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A remark on the Weierstrass points on open Riemann surfaces

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1. Introduction. Let us denote as usual by $\mathfrak{K}$ the class of canonical semi-exact differentials on an open Riemann surface, and by $\mathfrak{K}_0$ the class of exact differentials of class $\mathfrak{K}$. For an open Riemann surface R of genus g $(<\infty)$, we shall call a point P a *Weierstrass point* on R (for the integrals of class $\mathfrak{K}_0$) if there exists a non-constant function whose differential is of class $\mathfrak{K}_0$ and which has the only singularity of order at most g at P. In my previous paper, we obtained that the set of points which are not Weierstrass point is dense in R (Mori [5]). Main assertion to be proved in this paper is that if $\Gamma_{he}\cap\Gamma^*_{hse}\subset\Gamma^*_{he}$ holds on R, the number of Weierstrass point is at most $(g-1)g(g+1)$ as in the classical case. Moreover we shall show some properties of differentials of class $\mathfrak{K}$ on Riemann surfaces of class O_{KD} of finite genus.

2. At first we recall the definition of the principal operators L_0 and $(P)L_1$, where P denotes a regular partition of the ideal boundary (Ahlfors-Sario [1]).

Let R be a compact bordered surface with boundary β, and W a boundary neighborhood of R with relative boundary α which consists of a finite number of analytic curves. For a given real-valued function u on α, L_0u solves the boundary value problem in W with a vanishing normal derivative on β, and $(P)L_1u$ solves the boundary value problem in W with a constant value on each part of the partition P of β, the constants being chosen so that the flux along each part of P vanishes.

Now suppose that R is an arbitrary open Riemann surface. Let W be a regularly imbeded subregion with compact complement and with relative boundary α, and let Ω be a generic notation for a regular subregion which contains the complement of W, P_Ω the partition of $\partial\Omega$, which constitutes a consistent system $\{P_\Omega\}$ such that it induces the partition P. The operator L_0, as applied to $\Omega\cap W$ and acting on function u on α will be denoted by $L_{0\Omega}$, and similarly the operator $(P)L_1$ applied to $\Omega\cap W$ will be denoted by $(P)L_{1\Omega}$. The limit of $L_{0\Omega}u$ as Ω tends to R is L_0u and the limit of $(P)L_{1\Omega}u$ is $(P)L_1u$ on W.

Suppose that at a finite number of points $\zeta_j\in R$ there are given singularities of the form

$$(1)\qquad s = Re\sum_{n=1}^{\infty} a_n^{(j)}(z-\zeta_j)^{-n}+a^{(j)}\log|z-\zeta_j|\,,$$

where $a^{(j)}$ are real and subject to the condition $\sum_j a^{(j)}=0$. Then, there exist functions p_{0s} and p_{Ps}, harmonic on R except for the singularities (1), such that

$$L_0p_{os} = p_{os}\,,\qquad (P)L_1p_{Ps} = p_{Ps}$$

in W, if the complement of W contains all the ζ_j in its interior. These functions are uniquely determined save for additive constants. We say that p_{os} (p_{Ps}) has L_0-behavior ($(P)L_1$-behavior) in a boundary neighborhood of R.

Let Δ be a parametric disk on R and γ a 1-simplex contained in Δ. Consider a singularity

$$(2)\qquad \tau = \arg(z-\zeta_2)/(z-\zeta_1)\,,$$

where $\partial\gamma=\zeta_2-\zeta_1$. On the surface $R-\gamma$ we choose the normal operator which is composed of $(P)L_1$ for a boundary neighborhood of R and of the Dirichlet operator for $\Delta-\gamma$. This Dirichlet operator maps a continuous function on $\partial\Delta$ into the restriction to $\Delta-\gamma$ of the harmonic function in Δ with these boundary values. The direct sum of these operators yields a function $p_{P\tau}$ harmonic on $R-\gamma$. The differential $dp_{P\tau}$ can be extended harmonically to all of $R-(\zeta_1+\zeta_2)$, and we denote the extension by $dp_{P\tau}$, even though it is not exact. If γ is a finite 1-chain, it is homologous to a linear combination $\sum_j n_j\gamma_j$, where each γ_j is a 1-simplex contained in a parametric disk and each n_j is an integer. We extend the definition of $dp_{P\tau}$ to arbitrary γ by letting $dp_{P\tau} = \sum n_j dp_{P\tau_j}$. Similarly we can define the differential $dp_{o\tau}$ corresponding to the singularity (2), using the normal operator which is composed of L_0 for a boundary neighborhood of R and of the Dirichlet operator for $\Delta-\gamma$. Let δ be a 1-chain in R. Then we have

$$(3) \qquad \int_\delta (dp_{P\sigma}+idp_{P\tau}) = -2\pi i(\delta\times\gamma)$$

where σ denotes the singularity $\log|(z-\zeta_2)/(z-\zeta_1)|$ (Rodin [7]).

3. Let us denote by Q the canonical partition of the ideal boundary. *To each harmonic semi-exact differential ω with a finite number of singularities and periods, there corresponds a differential $\lambda(Q, \omega)$ with the same singularities and periods as ω and which, in a boundary neighborhood of R, is the differential of a function whose real and imaginary parts have $(Q)L_1$-behavior. Therefore we have*

$$(4) \qquad \lambda(Q, \omega) \equiv \omega$$

if and only if ω is distinguished (Rodin [7]). Moreover, *a meromorphic differential φ is of class $\mathfrak{K}$ if and only if*

$$(5) \qquad \lambda(Q, Re\, \varphi) \equiv Re\, \varphi\,.$$

Hence we know that *the real part of a meromorphic differential is distinguished if and only if it is of class $\mathfrak{K}$* (Mori [6]).

Similarly, *there corresponds a differential $\mu(\omega)$ with the same singularities and periods of ω and which, in a boundary neighborhood of R, is the differential of a function whose real and imaginary parts have L_0-behavior.* We can easily see that

$$(6) \qquad \mu(\omega) \equiv \omega$$

if and only if ω^* is distinguished.

It is known that *a Riemann surface is of class O_{KD} if and only if all the differentials $dp_{\varrho s}-dp_{Qs}$* vanish (Ahlfors-Sario [1]).

Then, by the use of above results we can easily show

Theorem 1. *The class $\mathfrak{K}$ is identical with $\mathfrak{K}^*$ if and only if the Riemann surface is of class O_{KD}. Every harmonic semi-exact differential which has at most a finite number of non-vanishing periods and is square integrable in a boundary neighborhood is distinguished if and only if the Riemann surface belongs to O_{KD}.*

4. From now on we restrict the Riemann surface R to be of finite genus unless otherwise stated, and let g be the genus of R. Moreover if R is of class O_{KD}, there exists a compact continuation $\tilde{R}$ of R which is conformally unique (A. Mori [4]). We identify R with the subregion on $\tilde{R}$ which is conformally equivalent to R. Then, a restriction to R of any differential on $\tilde{R}$ whose poles are all in R is a differential of class $\mathfrak{K}$ on R by Theorem 1. Conversely, let φ be a differential of class $\mathfrak{K}$ on R. Then, there exists a boundary neighborhood W of R such that the integral of φ on W is an AD-function on W. Since $\tilde{R}-R$ is an AD-removable set (Royden [8]), $\int\varphi$ can be extended analytically to $\tilde{R}$, and differential of $\int\varphi$ is an extension of φ to $\tilde{R}$. Thus we get

Corollary 1. *Suppose that a Riemann surface R of finite genus is of class O_{KD}, and $\tilde{R}$ is a compact continuation of R. Then the class $\mathfrak{K}$ is identical with the class of restrictions to R of differentials on $\mathfrak{K}$ whose poles are all in R.*

By this Corollary we see

Corollary 2. *If the genus of a Riemann surface of class O_{KD} is g, the degree of divisor of any differential of class $\mathfrak{K}$ is at most $2g-2$. Moreover, there does exist a differential of class $\mathfrak{K}$ such that the degree of whose divisor is strictly less than $2g-2$.*

5. Take a point P on a Riemann surface of genus g. Then, there always exists a non-constant function whose differential is of class $\mathfrak{K}_0$ and which has the only singularity of order at most $g+1$ at P, and hence there happens one of the following two cases: 1) there does not exist non-constant such function with the only singularity of order at most g at P, or 2) there exists a non-constant such function with the only singularity of order at most g at P. If the second case happens, we call the point P a *Weierstrass point* on R (for the integrals of class $\mathfrak{K}_0$) after the classical case. The set of points at which the first case happens is dense in R (Mori [5]).

If R is a Riemann surface of class O_{KD} and $\tilde{R}$ is a compact continuation of R, the restriction to R of a function on $\tilde{R}$ whose poles are all in R is a function whose differential is of class $\mathfrak{K}_0$, and any integral of class $\mathfrak{K}_0$ can be extended analytically to all of $\tilde{R}$ by Corollary 1. Therefore, a Weierstrass point on $\tilde{R}$ which is in R is a Weierstrass point on R and conversely, any Weierstrass point on R is a Weierstrass point on $\tilde{R}$.

Corollary 3. *If a Riemann surface R is of class O_{KD} and of genus g, the number of Weierstrass points on R is at most $(g-1)g(g+1)$. Further, for any integer n such that $0\leqq n\leqq(g-1)g(g+1)$, there exists an open Riemann surface of class O_{KD} and of genus g which is exactly with n Weierstrass points.*

By removing a suitable number of Weierstrass points from a compact Riemann surface, we get the last assertion.

6. Let a Riemann surface R of genus g be of class O_{KD} and let $\tilde{R}$ be a compact continuation of R. We denote by $\{A_k, B_k\}_{k=1,2,\cdots,g}$ a homology basis of R modulo dividing cycles. Then $\{A_k, B_k\}$ forms a homology basis of $\tilde{R}$ as well. Let φ_{A_k} and φ_{B_k} $(k=1,2,\cdots,g)$ be the canonical semi-exact differentials which are uniquely determined by the conditions $Re\int_{B_h}\varphi_{A_k}=-Re\int_{A_h}\varphi_{B_k}=\delta_{hk}$ and $Re\int_{A_h}\varphi_{A_k}=Re\int_{B_h}\varphi_{B_k}=0$. The space Γ_{kse} which is spanned by the φ_{A_k} and φ_{B_k} over the real number field is identical with Γ_{ase} if and only if R belongs to O_{KD}, and the space $\Gamma_{as}\cap\Gamma_{ase}$ which always contains Γ_{kse} is spanned by the φ_{A_k} if the genus of R is finite and $\Gamma_{he}\cap\Gamma^*_{hse}\subset\Gamma^*_{he}$ holds (Mori [6]). Therefore the φ_{A_k} span Γ_{ase} of dimension g and hence they are linearly independent even over the complex number field. Then by Corollary 1 we can easily see that extensions $\tilde{\varphi}_{A_k}$ of φ_{A_k}

to $\tilde{R}$ form a basis of the space of analytic differentials on $\tilde{R}$.

We consider the following form of Jacobi inversion problem on R. For a 1-chain γ (finite or infinite) on R, we set

$$H(\gamma) = \left(\int_\gamma \varphi_{A_1}, \int_\gamma \varphi_{A_2}, \cdots, \int_\gamma \varphi_{A_g}\right).$$

For arbitrary given complex numbers $c_1, c_2, \cdots, c_g$, we try to find n paths γ_j starting from a given point and satisfying

$$(7) \qquad H(\sum_j \gamma_j) = (c_1, c_2, \cdots, c_g) \qquad \text{(mod periods)}.$$

On the compact surface $\tilde{R}$, if we set

$$\tilde{H}(\tilde{\gamma}) = \left(\int_{\tilde{\gamma}} \tilde{\varphi}_{A_1}, \int_{\tilde{\gamma}} \tilde{\varphi}_{A_2}, \cdots, \int_{\tilde{\gamma}} \tilde{\varphi}_{A_g}\right)$$

for a 1-chain $\tilde{\gamma} \subset \tilde{R}$, we can always find n paths $\tilde{\gamma}_j$ starting from a given point on $\tilde{R}$ and satisfying

$$\tilde{H}(\sum_j \tilde{\gamma}_j) = (c_1, c_2, \cdots, c_g) \qquad \text{(mod periods)}$$

if $n \geqq g$. Each $\tilde{\gamma}^j$ may not be contained in R. If $\tilde{\gamma}_j \subset R$, we let it be γ_j. Suppose that $\tilde{\gamma}_j$ is not contained in R. Then we take a planar boundary neighborhood W_j so that it does not contain the starting point of $\tilde{\gamma}_j$, and let P_j be the first point of $\partial W_j \cap \tilde{\gamma}_j$ where one meets when one moves along $\tilde{\gamma}_j$ from the starting point. We can choose a path γ_j'' connecting P_j and the end point Q_j of $\tilde{\gamma}_j$ in W_j except for Q_j if $Q_j \notin R$. By the assumption that $R \in O_{KD}$, any component of $\tilde{R}-R$ is a point on $\tilde{R}$ and it is accessible from the interior of R. Let us denote by $\tilde{\gamma}_j'$ the part of $\tilde{\gamma}_j$ which connects the starting point and P_j, and let $\gamma_j = \tilde{\gamma}_j' + \gamma_j''$. Because of planar character of W_j, we have

$$H(\gamma_j) = \tilde{H}(\tilde{\gamma}_j),$$

and we get n paths $\gamma_1, \gamma_2, \cdots, \gamma_n$ which satisfy (7).

Corollary 4. (Jocobi inversion problem) *We can always find n paths starting from an arbitrary given point on R which satisfy* (7) *on a Riemann surface R of genus g, if R is of class O_{KD} and $n \geqq g$.*

7. Now we consider a Riemann surface R on which $\Gamma_{he}\cap\Gamma^*_{hse}\subset\Gamma^*_{he}$ holds (Mori [6]). If R is of class O_{KD}, $\Gamma_{he}\cap\Gamma^*_{hse}\subset\Gamma^*_{he}$ holds, but we can not guarantee the converse. Let $\tilde{R}$ be a compact continuation of R which is the same genus as R, and $\tilde{f}$ a meromorphic function on $\tilde{R}$ such that whose poles are all in R. The differentials

$$\varphi_1(\tilde{f}) = \lambda(Q, Re\, d\tilde{f})+i\lambda(Q, Re\, d\tilde{f})^*$$
$$\varphi_2(\tilde{f}) = \lambda(Q, Im\, d\tilde{f})+i\lambda(Q, Im\, d\tilde{f})^*$$

are of class $\mathfrak{K}$ on R, and

$$Re\, d\tilde{f}-\lambda(Q, Re\, d\tilde{f})$$
$$Im\, d\tilde{f}-\lambda(Q, Im\, d\tilde{f})$$

belong to $\Gamma_{he}\cap\Gamma^*_{hse}=\Gamma_{he}\cap\Gamma^*_{he}$. Therefore $\int\varphi_1(\tilde{f})$ and $\int\varphi_2(\tilde{f})$ are single-valued on R. Thus we conclude that *if R is a Riemann surface of genus g on which $\Gamma_{he}\cap\Gamma^*_{hse}\subset\Gamma^*_{he}$ is valid and if $\tilde{R}$ is a compact continuation of R, then a Weierstrass point on $\tilde{R}$ which is in the interior of R is a Weierstrass point on R.*

Conversely, let P be a Weierstrass point on R and f a function whose differential is of class $\mathfrak{K}_0$ and which has the only singularity of order at most g at P. We take an open Riemann surface $\hat{R}$ of class O_{KD} on $\tilde{R}$ so that it contains R, and consider the differential $\lambda(Q, Re\, df)$ on $\hat{R}$. We have

$$Re\, df-\lambda(Q, Re\, df)\in\Gamma_{he}\cap\Gamma^*_{hse} = \Gamma_{he}\cap\Gamma^*_{he}$$

on R. Moreover $\lambda(Q, Re\, df)$ and its conjugate $\lambda(Q, Re\, df)^*$ have no periods along any dividing cycle on $\hat{R}$. Therefore the function

$$\int\lambda(Q, Re\, df)+i\lambda(Q, Re\, df)^*$$

which has the same singularity as f is an integral of class $\mathfrak{K}_0$ on $\hat{R}$. This implies that P is a Weierstrass point on $\hat{R}$. Then, by the same way as the proof of Corollary 3, we can show that P is a Weierstrass point on $\tilde{R}$. Thus we have proved

Theorem 2. *Suppose that R is a Riemann surface of genus g*

on which $\Gamma_{he} \cap \Gamma^*_{hse} \subset \Gamma^*_{he}$ *holds. Then the number of Weierstrass points on R is at most* $(g-1)g(g+1)$.

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