J. Math. Kyoto Univ.
5-2 (1966) 177-184

Connected fields of arbitrary characteristic*

By

Alan G. WATERMAN and George M. BERGMAN

(Communicated by Prof. M. Nagata)

(Received Jan. 10, 1966)

The question of whether there exists a connected field of non-zero characteristic was posed to us by Paul Chernoff. The authors wish to thank Professor Nagata for several observations and suggestions which helped us to find this construction.

§1. Adjoining an "arc of indeterminates"

Let k be an arbitrary integral domain (without topology). Let R be the polynomial ring gotten by adjoining to k an independent indeterminate T_α for each α in the open interval $(0, 1)$. We also define $T_0=0$, $T_1=1$, these not being indeterminates, of course.

Given any $\varepsilon>0$, let us define $U_\varepsilon\subset R$ to be the union, over all finite sequences $\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n$ in $[0, 1]$ such that $\sum|\alpha_i-\beta_i|<\varepsilon$, of the ideals $(T_{\alpha_1}-T_{\beta_1}, \cdots, T_{\alpha_n}-T_{\beta_n})$. For $f\in R$, let us define $|f|=\inf_{f\in U_\varepsilon}\varepsilon$. Note that for all f, $f\in(T_1-T_0)$, hence $|f|\leqslant 1$.

Let us say that an $f\in R$ "involves" a $T_\gamma(\gamma\neq 0, 1)$ if $f\notin k[T_\alpha]_{\alpha\neq\gamma}$. We shall consider all f's to involve T_0 and T_1.

We shall make use of the following fact, the verification of which we leave to the reader: Suppose $f\neq 0$ belongs to the ideal $(T_{\alpha_1}-T_{\beta_1}, \cdots, T_{\alpha_n}-T_{\beta_n})$. Then f must involve two distinct elements

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* This work was done while the first author held National Science Foundation grant GP 2184, and the second author held an Intermediate Year National Science Foundation Graduate Fellowship.

T_γ, $T_{\gamma'}$ which are identified under the equivalence relation generated by the conditions $T_{\alpha_1} \sim T_{\beta_1}, \cdots, T_{\alpha_n} \sim T_{\beta_n}$. (For example, $T^2_{3/4} - T_{3/4} \in (T_{3/4} - T_{1/2}, T_1 - T_{1/2})$, and involves $T_{3/4}$ and T_1.)

Clearly, in the above situation, $|\gamma - \gamma'| \leqslant \sum |\alpha_i - \beta_i|$, since one can go from γ to γ' by a series of "$\alpha - \beta$" steps; i.e., the difference of some two subscripts that f involves is $\leqslant \sum |\alpha_i - \beta_i|$. Hence, given any $f \neq 0$, the minimum difference among the finitely many subscripts of T's that it involves gives a lower bound for $|f|$. Thus for $f \neq 0$, $|f| > 0$.

It is easy to see from the definition of U_ε that $U_\varepsilon + U_{\varepsilon'} \subset U_{\varepsilon+\varepsilon'}$. It follows from this "triangle inclusion" that "$|\ \ |$" is a norm on the additive group of R, i.e., that $|f - f'|$ is a metric. In the induced topology, addition will be continuous. The metric is translation-invariant, and U_ε is the open ball of radius ε about 0.

Each U_ε, as constructed, is a union of prime ideals. Hence the product of two elements belongs to U_ε if and only if one of them does. Hence $|fg| = \inf |f|, |g|$. It is easily deduced that multiplication is continuous in our topology. ▌

Finally, we note that the map $\alpha \to T_\alpha$ of $[0, 1]$ into R is an isometry. Hence 1 and 0 are connected by an arc in R, hence so are any two points. We conclude:

Lemma 1. *R is an arcwise connected topological ring.*

§2. A topology for the field of fractions

Assume K, in the above, to be a field. Let K be the field of fractions of R.

Lemma 2. *Let $e \in R$. Suppose that in K we have $e = \sum_{i=1}^{n} f_i/g_i$ $(f_i, g_i \in R)$. Then $\sum |f_i|/|g_i| \geqslant |e|$.*

Proof. Let us suppose we have a counterexample $\sum_{i=1}^{n} f_i/g_i$ with smallest possible n for some e.

We make an observation: Let f, g be two elements of R with no common factor. Any fraction f'/g' equal to f/g must be of the form fh/gh. Hence $|f'|/|g'| = \inf|f|, |h|/\inf|g|, |h|$, which will be $\geqslant |f|/|g|$ if $|f|/|g| \leqslant 1$. Hence in our expression $\sum f_i/g_i$, we can assume all fractions in lowest terms, without losing our assumed inequality.

It is now clear that n must be >1.

Assume $|g_1| \geqslant \cdots \geqslant |g_{n-1}| \geqslant |g_n|$. In order for the sum $\sum f_i/g_i$ to be a member of R, g_n must divide the product $g_1 \cdots g_{n-1}$. Hence $|g_n| \geqslant |g_1 \cdots g_{n-1}| = \inf|g_1|, \cdots, |g_{n-1}| = |g_{n-1}|$. Hence $|g_n| = |g_{n-1}| = |g_n g_{n-1}|$. Hence if we now bring each of the last two terms of our sum to the denominator $g_n g_{n-1}$, the ratios |numerator|/|denominator| will not be affected. If we write these two terms as one fraction, the triangle inequality in R, applied to the numerator, shows that the resulting $n-1$-term series still satisfies $\sum$ |numerator|/|denominator|$<|e|$, contradicting our choice of n. ▌

This result might make us hope to prove a general triangle inequality for the function "|numerator|/|denominator|" on K, but this does not hold. (For instance, if $0<\alpha<\beta<\gamma$, we find $T_\gamma + (T_\alpha/T_\beta) = (T_\beta T_\gamma + T_\alpha)/T_\beta$, which has |numerator|/|denominator|$=1$, though $|T_\gamma|/|1|$ and $|T_\alpha|/|T_\beta|$ can be made simultaneously as small as desired.) We shall have to be more clever to get our topology.

Let X be the set of all monotone increasing functions $\varepsilon: (0, 1] \rightarrow (0, 1]$. Let Y be the set of all monotone decreasing sequences of members of X: $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots)$, where $\forall i \ \varepsilon_i \in X$, and $\forall x \in (0, 1]$, $\varepsilon_{i+1}(x) \leq \varepsilon_i(x)$.

For all $\varepsilon \in Y$, define V_ε to be the set of elements of K that can be written $\sum_{i=1}^{n} f_i/g_i$ with $|f_i| \leqslant \varepsilon_i(|g_i|)$ $(f_i, g_i \in R,\ i=1, \cdots, n)$. We shall call such a way of writing an element an "ε-representation". (Note that any ε-representation can be augmented to arbitrary length by adding terms $f_{n+j}/g_{n+j} = 0/1$.)

Lemma 3. *The* V_ε's *form a neighborhood basis of zero for a structure of topological group on the additive group of* K.

The inclusion of R in K is a homeomorphism into.

Proof. The V_ε's are symmetric: $-V_\varepsilon = V_\varepsilon$. To show that they define a structure of topological group, it remains to show that for every $\varepsilon \in Y$ there exists a $\delta \in Y$ such that the "triangle inclusion" $V_\delta + V_\delta \subset V_\varepsilon$ holds. Let us define δ by $\delta_i = \varepsilon_{2i}$. Given two δ-represented elements $\sum_{i=1}^{n} f_i/g_i$ and $\sum_{i=1}^{n} f'_i/g'_i$ (we can assume the "n's" are the same), it is clear that $f_1/g_1 + f'_1/g'_1 + \cdots + f_n/g_n + f'_n/g'_n$ is an ε-representation of their sum.

To get the homeomorphism statement, it suffices to prove the equivalence (on R) of our neighborhood bases of zero: $\forall \varepsilon \in Y$, take the real number $\delta = \varepsilon_1(1)$. Then clearly, $U_\delta \subset V_\varepsilon$. $\forall$ real number $\varepsilon > 0$, define $\delta \in Y$ by $\delta_i(x) = \varepsilon x / 2^i$. It follows from lemma 2 that $V_\delta \cap R \subset U_\varepsilon$. ▌

Lemma 4. *Multiplication by any element f/g is a continuous map: $K \to K$.*

Proof. It suffices to prove continuity at 0. Given $\varepsilon \in Y$, define $\delta_i(x) = \varepsilon_i(\inf|g|, x)$. Then if $\sum f_i/g_i$ is a δ-representation of $u \in K$, we claim $\sum ff_i/gg_i$ is an ε-representation of $(f/g)u$. Indeed, $|ff_i| \leqslant |f_i| \leqslant \delta_i(|g_i|) = \varepsilon_i(\inf|g|, |g_i|) = \varepsilon_i(|gg_i|)$. So $(f/g) V_\delta \subset V_\varepsilon$. ▌

An element $\delta \in Y$ will be called "regular" if there exist $\eta \in X$, and real numbers $c_i \downarrow 0$ such that δ is defined by the equation:

$$\delta_i(x) = \eta(\inf x, c_i). \tag{1}$$

Lemma 5. *The regular elements form a cofinal subfamily in Y.*

Proof. Choose arbitrary $c_i \downarrow 0$. Given $\varepsilon \in Y$, let $\eta(x) = \inf_{c_j \geqslant x} \varepsilon_j(x)$. (If x is greater than all the c's we define this infimum to be 1.) Define δ by equation (1) using these η and c's. For any i, x, we have $\delta_i(x) = \eta(\inf x, c_i)$; the infimum which *defines* η here clearly includes i among the allowed indices "j", hence $\eta(\inf x, c_i) \leqslant \varepsilon_i(\inf x, c_i) \leqslant \varepsilon_i(x)$. So $\delta \leqslant \varepsilon$. ▌

Given $\varepsilon \in Y$, define W_ε to be the set of elements of K that can be written $\prod_{i=1}^{n}(1+f_i/g_i)$ with $|f_i| \leqslant \varepsilon_i(g_i)$.

Lemma 6. *The two families* $\{W_\varepsilon\}$ *and* $\{1+V_\varepsilon\}$ *generate the same filter on K. I.e., the W's form a neighborhood basis of* 1 *in our topology.*

Proof. First, given any $\varepsilon \in Y$, we shall find $\delta \in Y$ such that $1+V_\delta \subset W_\varepsilon$.

Choose δ satisfying $\delta_i(x) \leqslant \varepsilon_i(x/2)$ and $\delta_i(x) \leqslant x/2^{i+1}$, and regular, represented as in equation (1).

Let $\sum_{i=1}^{n} f_i/g_i$ be an element of V_δ, δ-represented, but with the terms rearranged so that the $|f_i|$ are in descending order. We claim that in this arrangement, it is not only still δ-represented, but satisfies the stronger relation $|f_i| \leqslant \delta_i(|g_1 \cdots g_i|)$:

Since $\delta_i(|g_1 \cdots g_i|) = \eta(\inf |g_1 \cdots g_i|, c_i)$, what we must prove is $|f_i| \leqslant \eta(|g_j|)$ $(j \leqslant i)$, and $|f_i| \leqslant \eta(c_i)$. For every $j \leqslant n$, let j' designate the index the term f_j/g_j had in the original δ-representation of our element. Then the first inequality holds because for $j \leqslant i$, $|f_i| \leqslant |f_j| \leqslant \delta_{j'}(|g_j|) \leqslant \eta(|g_j|)$. To get the second, we note that among the i indices $1, \cdots, i$, there must be at least one j whose original index j' was $\geqslant i$; $|f_i| \leqslant |f_j| \leqslant \delta_{j'}(|g_j|) \leqslant \eta(c_{j'}) \leqslant \eta(c_i)$.

We are now ready to prove that $1+\sum f_i/g_i \in W_\varepsilon$.

$$1+\sum_{i=1}^{n} f_i/g_i = \prod_{j=1}^{n}\left(\frac{1+\sum_{i=1}^{j} f_i/g_i}{1+\sum_{i=1}^{j-1} f_i/g_i}\right)$$

$$= \prod_{j=1}^{n}\left(1+\frac{f_j/g_j}{1+\sum_{i=1}^{j-1} f_i/g_i}\right)$$

(assuming none of these terms involve zero denominators.)

Let $1+\sum_{i=1}^{j-1} f_i/g_i = h_j/g_1 \cdots g_{j-1}$. Then $h_j/g_1 \cdots g_{j-1} + \sum_{i=1}^{j-1} -f_i/g_i = 1$. From the condition $\delta_i(x) \leqslant x/2^{i+1}$, it follows that $\sum_{i=1}^{j-1} |-f_i|/|g_i| < 1/2$, hence by lemma 2, $|h_j|/|g_1\cdots g_{j-1}|>1/2$. (Hence $h_j\neq 0$, and we have no zero denominators in our product.)

Simplifying fractions, the j^{th} term of our product becomes $1+f_j g_1\cdots g_{j-1}/g_j h_j$. We now verify that our product is now represented as a member of W_ε:

$$\begin{aligned}|f_j g_1\cdots g_{j-1}| &\leqslant |f_j|\\ &\leqslant \delta_j(|g_1\cdots g_j|) \quad \text{(by the argument concerning the rearrangement of the terms)}\\ &= \delta_j(\inf|g_1\cdots g_{j-1}|,\ |g_j|)\\ &\leqslant \delta_j(\inf 2|h_j|,\ |g_j|) \quad \text{(by our last computation)}\\ &\leqslant \delta_j(2|g_j h_j|)\\ &\leqslant \varepsilon_j(|g_j h_j|) \quad \text{(we took } \delta_i(x)\leqslant\varepsilon_i(x/2).)\end{aligned}$$

Thus $1+V_\delta\subset W_\varepsilon$.

The converse is easier. Given $\varepsilon\in Y$, we choose any regular $\delta\leqslant\varepsilon$. Taking an element $\prod^n(1+f_i/g_i)\in W_\delta$, we again assume the terms arranged so that the $|f_i|$ are decreasing, and thus have $|f_i|\leqslant\delta_i(|g_1\cdots g_i|)$.

$$\begin{aligned}\prod_{i=1}^{n}(1+f_i/g_i) &= 1+\sum_{j=1}^{n}\Big(\prod_{i=1}^{j}(1+f_i/g_i)-\prod_{i=1}^{j-1}(1+f_i/g_i)\Big)\\ &= 1+\sum_{j=1}^{n} f_j/g_j\prod_{i=1}^{j-1}(1+f_i/g_i)\end{aligned}$$

If we write the j^{th} term of this sum as a single fraction, it has numerator divisible by f_j, and denominator $g_1\cdots g_j$. So $|\text{numerator}|\leqslant|f_j|\leqslant\delta_j(|g_1\cdots g_j|)\leqslant\delta_j(|\text{denominator}|)\leqslant\varepsilon_j(|\text{denominator}|)$. So the sum is ε-represented, and $W_\delta\subset 1+V_\varepsilon$. ▌

Corollary 1. *Multiplication is continuous on $K\times K$.*

Proof. Given continuity of addition and of multiplication by constants, joint continuity of multiplication everywhere can be shown to be equivalent to joint continuity at 1, 1. To obtain continuity at 1, 1: given $\varepsilon\in Y$, we define δ by $\delta_i=\varepsilon_{2i}$. The "triangle inclusion" $W_\delta W_\delta\subset W_\varepsilon$ is immediate! (analog of proof of $V_\delta+V_\delta\subset V_\varepsilon$ in Lemma 3.) ▌

Corollary 2. *The inverse operation is continuous.*

Proof. It suffices to prove continuity at 1.

We note that $1/(1+f/g)=1+(-f)/(g+f)$. Given $\varepsilon\in X$, consider a $\delta\in X$ such that $\delta(x)\leqslant\varepsilon(x/2)$, $\delta(x)\leqslant x/2$. If $|f|\leqslant\delta(|g|)$, then $|-f|\leqslant\delta(|g|)\leqslant\varepsilon(|g|/2)\leqslant\varepsilon(|g+f|)$.

It is thus clear that given any $\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots)\in Y$, we can take a $\delta=(\delta_1,\delta_2,\cdots)\in Y$ such that if $\overset{n}{\Pi}(1+f_i/g_i)$ is a "δ-represented product" then $\overset{n}{\Pi}(1+(-f_i)/(g_i+f_i))$ is an ε-representation of its inverse. Hence $1/W_\delta\subset W_\varepsilon$. ∎

In summary:

Proposition. *K is a topological field with R homeomorphically embedded. Hence K is an arcwise connected topological field, having the same characteristic as our original arbitrary domain k.* ∎

§3. Further notes

The directed set Y has no countable cofinal subset, hence, almost certainly, $\{V_\varepsilon\}_{\varepsilon\in Y}$ does not either, and the topology we have constructed is nonmetrizable. However we can construct a countable subfamily $Y'\subset Y$ such that $\{V_{\varepsilon}{}_{\varepsilon\in Y'}\}$ defines a structure of topological field on K; this will be metrizable. If we look back over §2, we see that Y' merely has to contain certain countably many elements, and satisfy certain finite and countable families of closure conditions. For instance, in the proof of continuity of multiplication by f/g, we need, given any ε, to have a δ with $\delta_i(x)\leqslant\varepsilon(\inf x, |g|)$. Thus it will suffice if $\forall\varepsilon\in Y'$, $\forall$ positive integers n, $\exists\delta\in Y'$ such that $\delta_i(x)\leqslant\varepsilon_i(\inf x, 1/n)$.

We display without proof a countable family Y' constructed to satisfy all these conditions: It consists of regular elements $\delta^{(k)}$, $(k=1,2,\cdots)$, defined using (1), with the sequence $c_i=2^{-i}$ in every case, and based on the countable family of functions $\eta^{(k)}(x)=x^k$. (Thus $\delta^{(1)}>\delta^{(2)}>\cdots$)

(The arguments we have just used do not allow us to reach a general conclusion of the nature of "If a topological field K is generated as a field by a subring R with metrizable topology, then the topology of K can be weakened to a metrizable topology." One can get a topology in which addition is continuous, multiplication is bicontinuous at 1, 1, and inverse is continuous at 1, all using countable and finite-closure type conditions; but to make multiplication by all constants continuous at 0, we have to deal with uncountably many possible constants. In the above case, it turned out that for any multiplier f/g, we merely needed a certain closure condition involving $|g|$, which could be replaced by one of a countable set of numbers; but there seems no reason why this sort of simplification should occur in general.)

James Ax has pointed out a fact of the opposite nature to our results, which, after a generalization by Nagata and a further generalization by the present authors, is as follows:

Proposition. *Let R be a commutative ring with unit, having a topology which makes R a topological additive group. Suppose there are a prime number p and a non-zero $x\in R$ such that* (1) $px=0$, *and* (2) *for all f in a neighborhood of* 0 *in R,* $\lim_{n\to\infty} f^{p^n}x=0$. *Then R is not connected.*

Proof. Let S be the set of $f\in R$ such that $\lim_{n\to\infty} f^{p^n}x=0$. From the fact that $px=0$, we have the "Frobenius" result, $(f\pm g)^{p^n}x=f^{p^n}x\pm g^{p^n}x$. Hence S is an additive subgroup of R: if $f, g\in S$, $\lim_{n\to\infty}(f\pm g)^{p^n}x=\lim_{n\to\infty} f^{p^n}x\pm g^{p^n}x=0\pm 0$, hence $f\pm g\in S$.

Since S contains a neighborhood of 0, it is an open subgroup of R; and clearly, it does not contain 1, so R is not connected. ∎

(In the above, "$\lim_{n\to\infty}$" can clearly be replaced by a limit along any filter on the integers.)

In particular, in §1, we could not have gotten a connected ring of characteristic p—or even a connected ring in which p is a zero-divisor—using a metric such that $|fg|\leqslant|f|\,|g|$.