{2m-1}; Z_p)$, *then*

$$H^*(Q_2^{2m-1}; Z_p) = \Lambda(a_0) \otimes Z_p[\Delta a_0]$$

for dimensions less than $p(2mp-2)-2$, where Δ stands for the Bockstein operator δ/p.

Proof. By (2. 2)′ and (2. 8)′, $H^*(Q'_m;\ Z_p)\approx H^*(S^{2mp-3};\ Z_p)$ for dimensions less than $p(2mp-2)-2$. By (2. 1)′ and (2. 8)′, $H^*(Q_m;\ Z_p)\approx H^*(S^{2mp-2}_\infty;\ Z_p)$ for dimensions less than $2mp^2-3$. By (2. 8) and (1. 8), $H^{2mp-3}(Q_2^{2m-1};\ Z_p)\approx Z_p$ and $\Delta a_0\neq 0$. Consider the spectral sequence associated with the fibering (2. 4), then the transgression is trivial for $1\otimes a_0$ hence the spectral sequence is trivial for dimensions $<p(2mp-2)-2$. Then the lemma follows easily. q. e. d.

A similar discussion yields us to verify $H^*(\Omega^h Q_2^{2m-1};\ Z_p)$ by use of (2. 8)′.

Corollary 2. 4. *$H^*(\Omega^{2k} Q_2^{2m-1};\ Z_p)=\Lambda(\sigma^{2k}a_0)\otimes Z_p[\Delta\sigma^{2k}a_0]$ for dimension less than $p(2mp-2k-2)-2$, where σ^{2k} indicates the 2k-fold iterated suspensions in cohomology. $H^*(\Omega^{2k-1}Q_2^{2m-1};\ Z_p)=Z_p[\sigma^{2k-1}a_0]\otimes\Lambda(\Delta\sigma^{2k-1}a_0)$ for dimensions less than $p(2mp-2k-2)$.*

Lemma 2. 5. *Assume that $2mp-h\geq 6$. Then there exists a map $g: Y_p^{2mp-h-2}\to\Omega^h Q_2^{2m-1}$, uniquely up to homotopy equivalences, such that g^* is an isomorphism of $H^{2mp-h-3}(;\ Z_p)$. For such a map g the following diagram is commutative for some integers $x, y\not\equiv 0 \pmod p$:*

$$\begin{array}{ccccc}
\pi_i(S^{2mp-h-3}:p) & \xrightarrow{i_*} & \pi_i(Y_p^{2mp-h-2}:p) & \xrightarrow{\pi_*} & \pi_i(S^{2mp-h-2}:p)\\
\Big\downarrow x\cdot S^{h+2} & & \Big\downarrow g_* & & \Big\downarrow y\cdot S^{h+3}\\
 & & \pi_i(\Omega^h Q_2^{2m-1}:p) & & \\
 & & \approx\Big\uparrow \Omega^h(=id.\ \text{if}\ h=0) & & \\
\pi_{i+h+2}(S^{2mp-1}:p) & \xrightarrow{I'} & \pi_{i+h}(Q_2^{2m-1}:p) & \xrightarrow{I} & \pi_{i+h+3}(S^{2mp+1}:p).
\end{array}$$

Proof. The existence and the uniqueness of g follow from Corollary 2. 4, Theorem 1. 1 and Corollary 1. 3. Consider the case $h=0$ and put $g_0=g|S^{2mp-3}$. The class $\{g_0\}$ is a generator of $\pi_{2mp-3}(Q_2^{2m-1}:p)\approx Z_p$. By (2. 1)′ and (2. 4)′, $i'_*:\pi_{2mp-3}(Q'_m)\to\pi_{2mp-3}(Q_2^{2m-1})$ is a C_p-epimorphism. Thus we may assume that $g(S^{2mp-3})\subset Q'_m$.

Then there is a map $\bar{g}: S^{2mp-2} \to Q_m$ such that $i \circ g: Y_p^{2mp-2} \to Q_2^{2m-1} \to Q_m$ is homotopic to $\bar{g} \circ \pi: Y_p^{2mp-2} \to S^{2mp-2} \to Q_m$. We have obtained the following commutative diagram:

$$\begin{array}{ccccc} \pi_i(S^{2mp-2}) & \xrightarrow{i_*} & \pi_i(Y_p^{2mp-2}) & \xrightarrow{\pi_*} & \pi_i(S^{2mp-2}) \\ \downarrow g_{0*} & & \downarrow g_* & & \downarrow \bar{g}_* \\ \pi_i(Q'_m) & \xrightarrow{i_*'} & \pi_i(Q_2^{2m-1}) & \xrightarrow{i_*} & \pi_i(Q_m). \end{array}$$

By (2. 2)$'$, $I'_0\{g_0\} = x\iota_{2mp-1}$ for some $x \not\equiv 0 \pmod p$. It is easily seen that $\bar{g}^*$ is an isomorphism of $H^{2mp-2}(\,;Z_p)$. Thus $I_0\{\bar{g}\} = y\iota_{2mp+1}$ for some $y \not\equiv 0 (\mathrm{mod}\, p)$. Then the commutativity of the diagram of the lemma is verified by use of (2. 6). In the case $h>0$, g defines a map $g': Y_p^{2mp-2} \to Q_2^{2m-1}$ by the isomorphism $\Omega^h: \pi(Y_p^{2mp-1};\, Q_2^{2m-1}) \approx \pi(Y_p^{2mp-h-2};\, \Omega^h Q_2^{2m-1})$. This g' satisfies the assumption for $h=0$. Then the conclusion for g is proved by use of (1. 3), (ii) q. e. d.

Finally we shall make some remarks for the case $m=1$. Let X_3 be a 3-connective fibre space over S^3. The fibering induces a map: $\Omega^3 X_3 \to Q_2^1 = \Omega(\Omega^2 S^3, S^1)$ which induces isomorphisms of homotopy groups.

(2. 10) *$\Omega^3 X_3$ is singular homotopy equivalent to Q_2^1, and*
$\pi_i(Q_2^1) \approx \pi_{i+3}(X_3)$ *for all i.*

Then the sequence (2. 5) becomes

(2. 11)$'$ $\cdots \to \pi_{i+4}(S^{2p+1}: p) \xrightarrow{\Delta} \pi_{i+2}(S^{2p-1}: p) \xrightarrow{\bar{I}'} \pi_{i+3}(X_3: p) \xrightarrow{\bar{I}}$
$\pi_{i+3}(S^{2p+1}: p) \to \cdots$.

Since $\pi_{i+3}(X_3) \approx \pi_{i+3}(S^3)$ for $i>0$, we have the following exact sequence.

(2. 11) $\cdots \to \pi_{i+4}(S^{2p+1}: p) \xrightarrow{\Delta} \pi_{i+2}(S^{2p-1}: p) \xrightarrow{G} \pi_{i+3}(S^3: p) \xrightarrow{H_p}$
$\pi_{i+3}(S^{2p+1}: p) \to \cdots$, $i>0$.

In general H_p is defined by

$$H_p: \pi_{i+1}(S^{2m+1}) \approx \pi_i(S_\infty^{2m}) \xrightarrow{h_{p*}} \pi_i(S_\infty^{2mp}) \approx \pi_{i+1}(S^{2mp+1}),$$

and has the following properties:

(2. 12), (i) $H_p(\alpha\circ S\beta)=H_p\alpha\circ S\beta$,

(ii) $H_p=I\circ H^{(2)}:\pi_{i+1}(S^{2m+1};p)\to\pi_{i-2}(Q_2^{2m-1}:p)$
$\to\pi_{i+1}(S^{2mp+1}:p)$.

The sequence (2. 11)$'$ is also obtained as follows. ΩX_3 is a 2-connective fibre space over ΩS^3. The inclusion of S_{p-1}^2 into ΩS^3 induces a 2-connective fibering $X'\to S_{p-1}^2$. It is computed that $H^*(X';Z_p)\approx H^*(S^{2p-1};Z_p)$. By use of (1. 8), we have a map

$$g:S^{2p-1}\to X'$$

which induces $\mathcal{C}_p$-isomorphisms $g_*:\pi_i(S^{2p-1})\to\pi_i(X')$. From the homotopy exact sequence of the pair $(\Omega S^3, S_{p-1}^2)$, we have the following exact sequence:

$$\cdots\to\pi_{i+3}(\Omega S^3, S_{p-1}^2)\xrightarrow{\partial}\pi_{i+2}(X')\xrightarrow{\bar{i}_*}\pi_{i+2}(\Omega X_3)\to\cdots.$$

By use of (2. 1), (2. 10) and the above homomorphism g_* we have the exact sequences (2. 11)$'$ and (2. 11), where G is equivalent to $\bar{i}_*\circ g_*$. Thus G satisfies

$$G(\alpha\circ\beta)=G\alpha\circ S\beta.$$

In particular

(2. 13) *G is given by the formula $G(\beta)=\alpha_1(3)\circ S\beta$, $\beta\in\pi_{i+2}(S^{2p-1}:p)$, for some generator $\alpha_1(3)$ of $\pi_{2p}(S^3:p)\approx Z_p$.*

If $S^2\beta=0$, then $SG(\beta)=S\alpha_1(3)\circ S^2\beta=0$. By (1. 10), (i), $G(\beta)=0$. By the exactness of the sequence (2. 11) we have

Lemma 2. 6. *If $S^2\beta=0$ for $\beta\in\pi_{i+2}(S^{2p-1}:p)$, then β is in the image of $\Delta:\pi_{i+4}(S^{2p+1}:p)\to\pi_{i+2}(S^{2p-1}:p)$,* [18: Lemma 13. 7].

As an analogy of Lemma 2. 5 we have

Lemma 2. 7. *There exists a map $g:Y_p^{2p+1}\to X_3$, uniquely up to homotopy equivalences, which induces an isomorphism g^* of $H^{2p}(\ ;Z_p)$. For such a map g, the following diagram is commutative for some integers $x, y\not\equiv 0$* (mod p).

$$\begin{tikzcd}
\pi_i(S^{2p-1}:p) \arrow[r, "i_*"] \arrow[dd, "\approx"', "\cdot x"] & \pi_i(Y_p^{2p}:p) \arrow[r, "\pi_*"] \arrow[d, "S"] & \pi_i(S^{2p}:p) \arrow[dd, "y\cdot S"] \\
 & \pi_{i+1}(Y_p^{2p+1}:p) \arrow[d, "g_*"] & \\
\pi_i(S^{2p-1}:p) \arrow[r, "\bar{I}'"] \arrow[dr, "G"'] & \pi_{i+1}(X_3:p) \arrow[r, "\bar{I}"] \arrow[d, "p_*"] & \pi_{i+1}(S^{2p+1}:p). \\
 & \pi_{i+1}(S^3:p). \arrow[ur, "H_p"'] &
\end{tikzcd}$$

3. Cohomology of Q_{2k}^{2m-1}.

The main purpose of this section is to prove the fol owing

Theorem 3.1. *There are elements* $a_i \in H^{2mp+2i(p-1)-3}(Q_{2k}^{2m-1}; Z_p)$, $i=0, 1, \cdots, k-1$, *such that*

$$H^*(Q_{2k}^{2m-1}; Z_p) = \Lambda(a_0, a_1, \cdots, a_{k-1}) \otimes Z_p[\Delta a_0, \Delta a_1, \cdots, \Delta a_{k-1}]$$

for dimensions less than $p(2mp-2)-2$ *and the relations*

$$\mathcal{P}^1 a_i = (m+i+1) a_{i+1} \quad \textit{and} \quad \mathcal{P}^1 \Delta a_i = (m+i) \Delta a_{i+1}$$

hold for $i=0, 1, \cdots, k-2$. $(k<mp)$.

We prepare the following lemmas.

Lemma 3.2. *Let* $p: E \to B$ *be a fibre space with a fibre* F. *Assume that* E, F *are arcwise connected and* B *is simply connected having finitely generated homology groups. Let* u_α, $v_\beta \in H^*(B; Z_p)$ *and* $w_\gamma \in H^*(E; Z_p)$ *be of odd dimensionlities and* x_γ, $y_\delta \in H^*(B; Z_p)$ *and* $z \in H^*(E; Z_p)$ *be of even ones. If* $H^*(B; Z_p) = [u_\alpha, v_\beta] \otimes Z_p[x_\gamma, y_\delta]$ *for dimensions less than* N, $H^*(E; Z_p) = \Lambda(p^* v_\beta, w_\lambda) \otimes Z_p[p^* y_\delta, z_\mu]$ *for dimensions less than* $N-1$ *and if* $p^*(u_\alpha) = p^*(x_\gamma) = 0$, *then we have* $H^*(F; Z_p) = \Lambda(\sigma x_\gamma, i^* w_\lambda) \otimes Z_p[\sigma u_\alpha, i^* z_\mu]$ *for dimensions less than* $\mathrm{Min}(N-2, p\cdot \mathrm{deg}(\sigma u_\alpha)-1)$.

Proof. Let $\{E_r\}$ be the cohomology spectral sequence associated with the fibering. Then $E_2 = H^*(B; Z_p) \otimes H^*(F; Z_p)$ and E_∞ is associated to $H^*(E; Z_p)$. Consider the following formal spectral sequence $\{'E_r\}$. Put $B' = \Lambda(u_\alpha, v_\beta) \otimes Z_p[x_\gamma, y_\delta]$ and $F' = \Lambda(\sigma x_\gamma, i^* w_\lambda)$

$\otimes Z_p[\sigma u_\alpha, i^*z_\mu]$ for all dimensions. $\{'E_r\}$ is a spectral sequence determined by the conditions: $'E_2=B'\otimes F'$, $'d_r=0$ for the generators except $'d_r(1\otimes\sigma u_\alpha)=u_\alpha\otimes 1$, $'d_r(1\otimes\sigma x_\gamma)=x_\gamma\otimes 1$ and $'d_r$ is anti-derivative. Then $'E_r$ are computed easily, in particular we have $'E_\infty=\Lambda(v_\beta\otimes 1, 1\otimes i^*w_\lambda, u_\alpha\otimes(\sigma u_\alpha)^{p-1})\otimes Z_p[y_\delta\otimes 1, 1\otimes i^*z_\mu, 1\otimes(\sigma u_\alpha)^p]$. We consider also a spectral sequence $\{\bar{E}_r\}$ determined by the conditions: $\bar{E}_r^{p,q}=E_r^{p,q}$ and $\bar{d}_r=d_r$ for $p+q<N_0$, $\bar{E}_2^{p,q}=E_2^{p,q}$ for $p+q=N_0$ and $\bar{E}_r^{p,q}=0$ for $p+q>N_0$, where $N_0=\mathrm{Min}(N-1, p\cdot\deg\sigma u_\alpha)$. Remark that $E_r^{p,q}\subset\bar{E}_r^{p,q}$ for $p+q=N_0$. Let $f_2:'E_2\to\bar{E}_2$ be a homomorphism which is given by the correspondence of the generators of the same symbols. Then we see that f_2 defines a homomorphism $f:\{'E_r\}\to\{\bar{E}_r\}$ of the spectral sequences such that $f_2:'E_2^{p,0}\to\bar{E}_2^{p,0}$ are isomorphisms for $p<N_0+1$ and $f_\infty:'E_\infty^{p,q}\to\bar{E}_\infty^{p,q}$ are isomorphism for $p+q<N_0$. Apply the (cohomological) comparision theorem [20] to the homomorphism f, then we obtain that $f_2:'E_2^{0,q}\to\bar{E}_2^{0,q}$ are isomorphisms for $q<N_0-1$. Thus $H^*(F; Z_p)=F'$ for dimensions less than $N_0-1=\mathrm{Min}(N-2, p\cdot\deg\sigma u_\alpha-1)$. q. e. d.

Corollary 3.3. *Let X be an arcwise connected and simply connected topological space. Assume that $H^*(X; Z_p)=\Lambda(u_\alpha)\otimes Z_p[x_\beta]$ for dimensions less than N. Then $H^*(\Omega X; Z_p)=\Lambda(\sigma v_\beta)\otimes Z_p[\sigma u_\alpha]$ for dimensions less than* $\mathrm{Min}(N-2, p\cdot\deg\sigma u_\alpha-1)$.

Let K_{2m-1} be an Eilenberg-MacLane space of type $(Z, 2m-1)$ which is so chosen that K_{2m-1} contains S^{2m-1} and the inclusion of S^{2m-1} represents a generator of $\pi_{2m-1}(K_{2m-1})\approx Z$. Put $X_{2m-1}=\Omega(K_{2m-1}, S^{2m-1})$. Then X_{2m-1} is a $(2m-1)$-connective fibre space over S^{2m-1} and there are two fiberings:

(3.1), (i) $X_{2m-1}\to S^{2m-1}$ *with a fibre* $\Omega K_{2m-1}=K_{2m-2}$,
(ii) $'S^{2m-1}\to K_{2m-1}$ *with a fibre* X_{2m-1},

where $'S^{2m-1}=\Omega(K_{2m-1}; K_{2m-1}, S^{2m-1})\supset S^{2m-1}$ (deformation retract).

Lemma 3.4. *Let $\sigma: H^j(K_{2m-1}; Z_p)\to H^{j-1}(X_{2m-1}; Z_p)$, $j\neq 2m-1$,* *be the suspension homomorphism with respect to the fibering* (3. 1), (ii) *and let* $u \in H^{2m-1}(K_{2m-1}; Z_p)$ *be the fundamental class. Then we have*

$$H^*(X_{2m-1}; Z_p) = Z_p[\sigma\mathcal{P}^{m-1}u, \sigma\mathcal{P}^I u] \otimes \Lambda(\sigma\Delta\mathcal{P}^{m-1}u, \sigma\mathcal{P}^J u),$$

where $\mathcal{P}^I$, $\mathcal{P}^J$ *are admissible cohomological operations* $\Delta^{\varepsilon_0}\mathcal{P}^{a_1}\Delta^{\varepsilon_1}\cdots \mathcal{P}^{a_{r-1}}\Delta^{\varepsilon_{r-1}}\mathcal{P}^{a_r}$ *of excess* $e = \sum_{i=1}^{r-1}(2(a_{i+1} - p\cdot a_i) - \varepsilon_i) + 2a_r < 2m-2$, $\deg(\mathcal{P}^I u)$ *is odd and* $\deg(\mathcal{P}^J u)$ *is even.*

Proof. Consider the spectral sequence $\{E_r\}$ associated with the fibering (3. 1), (i). $E_2 = H^*(S^{2m-1}; Z_p) \otimes H^*(K_{2m-2}; Z_p)$. By Cartan [3], $H^*(K_{2m-2}; Z_p) = Z_p[v, \mathcal{P}^I v] \otimes \Lambda(\mathcal{P}^J v)$ for the fundamental class $v \in H^{2m-2}$. Obviously, $d_{2m-1}(1 \otimes v) = s \otimes 1$ for a generator s of $H^{2m-1}(S^{2m-1}; Z_p)$ and $d_r(1 \otimes \mathcal{P}^I v) = d_r(1 \otimes \mathcal{P}^J v) = 0$. It follows that $E_\infty^* = Z_p[1 \otimes v^p, 1 \otimes \mathcal{P}^I v] \otimes \Lambda(s \otimes v^{p-1}, 1 \otimes \mathcal{P}^J v)$. It is verfied without difficulty that $1 \otimes v^p = 1 \otimes \mathcal{P}^{m-1} v$, $1 \otimes \mathcal{P}^I v$ and $1 \otimes \mathcal{P}^J v$ correspond to $\sigma\mathcal{P}^{m-1}u$, $\sigma\mathcal{P}^I u$ and $\sigma\mathcal{P}^J u$ respectively. Then it is sufficient to prove that the element $\sigma\Delta\mathcal{P}^{m-1}u$ is an independent generator. To prove this, we apply Lemma 3. 1 to the fibering (3. 1), (ii). Then we see that the lemma is true for dimensions less than $p\cdot\deg(\sigma\mathcal{P}^1 u) - 2 = p(2m+2p-4) - 2$. This completes the proof since $\deg(\sigma\Delta\mathcal{P}^{m-1}u) = p(2m-2)+1$.

Lemma 3. 5. *Let* $k=2$, *then Theorem* 3. 1 *holds for dimensions less than* $p(2m+2p-4)-3$. *In particular the relations* $\mathcal{P}^1 a_0 = (m+1)a_1$ *and* $\mathcal{P}^1\Delta a_0 = m\Delta a_1$ *hold in* $H^*(Q_4^{2m-1}; Z_p)$.

Proof. Consider a $(2m+3)$-connective fibre space X_{2m+3} over S^{2m+3} constructed as above. Then $\Omega^4 X_{2m+3}$ is a $(2m-1)$-connective fibre space over $\Omega^4 S^{2m+3}$, from which a $(2m-1)$-connective fibre space X_{2m-1} over S^{2m-1} is induced by the inclusion of S^{2m-1} into $\Omega^4 S^{2m+3}$. It is easy to see that the fiberings induce a homotopy equivalence:

$$\Omega(\Omega^4 X_{2m+3}, X_{2m-1}) \to Q_4^{2m-1} = \Omega(\Omega^4 S^{2m+3}, S^{2m-1}).$$

By Lemma 3. 4, we have

$$H^*(X_{2m-1};\ Z_p)=Z_p[\sigma\mathcal{P}^{m-1}u,\ \sigma\mathcal{P}^I u]\otimes\Lambda(\sigma\Delta\mathcal{P}^{m-1}u,\ \sigma\mathcal{P}^J u)$$

and $$H^*(X_{2m+3};\ Z_p)=Z_p[\sigma\mathcal{P}^{m+1}w,\ \sigma\mathcal{P}^{I'}w]\otimes\Lambda(\sigma\Delta\mathcal{P}^{m+1}w,\ \sigma\mathcal{P}^{J'}w),$$

where $u\in H^{2m-1}$ and $w\in H^{2m+3}$ are the fundamental classes, $e(I)$, $e(J)<2m-2$ and $e(I')$, $e(J')<2m+2$. Apply Corollary 3.3 to X_{2m+1} four times, then we have $H^*(\Omega^4X_{2m+3};\ Z_p)=Z_p[\sigma^5\mathcal{P}^{m+1}w,\ \sigma^5\mathcal{P}^{I'}w]\otimes\Lambda(\sigma^5\Delta\mathcal{P}^{m+1}w,\ \sigma^5\mathcal{P}^{J'}w)$ for dimensions less than $p\cdot\deg(\sigma^5\mathcal{P}^1 w)-1=p(2m+2p-4)-1$. By checking the excesses of I' and J', we see that

$$H^*(\Omega^4X_{2m+3};\ Z_p)=Z_p[\sigma^5\mathcal{P}^{m+j}w(j=1,0,-1),\ \sigma^5\mathcal{P}^I w]$$
$$\otimes\Lambda(\sigma^5\Delta\mathcal{P}^{m+j}w(j=1,0,-1),\ \sigma^5\mathcal{P}^J w)$$

for dimensions less than $p(2p+2m-4)-2$. It follows from the naturality of $\mathcal{P}^I$ with respect to the suspensions that $i^*(\sigma^5\mathcal{P}^I w)=\sigma\mathcal{P}^I u$ for the injection homomorphism $i^*: H^*(\Omega^4X_{2m+3};\ Z_p)\to H^*(X_{2m-1};\ Z_p)$. The same is true for $\mathcal{P}^{m+j}$, $\Delta\mathcal{P}^{m+j}$ and $\mathcal{P}^J$. If $j=0$, 1, then $i^*(\sigma^5\mathcal{P}^{m+j}w)=i^*(\sigma^5\Delta\mathcal{P}^{m+j}w)=0$ since $\mathcal{P}^t u=0$ for $t>m-1$.

The inclusion $i: X_{2m-1}\to\Omega^4X_{2m+3}$ is equivalent to a fibering which has a fibre $\Omega(\Omega^4X_{2m+3},X_{2m-1})$. Apply Lemma 3.2 to the fibering, then we have

$$H^*(Q_4^{2m-1};\ Z_p)=H^*(\Omega(\Omega^4X_{2m+3},\ X_{2m-1});\ Z_p)$$
$$=\Lambda(\sigma^6\mathcal{P}^m w,\ \sigma^6\mathcal{P}^{m+1}w)\otimes Z_p[\sigma^6\Delta\mathcal{P}^m w,\ \sigma^6\Delta\mathcal{P}^{m+1}w]$$

for dimensions less than $p(2m+2p-4)-3$. Put $a_0=\sigma^6\mathcal{P}^m w$ and $a_1=\sigma^6\mathcal{P}^{m+1}w$, then $\Delta a_0=\sigma^6\Delta\mathcal{P}^m w$ and $\Delta a_1=\sigma^6\Delta\mathcal{P}^{m+1}w$. The relations of the lemma follow from Adem's relations $\mathcal{P}^1\mathcal{P}^m=(m+1)\mathcal{P}^{m+1}$ and $\mathcal{P}^1\Delta\mathcal{P}^m=m\Delta\mathcal{P}^{m+1}+\mathcal{P}^{m+1}\Delta$. q. e. d.

Applying (1.6) to the triple $(\Omega^{2k}S^{2m+2k-1},\ \Omega^{2h}S^{2m+2h-1},\ S^{2m-1})$ we have

(3.2). *There is a sequence of inclusions* $Q_{2h}^{2m-1}\xrightarrow{i}Q_{2k}^{2m-1}\xrightarrow{j}\Omega^{2h}Q_{2(k-h)}^{2m+2h-1}$ *equivalent to a fibering.*

Here remark that $\Omega(\Omega^t X,\Omega^t A)$ is homeomorphic to $\Omega^{t+1}(X,A)$.

Proof of Theorem 3.1. Lemma 2.3 shows that the theorem is true for $k=1$. Let $k>1$ and assume that the theorem is true for Q_{2h}^{2m-1} of $h<k$. Consider the spectral sequence $\{E_r\}$ associated with the fibering (3.2) of the case $h=k-1$. Then $E_2=H^*(\Omega^{2k-2}Q_2^{2m+2k-3};Z_p)\otimes H^*(Q_{2k-2}^{2m-1};Z_p)=(\Lambda(\sigma^{2k-2}a_0')\otimes Z_p[\Delta\sigma^{2k-2}a_0'])\otimes(\Lambda(a_0,\cdots,a_{k-2})\otimes Z_p[\Delta a_0,\cdots,\Delta a_{k-2}])$ for dimensions less than $p(2mp-2)-2$, by the assumption and Corollary 2.4. It follows that

$$E_2=E_\infty\approx H^*(Q_{2k}^{2m-1};Z_p)=\Lambda(a_0,\cdots,a_{k-1})\otimes Z_p[\Delta a_0,\cdots,\Delta a_{k-1}]$$

for dimensions lass than $p(2mp-2)-2$, where $i^*a_j=a_j$, $j=0,\cdots,k-2$ and $a_{k-1}=j^*\sigma^{2k-2}a_0'$. This proves the first assertion of the theorem. We remark that the generators a_i are chosen as above inductively. Then we have, with respect to the inclusions i and j of (3.2),

$$i^*a_j=a_j,\ j=0,\cdots,h-1 \text{ and } a_l=j^*\sigma^{2h}a_{l-h}',\ l=h,\cdots,k-1.$$

In particular, $a_j=j^*\sigma^2a_{j-1}'$, $j=1,\cdots,k-1$, for the inclusion $j:Q_{2k}^{2m-1}\to\Omega^2Q_{2k-2}^{2m+1}$. By the assumption the relations $\mathcal{P}^1a_{j-1}'=((m+1)-(j-1)+1)a_j'$ and $\mathcal{P}^1\Delta a_{j-1}'=((m+1)-(j-1))\Delta a_j'$ hold in $H^*(Q_{2k-2}^{2m-1};Z_p)$. Then, by the naturality of $\mathcal{P}^1$, we have the relations of the theorem for $i=1,\cdots,k-1$. The relations for $i=0$ follow from Lemma 3.5 and from that $i^*:H^*(Q_{2k}^{2m-1};Z_p)\to H^*(Q_4^{2m-1};Z_p)$ is an isomorphism for dimensions less than $\deg a_2$. q. e. d.

Consider the homotopy exact sequence

$$(3.3).\quad \cdots\to\pi_{i+1}(\Omega^{2h}Q_{2(k-h)}^{2m+2h-1})\xrightarrow{\partial}\pi_i(Q_{2h}^{2m-1})\xrightarrow{i_*}\pi_i(Q_{2k}^{2m-1})\xrightarrow{i_*}\cdots$$

associated with the fibering (3.2). It is known from the last half of (1.6)

(3.4) *There is a map* $d:\Omega^{2h+1}Q_{2(k-h)}^{2m+2h-1}\to Q_{2h}^{2m-1}$ *such that* $d_*:\pi_i(\Omega^{2h+1}Q_{2(k-h)}^{2m+2h-1})\to\pi_i(Q_{2h}^{2m-1})$ *is equivalent to* ∂ *of* (3.3).

In fact, i of (3.2) is equivalent to a fibreing with a fibre $\Omega(Q_{2k}^{2m-1},Q_{2h}^{2m-1})$ and the fibre is homotopy equivalent to $\Omega^{2h+1}Q_{2(k-h)}^{2m+2h-1}$.

Proposition 3.6. *Assume that* $k(p-1)<mp-1$. *There exists a sequence of finite CW-complexes* $K(m,j)$, $j=1,2,\cdots,k$, *satisfying the following conditions*:

(i). *$K(m,1)=Y_p^{2mp-2}$. For $1\leq j<k$, $K(m,j+1)$ is a mapping-cone of a map $h_j: Y_p^{2mp-3+2j(p-1)}\to K(m,j)$. Thus $K(m,j+1)/K(m,j)$ $=Y_p^{2mp-2+2j(p-1)}$.*

(ii). *There exists a map $G: K(m,k)\to Q_{2k}^{2m-1}$ such that $G(K(m,j))\subset Q_{2j}^{2m-1}$ and $G^*: H^*(Q_{2k}^{2m-1}; Z_p)\to H^*(K(m,k); Z_p)$ is an isomorphism for dimensions less than $4mp-3$.*

(iii). *Put $G_j=G\,|\,K(m,j): K(m,j)\to Q_{2j}^{2m-1}$. Then there exist maps g and g' such that the following diagram is homotopy commutative:*

$$\begin{array}{ccccccc}
Y_p^{2mp-3+2j(p-1)} & \xrightarrow{h_j} & K(m,j) & \xrightarrow{i} & K(m,j+1) & \xrightarrow{\pi} & Y_p^{2mp-2+2j(p-1)} \\
\downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle G_j} & & \downarrow{\scriptstyle G_{j+1}} & & \downarrow{\scriptstyle g} \\
\Omega^{2j+1}Q_2^{2m+2j-1} & \xrightarrow{d} & Q_{2j}^{2m-1} & \xrightarrow{i} & Q_{2j+2}^{2m-1} & \xrightarrow{j} & \Omega^{2j}Q_2^{2m+2j-1}.
\end{array}$$

The maps g and g' satisfy Lemma 2.5.

This follows from Theorem 1.1 and Theorem 3.1.

For the case $m=1$, the following Propositions 3.7 and 3.8 will be used in place of Theorem 3.1 and Proposition 3.6.

(2.10) is easily generalized to

(3.5). *$\Omega^{2k+1}X_{2k+1}$ is singular homotopy equivalent to Q_{2k}^1 and $\pi_i(Q_{2k}^1)\approx\pi_{i+2k+1}(X_{2k+1})$ for all i.*

We have the following commutative and exact diagram.

$$\begin{array}{ccccccccc}
 & & \pi_{i+3}(X_3) & \xrightarrow{i_*} & \pi_{i+3}(\Omega^2X_5) & & \\
 & \nearrow{\scriptstyle p_*'} & \downarrow{\scriptstyle p_{3*}} & & \downarrow{\scriptstyle \Omega^2p_{5*}} & \searrow{\scriptstyle \partial'} & \\
(3.6)\quad \cdots\to\pi_{i+3}(Q_2^3) & \xrightarrow{p_*} & \pi_{i+3}(S^3) & \xrightarrow{i_*} & \pi_{i+3}(\Omega^2S^5) & \xrightarrow{\partial} & \pi_{i+2}(Q_2^3)\cdots, \\
 & & & \searrow{\scriptstyle S^2} & \approx\uparrow{\scriptstyle \Omega^2} & \nearrow{\scriptstyle H^{(2)}} & \\
 & & & & \pi_{i+5}(S^5) & &
\end{array}$$

where p_{3*} and Ω^2p_{5*} are isomorphisms for $i>0$.

From Lemma 3.4 and Corollary 3.3 it follows

Proposition 3.7. *$H^*(\Omega^{2k-2}X_{2k+1}; Z_p)=Z_p[a_0, a_1,\cdots, a_{k-1}]\otimes\Lambda(\Delta a_0,\cdots,\Delta a_{k-1})$ for dimensions less than $4p^2-2$ where the elements*

$a_i \in H^{2(i+1)(p-1)+2}(\Omega^{2k-2}X_{2k+1}; Z_p)$ *satisfy the relations*

$$\mathcal{P}^1 a_i = (i+2)a_{i+1} \quad \text{and} \quad \mathcal{P}^1 \Delta a_i = (i+1)\Delta a_{i+1}\,.$$

Proposition 3.8. *There exists a complex* $\bar{K} = Y_p^{2p} \cup_h CY_p^{4p-3}$, $h: Y_p^{4p-3} \to Y_p^{2p}$, *and there exists a map* $G: (S\bar{K}, Y_p^{2p+1}) \to (\Omega^2 X_5, X_3)$ *inducing an isomorphism of* $H^*(\ ; Z_p)$ *for dimensions less than* $4p-1$. *Put* $g = G \mid Y_p^{2p+1}$ *then there exist maps* g'' *and* g' *such that the following diagrams are homotopy commutative.*

$$\begin{array}{ccccc} Y_p^{4p-2} & \xrightarrow{Sh} & Y_p^{2p+1} & \xrightarrow{i} & S\bar{K} \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle G} \\ Q_2^3 & \xrightarrow{d'} & X_3 & \longrightarrow & \Omega^2 X_5\,, \end{array} \qquad \begin{array}{ccc} \bar{K} & \xrightarrow{\pi} & Y_p^{4p-3} \\ \downarrow{\scriptstyle \Omega G} & & \downarrow{\scriptstyle g''} \\ \Omega^3 X_5 & \xrightarrow{j} & Q_2^3\,. \end{array}$$

The maps g'' *and* g' *satisfy Lemma* 2.5 *and* g *does Lemma* 2.7.

4. Homotopy of Moore spaces.

We denote

$$\pi_k^S(Y_p;\ Y_p) = \pi^S(Y_p^{n+k};\ Y_p^n), \quad \pi_k^S(Y_p;\ S) = \pi^S(Y_p^{n+k};\ S^n)$$

and $\pi_k^S(Y_p) = \pi^S(S^{n+k};\ Y_p^n)$.

These groups are Z_p-modules by (1.5). Thus we have the following split exact sequences:

(4.1), (i) $0 \to \pi_{k+1}^S(Y_p;\ S) \xrightarrow{i_*} \pi_k^S(Y_p;\ Y_p) \xrightarrow{\pi_*} \pi_k^S(Y_p;\ S) \to 0,$

(ii) $0 \to \pi_k^S(Y_p) \xrightarrow{\pi^*} \pi_k^S(Y_p;\ Y_p) \xrightarrow{i^*} \pi_{k-1}^S(Y_p) \to 0,$

(iii) $0 \to \pi_{k+1}^S \otimes Z_p \xrightarrow{i_*} \pi_k^S(Y_p) \xrightarrow{\pi_*} \mathrm{Tor}(\pi_k^S, Z_p) \to 0,$

(iv) $0 \to \pi_k^S \otimes Z_p \xrightarrow{\pi^*} \pi_k^S(Y_p;\ S) \xrightarrow{i^*} \mathrm{Tor}(\pi_{k-1}^S, Z_p) \to 0.$

The sum $\pi_*^S(Y_p;\ Y_p) = \sum_k \pi_k^S(Y_p;\ Y_p)$ forms a Z_p-algebra with the multiplication given by the composition. Yamamoto has obtained the following relations.

(4.2) *Let* $\delta = \pi^* i_*(\iota) \in \pi_{-1}^S(Y_p;\ Y_p)$ *be the class of the composition* $i \circ \pi: Y_p^{n-1} \to Y_p^n$ *and let* $\alpha \in \pi_{2p-2}^S(Y_p;\ Y_p) \approx Z_p$ *be a generator* *which is characterized by the coefficient* 1 mod p *of its functional* $\mathcal{P}^1$*-operation. Then the relations*

$$\delta\delta=0 \quad \textit{and} \quad 2\alpha\delta\alpha=\alpha^2\delta+\delta\alpha^2$$

hold [19: proposition 5.1].

The elements

$$\alpha_k=\pi_* i^*(\alpha^k)=i^*\pi_*(\alpha^k)\in\pi^S_{2k(p-1)-1}, \quad k=1, 2, \cdots,$$

coincide with those in [17] and [16]. Recently Adams has proved

(4.3). *If* $k=a\cdot p^t$, $a\not\equiv 0$ (mod p), *then* α_k *cannot be divisible by* p^{t+1}. *In particular* α_k *cannot be divisible by* p *if* $k\not\equiv 0$ (mod p).

In fact, he has defined a homomorphism $e:\pi^S_{2k(p-1)-1}\rightarrow Q/Z$ which has the value $-(1/p)$ mod 1 on α_k [1: Proposition 12.7]. On the other hand each image of e is of a form $z/m(k(p-1))$, $z\in Z$, where $m(k(p-1))=b\cdot p^{t+1}$ for some integer $b\not\equiv 0$ (mod p) [1: Proposition 7.9]. Then (4.2) follows immediately.

It follows from (4.3) and (4.1)

(4.4) $\quad \delta\alpha^k\delta=\pi^* i_*(\alpha_k)\neq 0 \quad$ *if* $k\not\equiv 0$ (mod p).

From this Yamamoto has proved the following

(4.5). *Let* $Z_p(\delta, \alpha)$ *be a subalgebra of* $\pi^S_*(Y_p;\, Y_p)$ *generated by* δ *and* α, *then the relations in* $Z_p(\delta, \alpha)$ *are generated by those in* (4.2), *and* $Z_p(\delta, \alpha)$ *has a* Z_p*-base* $\{1, \delta, \alpha^k, \alpha^k\delta, \alpha^{k-1}\delta\alpha, \alpha^{k-1}\delta\alpha\delta;\ k=1, 2, \cdots\}$ [19: Theorem III].

Every element of $Z_p(\delta, \alpha)$ becomes a linear combination of the above base by use of the following relations (4.6) which is obtained from (4.2).

(4.6) $$\alpha^s\delta\alpha^t=t\cdot\alpha^{s+t-1}\delta\alpha+(1-t)\alpha^{s+t}\delta,$$
$$\alpha^s\delta\alpha^t\delta=\delta\alpha^t\delta\alpha^s=t\cdot\alpha^{s+t-1}\delta\alpha\delta.$$

Now, we consider unstable cases. By (2.8) there exists uniquely an element $\alpha_1(3)\in\pi_{2p}(S^3:p)\approx Z_p$ such that

$$S^\infty\alpha_1(3)=\alpha_1\in\pi^S_{2p-3}.$$

We denote that $\alpha_1(n)=S^{n-3}\alpha_1(3)\in\pi_{n+2p-3}(S^n:p)$ for $n\geqq 3$. Since $p\cdot\alpha_1(3)=0$, there exists an extension $\bar{\alpha}_1(3)\in\pi(Y_p^{2p+1};\ S^3)$ of $\alpha_1(3)$, i.e., $i^*(\bar{\alpha}_1(3))=\alpha_1(3)$. Consider $p\iota_3\circ\bar{\alpha}_1(3)$. Since S^3 is an H-space we have $p\iota_3\circ\bar{\alpha}_1(3)=p\cdot\bar{\alpha}_1(3)=\bar{\alpha}_1(3)\circ p\iota_Y=0$. Thus there exists a coextension $\alpha(4)\in\pi(Y_p^{2p+2};\ Y_p^4)$ of $\bar{\alpha}_1(3)$, in the sense of [18]. Since $\pi_* i^*(\alpha(4))=i^*\pi_*(\alpha(4))=i^*S\bar{\alpha}_1(3)=S\alpha_1(3)=\alpha_1(4)$, and since $\alpha_1=\pi_* i^*(\alpha)$ characterizes α, we have $S^\infty\alpha(4)=\alpha$. We have obtained

(4.7). *There exists an element* $\alpha(4)$ *of* $\pi(Y_p^{2p+2};\ Y_p^4)$ *which is a coextension of* $\bar{\alpha}(3)$ *and* $S^\infty\alpha(4)=\alpha$. $\pi_*\alpha(4)=S\bar{\alpha}_1(3)$ *and* $i^*\bar{\alpha}_1(3)=\alpha_1(3)$.

We put for $n\geqq 4$ and for $k=1, 2, \cdots$ by induction on k:

$$\alpha(n)=S^{n-4}\alpha(4)=\alpha^1(n),\ \alpha^k(n)=\alpha(n)\circ\alpha^{k-1}(n+2p-2)$$

and $\quad\alpha_k(n)=\pi_* i^*(\alpha^k(n)),\ \alpha_k(3)=\bar{\alpha}_1(3)\circ(i^*\alpha^{k-1}(2p+1))$.

Obviously, $S^\infty\alpha^k(n)=\alpha^k$, $S^\infty\alpha_k(n)=\alpha_k$ and $S\alpha_k(3)=\alpha_k(4)$.

Lemma 4.1. *Let* $t>0$. *If an element* $\gamma\in\pi_t^S$ *is in the image of* $S^\infty:\pi_{t+2p-3}(S^{2p-3})\to\pi_t^S$, *then* $\alpha_1\gamma=0$. *In particular*, $\alpha_1\alpha_k=0$.

Proof. α_1 is an image of the stable J-homomorphism $J:\pi_{2p-3}(SO(\infty))\to\pi_{2p-3}^S$. Let γ' be an element of $\pi_{t+2p-3}(S^{2p-3})$ such that $S^\infty\gamma'=\gamma$ and $\beta\in\pi_{2p-3}(SO(\infty))$ satisty $J(\beta)=\alpha_1$. Then $\alpha_1\gamma=J(\beta\circ\gamma')$. Since $\pi_{t+2p-3}(S^{2p-3})$ is finite, $\beta\circ\gamma'$ is of finite order. Since $\pi_{2p-3+t}(SO(\infty))\approx Z, Z_2$ or 0, we have $2(\beta\circ\gamma')=0$ and $2(\alpha_1\gamma)=p(\alpha_1\gamma)=0$. Thus $\alpha_1\gamma=0$. Since $\alpha_k=S^\infty\alpha_k(3)$, $\alpha_1\alpha_k=0$. q. e. d.

The class of the composition $i\circ\pi: Y_p^{n-1}\to Y_p^n$ is denote by $\delta(n)\in\pi(Y_p^{n-1};\ Y_p^n)$, $n\geqq 3$. Obviously

$$\delta(n)\circ\delta(n-1)=0\quad \textit{for}\ n\geqq 4.$$

We shall use the following notations:

$$\bar{\alpha}_1(n)=(-1)^{n-1}S^{n-3}\bar{\alpha}_1(3),\ \alpha^k\delta(n)=\alpha^k(n)\circ\delta(n+2k(p-1)-1),$$
$$\delta\alpha^k(n)=\delta(n)\circ\alpha^k(n-1),\ \alpha^s\delta\alpha^t(n)=\alpha^s\delta(n)\circ\alpha^t(n+2s(p-1)-2),$$

etc.

In general, if $\tilde{\gamma}$ is a coextension of γ then $\tilde{\gamma}\circ S\beta$ is a coextension of $\gamma\circ\beta$ and $S\tilde{\gamma}$ is a coextension of $-\gamma$, and if $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are coextensions of the same element γ then $\tilde{\gamma}_1-\tilde{\gamma}_2$ is in the i_*-image. For example, $\alpha(n)$ is a coextension of $\bar{\alpha}_1(n-1)$. $\pi_*\alpha(n)=\bar{\alpha}_1(n)$.

Proposition 4.2. *The second relation of* (4.1) *holds for unstable case* $n\geqq 6$, i.e.,

$$2\cdot\alpha\delta\alpha(n)=\alpha^2\delta(n)+\delta\alpha^2(n) \quad \textit{for } n\geqq 6.$$

Proof. $\alpha\delta\alpha(n)$ and $\alpha^2\delta(n)$ are coextensions of $\bar{\alpha}_1(n-1)\circ\delta\alpha(n+2p-3)$ and $\bar{\alpha}_1(n-1)\circ\alpha\delta(n+2p-3)$ respectively. We have

$$\begin{aligned}\bar{\alpha}_1(n-1)\circ\delta\alpha(n+2p-3)&=-i^*\pi_*\alpha(n-1)\circ\pi_*\alpha(n+2p-4)\\&=\alpha_1(n-1)\circ\bar{\alpha}(n+2p-4)\end{aligned}$$

and this is an element of $\pi^*\{\alpha_1(n-1),\ \alpha_1(n+4p-4),\ p\iota_{n+4p-7}\}$ by Proposition 1.9 of [18]. Here we remark that $\alpha_1(n-1)\circ\alpha(n+2p-4)=0$ for $n\geqq 6$ since this is a stable case. Thus

$$\bar{\alpha}_1(n-1)\circ\delta\alpha(n+2p-3)=\pi^*(x\cdot\alpha_2(n-1))$$

for some $x\in Z_p$. We have also

$$\begin{aligned}\bar{\alpha}_1(n-1)\circ\alpha\delta(n+2p-3)&=-\pi_*\alpha(n-1)\circ\pi^*i^*\alpha(n+2p-3)\\&=-\pi^*i^*\pi_*\alpha^2(n-1)=-\pi^*\alpha_2(n-1).\end{aligned}$$

The proposition is true for sufficiently large n by (4.2). Then we see that $x=-1/2$ and that $2\cdot\alpha\delta\alpha(n)$ and $\alpha^2\delta(n)$ are coextensions of the same element $-\pi^*\alpha_2(n-1)$. It follows that $2\cdot\alpha\delta\alpha(n)-\alpha^2\delta(n)$ is in $i_*\pi(Y_p^{n+4p-5};\ S^{n-1})$. It is computed easily that $\pi(Y_p^{n+4p-5};\ S^{n-1})$ is generated by $\pi_*\alpha^2(n-1)$. Thus $2\cdot\alpha\delta\alpha(n)-\alpha^2\delta(n)=y\cdot i_*\pi_*\alpha^2(n-1)=y\cdot\delta\alpha^2(n)$ for some $y\in Z_p$. Taking n sufficiently large, we see that $y=1$. q. e. d.

Corollary 4.3. *The relations* (4.6) *hold for unstable case* $n\geqq 6$.

Proposition 4.4. (i) $i^*\alpha^k\in\pi^S_{2k(p-1)-1}(Y_p)$ *and* $i^*(\alpha^{k-1}\delta\alpha)\in\pi^S_{2k(p-1)-2}(Y_p)$ *do not vanish.*

(ii) *For* $n\geqq 4$, $i^*\alpha^k(n)\in\pi_{n+2k(p-1)-1}(Y^n_p)$ *is a coextension of* $(-1)^n\alpha_k(n-1)$.

(iii) *Assume that* $\alpha_{k-1}(n)\circ\alpha_1(n+2(k-1)(p-1)-1)=0$ *for some odd n, then there exists an element*

$$\alpha'_k(n)\in\pi_{n+2k(p-1)-1}(S^n:p)$$

such that $i_*\alpha'_k(n)=i^*(\alpha^{k-1}\delta\alpha(n+1))$. *Thus* $\alpha'_k(m)=S^{m-n}\alpha'_k(n)$, $m\geq n$, *and* $\alpha'_k=S^\infty\alpha'_k(n)\in\pi^S_{2k(p-1)-1}$ *are not divisible by* p. *The assumption holds for* $n\geq 2k+1$.

(iv) *If* $k\not\equiv 0$ (mod p), *then by putting* $\alpha'_k(n)=(1/k)\alpha_k(n)$ *the assertion of* (iii) *holds for* $n\geq 6$.

Proof. (i) $\pi^*i^*\alpha^k=\alpha^k\delta\neq 0$ and $\pi^*i^*(\alpha^{k-1}\delta\alpha)=\alpha^{k-1}\delta\alpha\delta\neq 0$ by (4.5). By the exactness of (4.1), ii) we have $i^*\alpha^k\neq 0$ and $i^*(\alpha^{k-1}\delta\alpha)\neq 0$.

(ii) Since $\alpha(4)$ is a coextension of $\bar{\alpha}_1(3)$, $\alpha^k(4)=\alpha\circ S\alpha^{k-1}(2p+1)$ is a coextension of $\bar{\alpha}_1(3)\circ\alpha^{k-1}(2p+1)$. Then $i^*\alpha^k(4)$ is a coextension of $i^*(\bar{\alpha}_1(3)\circ\alpha^{k-1}(2p+1))=\bar{\alpha}_1(3)\circ(i^*\alpha^{k-1}(2p+1))=\alpha_k(3)$. Thus $i^*\alpha^k(n)=S^{n-4}i^*\alpha^k(4)$ is a coextension of $(-1)^{n-4}S^{n-4}\alpha_k(3)=(-1)^n\alpha_k(n-1)$.

(iii) It follows from (ii) that $i^*(\alpha^{k-1}\delta\alpha(n+1))=(i^*\alpha^{k-1}(n+1))\circ S(i^*\pi_*\alpha(n+2(k-1)(p-1)-1))$ is a coextension of $\alpha_{k-1}(n)\circ\alpha_1(n+2(k-1)(p-1)-1)=0$. This means that $i^*(\alpha^{k-1}\delta\alpha)$ is in $i_*\pi_{n+2k(p-1)-1}(S^n)$. Choose an element $\alpha'_k(n)$ of $\pi_{n+2k(p-1)-1}(S^n:p)$ such that $i_*\alpha'_k(n)=i^*(\alpha^{k-1}\delta\alpha(n+1))$, then the assertions of (iii), except the last one, are verified easily. By Lemma 4.1

$$S^\infty(\alpha_{k-1}(n)\circ\alpha_1(n+2(k-1)(p-1)))=\alpha_{k-1}\alpha_1=\alpha_1\alpha_{k-1}=0.$$

Let $n=2m-1$, then $S^\infty:\pi_{2m-1+2k(p-1)-2}(S^{2m-1}:p)\to(\pi^S_{2k(p-1)-1}:p)$ is an isomorphism for $2k(p-1)-2<2m(p-1)-2$, i.e., for $k\leq m-1$, by (2.8). Thus the assumption of (iii) holds for $n\geq 2k+1$.

(iv) By Corollary 4.3, $\pi^*i_*\alpha'_k(5)=\pi^*(1/k)i_*\pi_*i^*\alpha^k(5)=(1/k)\delta\alpha^k\delta(6)=\alpha^{k-1}\delta\alpha\delta(6)=\pi^*i^*\alpha^{k-1}\delta\alpha(6)$. By the exactness of Puppe's sequence we have $i^*\alpha'_k(5)\equiv i^*\alpha^{k-1}\delta\alpha(6)$ mod $p\cdot\pi_{2k(p-1)+4}(Y^6_p)$. It follows from (1.5) that $i^*\alpha'_k(n)=i^*\alpha^{k-1}\delta\alpha(n+1)$ for $n\geq 6$. q. e. d.

Proposition 4.5. *The complex* $K(m, 2)=Y_p^{2mp-2}\cup_h CY_p^{2mp+2p-5}$, $m\geq 2$, *of Proposition* 3.6 *can be chosen such that the attaching map* $h: Y_p^{2mp+2p-5}\rightarrow Y_p^{2mp-2}$ *represents* $(m+1)\delta\alpha(2mp-2)-m\cdot\alpha\delta(2mp-2)$.

The complex $\bar{K}=Y_p^{2p}\cup_h CY_p^{4p-3}$ *of Proposition* 3.8 *can be chosen such that the attaching map* $h: Y_p^{4p-3}\rightarrow Y_p^{2p}$ *represents* $2\cdot\delta\alpha(2p)-\alpha\delta(2p)$.

Proof. First remark that these elements belong to $\pi(Y_p^{n+2p-3}; Y_p^n)$, $n=2mp-2$ or $=2p$, and the group is in stable range by (1.4). Since the group is generated by $\delta\alpha(n)$ and $\alpha\delta(n)$, $\{h\}=x\cdot\delta\alpha(n)+y\cdot\alpha\delta(n)$ for some $x, y\in Z_p$. We may assume that h maps S^{n+2p-4} into S^{n-1}. Let $K=K(m, 2)$ or $=\bar{K}$, $h_0=h\,|\,S^{n+2p-4}: S^{n+2p-4}\rightarrow S^{n-1}$ and $\bar{h}: S^{n+2p-3}\rightarrow S^n$ satisfy $\pi\circ h=\bar{h}\circ\pi$. Then h_0 and $\bar{h}$ represent $x\cdot\alpha_1(n-1)$ and $y\cdot\alpha_1(n+2p-4)$ respectively by suitable orientations of the speheres. Let K_0 be a mapping cone of h_0 naturally imbedded in K as a subcomplex, then K/K_0 is a mapping cone of $\bar{h}$. Given orientations $u_0\in H^{n-1}(K_0; Z_p)$, $u_0'\in H^{n+2p-3}(K_0; Z_p)$ and $\bar{u}\in H^n(K/K_0; Z_p)$, $\bar{u}'\in H^{n+2p-2}(K/K_0; Z_p)$ by the construction of the mapping cones, we have $\mathcal{P}^1u_0=x\cdot u_0'$ and $\mathcal{P}^1\bar{u}=y\cdot\bar{u}'$. We denote by the same symbol u_0, u_0', $\bar{u}$, $\bar{u}'\in H^*(K; Z_p)$ the elements corresponding uniquely to the above orientations by the natural homomorphisms i^* and π^*. Obviously $\Delta u_0=\bar{u}$. But $\Delta u_0'=-\bar{u}'$ since the cell $CCS^{n+2p-4}=S^{n+2p-4}\wedge I\wedge I$ in K is oriented with respect to one of the coordinates I, the Bockstein operator Δ is defined with respect to the other coordinate hence the sign appears by interchanging the coordinates. We have the relation

$$x\cdot\mathcal{P}^1\Delta u_0=x\cdot\mathcal{P}^1\bar{u}=xy\cdot\bar{u}=-xy\cdot\Delta u_0'=-y\cdot\Delta\mathcal{P}^1u_0.$$

On the other hand the relations

$$(m+1)\mathcal{P}^1\Delta a_0=m(m+1)\Delta a_1=m\cdot\Delta\mathcal{P}^1a_0,\qquad \mathcal{P}^1\neq 0$$

hold in $Q_4^{2m-1}(m\geq 2)$ by Theorem 3.1 and in Ω^3X_5 $(m=1)$ by Proposition 3.7. By use of natural isomorphism G^* in Proposition

3. 6 or Proposition 3. 8, we see that the same relation holds replacing a_0 by u_0. Since $\mathcal{P}^1 \neq 0$ in K, x or $y \not\equiv 0 \pmod p$ and there is an integer $z \not\equiv 0 \pmod p$ such that

$$zx \equiv m+1 \quad \text{and} \quad zy \equiv -m \pmod p.$$

If $z \equiv 1 \pmod p$, K is the required complex. If $z \not\equiv 1 \pmod p$, replacing h by a representative h' of $z\{h\}$ and K by $K' = Y_p^n \cup_{h'} CY_p^{n+2p-4}$ we have a map $f: K' \to K$ which extends the identity of Y_p^n and induces isomorphisms of cohomology groups. Then f is a homotopy equivalence and K' is the required cpmplex. q. e. d.

Proposition 4. 6. *Let* $n \geq 7$, $k > 0$ *and let* $h: Y_p^{n+2p-3} \to Y_p^n$ *be a representative of* $(m+1)\delta\alpha(n) - m\alpha\delta(n)$. *For the element* $i^*\alpha^{k-1}(n+2p-3)$ *of* $\pi_{n+2k(p-1)-2}(Y_p^{n+2p-3})$ *we have*

$$h_*(i^*\alpha^{k-1}(n+2p-3)) = (m+k) \cdot i^*(\alpha^{k-1}\delta\alpha(n)) \in \pi_{n+2k(p-1)-2}(Y_p^n).$$

If $n=6$, *the relation holds* mod $p \cdot \pi_{2k(p-1)+4}(Y_p^6)$.

Proof. For $n \geq 6$, we have by Corollary 4. 3

$$\begin{aligned}\pi^* h_*(i^*\alpha^{k-1}(n+2p-3)) &= \{(m+1)\delta\alpha(n) - m\alpha(n)\} \circ \alpha^{k-1}\delta(n+2p-3) \\ &= (m+1)\delta\alpha^k\delta(n) - m \cdot \alpha\delta\alpha^{k-1}\delta(n) \\ &= \{(m+1)k - m(k-1)\}\alpha^{k-1}\delta\alpha\delta(n) = \pi^*(m+k)i^*(\alpha^{k-1}\delta\alpha(n)).\end{aligned}$$

From the exactness of Puppe's sequence

$$(4\cdot 8) \qquad \pi_j(Y_p^n) \xrightarrow{p} \pi_j(Y_p^n) \xrightarrow{\pi^*} \pi(Y_p^{j+1};\ Y_p^n)$$

it follows

$$h_*(i^*\alpha^{k-1}(n+2p-3)) \equiv (m+k) \cdot i^*(\alpha^{k-1}\delta\alpha(n)) \\ \mod p \cdot \pi_{n+2k(p-1)-2}(Y_p^n).$$

If $n \geq 7$, the relation holds mod $p \cdot S(\pi_{n+2k(p-1)-3}(Y_p^{n-1}))$ which vanishes by (1. 5). q. e. d.

Lemma 4 .7. *Let* $h: X \to Y$ *be a map and let* $\alpha \in \pi_i(X)$ *be an element of order* p *such that* $h_*\alpha = 0$. *Let* $\gamma \in \pi_{i+1}(Y \cup_h CX)$ *be a coextension of* α *and* $\bar{\alpha} \in \pi(Y_p^{i+1};\ X)$ *an extension of* α. *Then* *there exists an element* $\gamma_0\in\pi_{i+1}(Y)$ *such that*

$$p\gamma=j_*\gamma_0 \quad \textit{and} \quad \pi^*\gamma_0=-h^*\bar{\alpha},$$

where $j: Y\to Y\cup_h CX$ *is the inclusion.*

Proof. By Propositions 1.8 and 1.9 of [18] there exist elements $\gamma_0', \gamma_0''\in\{\{h\}, \alpha, p\iota_i\}$ such that $p\gamma=-j_*\gamma_0'$ and $\pi^*\gamma_0''=h_*\bar{\alpha}$. Since $\gamma_0'\equiv\gamma_0''$ mod $h_*\pi_{i+1}(X)+p\cdot\pi_{i+1}(Y)$, $\gamma_0'-\gamma_0''=h_*\delta+p\delta'$ for some $\delta\in\pi_{i+1}(X)$ and $\delta'\in\pi_{i+1}(Y)$. Put $\gamma_0=-\gamma_0'+h_*\delta=-\gamma_0''-p\delta'$ then we have

$$j_*\gamma_0=-j_*\gamma_0'+j_*h_*\delta=p\gamma$$

and $$\pi^*\gamma_0=-\pi^*\gamma_0''-\pi^*(p\delta')=-h_*\bar{\alpha}.$$ q. e. d.

Proposition 4.8. *Let* $n\geqq 7$, $h: Y_p^{n+2p-3}\to Y_p^n$ *be a representative of* $(m+1)\delta\alpha(n)-m\alpha\delta(n)$ *and let* $K=Y_p^n\cup_h CY_p^{n+2p-3}$ *be a mapping cone of* h. *Assume that* $m+k\equiv 0$ (mod p).

(i) *For* $k\geqq 2$, *there exists a coextension* $\gamma\in\pi_{n+2k(p-1)-2}(K)$ *of* $i^*(\alpha^{k-2}\delta\alpha(n+2p-3))\in\pi_{n+2k(p-1)-3}(Y_p^{n+2p-3})$ *such that*

$$p\cdot\gamma=j_*(i^*\alpha^{k-1}\delta\alpha(n)) \qquad \textit{if } n>7$$

and $$p\cdot\gamma\equiv j_*(i^*\alpha^{k-1}\delta\alpha(7)) \text{ mod } p\cdot j_*\pi_{2k(p-1)+5}(Y_p^7) \quad \textit{if } n=7.$$

(ii) *For* $k\geqq 1$, *there exists a coextension* $\gamma\in\pi_{n+2k(p-1)-1}(K)$ *of* $i^*\alpha^{k-1}(n+2p-3)\in\pi_{n+2k(p-1)-2}(Y_p^{n+2p-3})$ *such that*

$$p\cdot\gamma=-j_*(i^*\alpha^k(n)) \qquad \textit{if } n>7$$

and $$p\cdot\gamma\equiv -j_*(i_*\alpha^k(7)) \text{ mod } p\cdot j_*\pi_{2k(p-1)+6}(Y_p^7) \quad \textit{if } n=7.$$

Proof. (i) Put $q=2k(p-1)$. By Corollary 4.3 the relation

$$\pi^*h_*(i^*\alpha^{k-2}\delta\alpha(n+2p-3))=(m+1)\delta\alpha^{k-1}\delta\alpha\delta(n)-m\cdot\alpha\delta\alpha^{k-2}\delta\alpha\delta(n)=0$$

holds for $n\geqq 6$. By the exactness of (4.8) we have

$$h_*(i^*\alpha^{k-2}\delta\alpha(n+2p-3))\in p\cdot S\pi_{n+q-4}(Y_p^{n-1})=0 \quad \text{for } n\geqq 7.$$

Thus a coextension γ of $i^*(\alpha^{k-2}\delta\alpha(n+2p-3))$ exists. Obviously $\alpha^{k-2}\delta\alpha(n+2p-3)$ is an extension of $i^*(\alpha^{k-2}\delta\alpha(n+2p-3))$. By Lemma 4.7 there exists $\gamma_0\in\pi_{n+q-2}(Y_p^n)$ such that $p\gamma=j_*\gamma_0$ and $\pi^*\gamma_0=-h_*\alpha^{k-2}\delta\alpha(n+2p-3)$. By use of Corollary 4.3 we have

$$\begin{aligned}\pi^*\gamma_0 &= -\{(m+1)\delta\alpha(n)-m\cdot\alpha\delta(n)\}\circ\alpha^{k-2}\delta\alpha(n+2p-3)\\ &= -(m+1)\delta\alpha^{k-1}\delta\alpha(n)+m\cdot\alpha\delta\alpha^{k-2}\delta\alpha(n)\\ &= \{-(m+1)(k-1)+m(k-2)\}\alpha^{k-1}\delta\alpha\delta(n)\\ &= -\pi^*i^*(m+k-1)\alpha^{k-1}\delta\alpha(n)=\pi^*i^*\alpha^{k-1}\delta\alpha(n).\end{aligned}$$

Then (i) is proved by the exactness of (4.8) and $p\cdot S\pi_{n+q-3}(Y_p^{n-1})=0$.

(ii) By Proposition 4.6, $h_*(i^*\alpha^{k-1}(n+2p-3))=(m+k)i^*(\alpha^{k-1}\delta\alpha(n))=0$. Thus there exists a coextension γ of $i^*\alpha^{k-1}(n+2p-3)$. By Lemma 4.7 $p\gamma=j_*\gamma_0$ and $\pi^*\gamma_0=-h_*\alpha^{k-1}(n+2p-3)$ for some γ_0. By use of Corollary 4.3 we have

$$\begin{aligned}\pi^*\gamma_0 &= -\{(m+1)\delta\alpha(n)-m\alpha\delta(n)\}\circ\alpha^{k-1}(n+2p-3)\\ &= -(m+1)\delta\alpha^{k}(n)+m\cdot\alpha\delta\alpha^{k-1}(n)\\ &= -(m+1)\{k\cdot\alpha^{k-1}\delta\alpha(n)+(1-k)\alpha^{k}\delta(n)\}\\ &\qquad +m\{(k-1)\alpha^{k-1}\delta\alpha(n)+(2-k)\alpha^{k}\delta(n)\}\\ &= -(m+k)\alpha^{k-1}\delta\alpha(n)+(m+k-1)\alpha^{k}\delta(n)=-\alpha^{k}\delta(n)\\ &= -\pi^*i^*\alpha^{k}(n).\end{aligned}$$

Then (ii) is proved by the exactness of (4.8).

5. Unstable elements of the first and the second types.

An element γ of $\pi_i(S^{2m+1}:p)$ will be called as *an unstable element of the first type* if $S^2\gamma=0$ and $\gamma\notin \operatorname{Im} S^2$. Consider the exact sequence (1.7) of the case $n=2m-1$ and $k=2$:

$$(5.1)\qquad \cdots\to\pi_i(Q_2^{2m-1})\xrightarrow{p_*}\pi_i(S^{2m-1})\xrightarrow{S^2}\pi_{i+2}(S^{2m+1})\xrightarrow{H^{(2)}}\pi_{i-1}(Q_2^{2m-1})\xrightarrow{p_*}\cdots.$$

For the map $d:\Omega^3Q_2^{2m+1}\to Q_2^{2m-1}$ of (3.4) we have the following commutative diagram.

$$(5.2)\qquad \begin{array}{ccc}\pi_{i+3}(Q_2^{2m+1}) & \xrightarrow{p_*} & \pi_{i+3}(S^{2m+1})\\ \approx\Big\downarrow\Omega^3 & & \Big\downarrow H^{(2)}\\ \pi_i(\Omega^3Q_2^{2m+1}) & \xrightarrow{d_*} & \pi_i(Q_2^{2m-1}).\end{array}$$

Thus we have

(5. 3). *If* $d_*\Omega^3\gamma'=H^{(2)}p_*\gamma'\neq 0$ *for some* $\gamma'\in\pi_{i+3}(Q_2^{2m+1}:p)$, *then* $\gamma=p_*\gamma'$ *is an unstable element of the first type.*

Choose a complex $K(m,2)=Y_p^{2mp-2}\cup_h CY_p^{2(m+1)p-5}$ and a map G: $K(m,2)\to Q_4^{2m-1}$ which satisfy Proposition 3. 6 and Proposition 4. 5. Thus the map $h: Y_p^{2(m+1)p-5}\to Y_p^{2mp-2}$ represents

$$(m+1)\delta\alpha(2mp-2)-m\cdot\alpha\delta(2mp-2).$$

By Proposition 3. 6 we have a commutative diagram

$$(5.4)\qquad \begin{array}{ccc} \pi_i(Y_p^{2(m+1)p-5}) & \xrightarrow{h_*} & \pi_i(Y_p^{2mp-2}) \\ \big\downarrow g'_* & & \big\downarrow G_{1*} \\ \pi_i(\Omega^3 Q_2^{2m+1}) & \xrightarrow{d_*} & \pi_i(Q_2^{2m-1}). \end{array}$$

For the case $m=1$ a complex $\bar{K}=Y_p^{2p}\cup_h CY_p^{4p-3}$ of Proposition 3. 8 and the following commutative diagram will be also considered.

$$(5.4)'\qquad \begin{array}{ccc} \pi_{i+3}(Y_p^{4p-2}) & \xrightarrow{Sh_*} & \pi_{i+3}(Y_p^{2p+1}) \\ \big\downarrow g'_* & & \big\downarrow g_* \\ \pi_{i+3}(Q_2^3) & \xrightarrow{d'_*} & \pi_{i+3}(X_3)\approx\pi_i(Q_2^1). \end{array}$$

Theorem 5. 1. (i) *Let* $\beta\in\pi_i(S^{2(m+1)p-7})$ *be an element of order* p. *Then there exists an element* γ' *of* $\pi_{i+5}(Q_2^{2m+1}:p)$ *such that*

$p\cdot\gamma'=0$, $I(\gamma')=S^8\beta$ *and* $IH^{(2)}(p_*\gamma')=H_p(p_*\gamma')=xm\cdot\alpha_1(2mp+1)\circ S^5\beta$

for some $x\not\equiv 0$ (mod p). *Thus if* $m\not\equiv 0$ (mod p) *and* $\alpha_1(2mp+1)\circ S^5\beta\neq 0$ *then* $p_*\gamma'$ *is an unstable element of the first type.*

(ii) *Let* β *be an element of* $\pi_i(S^{2(m+1)p-7}:p)$. *Then the element* $\gamma'=I'(S^6\beta)\in\pi_{i+4}(Q_2^{2m+1}:p)$ *satisfies*

$$p\cdot\gamma'=0 \text{ and } H^{(2)}p_*\gamma'=y\cdot(m+1)\cdot I'(\alpha_1(2mp-1)\circ S^3\beta)$$

for some $y\not\equiv 0$ (mod p). *Thus if* $m\not\equiv -1$(mod p) *and* $\alpha_1(2mp-1)\circ S^3\beta\notin\Delta\pi_{i+5}(S^{2mp+1}:p)$ *then* $p_*\gamma'$ *is an unstable element of the first type.*

(iii) *In the case* $m=1$, *replacing* $\beta\in\pi_i(S^{2(m+1)p-7})$ *and* $S^t\beta$, $t\geq 3$, *by* $\beta\in\pi_{i+2}(S^{4p-5})$ *and* $S^{t-2}\beta$ *respectively,* (i) *and* (ii) *still hold.*

Proof. (i) Let $\beta_0\in\pi_{i+1}(Y_p^{2(m+1)p-6})$ be a coextension of β. $\pi_*\beta_0=S\beta$. By (1.5), $p\cdot S\beta_0=0$. By Lemma 2.5

$$I\Omega^{-3}g'_* S\beta_0=y\cdot S^6S(\pi_*\beta_0)=y\cdot S^8\beta$$

for some $y\not\equiv 0 \pmod p$. For an integer z such that $zy\equiv 1 \pmod p$, we put $\gamma'=z\cdot\Omega^{-3}g'_*S\beta_0$. Then $p\cdot\gamma'=0$ and $I\gamma'=S^8\beta$. We have

$$\begin{aligned} IH^{(2)}p_*\gamma' &= z\cdot IH^{(2)}p_*\Omega^{-3}g'_*S\beta_0 \\ &= z\cdot I(d_*g'_*S\beta_0) && \text{by (5.2)} \\ &= z\cdot I(G_{1*}h_*S\beta_0) && \text{by (5.4)} \\ &= y'z\cdot S^3(\pi_*h_*S\beta_0) \text{ for some } y'\not\equiv 0 \pmod p && \text{by Lemma 2.5,} \end{aligned}$$

and $\pi_*h_*S\beta_0=\pi_*\{(m+1)\delta\alpha(2mp-2)-m\cdot\alpha\delta(2mp-2)\}\circ S\beta_0$

$$\begin{aligned} &= \{(m+1)\pi_*i_*\pi_*\alpha(2mp-2)-m\cdot\pi^*i^*\pi_*\alpha(2mp-2)\}\circ S\beta_0 \\ &= -m\cdot\alpha_1(2mp-2)\circ\pi_*S\beta_0=-m\cdot\alpha_1(2mp-2)\circ S^2\beta. \end{aligned}$$

Thus $IH^{(2)}p_*\gamma'=-my'z\circ\alpha_1(2mp+1)\circ S^5\beta$ and (i) is proved by putting $x=-y'z$ and by (5.3).

(ii) By Lemma 2.5,

$$\gamma'=I'(S^6\beta)=x\cdot\Omega^{-3}g'_*i_*S\beta \quad \text{for some } x\not\equiv 0 \pmod p.$$

By (2.7) and (2.5), we have $p\gamma'=I'(p\cdot S^6\beta)=I'\Delta S^2(S^6\beta)=0$. For some $x'\not\equiv 0 \pmod p$ we have

$$\begin{aligned} H^{(2)}p_*\gamma' &= x\cdot H^{(2)}p_*\Omega^{-3}g'_*i_*S\beta=x\cdot d_*g'_*i_*S\beta && \text{by (5.2)} \\ &= x\cdot G_{1*}h_*i_*S\beta && \text{by (5.4)} \\ &= x\cdot G_{1*}\{(m+1)i^*i_*\pi_*\alpha(2mp-3)-m\cdot i^*\pi^*i^*\alpha(2mp-2)\}\circ S\beta \\ &= x(m+1)G_{1*}i_*\alpha_1(2mp-3)\circ S\beta \\ &= xx'(m+1)I'S^2(\alpha_1(2mp-3)\circ S\beta) && \text{by Lemma 2.5} \\ &= xx'(m+1)I'(\alpha_1(2mp-1)\circ S^3\beta). \end{aligned}$$

Thus (ii) is proved by putting $y=x'x$ and by (5.3), (2.5).

(iii) is proved similarly by use of (5.4)$'$ and Lemma 2.7.

Theorem 5.2. *For $k\geq 1$ there exists an element γ' of $\pi_{2mp+2k(p-1)-1}(Q_2^{2m+1}:p)$ satisfying the following conditions*:

(i) $p\cdot\gamma'=0$,

(ii) $I(\gamma')=\alpha_{k-1}(2(m+1)p+1)\in\pi_{2(m+1)p+2(k-1)(p-1)}(S^{2(m+1)p+1}:p)$ *for* $k>1$ *and* $\gamma'=I'\iota_{2(m+1)p-1}$ *for* $k=1$,

(iii) *if there exists an element* $\alpha'_k(2mp-3)$ *of* $\pi_{2mp+2k(p-1)-4}(S^{2mp-3}:p)$ *such that* $i_*\alpha'_k(2mp-3)=i^*(\alpha^{k-1}\delta\alpha(2mp-2))$ *then*

$$H^{(2)}p_*\gamma'=x\cdot(m+k)\cdot I'(S^2\alpha'_k(2mp-3))$$

for some $x\not\equiv 0 \pmod p$,

(iv) *in the case* $m=1$ *we may replace* $\alpha'_k(2mp-3)$ *by* $\alpha'_k(2p-1)$ *such that* $i_*\alpha'_k(2p-1)=i^*\alpha^{k-1}\delta\alpha(2p)$ *in* (iii).

Proof. By Lemma 2.5

$$I(\Omega^{-3}g'_*i^*\alpha^{k-1}(2(m+1)p-5))=y\cdot S^6\pi_*i^*\alpha^{k-1}(2(m+1)p-5)$$
$$=y\cdot S^6\alpha_{k-1}(2(m+1)p-5)=y\cdot\alpha_{k-1}(2(m+1)p+1)$$

for some $y\not\equiv 0 \pmod p$. Put $\gamma'=(1/y)\Omega^{-3}g'_*i^*\alpha^{k-1}(2(m+1)p-5)$, then (i) and (ii) hold. We have under the assumption of (iii)

$$\begin{aligned}H^{(2)}p_*\gamma'&=d_*\Omega^3\gamma'=(1/y)d_*g'_*i^*\alpha^{k-1}(2(m+1)p-5) && \text{by (5.2)}\\ &=(1/y)G_{1*}h_*i^*\alpha^{k-1}(2(m+1)p-5) && \text{by (5.4)}\\ &=(m+k)/y\cdot G_{1*}i^*\alpha^{k-1}\delta\alpha(2mp-2) && \text{by Proposition 4.6}\\ &=(m+k)/y\cdot G_{1*}i_*\alpha'_k(2mp-3) && \\ &=(m+k)x'/y\cdot I'(S^2\alpha'_k(2mp-3)) && \text{by Lemma 2.5}\end{aligned}$$

for some $x'\not\equiv 0 \pmod p$. Then (iii) holds for $x\equiv x'/y \pmod p$. Here the condition $2mp-2\geq 7$ is necessary in order to apply Proposition 4.6. So for $m=1$ we use Lemma 2.7 in place of Lemma 2.5. Then (iv) is obtained and (iii) of $m=1$ follows. q. e. d.

Theorem 5.3. *Assume that* $m+k\equiv 0 \pmod p$ *and* $m\geq 2$.

(i) *For* $k\geq 1$ *there exist elements*

$$\varepsilon\in\pi_{2m+2(m+k)(p-1)-1}(Q_2^{2m+1}:p),\ \varepsilon'\in\pi_{2m+2(m+k)(p-1)-3}(Q_2^{2m-1}:p)$$

and $\gamma\in\pi_{2m+2(m+k)(p-1)-3}(S^{2m-1}:p)$

such that

$p_*\varepsilon = S^2\gamma$, $p_*\varepsilon' = p\cdot\gamma$, $I(\varepsilon') = x\cdot\alpha_k(2mp+1)$ *for some* $x \not\equiv 0 \pmod p$ *and* $I(\varepsilon) = \alpha_{k-1}(2(m+1)p+1)$ *for* $k>1$, $\varepsilon = I'\iota_{2(m+1)p-1}$ *for* $k=1$.

(ii) *For* $k \geqq 2$ *there exist elements*

$$\varepsilon \in \pi_{2m+2(m+k)(p-1)-2}(Q_2^{2m+1}:p),\ \varepsilon' \in \pi_{2m+2(m+k)(p-1)-4}(Q_2^{2m-1}:p)$$

and
$$\gamma \in \pi_{2m+2(m+k)(p-1)-4}(S^{2m-1})$$

satisfying the following properties.

$$p_*\varepsilon = S^2\gamma \quad \text{and} \quad p_*\varepsilon' = p\cdot\gamma.$$

If there exists an element $\alpha'_{k-1}(2(m+1)p-3)$ *of Proposition* 4.4, (iii), *then* $\varepsilon = I'(S^2\alpha'_{k-1}(2(m+1)p-3))$. *If there exists an element* $\alpha'_k(2mp-3)$ *of Proposition* 4.4, (iii), *then* $I'(S^2\alpha'_k(2mp-3)) = x\cdot\varepsilon'$ *for some* $x \not\equiv 0 \pmod p$.

Proof. By Proposition 3.6 and Proposition 4.5 we have the following commutative diagram:

$$\begin{array}{ccccccc}
\pi_i(Y_p^{2mp-2}) & \xrightarrow{j_*} & \pi_i(K(m,2)) & \xrightarrow{\pi_*} & \pi_i(Y_p^{2(m+1)p-4}) & \xrightarrow{S^2} & \pi_{i+2}(Y_p^{2(m+1)p-2}) \\
\downarrow G_{1*} & & \downarrow G_* & & \downarrow g_* & & \downarrow g''_* \\
\pi_i(Q_2^{2m-1}) & \xrightarrow{i_*} & \pi_i(Q_4^{2m-1}) & \xrightarrow{j_*} & \pi_i(\Omega^2 Q_2^{2m+1}) & \underset{\approx}{\xleftarrow{\Omega^2}} & \pi_{i+2}(Q_2^{2m+1}) \\
& \searrow p_* & \downarrow p_* & & & p_* \swarrow & \\
& & \pi_i(S^{2m-1}) & \xrightarrow{S^2} & \pi_{i+2}(S^{2m+1}), & &
\end{array}$$

where $K(m,2)$ is a mapping cone of a representatives h of $(m+1)\delta\alpha(2mp-2) - m\cdot\alpha\delta(2mp-2)$ and $\{g\} = \Omega^2\{g''\}$. Translate the elements of Proposition 4.8, (ii), which are in the top sequence, to the middle one, then we see the existence of an element $\gamma' \in \pi_{2m+2(m+k)(p-1)-3}(Q_4^{2m-1})$ such that ($\gamma' = G_*\gamma$ for γ of Proposition 4.8, (ii))

$$-p\cdot\gamma' = i_*G_{1*}(i^*\alpha^k(2mp-2)) \text{ and } j_*\gamma' = g_*(i^*\alpha^{k-1}(2(m+1)p-4)).$$

As is seen in the proof of Theorem 5.2, (ii) these relations imply the properties of (i) for $\gamma = z\cdot p_*\gamma'$, $\varepsilon = z\cdot\Omega^{-2}j_*\gamma'$ and $\varepsilon' = -z\cdot G_{1*}i^*\alpha^k(2mp-2)$ with a suitable coefficient $z \not\equiv 0 \pmod p$.

The proof of (ii) is similar applying Proposition 4.8. (ii), and omitted.

Theorem 5.4. (i) *Assume that* $m+k\equiv 0 \pmod p$ *and there exist* $\alpha'_k(2mp-3)$ *and* $\alpha'_{k-1}(2(m+1)p-5)$ *of Proposition* 4.4, (iii). *If there exists an element* γ *of* $\pi_{2m+2(m+k)(p-1)+1}(S^{2m+3}:p)$ *such that* $H^{(2)}\gamma=I'\alpha'_{k-1}(2(m+1)p-1)$, $k\geqq 2$, *then there exists an element* $\gamma'\in\pi_{2m+2(m+k)(p-1)-1}(S^{2m+1}:p)$ *such that*

$$p\cdot\gamma=S^2\gamma'$$

and $$H^{(2)}\gamma'\equiv x\cdot I'\alpha'_k(2mp-1) \mod p\cdot\pi_{2m+2(m+k)(p-1)-4}(Q_2^{2m-1}:p)$$

for some $x\not\equiv 0 \pmod p$.

(ii) *Assume that* $m+k\equiv 0 \pmod p$ *and* $I'(\pi_{2m+2(m+k)(p-1)+2}(S^{2(m+1)p-1}:p))\cap H^{(2)}(\pi_{2m+2(m+k)(p-1)+2}(S^{2m+3}:p))=0$. *If there exists an element* γ *of* $\pi_{2m+(m+k)(p-1)+2}(S^{2m+3}:p)$ *such that* $H_p\gamma=I(H^{(2)}\gamma)=\alpha_{k-1}(2(m+1)p+1)$ *for* $k>1$ *and* $H^{(2)}\gamma=I'(\iota_{2(m+1)p-1})$ *for* $k=1$, *then there exists an element* γ' *of* $\pi_{2m+2(m+k)(p-1)}(S^{2m+1}:p)$ *such that*

$$p\cdot\gamma=S^2\gamma'$$

and for some $x\not\equiv 0 \pmod p$

$$H_p\gamma'=I(H^{(2)}\gamma')\equiv x\cdot\alpha_k(2mp+1) \mod p\cdot\pi_{2m+2(m+k)(p-1)}(S^{2mp+1}:p).$$

Proof. (i) $H^{(2)}(p\cdot\gamma)=I'(p\cdot\alpha'_{k-1}(2(m+1)p-1)=I'\Delta S^2\alpha'_{k-1}(2(m+1)p-1)=0$ by (2.7) and by the exactness of (2.5). It follows from the exactness of (5.1) the existence of an element γ'' such that $p\cdot\gamma=S^2\gamma''$.

Let $\bar\gamma\in\pi_{2m+2(m+k)(p-1)-4}(K(m,2))$ be a coextension of $i^*(\alpha^{k-2}\delta\alpha(2(m+1)p-5))$ then by Proposition 4.8, (i), Proposition 4.5 and by Proposition 4.4, (iii) we have

$$p\cdot\bar\gamma=-j_*(i^*\alpha^{k-1}\delta\alpha(2mp-2))=-j_*i_*\alpha'_k(2mp-3)$$
and $$\pi_*\bar\gamma=i^*(\alpha^{k-2}\delta\alpha(2(m+1)p-4))=i_*\alpha'_{k-1}(2(m+1)p-5).$$

By Proposition 3.6 we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & \pi_i(Y_p^{2mp-2}) & \xrightarrow{i_*} & \pi_i(K(m,2)) & \xrightarrow{\pi_*} & \pi_i(Y_p^{2(m+1)p-4}) \\
 & & \downarrow G_{1*} & & \downarrow G_* & & \downarrow g_* \\
\pi_i(\Omega^3 Q_2^{2m+1}) & \xrightarrow{d_*} & \pi_i(Q_2^{2m-1}) & \xrightarrow{i_*} & \pi_i(Q_4^{2m-1}) & \xrightarrow{i_*} & \pi_i(\Omega^2 Q_2^{2m+1}) \\
\approx\uparrow \Omega^3 & & \uparrow H^{(2)} & & \uparrow H^{(4)} & & \approx\uparrow \Omega^2 \\
\pi_{i+3}(Q_2^{2m+1}) & \xrightarrow{p_*} & \pi_{i+3}(S^{2m+1}) & \xrightarrow{S^2} & \pi_{i+5}(S^{2m+3}) & \xrightarrow{H^{(2)}} & \pi_{i+2}(Q_2^{2m+1}).
\end{array}$$

For some $x'\not\equiv 0 \pmod p$ we have, by Lemma 2.5,

$$\begin{aligned}
x'\cdot j_* G_*\bar{\gamma} &= x'\cdot g_*\pi_*\bar{\gamma} = x'\cdot g_* i_* \alpha'_{k-1}(2(m+1)p-5) \\
&= \Omega^2 I' S^4 \alpha'_{k-1}(2(m+1)p-5) \\
&= \Omega^2 I' \alpha'_{k-1}(2(m+1)p-1).
\end{aligned}$$

Thus $j_*(H^{(4)}\gamma - x'\cdot G_*\bar{\gamma})=0$. By the exactness of the above middle sequence there exists $\beta\in\pi_{2m+2(m+k)(p-1)-4}(Q_2^{2m-1}:p)$ such that

$$i_*\beta = H^{(4)}\gamma - x'\cdot G_*\bar{\gamma}$$

and
$$\begin{aligned}
i_*(p\cdot\beta) &= H^{(4)}(S^2\gamma'') + x'\cdot G_* j_* i_* \alpha'_k(2mp-3) \\
&= i_*(H^{(2)}\gamma'' + x'\cdot G_{1*} i_* \alpha'_k(2mp-3)).
\end{aligned}$$

Again by the exactness of the sequence we have

$$H^{(2)}\gamma'' = -x'\cdot G_{1*} i_* \alpha'_k(2mp-3) + p\cdot\beta + d_*\beta' \quad \text{for some } \beta'.$$

By Lemma 2.5

$$-x'\cdot G_{1*} i_* \alpha'_k(2mp-3) = x\cdot I' S^2 \alpha'_k(2mp-3) = x\cdot I'\alpha'_k(2mp-1)$$

for some $x\not\equiv 0 \pmod p$. Put $\gamma' = \gamma'' - p_*\Omega^{-3}\beta'$, then we have

$$S^2\gamma' = S^2\gamma'' - S^2 p_*\Omega^{-3}\beta' = p\cdot\gamma$$

and
$$H^{(2)}\gamma'' = x\cdot I'\alpha'_k(2mp-1) + p\cdot\beta.$$

Remark that the above proof breaks if $m=1$ and $p=3$ since $2mp-3<7$ and Proposition 4.8 cannot be applied for this case $n=2mp-3$. So, in the case $m=1$ we use Lemma 2.7 in place of Lemma 2.5. By Proposition 3.8 we have the following commutative diagram:

$$\begin{tikzcd}
 & \pi_{i+3}(Y_p^{2p+1}) \arrow[r, "j_*"] \arrow[d, "g_*"] & \pi_{i+3}(S\overline{K}) \arrow[r, "j'_*"] \arrow[d, "G_*"] & \pi_{i+3}(S\overline{K}, Y_p^{2p+1}) \arrow[d, "G_*"] & \pi_{i+3}(CY_p^{4p-2}, Y_p^{4p-2}) \arrow[l, "\chi_*"'] \arrow[d, "\partial", "\approx"'] \\
\pi_{i+3}(Q_2^3) \arrow[r, "d'_*"] \arrow[dr, "p_*"'] & \pi_{i+3}(X_3) \arrow[r, "i_*"] \arrow[d, "p_{3*}"] & \pi_{i+3}(\Omega^2 X_5) \arrow[r, "j_*"] \arrow[d, "p_{5*}\cdot\Omega^2"] \arrow[dr, "\partial"] & \pi_{i+3}(\Omega^2 X_5, X_3) \arrow[d, "\Omega", "\approx"'] & \pi_{i+2}(Y_p^{4p-2}) \arrow[dl, "g''_*"] \\
 & \pi_{i+3}(S^3) \arrow[r, "S^2"] & \pi_{i+5}(S^5) \arrow[r, "H^{(2)}"] & \pi_{i+2}(Q_2^3), &
\end{tikzcd}$$

where p_{3*} and $p_{5*}\circ\Omega^2$ are isomorphisms for $i>0$ and the inverses are equivalent to $H^{(2)}$ and $H^{(4)}$. By Proposition 4.8, there exists a coextension $\bar{\gamma}\in\pi_{2p+2k(p-1)-1}(S\overline{K})$ of $i^*(\alpha^{k-2}\delta\alpha(4p-2))=i_*\alpha'_{k-1}(4p-3)$, such that

$$j'_*\bar{\gamma}=\chi_*\partial^{-1}(i_*\alpha'_{k-1}(4p-3))$$

and
$$p\cdot\bar{\gamma}=-j^*i_*\alpha'_k(2p)+p\cdot j_*\varepsilon \quad \text{for some } \varepsilon\in\pi_{2p+2k(p-1)-1}(Y_p^{2p+1}),$$

where $\varepsilon=0$ if $p\neq 3$. Then (i) for $m=1$ is proved similarly as above and the details are left to the reader.

(ii) is proved by use of Proposition 4.8, (ii) in place of Proposition 4.8, (i) in the proof of (i). $IH^{(2)}(p\cdot\gamma)=p\cdot\alpha_{k-1}(2(m+1)p+1)=0$. By the assumption and (2.5) we have $H^{(2)}(p\cdot\gamma)=0$ and $S^2\gamma''=p\cdot\gamma$ for some γ''. Let $\bar{\gamma}$ be a coextension of $i^*\alpha^{k-1}(2(m+1)p-5)$, then by Proposition 4.8, (ii) and Proposition 4.5 we have

$$p\cdot\bar{\gamma}=-j_*(i^*\alpha^k(2mp-2)) \quad \text{and} \quad \pi_*\bar{\gamma}=i^*\alpha^{k-1}(2(m+1)p-4).$$

By use of Lemma 2.5 we have $I\Omega^{-2}j_*G_*\bar{\gamma}=y\cdot\alpha_{k-1}(2(m+1)p+1)$ if $k>1$ and $\Omega^{-2}j_*G_*\bar{\gamma}=y\cdot I'\iota_{2(m+1)p-1}$ if $k=1$, for some $y\not\equiv 0 \pmod p$. By the exactness of (2.5) and by the assumption we have $H^{(2)}\gamma=(1/y)\Omega^{-2}j_*G_*\bar{\gamma}$. By a proof parallel to (i) we obtain

$$H^{(2)}\gamma''=(1/y)G_{1*}i^*\alpha^k(2mp-2)+p\beta+d_*\beta'$$

for some β and β'. By Lemma 2.5, $I((-1/y)G_{1*}i^*\alpha^k(2mp-2))=-y'/y\cdot\alpha_k(2mp+1)$, $y'\not\equiv 0 \pmod p$. Then the assertion of (ii) is obtained by putting $x=-y'/y$ and $\gamma'=\gamma''-p_*\Omega^3\beta'$. q. e. d.

6. Meta-stable groups I.

We introduce the following results of stable groups from [16].

(6.1) $(\pi_k^S:p)\approx Z_p$ *for* $k=2r(p-1)-1,\ 1\leq r<p^2,$
$r\not\equiv 0\pmod p)$: *generator* α_r,
$\approx Z_{p^2}$ *for* $k=2tp(p-1)-1,\ 1\leq t<p-1$:
generator α'_{tp},
$\approx Z_{p^2}+Z_p$ *for* $k=2(p^2-p)(p-1)-1$:
generators $\alpha'_{(p-1)p},\ \alpha_1\beta_1^{p-1}$,
$\approx Z_p$ *for* $k=2(sp+r)(p-1)-2(s-r),\ 0\leq r<s,$
$sp+r<p^2$: *generator* $\beta_1^{s-r-1}\beta_{r+1}$,
$\approx Z_p$ *for* $k=2(sp+r+1)(p-1)-2(s-r)-1,$
$0\leq r<s,\ sp+r+1<p^2,\ (r,s)\neq(0,p-1)$:
generator $\alpha_1\beta_1^{s-r-1}\beta_{r+1}$,
$=0$ *otherwise for* $k<2p^2(p-1)-3.$

By Proposition 4.17 of [16], the above element α'_{tp} satisfies

$$(1/t)\alpha'_{tp}\in\{p\iota,\ \alpha_{k-1},\ \alpha_1\},\ k=tp.$$

We see also the element α'_k of Proposition 4.4, (iii) satisfies

(6.2)′ $\alpha'_k\in\pm\{p\iota,\ \alpha_{k-1},\ \alpha_1\}$.

For $i_*\alpha'_k=i^*(\alpha^{k-1}\delta\alpha)=i^*\alpha^{k-1}i^*\pi_*\alpha=i^*\alpha^{k-1}\circ\alpha_1$. Here $i^*\alpha^{k-1}$ is a coextension of $\pm\alpha_{k-1}$. By Proposition 1.8 of [18], $i^*\alpha^{k-1}\circ\alpha_1\in\pm i_*\{p\iota,\ \alpha_{k-1},\ \alpha_1\}$. Since the kernel of i_* is $p\cdot\pi^S_{2k(p-1)-1}$ and it is contained in the indeterminacy, we have (6.2)′. We have also

(6.2). *In* (6.1), *the generators* α'_{tp} *may be replaced by the corresponding elements in Proposition* 4.4, (iii). *If* $r\not\equiv 0\pmod p)$ α_r *may be replaced by* α'_r (*Proposition* 4.4, (iv)).

We shall use the following natations.

(6.3) (i) *For* $\gamma\in S^\infty\pi_{i+2}(S^{2mp-1}:p)\subset\pi^S_{i-2mp+3}$,
$Q^m(\gamma)\in\pi_i(Q_2^{2m-1}:p)$

denotes an element such that

$$Q^m(\gamma)=I'(\gamma') \quad and \quad S^\infty\gamma'=\gamma$$

for some $\gamma'\in\pi_{i+2}(S^{2mp-1}:p)$.

(ii) *For* $\gamma\in\pi^S_{i-2mp+2}$,

$$\bar{Q}^m(\gamma)\in\pi_i(Q_2^{2m-1}:p)$$

denotes an element (if exists) such that $S^\infty I(\bar{Q}^m(\gamma))=\gamma$.

In meta-stable case, $i<2mp^2-5$, $Q^m(\gamma)$ exists uniquely for any $\gamma\in(\pi^S_{i-2mp+3}:p)$ and if $p\cdot\gamma=0$ then $\bar{Q}^m(\gamma)$ exists and is unique mod $\mathrm{Im}\, I'$.

It follows from Theorem 2.2 and (6.1)

(6.4). *Assume* $i<2mp^2-5$ *and* $i-2mp<2p^2(p-1)-6$, *then* $\pi_i(Q_2^{2m-1}:p)$ *is a* Z_p*-module having the following base*:

$\{Q^m(\alpha'_p),\ \bar{Q}^m(\beta_1)\}$	*if* $i-2mp=2p(p-1)-4$,
$\{Q^m(\alpha_{sp+s-1}),\ \bar{Q}^m(\beta_s)\}$	*if* $i-2mp=2(sp+s-1)(p-1)-4,\ 2\leq s<p$,
$\{\bar{Q}^m(\alpha_{(p-1)p-1}),\ Q^m(\beta_1^{p-1})\}$	*if* $i-2mp=2(p^2-p-1)(p-1)-3$,
$\{Q^m(\alpha'_{(p-1)p}),\ Q^m(\alpha_1\beta_1^{p-1})\}$	*if* $i-2mp=2(p^2-p)(p-1)-4$,
$\{\bar{Q}^m(\alpha_{(p-1)p}),\ \bar{Q}^m(\alpha_1\beta_1^{p-1})\}$	*if* $i-2mp=2(p^2-p)(p-1)-3$,
$\{I'\iota_{2mp-1}\}$	*if* $i-2mp=-3$,
$\{\bar{Q}^m(\alpha_r)\}$	*if* $i-2mp=2r(p-1)-3,\ 1\leq r<p^2$, $r\neq p^2-p,\ r\neq p^2-p-1$,
$\{Q^m(\alpha_r)\}$	*if* $i-2mp=2r(p-1)-4,\ 1\leq r<p^2$, $r\not\equiv 0 \pmod p,\ r\not\equiv -1 \pmod{p+1}$,
$\{Q^m(\alpha'_{tp})\}$	*if* $i-2mp=2tp(p-1)-4,\ 2\leq t<p-1$,
$\{Q^m(\beta_1^{s-r-1}\beta_{r+1})\}$	*if* $i-2mp=2(sp+r)(p-1)-2(s-r)-3$, $0\leq r<s,\ sp+r\leq p^2,\ (r,s)\neq(0,p-1)$,
$\{\bar{Q}^m(\beta_1^{s-r-1}\beta_{r-1})\}$	*if* $i-2mp=2(sp+r)(p-1)-2(s-r)-2$, $0\leq r,\ r+1<s,\ sp+r<p^2,\ (r,s)\neq(1,0)$,
$\{Q^m(\alpha_1\beta_1^{s-r-1}\beta_{r+1})\}$	*if* $i-2mp=2(sp+r+1)(p-1)-2(s-r)-4$, $0\leq r<s,\ sp+r+1<p^2,\ (r,s)\neq(0,p-1)$,
$\{\bar{Q}^m(\alpha_1\beta_1^{s-r-1}\beta_{r+1})\}$	*if* $i-2mp=2(sp+r+1)(p-1)-2(s-r)-3$, $0\leq r<s,\ sp+r+1<p^2,\ (r,s)\neq(0,p-1)$,

$\{\phi\}$ *if otherwise.*

Consider the composition

$$H^{(2)}\circ p_* = \partial : \pi_i(Q_2^{2m+1}:p)\to\pi_i(S^{2m+1}:p)\to\pi_{i-3}(Q_2^{2m-1}:p),\ m\geq 1.$$

Lemma 6.1. (**i**) *Let $r\geq 0$, $p>s\geq 1$ and assume $m\not\equiv 0$ (mod p). For $r\geq 1$ and for $(r,s)=(0,1)$ we have a relation*

$$H^{(2)}(p_*(\overline{Q}^{m+1}(\beta_1^r\beta_s)))=x\cdot\overline{Q}^m(\alpha_1\beta_1^r\beta_s),\quad x\not\equiv 0 \pmod p.$$

For $r=0$ and $s>1$, the relation holds provided that $\beta_s=S^\infty\beta_s(n)$ for some $\beta_s(n)\in\pi_{n+2(sp+s-1)(p-1)-2}(S^n:p)$ of order p and for some $n\leq 2(m+1)p-3$.

(**ii**) *Let $r\geq 0$, $p>s\geq 1$ and assume $m\not\equiv -1$ (mod p). For $r\geq 1$ we have a relation*

$$H^{(2)}(p_*(Q^{m+1}(\beta_1^r\beta_s)))=x\cdot Q^m(\alpha_1\beta_1^r\beta_s),\quad x\not\equiv 0 \pmod p.$$

For $r=0$, the relation holds provided the same one as in (i).

(**iii**) *Assume $k+m\not\equiv 0$ (mod p). For $k\geq 2$ we have, if $\alpha_k'(2mp-3)$ exists,*

$$H^{(2)}(p_*(\overline{Q}^{m+1}(\alpha_{k-1})))=x\cdot Q^m(\alpha_k'),\quad x\not\equiv 0 \pmod p.$$

Also we have $H^{(2)}(p_*(I'\iota_{2(m+1)p-1}))=x\cdot Q^m(\alpha_1)$, $x\not\equiv 0$ (mod p) if $m\not\equiv -1$ (mod p).

Proof. (i) By Theorem 5.1, it is sufficient to prove that there exists an element $\beta_1^r\beta_s(n)\in\pi_{n+k}(S^n:p)$, $k=2((r+s)p+s-1)(p-1)-2(r+1)$ such that $n\leq 2(m+1)p-3$, $p\cdot\beta_1^r\beta_s(n)=0$ and $S^\infty\beta_1^r\beta_s(n)=\beta_1^r\beta_s$. For $r=0$ and $s>1$ this is assumed. Let $r\geq 1$ or $(r,s)=(0,1)$. We shall prove the existence of such $\beta_1^r\beta_s(n)$ for $n=2p+1$. By (2.8), if $n\geq 2(sp+s-1)-1$, $\beta_s=S^\infty\beta_s(n)$ for some $\beta_s(n)\in\pi_{n+k}(S^n:p)$ and further if $n\geq 2(sp+s-1)+1$ $\beta_s(n)$ is of order p. Define $\beta_1^r\beta_s(2p-1)$ by $\beta_1^r(2p-1)=\beta_1(2p-1)\circ S^{2p(p-1)-2}\beta_1^{r-1}(2p-1)$, $r=2, 3, \cdots$, and $\beta_1^r\beta_s(2p-1)=\beta_1^r(2p-1)\circ\beta_s(2p-1+2rp(p-1)-2r)$. Then $\beta_1^r\beta_s(2p+1)=S^2(\beta_1^r\beta_s(2p-1))$ is the required element.

(ii) The proof of (ii) is similar to that of (i) with a remark that for the case $m=1$ the existence of the element $\beta_1^r\beta_s(2p-1)$

satisfying the required properties has to be established. The only difference with (i) is that $p\cdot\beta_1(2p-1)$ is not necessarily zero.

(iii) follows immediately from Theorem 5.2. q. e. d.

We have also from Theorem 5.2

(6.5). *If* $k+m\equiv 0 \pmod p$ *then* $H^{(2)}p_*\overline{Q}^{m+1}(\alpha_{k-1})=0$ *for* $k>1$ *and* $H^{(2)}p_*(I'\iota_{2(m+1)p-1})=0$ *for* $k=1$.

Proposition 6.2. *Assume* $k<2(m+p^2)(p-1)-1$ *and* $k<2m(p+1)(p-1)-3$ *then* $\pi_{2m+1+k}(S^{2m+1}:p)$ *has a direct summand* $U(k,m)$ *such that* $S^2U(k,m)=0$, $U(k,m)\cap \mathrm{Im}\, S^2=0$ *and*

$U(k,m)\approx Z_p+Z_p$ *if* $m\not\equiv -1 \pmod p$ *and* $k=2(m+p^2-p)(p-1)-2$,

$U(k,m)\approx Z_p$ *if* $k=2r(p-1)-2$, $r\geq 1$, $r\not\equiv 0 \pmod p$ *and not the above case*,

$U(k,m)\approx Z_p$ *if* $m\not\equiv 0 \pmod p$, $k=2(m+(s+r)p+s)(p-1)-2(r+1)-1$, $0\leq r$, $1\leq s$,

$U(k,m)\approx Z_p$ *if* $m\not\equiv -1 \pmod p$, $k=2(m+(s+r)(p+s)(p-1)-2(r+1)-2$, $0\leq r$, $1\leq s$, $(r,s)\neq(p-2,1)$,

$U(k,m)=0$ *if otherwise.*

This follows from (5.3), (6.4) and Lemma 6.1. $U(k,m)$ are generated by corresponding elements in $H^{(2)}(\ \ )$ of Lemma 6.1.

Proposition 6.3. *Assume* $k<2(m+p^2-1)(p-1)-4$ *and* $2m(p-1)-2<k<2(m-1)(p^2-1)-3$ *then we have an exact sequence*

$$\cdots\xrightarrow{H^{(2)}}V(k+1,m)\xrightarrow{p_*}\pi_{2m-1+k}(S^{2m-1}:p)/U(k,m-1)$$
$$\xrightarrow{S^2}\pi_{2m+1+k}(S^{2m+1}:p)/U(k,m)\xrightarrow{H^{(2)}}V(k,m)\to\cdots,$$

where $V(k,m)$ *are given as follows*:

$V(k,m)\approx Z_p+Z_p$ *if* $k=2tp(p-1)-1$ *and* $m=(t-p+1)p+1$,

$V(k,m)\approx Z_p$ *if* $k=2tp(p-1)-1$ *and* $m\neq(t-p+1)p+1$,

$V(k,m)\approx Z_p$ *if* $k=2tp(p-1)-2$,

$V(k, m)\approx Z_p$ *if* $k=2tp(p-1)-2s-1,\ p>s\geq 1,\ m=(t-s)p,$

$V(k, m)\approx Z_p$ *if* $k=2tp(p-1)-2s-2,\ p>s\geq 1,$
$m=(t-s)p-1,$

$V(k, m)\approx Z_p$ *if* $k=2(ap+b)(p-1)-2,\ p>b>0,$
$m=(a-b)p+1,$

$V(k, m)\approx Z_p$ *if* $k=2(ap+b)(p-1)-2s-1,\ p>b>0,\ p>s\geq 1,$
$(b, s)\neq(1, p-1), m=(a-b-s+1)p$ or $=(a-b-s)p,$

$V(k, m)\approx Z_p$ *if* $k=2(ap+b)(p-1)-2s-2,\ p>b>0,\ p>s\geq 1,$
$m=(a-b-s)p+1$ or $=(a-b-s)p-1,$

$V(k, m)=0$ *if otherwise.*

Proof. $V(k, m)$ are generated by the elements in (6. 3) which does not appear in Lemma 6. 1. $\pi_{2m+k-2}(Q_2^{2m-1}: p)$ is a direct sum of $V(k, m)$, $H^{(2)}U(k, m)$ and a submodule $U'(k, m)$ which is mapped isomorphically onto $U(k-1, m-1)$ under p_*. Then the proposition follows from the exactness of (5. 1). q. e. d.

Corollary 6. 4. *Assume* $k<2(m+p^2-1)(p-1)-4$ *and* $k<2(m-1)(p+1)(p-1)-3$ *then we have isomorphisms*

$$\pi_{2m-1+k}(S^{2m-1}: p)\approx(\pi_k^S: p)+U(k, m-1)$$

in the following cases:

(i) $k=2r(p-1)-1,\ r\not\equiv 0 \pmod p,$

(ii) $k=2(ap+b)(p-1)-2,\ p>b>0,\ m>(a-b)p+1,$

(iii) $k=2(ap+b)(p-1)-3,\ p>b>0,\ m>(a-b)p,$

(iv) $k=2(ap+b)(p-1)-2s-2,\ p>b\geq 0,\ p>s\geq 1,$
$m>(a-b-s+1)p-1,\ (b, s)\neq(p-1, p-1),$

(v) $k=2(ap+b)(p-1)-2s-1,\ p>b\geq 0,\ p>s\geq 2,$
$m>(a-b-s+1)p,\ (b, s)\neq(p-1, p-1).$

Here the subgroup of $\pi_{2m-1+k}(S^{2m-1}: p)$ corresponding to $(\pi_k^S: p)$ is mapped isomorphically onto $(\pi_k^S: p)$ by S^∞.

Proof. In these cases, we have $V(k', m)=0$ for $k'\geq k$. Then the corollary follows from Proposition 6. 3 and (2. 8).

7. Unstable groups I.

We shall compute the groups $\pi_{2m-1+t}(S^{2m-1}:p)$ for $t<6p(p-1)-7$ if $p\geq 5$ and for $t<33$ if $p=3$. The basic tool of the computation is the exact sequence (5.1).

Theorem 7.1.

$$\pi_{2m-1+k}(S^{2m-1}:p)\approx\begin{cases} Z_p & \text{for } k=2r(p-1)-1,\ r=1,2,\cdots,p-1 \\ & \quad \text{and } m\geq 2, \\ Z_p & \text{for } k=2r(p-1)-2,\ r=2,\cdots,p-1 \\ & \quad \text{and } r\geq m\geq 2, \\ 0 & \text{otherwise for } k<2p(p-1)-2, \end{cases}$$

$$\pi_{2m-1+2p(p-1)-2}(S^{2m-1}:p)\approx\begin{cases} Z_{p^2} & \text{for } p\geq m\geq 3, \\ Z_p & \text{for } m=2 \text{ and for } m>p, \end{cases}$$

$$\pi_{2m-1+2p(p-1)-1}(S^{2m-1}:p)\approx\begin{cases} Z_{p^2} & \text{for } m\geq 3, \\ Z_p & \text{for } m=2. \end{cases}$$

Proof. For the case $k<2p(p-1)-3$, the results follow immediately from Corollary 6.4, (6.1) and Proposition 6.2. We remark

(7.1). (i) $\pi_{2m-1+2r(p-1)-1}(S^{2m-1}:p)$, $1\leq r<p$, $m\geq 2$, *is generated by* $\alpha_r(2m-1)$.

(ii) $\pi_{2m-1+2r(p-1)-2}(S^{2m-1}:p)$, $p>r\geq m\geq 2$, *is generated by an unstable element of the first type.*

Put $q=2p(p-1)$. Consider the exact sequences (5.1):

$$\pi_{2m+q-4}(S^{2m-1}:p)\rightarrow\pi_{2m+q-2}(S^{2m+1}:p)\rightarrow\pi_{2m+q-5}(Q_2^{2m-1}:p),$$

for $m=1,2,\cdots$, where $\pi_{q-2}(S^1:p)=0$ obviously and $\pi_{2m+q-5}(Q_2^{2m-1}:p)=0$ by (6.4). It follows that $\pi_{2m-1+q-3}(S^{2m-1}:p)=0$ for all m. This completes the first statement of the theorem.

By (5.1) and by the result just obtained, we have exact sequences

$$\pi_{2m+q-3}(Q_2^{2m-1}:p)\xrightarrow{p_*}\pi_{2m+q-3}(S^{2m-1}:p)\xrightarrow{S^2}\pi_{2m+q-1}(S^{2m+1}:p)$$
$$\xrightarrow{H^{(2)}}\pi_{2m+q-4}(Q_2^{2m-1}:p)\rightarrow 0,$$

where

$$\pi_{2m+q-4}(Q_2^{2m-1}:p)\approx\begin{cases}Z_p & \text{generated by } Q^m(\alpha_{p-m}) \text{ for } 1\leq m<p,\\ 0 & \text{for } m\geq p,\end{cases}$$

$$\pi_{2m+q-3}(Q_2^{2m-1}:p)\approx\begin{cases}Z_p & \text{generated by } \overline{Q}^m(\alpha_{p-m}) \text{ for } 1\leq m<p,\\ Z_p & \text{generated by } I'\iota_{2p^2-1} \quad \text{for } m=p,\\ 0 & \text{for } m>p,\end{cases}$$

by (6. 4). Let β be an element of $\pi_{2p+q-3}(S^{2p-1}:p)$ such that $H^{(2)}\beta=Q^{p-1}(\alpha_1)$. By Theorem 5.4, (i), there exists an element β' of $\pi_{2p+q-5}(S^{2p-3}:p)$ such that $S^2\beta'=p\cdot\beta$ and $H^{(2)}\beta'=x\cdot Q^{p-2}(\alpha_2)$, $x\not\equiv 0$ (mod p). β' does not vanish under S^2 since the kernel of the S^2 is generated by $p_*\overline{Q}^{p-1}(\alpha_1)$ and $H^{(2)}p_*(\overline{Q}^{p-1}(\alpha_1))=0$ by (6. 5). It follows that the order of β is a multiple of p^2. But, $S^2(p\cdot\beta)=0$ since $\pi_{2p+q-1}(S^{2p+1}:p)\approx(\pi_{q-2}^s:p)\approx Z_p$ by (2. 8) and (6. 1). This shows that $p\cdot\beta=y\cdot p_*(I'\iota_{2p^2-1})$ and $p^2\cdot\beta=0$, $y\not\equiv 0$ (mod p). We have obtained that β generates $\pi_{2p+q-3}(S^{2p-1}:p)\approx Z_{p^2}$ and $S^2\pi_{2p+q-5}(S^{2p-3}:p)\approx Z_p$. If $p>3$, we repeat this discussion for β' in place of β and so on. Then the second assertion of the theorem is obtained. We have obtained also

(7. 2). (i) *For* $S^2:\pi_{2m-1+2p(p-1)-2}(S^{2m-1}:p)\rightarrow\pi_{2m+1+2p(p-1)-2}(S^{2m+1}:p)$ *we have*

$$\operatorname{Ker} S^2\approx\begin{cases}Z_p & \textit{for } p\geq m\geq 3\\ 0 & \textit{for } m<3 \textit{ and for } m>p\end{cases}$$

and

$$\operatorname{Coker} S^2\approx\begin{cases}Z_p & \textit{for } p-1>m\geq 1\\ 0 & \textit{for } m\geq p-1.\end{cases}$$

(ii) *For* $m\geq 3$, $S^2:\pi_{2m-1+2p(p-1)-1}(S^{2m-1}:p)\rightarrow\pi_{2m+1+2p(p-1)-1}(S^{2m+1}:p)$ *is an epimorphism.*

Then the last assertion of the theorem follows easily from (5. 1), (6. 1), (6. 4) and (7. 2), (ii). q. e. d.

Theorem 7. 2. *Let* $p>r\geq 2$.

$$\pi_{2m-1+2(p+1)(p-1)-3}(S^{2m-1}:p)\approx Z_p \quad \textit{for } m\geq 2,$$

$$\pi_{2m-1+2(p+1)(p-1)-2}(S^{2m-1}:p)\approx\begin{cases}Z_p & \textit{for } p+1\geq m\geq 2,\\ 0 & \textit{for } m>p+1,\end{cases}$$

$$\pi_{2m-1+2(p+1)(p-1)-1}(S^{2m-1}:p)\approx Z_p \quad \textit{for } m\geq 2,$$

$$\pi_{2m-1+2(p+r)(p-1)-4}(S^{2m-1}:p)$$

$$\approx\pi_{2m-1+2(p+r)(p-1)-3}(S^{2m-1}:p)\approx\begin{cases}Z_p & \textit{for } m=r,\\ 0 & \textit{for } m\neq r,\end{cases}$$

$$\pi_{2m-1+2(p+r)(p-1)-2}(S^{2m-1}:p)\approx\begin{cases}Z_p & \textit{for } p+r\geq m\geq 2,\\ 0 & \textit{for } m>p+r,\end{cases}$$

$$\pi_{2m-1+2(p+r)(p-1)-1}(S^{2m-1}:p)\approx Z_p \quad \textit{for } m\geq 2,$$

$$\pi_{2m-1+k}(S^{2m-1}:p)=0 \quad \textit{for } k=2(p+1)(p-1)-4 \textit{ and}$$

$$\textit{for } k\not\equiv -1,\ -2,\ -3,\ -4 \pmod p,\ 2p(p-1)\leq k<4p(p-1)-5.$$

Proof. For the case $m\geq 3$ the results follow immediately from Corollary 6.4, (6.1) and Proposition 6.2.

For the case $m=2$ the results will be computed by use of the exact sequence (2.11). Consider the homomorphisms

$$\varDelta:\pi_{i+2}(S^{2p+1}:p)\rightarrow\pi_i(S^{2p-1}:p)$$

for $2p(p-1)+1\leq i\leq 4p(p-1)$. The two groups vanish except the following cases:

a) $i=2p+2(p+j)(p-1)-2,\ \ j=0,1,\cdots,p-1,$

b) $i=2p+2(p+j)(p-1)-3,\ \ j=0,1,\cdots,p-1,$

c) $i=2p+2(p+1)(p-1)-4.$

In the cases a) and c) the groups are cyclic and $S^2:\pi_i(S^{2p-1}:p)\rightarrow\pi_{i+2}(S^{2p+1}:p)$ are isomorphisms since $U(i-2p+1,\ p-1)=U(i-2p+1,p)=0$. By use of the relation (2.7) it follows

$$\operatorname{Ker}\varDelta\approx\operatorname{Coker}\varDelta\approx Z_p \quad \text{for the cases a) and c).}$$

Next consider the case b) of $j=0$, i.e., $i=2p+2p(p-1)-3$. In this case the groups are isomorphic to Z_{p^2} and Z_p respectively and S^2 is an epimorphism, by (7.2), (i). By use of (2.7), we see that $\varDelta$ is a monomorphism. Thus

$$\mathrm{Ker}\,\Delta=0 \text{ and } \mathrm{Coker}\,\Delta\approx Z_p \quad \text{for the case b) of } j=0.$$

Consider the case b) of $j>0$. Then the two groups are isomorphic to Z_p and $S^2\pi_i(S^{2p-1}:p)=S^2U(i-2p+1, p-1)=0$. By Lemma 2.6, we have that Δ are epimorphisms hence isomorphisms. Thus

$$\mathrm{Ker}\,\Delta=\mathrm{Coker}\,\Delta=0 \quad \text{for the case b) of } j>0.$$

By the exactness of (2.11) the results for $m=2$ are computed easily. q. e. d.

Lemma 7.3. *If* $1\leq t<p$, *then* $S^\infty:\pi_{2tp(p-1)+4}(S^5:p)\to(\pi^S_{2tp(p-1)-1}:p)$ *is an epimorphism.*

Proof. By (7.2), (ii) this is true for $t=1$. Thus there exists an element $\alpha'_p(5)$ such that $p\cdot\alpha'_p(5)=\alpha_p(5)$. Choose elements $\alpha'_{tp}(5)$ from the secondary composition $\{\alpha'_p(5), p^2\iota_{2p(p-1)+4}, S^{2p(p-1)-1}\alpha'_{(t-1)p}(5)\}$ inductively as same as the construction of $\alpha_r(3)$. Then it is verified that $p\cdot S^\infty\alpha'_{tp}(5)=\alpha'_{tp}$, $t\leq p$ [16: (4.15)]. This shows that $S^\infty(\alpha'_{tp}(5))$ is of order p^2, and it generates $(\pi^S_{2tp(p-1)-1}:p)$ if $t<p-1$. In the case $t=p-1$, the other generator $\alpha_1\beta_1^{p-1}$ is the image of $\alpha_1(3)\circ\beta_1^{p-1}(2p)\in\pi_{2p(p-1)^2+2}(S^3:p)$. q. e. d.

Theorem 7.4. *Let* $p\geq 5$. *Then we have*

$$\pi_{2m-1+4p(p-1)-5}(S^{2m-1}:p)=0$$

$$\pi_{2m-1+4p(p-1)-4}(S^{2m-1}:p)\approx\begin{cases}Z_p & \textit{for } m\geq p,\\ 0 & \textit{for } m<p,\end{cases}$$

$$\pi_{2m-1+4p(p-1)-3}(S^{2m-1}:p)\approx\begin{cases}Z_p & \textit{for } 2p>m\geq p,\\ 0 & \textit{for } m\geq 2p \textit{ and for } m<p.\end{cases}$$

$$\pi_{2m-1+4p(p-1)-2}(S^{2m-1}:p)\approx\begin{cases}Z_{p^2} & \textit{for } 2p>m>2,\\ Z_p & \textit{for } m=2p \textit{ and for } m=2,\\ 0 & \textit{for } m>2p,\end{cases}$$

$$\pi_{2m-1+4p(p-1)-1}(S^{2m-1}:p)\approx\begin{cases}Z_{p^2} & \textit{for } m>2,\\ Z_p & \textit{for } m=2.\end{cases}$$

Proof. For the case $m=2$ the same proof as Theorem 7.2 is valid.

Put $q=4p(p-1)$ and consider the exact sequences (5.1):

$$(*)\cdots\longrightarrow\pi_{2m-1+q-i}(Q_2^{2m-1}:p)\xrightarrow{p_*}\pi_{2m-1+q-i}(S^{2m-1}:p)\xrightarrow{S^2}$$
$$\pi_{2m+1+q-i}(S^{2m+1}:p)\xrightarrow{H^{(2)}}\pi_{2m-1+q-i-1}(Q_2^{2m-1}:p)\xrightarrow{p_*}\cdots,$$

for $m=2, 3, \cdots, i=5, 4, 3, 2, 1$.

By (6.4), $\pi_{2m-1+q-6}(Q_2^{2m-1}:p)=0$ for $m\geqq 2$, and $\pi_{2m-1+q-5}(Q_2^{2m-1}:p)$ $=0$ for $m\geqq 2$ and $m\neq p-1$, $\approx Z_p$ for $m=p-1$. Then the first assection follows immediately by the exactness of (*), $i=5$. Also the exactness of (*), $i=4$, implies that $\pi_{2m-1+q-4}(S^{2m-1}:p)=0$ for $m<p$, $\approx Z_p$ for $m=p$ and $S^2:\pi_{2m-1+q-4}(S^{2m-1}:p)\rightarrow\pi_{2m+1+q-4}(S^{2m+1}:p)$ are epimorphisms for $m\geqq p$. These S^2 are isomorphisms since $(\pi_{q-4}^S:p)$ $\approx Z_p$. Thus the second assertion has been proved. Remark that

(7.3) $\pi_{2m-1+4p(p-1)-4}(S^{2m-1}:p)$, *$m\geqq p$, are generated by* $\beta_1^2(2m-1)$.

Since S^2 in (*) of $i=4$ are monomorphisms, $H^{(2)}$ in (*) of $i=3$ are epimorphisms. By (6.4), $\pi_{2m-1+q-4}(Q_2^{2m-1}:p)\approx Z_p$ for $m=p-1, p$ and $=0$ for $m\neq p-1, p$. We have also $U(q-3, p-1)\approx Z_p$ by Proposition 6.2. It follows that $\pi_{2m-1+q-3}(S^{2m-1}:p)=0$ for $m<p$, $\approx Z_p$ for $m=p$, $p+1$ and $S^2:\pi_{2m-1+q-3}(S^{2m-1}:p)\rightarrow\pi_{2m+1+q-3}(S^{2m+1}:p)$ are epimorphisms for $m\geqq p+1$. Assume that $\pi_{4p-3+q-3}(S^{4p-3}:p)=0$ in which the element $p_*Q^{2p-2}(\alpha_1)$ is. Then there exists an element γ of $\pi_{4p-1+q-2}(S^{4p-1}:p)$ such that $H^{(2)}\gamma=Q^{2p-2}(\alpha_1)\neq 0$. $S^2\gamma$ does not vanish since the kernel of this S^2 is generated by $p_*(I'\iota_{4p^2-1})$ and $H^{(2)}p_*(I'\iota_{4p^2-1})=0$. But $S^2\gamma$ is already a stable element and this contradicts to $(\pi_{q-2}^S:p)=0$. Thus we concludes that $\pi_{4p-3+q-3}(S^{4p-3}:p)\approx Z_p$ and this is generated by $p_*Q^{2p-2}(\alpha_1)$. Then the third assertion is proved.

Next consider the sequences (*) for $i=1$. By (6.4), $\pi_{2m-1+q-1}$ $(Q_2^{2m-1}:p)=0$ for $m\geqq 2$. It follows that S^2 in (*) of $i=1$ are monomorphisms for $m\geqq 2$. Then these S^2 are isomorphisms for $m\geqq 3$ by Lemma 7.3. We have proved the last assertion.

It follows also from the exactness of (*) of $i=1, 2$ that $p_*:\pi_{2m-1+q-2}(Q_2^{2m-1}:p)\rightarrow\pi_{2m+1+q-2}(S^{2m-1}:p)$ are monomorphisms for $m\geqq 3$. Put $\gamma=p_*(I'\iota_{4p^2-1})$. Then $\gamma\neq 0$ and γ generates the group $\pi_{4p-1+q-2}$

$(S^{4p-1}:p)$ since $\pi_{4p+1+q-2}(S^{4p+1}:p)=(\pi_{q-2}^S:p)=0$. Thus $\pi_{4p-1+q-2}(S^{4p-1}:p)$ $\approx Z_p$. By Theorem 5.3, (i), there exists an element γ_1 of $\pi_{4p-3+q-2}$ $(S^{4p-3}:p)$ such that $S^2\gamma_1=\gamma$ and $p\,\gamma_1=x\cdot p_*\bar{Q}^{2p-2}(\alpha_1)\neq 0$ for some $x\not\equiv 0$ (mod p). Repeating this we have elements γ_i of $\pi_{4p-2i-1+q-2}(S^{4p-2i-1}:p)$ for $i=2, 3, \cdots, 2p-3$ such that $S^2\gamma_i=p\gamma_{i-1}$ and $p\gamma_i=x_i\cdot p_*\bar{Q}^{2p-i-1}(\alpha_i)\neq 0$ for some $x_i\not\equiv 0$ (mod p). On the other hand from the computation of the third assertion we see that the cokernel of S^2 in (*) of $i=2$ are isomorphic to Z_p for $2p-2\geqq m\geqq 2$. Then the fourth assertion of the theorem is proved easily. q. e. d.

In the above proof we have seen

(7.4). (i) *The groups* $\pi_{2m-1+4p(p-1)-3}(S^{2m-1}:p)$ *of* $2p>m>p$ *are generated by unstable elements of the third type, i.e., these groups are isomorphic under iterated suspensions.*

(ii) *The groups* $\pi_{2m-1+4p(p-1)-2}(S^{2m-1}:p)$ *of* $2p\geqq m\geqq 2$ *are generated by unstable elements of the second type, i.e., a similar assertion as in* (7.2), (i) *holds.*

Theorem 7.5. *Let* $p\geqq 5$ *and* $p>j\geqq 3$.

$$\pi_{2m-1+2(2p+1)(p-1)-5}(S^{2m-1}:p)\approx Z_p \quad \text{for } m\geqq 2,$$

$$\pi_{2m-1+2(2p+1)(p-1)-4}(S^{2m-1}:p)\approx\begin{cases} Z_p & \text{for } p+1\geqq m\geqq 2,\\ 0 & \text{for } m>p+1,\end{cases}$$

$$\pi_{2m-1+2(2p+1)(p-1)-3}(S^{2m-1}:p)=0,$$

$$\pi_{2m-1+2(2p+1)(p-1)-2}(S^{2m-1}:p)\approx\begin{cases} Z_p+Z_p & \text{for } 2p+1\geqq m>p+1,\\ Z_p & \text{for } m>2p+1 \text{ and}\\ & \text{for } p+1\geqq m\geqq 2,\end{cases}$$

$$\pi_{2m-1+2(2p+1)(p-1)-1}(S^{2m-1}:p)\approx Z_p \quad \text{for } m\geqq 2,$$

$$\pi_{2m-1+2(2p+2)(p-1)-6}(S^{2m-1}:p)$$
$$\approx\pi_{2m-1+2(2p+2)(p-1)-5}(S^{2m-1}:p)\approx\begin{cases} Z_p & \text{for } m=2,\\ 0 & \text{for } m>2,\end{cases}$$

$$\pi_{2m-1+2(2p+2)(p-1)-4}(S^{2m-1}:p)\approx\begin{cases} Z_p & \text{for } m=p+2,\\ 0 & \text{for } m\neq p+2,\end{cases}$$

$$\pi_{2m-1+2(2p+2)(p-1)-3}(S^{2m-1}:p)\approx\begin{cases}Z_p+Z_p & \text{for } m=p+2,\\ Z_p & \text{for } m\neq p+2 \text{ and } m>2,\\ 0 & \text{for } m=2,\end{cases}$$

$$\pi_{2m-1+2(2p+2)(p-1)-2}(S^{2m-1}:p)\approx\begin{cases}Z_p & \text{for } 2p+2\geqq m\geqq 2,\\ 0 & \text{for } m>2p+2,\end{cases}$$

$$\pi_{2m-1+2(2p+2)(p-1)-1}(S^{2m-1}:p)\approx Z_p \quad \text{for } m\geqq 2,$$

$$\pi_{2m-1+2(2p+j)(p-1)-6}(S^{2m-1}:p)$$
$$\approx\pi_{2m-1+2(2p+j)(p-1)-5}(S^{2m-1}:p)\approx\begin{cases}Z_p & \text{for } m=j,\\ 0 & \text{for } m\neq j,\end{cases}$$

$$\pi_{2m-1+2(2p+j)(p-1)-4}(S^{2m-1}:p)$$
$$\approx\pi_{2m-1+2(2p+j)(p-1)-3}(S^{2m-1}:p)\approx\begin{cases}Z_p & \text{for } m=j-1,\ p+j,\\ 0 & \text{for } m\neq j-1,\ p+j,\end{cases}$$

$$\pi_{2m-1+2(2p+j)(p-1)-2}(S^{2m-1}:p)\approx\begin{cases}Z_p & \text{for } 2p+j\geqq m\geqq 2,\\ 0 & \text{for } m>2p+j\end{cases}$$

$$\pi_{2m-1+2(2p+j)(p-1)-1}(S^{2m-1}:p)\approx Z_p \quad \text{for } m\geqq 2,$$

$$\pi_{2m-1+k}(S^{2m-1}:p)=0 \textit{ otherwise for } 4p(p-1)\leqq m<6p(p-1)-7.$$

Proof. Except the cases $k-2(2p+1)(p-1)=-1,\ -2,\ -3,\ -4$, the above results for $m>3$ are obtained directly from Corollary 6.4, the case $m=3$ they are computed from the case $m=4$ by use of (5.1) and (6.4) and for the case $m=2$ they are computed by use of (2.5), (2.7) and Lemma 2.6. Remark that in the computation it appears only two non-vanishing $V(m,k)$: $V(2(2p+1)(p-1)-3,\ p)\approx V(2(2p+1)(p-1)-2,\ p+1)\approx Z_p$. Then the remaining cases are computed from the stable cases by use of Proposition 6.3. The details are left to the reader. q. e. d.

Remark. It seems there is no reason to stop the computation at the above range of k. In fact one can compute up to $k<6p^2-10$ in which case $\pi_{2m-1+k}(S^{2m-1}:p)$ of $m>3$ are in meta-stable ranges. For example unstable elements of the fourth type appear in the groups $\pi_{2m-1+(3p+1)(p-1)-3}(S^{2m-1}:p)$, $2p+1\geqq m\geqq p+1$.

Theorem 7.6. (i) *Theorem* 7.4 *holds for the case* $p=3$ *and*

$2(p+2)(p-1)-1=4p(p-1)-5<k<4p(p-1)-1(=23)$.

(ii)

$$\pi_{2m-1+23}(S^{2m-1}:3)\approx\begin{cases}Z_9+Z_3 & \textit{for } m>2,\\ Z_3+Z_3 & \textit{for } m=2,\end{cases}$$

$$\pi_{2m-1+24}(S^{2m-1}:3)\approx\begin{cases}Z_3 & \textit{for } 4\geq m\geq 2,\\ 0 & \textit{for } m>4,\end{cases}$$

$$\pi_{2m-1+25}(S^{2m-1}:3)=0,$$

$$\pi_{2m-1+26}(S^{2m-1}:3)\approx\begin{cases}Z_3+Z_3 & \textit{for } m=2 \textit{ and for } 7\geq m>4.\\ Z_3 & \textit{for } 4\geq m>2 \textit{ and for } m>7,\end{cases}$$

$$\pi_{2m-1+27}(S^{2m-1}:3)\approx\begin{cases}Z_3+Z_3 & \textit{for } m=2,\\ Z_3 & \textit{for } m>2,\end{cases}$$

$$\pi_{2m-1+28}(S^{2m-1}:3)\approx\begin{cases}Z_3 & \textit{for } m=5,\\ 0 & \textit{for } m\neq 5.\end{cases}$$

(iii)

$$\pi_{2m-1+29}(S^{2m-1}:3)\approx\begin{cases}Z_3+Z_3 & \textit{for } m=5,\\ Z_3 & \textit{for } m\neq 5 \textit{ and } m>2,\\ 0 & \textit{for } m=2,\end{cases}$$

$$\pi_{2m-1+30}(S^{2m-1}:3)\approx\begin{cases}Z_3+Z_3 & \textit{for } 8\geq m>2,\\ Z_3 & \textit{for } m=2 \textit{ and for } m>8,\end{cases}$$

$$\pi_{2m-1+31}(S^{2m-1}:3)\approx\begin{cases}Z_3+Z_3 & \textit{for } 5\geq m>2,\\ Z_3 & \textit{for } m=2 \textit{ and for } m>5,\end{cases}$$

$$\pi_{2m-1+32}(S^{2m-1}:3)\approx\begin{cases}Z_3 & \textit{for } m=2,\\ 0 & \textit{for } m>2.\end{cases}$$

Proof. The proof of (i) is same as that of Theorem 7.4. The proof of (ii) is also similar to that of Theorem 7.5. Look at the last assertion of Theorem 7.4 and the results of Theorem 7.5 and put $p=3$, then some of them are overlapped and by making the direct sums of these overlapping groups the results of (ii) are obtained.

The essential difference occurs in the cases of (iii).

The first diffrence is the appearance of a new generator β_1^3 of the stable group $(\pi^S_{30}:3)\approx Z_3$. The second difference is the existence of non-vanishing $V(k, m)$: $V(31, 4)$, $V(32, 6)$ and $V(33, 7)$ which are isomorphic to Z_3 and generated by $Q^3(\beta_1^2)=I'\beta_1^2(17)$, $Q^5(\alpha_1\beta_1)=I'(\alpha_1(29)\circ\beta_1(31))$ and $Q^6(\alpha_1)=I'\alpha_1(35)$ respectively.

To simplify the notations we put $\pi_i(S^n:3)=\pi_i^n$. The groups π_i^3 in (iii) are computed easily by use of (2.5), (2.7) and Lemma 2.6. Some meta-stable groups of (iii) are determined by Corollary 6.4. Then it remains the following groups: π^5_{25}, $\pi^{2m-1}_{2m-1+31}$ for $m=3, 4, 5$, and $\pi^{2m-1}_{2m-1+32}$ for $m=3, 4, 5, 6$.

By (6.4), $\pi_{34}(Q_2^5:3)$ is generated by $Q^4(\alpha_5)$. By Lemma 6.1, $Q^4(\alpha_5)$ is an image of $H^{(2)}p_*$, hence $p_*Q^4(\alpha_5)=0$ by the exactness of (5.1). Then we have an exact sequence $0\rightarrow\pi^5_{34}\rightarrow\pi^7_{36}\rightarrow\pi_{33}(Q_2^5:3)=0$ by (5.1) and (6.4). Thus $\pi^5_{34}\approx\pi^7_{36}\approx Z_3$.

Also we have an exact sequence

$$*)\qquad \pi_{35}(Q_2^5:3)\xrightarrow{p_*}\pi^5_{35}\xrightarrow{S^2}\pi^7_{37}\xrightarrow{H^{(2)}}Z_3\longrightarrow 0,$$

where $\pi^7_{37}\approx Z_3+Z_3$ and $\pi_{35}(Q_2^5:3)$ is generated by $\overline{Q}^3(\alpha_5)$ and $Q^3(\beta_1^2)$. By Lemma 6.1, $H^{(2)}p_*\overline{Q}^3(\alpha_5)=\pm Q^2(\alpha'_6)\in\pi_{32}(Q_2^3:3)$. $S^2:\pi^{11}_{34}\rightarrow\pi^{13}_{35}$ is an isomorphism since these groups are isomorphic to Z_9+Z_3 and S^∞ maps these groups onto $(\pi^S_{23}:3)\approx Z_9+Z_3$ by Lemma 7.3. Then it follows from the exactness of (2.5) and from (2.7) that $\pi_{32}(Q_2^3:3)\approx Z_3+Z_3$ and $Q^2(\alpha'_6)$ is a generator. Thus $p_*\overline{Q}^3(\alpha_5)$ generates a direct factor isomorphic to Z_3. Next consider $p_*Q^3(\beta_1^2)$. Since $\pi^5_{25}\approx Z_3$ and $S^\infty\beta_1^2(5)=\beta_1^2\neq 0$, $S^2:\pi^5_{25}\rightarrow\pi^7_{27}$ is a monomorphism. From the exactness of (5.1) it follows that $H^{(2)}:\pi^7_{28}\rightarrow\pi_{25}(Q_2^5:3)$ is an epimorphism. Let γ be an element of π^7_{28} such that $H^{(2)}\gamma=Q^3(\beta_1)\in\pi_{25}(Q_2^5:3)$. By use of (2.6) and (1.3), (ii) we have $Q^3(\beta_1^2)=Q^3(\beta_1)\circ\beta_1(25)=H^{(2)}\gamma\circ\beta_1(25)=H^{(2)}(\gamma\circ\beta_1(28))$. It follows that $p_*Q^3(\beta_1^2)=p_*H^{(2)}(\gamma\circ\beta_1(28))=0$. Then the result $\pi^5_{35}\approx Z_3+Z_3$ follows from the exactness of *).

We have exact sequences

$$\pi_{36}(Q_2^5:3)\xrightarrow{p_*}\pi^5_{36}\xrightarrow{S^2}\pi^7_{38}\xrightarrow{H^{(2)}}\{Q^3(\beta_1^2)\}\longrightarrow 0,$$

$$\pi_{38}(Q_2^7:3)\xrightarrow{p_*}\pi_{38}^7\xrightarrow{S^2}\pi_{40}^9\xrightarrow{H^{(2)}}\pi_{37}(Q_2^7:3)\xrightarrow{p_*},$$

where $\pi_{38}(Q_2^7:3)=0$, $\bar{Q}^3(\beta_1^2)$ and $\bar{Q}^4(\alpha_4)$ generate $\pi_{36}(Q_2^5:3)$ and $\pi_{37}(Q_2^7:3)$ respectively. By Lemma 6.1, $H^{(2)}p_*$-images of $\bar{Q}^3(\beta_1^2)$ and $\bar{Q}^4(\alpha_4)$ do not vanish. It follows that $\pi_{38}^7\approx\pi_{40}^9$, $\pi_{36}^5\approx Z_3+S^2\pi_{36}^5$ and $\pi_{38}^7/S^2\pi_{36}^5\approx Z_3$. The elements $\alpha_8(2m-1)$, $m=3, 4, 5$, generate direct factors of these groups isomorphic to Z_3. Thus π_{38}^7 has at least 9 elements. In the exact sequence

$$\pi_{40}((Q_2^9:3)\to\pi_{40}^9\to\pi_{42}^{11}$$

$\pi_{42}^{11}\approx Z_3$ and $\pi_{40}(Q_2^9:3)\approx Z_3$. It follows then $\pi_{38}^7\approx\pi_{40}^9\approx\pi_{36}^5\approx Z_3+Z_3$.

From the above discussion and from (6.4) we see that in the exact sequences

$$\pi_{2m-1+32}^{2m-1}\xrightarrow{S^2}\pi_{2m+1+32}^{2m+1}\xrightarrow{H^{(2)}}\pi_{2m+30}(Q_2^{2m-1}:p)\xrightarrow{p_*}\pi_{2m-1+31}^{2m-1}$$

p_* are monomorphisms for $m\geqq 3$. Thus S^2 are epimorphisms for $m\geqq 3$. Then in order to prove the last assertion of the theorem it is sufficient to prove $\pi_{37}^5=0$. By use of the exactness of (2.5) we have directly $\pi_{34}(Q_2^3:3)=0$. Thus $\pi_{37}^5=S^2\pi_{35}^3$. The result $\pi_{35}^3\approx Z_3$ is verified from an exact sequence

$$\pi_{36}^7\xrightarrow{\Delta}\pi_{34}^5\xrightarrow{G}\pi_{35}^3\longrightarrow\pi_{35}^7=0.$$

The group π_{34}^5 is isomorphic to Z_3. By use of Proposition 6.3 we see that $S^\infty:\pi_{35}^9\to(\pi_{26}^S:3)$ is an epimorphism. Let $\beta_2(9)$ be an element of π_{35}^9 such that $S^\infty\beta_2(9)=\beta_2$. By (2.11), (ii) there exists an element ε of π_{34}^5 such that $S^2\varepsilon=S(\alpha_1(6)\circ\beta_2(9))$. Then $S^\infty\varepsilon=\alpha_1\beta_2\neq 0$. Thus ε is a generator of π_{34}^5. By (2.13), π_{35}^3 is generated by $G(\varepsilon)=\alpha_1(3)\circ S\varepsilon$. Since $\alpha_1(5)\circ\alpha_1(8)\in\pi_{11}^5=0$, we have $S^2(\alpha_1(3)\circ\varepsilon)=\alpha_1(5)\circ\alpha_1(8)\circ\beta_2(11)=0$. Thus $\pi_{37}^5=S^2\pi_{35}^3=0$. This completes the proof of the theorem.

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