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Remarks on generalized rings of quotients II

By

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Introduction. In a recent paper of the writer we defined the notion of generalized rings of quotients; they are certain rings contained in the total quotient ring of the original ring. But rings of quotients are not necessarily subrings of the total quotient ring. Therefore in §1 of this paper, we generalize the notion of generalized rings of quotients so that we cover completely the rings of quotients and previous generalization of rings of quotients. Thus those we shall treat are rings which are contained in the total quotient ring of a homomorphic image of a ring having similar properties as previously defined generalized rings of quotients. We prove that the kernel of such a homomorphism is also the kernel of a ring of quotients (in the usual sence). In §2 we shall prove a theorem on the weak global dimension of a ring which is a slight generalization of a result contained in [6]. Rings are always commutative rings with units.

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§1. First, we give a few results on flatness which are well-known.

Lemma 1. *Let R, R' be rings and let f: $R \rightarrow R'$ be a homomorphism. Then R' is flat as an R-module if and only if for every maximal ideal $\mathfrak{m}'$ of R', $R'_{\mathfrak{m}'}$ is $R_{\mathfrak{m}}$-flat with* $\mathfrak{m}=f^{-1}(\mathfrak{m}')$.

(cf. Prop. 15 of §3, Chap. II in [2]).

Lemma 2. *Let R and R' be rings such that R' is an R-module and assume that R' is flat. Then for ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R we have $(\mathfrak{a}:\mathfrak{b})R'=\mathfrak{a}R':\mathfrak{b}R'$, provided that $\mathfrak{b}$ is finitely generated* (cf. (18.1) in [5] or *Remarque* in §2, Chap. 1 in [2]).

Now, let R and R^* be rings, and let f: $R\rightarrow R^*$ be a homomorphism with kernel $\mathfrak{a}$. In this section we assume that R^* is contained in the total quotient ring of $f(R)$. For any ideal $\mathfrak{b}^*$ of R^* we denote by $\mathfrak{b}^*\cap R$ the inverse image of $\mathfrak{b}^*$ by f.

Theorem 1. *The following conditions are equivalent to each other:*

(1) *R^* is R-flat.*

(2) *R^* is $f(R)$-flat and $(0:aR)R^*=R^*$ for every element a in $\mathfrak{a}$.*

(3) *For every maximal ideal $\mathfrak{m}^*$ of R^* we have $R^*_{\mathfrak{m}^*}=R_{\mathfrak{m}}$ with $\mathfrak{m}=\mathfrak{m}^*\cap R$.*

Proof. Assume, first, that R^* is R-flat. Then it is obvious that R^* is $f(R)$-flat and using Lemma 2, we have $(0:aR)R^*=0R^*:aR^*=R^*$ since a is in the kernel of f, which shows that (1) implies (2). If (2) is valid, then for any maximal ideal $\mathfrak{m}^*$ of R^* we have $R^*_{\mathfrak{m}^*}=f(R)_{\mathfrak{m}^*\cap f(R)}$ by Theorem 1 in [1], because R^* is contained in the total quotient ring of $f(R)$. On the other hand, $\mathfrak{m}=\mathfrak{m}^*\cap R$ can not contain $0:aR$ for any a in $\mathfrak{a}$ by the assumption that $(0:aR)R^*=R^*$. Therefore we have $\mathfrak{a}R_{\mathfrak{m}}=0$ and $R^*_{\mathfrak{m}^*}=f(R)_{\mathfrak{m}^*\cap f(R)}=R_{\mathfrak{m}}/\mathfrak{a}R_{\mathfrak{m}}=R_{\mathfrak{m}}$, which shows that (3) follows from (2). Implication (3)$\rightarrow$(1) is shown by Lemma 1.

Applying Theorem 1 in [1], we have

Corollary 1. *R^* is R-flat if and only if for every maximal ideal $\mathfrak{m}^*$ of R^*, we have $R^*_{\mathfrak{m}^*}=f(R)_{\mathfrak{m}^*\cap f(R)}$ and $(0:a)R^*=R^*$ for every element a in $\mathfrak{a}$.*

Theorem 2 in [1], replaced A and B by R and R^* respectively, is valid without any modification, that is;

Corollary 2. *Assume that R^* is R-flat. Then*:

(1) *For any ideal $\mathfrak{b}^*$ of R^*, we have $(\mathfrak{b}^* \cap R)R^* = \mathfrak{b}^*$.*

(2) *Let $\mathfrak{q}$ be a primary ideal of R belonging to a prime ideal $\mathfrak{p}$ and such that $\mathfrak{q}R^* \neq R^*$. Then $\mathfrak{p}R^* \neq R^*$, $\mathfrak{p}R^*$ is a prime ideal, $\mathfrak{q}R^*$ is primary to $\mathfrak{p}R^*$, $\mathfrak{p}R \cap R = \mathfrak{p}$ and $\mathfrak{q}R^* \cap R = \mathfrak{q}$.*

Proof. (1) follows directly from Theorem 2 in [1]. For (2), it is sufficient to show that in this case $\mathfrak{p}$ and $\mathfrak{q}$ contain $\mathfrak{a}$. Let $\mathfrak{m}^*$ be a maximal ideal of R^* containing $\mathfrak{p}R^*$ and let $\mathfrak{m} = \mathfrak{m}^* \cap R$. Then we have $\mathfrak{m} \supseteq \mathfrak{p}$ and $\mathfrak{a}R_{\mathfrak{m}} = 0$ as was shown in the proof of Theorem 1, therefore we have $\mathfrak{a}R_{\mathfrak{p}} = 0$ which implies that $\mathfrak{p} \supseteq \mathfrak{a}$. From this it follows immediately that $\mathfrak{q}$ contains $\mathfrak{a}$.

Corollary 3. *Assume that R^* is R-flat. Then the total quotient ring of R^* is a ring of quotients of R with respect to a multiplicatively closed subset S of R and $\mathfrak{a}$ is the kernel of the canonical homomorphism $R \to R_S$, that is, if we set $N(S) = \{x \in R;\ sx = 0$ for a suitable s in $S\}$, then $\mathfrak{a} = N(S)$.*

Proof. Let S be the set of elements s of R such that s is not a zero-divisor modulo $\mathfrak{a}$. It is trivial that S is multiplicatively closed, and we show that $\mathfrak{a} = N(S)$. From the construction of S, it follows easily that $\mathfrak{a}$ contains $N(S)$. Let a be an element of $\mathfrak{a}$. By Theorem 1, we have $(0:a)R^* = R^*$. Then there exist finite subsets (a_i) and (b_i) of $0:aR$ and R respectively, and an element s of S such that $\sum_i a_i b_i - s = a'$ is in $\mathfrak{a}$. Therefore we have $a(s+a') = 0$ with $s + a' \in S$, that is, a is contained in $N(S)$. Thus we have $\mathfrak{a} = N(S)$. From this, we see that the total quotient ring of $f(R)$ is the ring of quotients R_S of R with respect to S and the total quotient ring of R^* is also R_S.

The following corollary is a characterization of the ring R such that w. gl. dim $R = 0$.

Corollary 4. *For a ring R we have* w. gl. dim $R = 0$ *if and* *only if $R/\mathfrak{m}$ is R-flat for every maximal ideal $\mathfrak{m}$ of R.*

Proof. *Only if* part is clear and *if* part follows easily from Corollary 3, because in this case we have $R_{\mathfrak{m}}=R/\mathfrak{m}$, that is, $R_{\mathfrak{m}}$ is a field for every maximal ideal $\mathfrak{m}$ of R, whence we have w. gl. dim $R=0$ as is well-known (cf. [4]).

§2. Let R be a ring with the total quotient ring K. Richman proved the following theorem (Theorem 4 in [6]):

If R is an integral domain and satisfies the following condition (F), then R is a Prüfer ring, that is, w. gl. dim $R\leqq 1$.

(*F*) *For any ring R' such that $R\subseteq R'\subseteq K$, R' is R-flat.*

In this section, we shall prove the above theorem in a slightly generalized form, where R is not necessarily an integral domain but the total quotient ring K of R satisfies the condition that w. gl. dim $K=0$ if R is not an integral domain. The proof follows easliy from some results about quasi-von Neumann-regular rings contained in [4].

A ring R is said to be a *von Neumann regular ring* if for any $a\in R$, there is an element b of R with $aba=a$. If the total quotient ring K of a ring R is von Neumann regular, then we say that R is a *quasi-von Neumann-regular ring*. It is known that R is von Neumann regular if and only if w. gl. dim $R=0$ (cf. [4]). Therefore a ring R with the total quotient ring K is quasi-von Neumann-regular if and only if w. gl. dim $K=0$.

First we refer to some properties of quasi-von Neumann-regular rings without proofs (see [4]).

(1) Let R be a quasi-von Neumann-regular ring with the total quotient ring K and let S be a multiplicatively closed subset of R. Then the ring of quotients R_S of R with respect to S is also a quasi-von Neumann-regular ring and the ring of quotients K_S of K with respect to S is the total quotient ring of R_S (Prop. 2 in [4]).

(2) Let R be a quasi-von Neumann-regular ring. If R is quasi-local and integrally closed in its total quotient ring, then R is an

integral domain (Prop. 6 in [4]).

(3) For a ring R we have w. gl. dim $R \leqq 1$ if and only if the ring of quotients $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is a valuation ring (Prop. 11 in [4]).

(4) For a ring R with the total quotient ring K, the following conditions are equivalent:

a) R is a semi-hereditary ring, that is, every finitely generated ideal of R is projective (It is equivalent to say that R is a Prüfer ring if R is an integral domain).

b) w. gl. dim $R \leqq 1$ and w. gl. dim $K=0$.

c) For any torsion-free R-module M, M is R-flat (Theorem 5 in [4]).

Theorem 2. *Let R be a ring with the total quotient ring K. If R satisfies the condition (F) and if* w. gl. dim $K=0$, *then we have* w. gl. dim $R \leqq 1$, *that is, R is a semi-hereditary ring.*

Before proving this theorem, we give some preliminary results.

Lemma 3. *If a quasi-von Neumann-regular ring R with the total quotient ring K satisfies the condition (F), then the ring of quotients R_S of R with respect to any multiplicatively closed subset S of R satisfies the condition (F).*

Proof. The total quotient ring of R_S is K_S by virtue of (1). Therefore for any ring R^* such as $R_S \subseteq R^* \subseteq K_S$, there is a ring R' such that $R \subseteq R' \subseteq K$ and $R^* = R'_S$. Since R' is R-flat, we see that R^* is R_S-flat, which shows that R_S satisfies the condition (F).

Lemma 4. *If a ring R satisfies the condition (F), then R is integrally closed in its total quotient ring.*

Proof. Let R^* be the integral closure of R in its total quotient ring. Then R^* is R-flat and integral over R, and we have $R^* = R$ by Coroll. 2 of Theorem 1 in [1].

Lemma 5. *If a quasi-local integral domain R satisfies the condition (F), then R is a valuation ring.*

For the proof of this lemma, see [6].

Proof of Theorem 2: Let $\mathfrak{m}$ be an arbitrary maximal ideal of R. Then $R_{\mathfrak{m}}$ satisfies the condition (F) and for the total quotient ring $K_{\mathfrak{m}}$ of $R_{\mathfrak{m}}$, we have w. gl. dim $K_{\mathfrak{m}}=0$. Since $R_{\mathfrak{m}}$ is integrally closed by Lemma 4, $R_{\mathfrak{m}}$ is an integral domain by virtue of (2). Then Lemma 5 implies that $R_{\mathfrak{m}}$ is a valuation ring, and we have w. gl. dim $R\leqq 1$ by (3), that is, R is a semi-hereditary ring by (4)

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