Rational sections and Chern classes of vector bundles*

by

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Let $\mathcal{E}$ be a quasi-coherent sheaf, of finite type, on an integral prescheme $X$, and denote by $V(\mathcal{E})$, $P(\mathcal{E})$ the vector and projective fibres of $\mathcal{E}$ respectively. Then each non-zero rational section $\omega$ of $V(\mathcal{E})$ over $X$ defines a rational section $\bar{\omega}$ of $P(\mathcal{E})$ over $X$ (section 2), and we can construct a closed subscheme $\langle \omega \rangle$ of $X$ whose points are the non-regular points of $\bar{\omega}$ (Prop. 5). Denote by $[\omega]$ the $X$-prescheme obtained by blowing up centered at $\langle \omega \rangle$. On the other hand we can construct a quasi-coherent fractional Ideal $\mathcal{O}_X(\omega)$ of the sheaf of rational functions $\mathcal{R}(X)$ of $X$ which is invertible when $X$ is UFD (Cor. of Prop.4) and which corresponds to the Cartier divisor of the rational section $\omega$.

In this note, we shall prove some relations between these schemes or sheaves (Th. 1.2). In the case that $X$ is a non-singular quasi-projective algebraic scheme, they give an explicite formula of Chern classes of vector bundles of rank 2 (Cor. of Th.2'). And, as a special case, if $X$ is a surface and $V(\mathcal{E})$ is the bundles of simple differentials, then our formula proves that the Severi-series of $X$ coincides with the second Chern class $c_2(X)$ of $X$ (last Remark).

1. Rational maps and rational functions (EGA. I.7) Let $X$ and $Y$ be $S$-preschemes, and $\mathcal{U}_X$ the set of dense open subsets of $X$; then the family of sets of $S$-morphisms $(\text{Hom}_S(U, Y))_{U \in \mathcal{U}_X}$

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forms an inductive system (with natural restriction of morphisms), and each element of the set $\text{Rat}_S(X, Y) = \lim_{U \subseteq X} \text{Hom}_S(U, Y)$ is called a \textit{rational map} from $X$ to $Y$ over $S$ (or a \textit{rational S-map} form $X$ to $Y$). We shall call the rational S-maps from $S$ to $X$ the \textit{rational sections} of $S$-prescheme $X$: $\text{Rat}_S(S, X) = \Gamma_{\text{rat}}(X/S)$. Let $\mathcal{F}$ be a sheaf (of sets) on a prescheme $X$, for each open subset $U$ of $X$, put $\Gamma_{\text{rat}}(U, \mathcal{F}) = \lim_{V \subseteq U} \text{Horns}(U, V)$; each element of $\Gamma_{\text{rat}}(U, \mathcal{F})$ is called a \textit{rational section} of $\mathcal{F}$ on $U$. It is easy to see that, for two open subsets $U$ and $V$ of $X$, if $V \subseteq U$ and $V$ is dense in $U$, then $\Gamma_{\text{rat}}(U, \mathcal{F}) = \Gamma_{\text{rat}}(V, \mathcal{F})$, and that, if $U$ is irreducible, then $\Gamma_{\text{rat}}(U, \mathcal{F})$ is nothing but the stalk at the generic point $x$ of $U$. In case of $\mathcal{F} = \mathcal{O}_X$, the structure sheaf of $X$, the rational sections of $\mathcal{O}_X$ on $U$ are called the \textit{rational functions} of $X$ on $U$, and we denote $\mathcal{R}(U) = \Gamma_{\text{rat}}(U, \mathcal{O}_X)$. The sheaf associated with the presheaf $U \mapsto \mathcal{R}(U)$ is called the \textit{sheaf of rational functions} on $X$ and we denote it $\mathcal{R}(X)$. The canonical map $\Gamma(U, \mathcal{O}_X) \to \mathcal{R}(U)$ defines the canonical homomorphism $\iota: \mathcal{O}_X \to \mathcal{R}(X)$, and, by means of it, $\mathcal{R}(X)$ is considered as an $\mathcal{O}_X$-Algebra.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-Module, $U$ a dense open subset of $X$ and $f: \mathcal{F}|U \to \mathcal{O}_X|U$ an $(\mathcal{O}_X|U)$-homomorphism. Then, for each open subset $W$ of $X$, consider the following $\Gamma(W, \mathcal{O}_X)$-homomorphism obtained as the composition map:

\[
(1) \quad \bar{f}(W): \Gamma(W, \mathcal{F}) \xrightarrow{\text{rest.}} \Gamma(W \cap U, \mathcal{F}) \xrightarrow{f(W \cap U)} \Gamma(W \cap U, \mathcal{O}_X) \xrightarrow{\text{(rest.)}^{-1}} \Gamma(W, \mathcal{R}(X))
\]

(note that $W \cap U$ is dense in $W$, hence the restriction $\Gamma(W, \mathcal{R}(X)) \to \Gamma(W \cap U, \mathcal{R}(X))$ is an isomorphism). Obviously $f(W)$ commutes with the restriction maps of the sections of $\mathcal{F}$ and $\mathcal{R}(X)$, hence the collection $(\bar{f}(W))_{W \subseteq X}$ gives an $\mathcal{O}_X$-homomorphism $\bar{f}: \mathcal{F} \to \mathcal{R}(X)$, and, thus, we get a map

\[
\alpha_U: \text{Hom}_{\mathcal{O}_X|U} (\mathcal{F}|U, \mathcal{O}_X|U) = \Gamma(U, \mathcal{F}) \to \text{Hom}_{\mathcal{O}_X} (\mathcal{F}, \mathcal{R}(X)),
\]
Rational sections and Chern classes of vector bundles

$\mathcal{F} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, the dual of the $\mathcal{O}_X$-Module $\mathcal{F}$. Moreover, it is clear that $\alpha_U$ is a $\Gamma(X, \mathcal{O}_X)$-homomorphism and commutes with the restriction map $\alpha^U_V : \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})(U, V \subseteq U_x, U \supseteq V) : \alpha_U = \alpha_V \circ \alpha^U_V$. Therefore, passing to the inductive limit, we have the canonical $\Gamma(X, \mathcal{R}(X))$-homomorphism

$\alpha : \Gamma_{\mathcal{R}_x}(X, \mathcal{F}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}(X)).$

The following proposition is well known (EGA.I.7.3).

**Proposition 1.** Let $X$ be an integral (i.e., reduced and irreducible) prescheme. Then, (i) $\mathcal{R}(X)$ is a quasi-coherent $\mathcal{O}_X$-Module, (ii) $\mathcal{R}(X)$ is a constant sheaf, hence $\Gamma(U, \mathcal{R}(X)) = R(U) = R(X)$ for each open subset $U$ of $X$, (iii) the canonical homomorphism $\mathcal{O}_X \to \mathcal{R}(X)$ is injective, (iv), for each point $x$ of $X$, $\mathcal{R}(X)_x = R(X)$ is the quotient field of $\mathcal{O}_x$, and at the generic point $x$, $\mathcal{R}(X)_x = R(X) = \mathcal{O}_{x; x}$, and (v), for any quasi-coherent $\mathcal{O}_X$-Module $\mathcal{F}$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R}(X) = R(X)^{(1)}$ (direct sum).

**Corollary.** If $X$ is integral, then, for each $\mathcal{O}_X$-Module $\mathcal{F}$ of finite type, the canonical homomorphism $\alpha : \Gamma_{\mathcal{R}_x}(X, \mathcal{F}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}(X))$ is injective. Moreover, if $\mathcal{F}$ is quasi-coherent, then $\alpha$ is an isomorphism.

**Proof.** Since $X$ is irreducible, $\Gamma_{\mathcal{R}_x}(X, \mathcal{F}) = \mathcal{F}_{\overline{x}} = \mathcal{H}om_{\mathcal{O}_{X; x}}(\mathcal{F}_{\overline{x}}, \mathcal{O}_{X; x})$, where $\overline{x}$ is the generic point of $X$. Since $\mathcal{R}(X)_x = \mathcal{O}_{x; x}$ (Prop. 1 (iv)), by the definition of $\alpha$, it is easy to check that the composition map

$\Gamma_{\mathcal{R}_x}(X, \mathcal{F}) = \mathcal{F}_{\overline{x}} \xrightarrow{\alpha} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}(X)) \to \mathcal{H}om_{\mathcal{O}_{X; x}}(\mathcal{F}_{\overline{x}}, \mathcal{R}(X)_x) = \mathcal{F}_{\overline{x}}$

is the identity map, where the last arrow is the map which corresponds each sheaf homomorphism $f$ to its restriction $f_x$ at the generic point $\overline{x}$. Hence $\alpha$ is injective. Moreover, assume that $\mathcal{F}$ is quasi-coherent. When that is so, in order to prove that $\alpha$ is surjective, it is sufficient to prove that the last arrow is injective, i.e., for $f \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}(X))$, $f_x = 0$ implies $f = 0$. To show this, we may
assume $X$ to be affine. Let $X = \text{Spec}(A)$, $\mathcal{F} = \widehat{M}$, $M$ is an $A$-module; then $A$ is integral and $\mathcal{R}(X)$ is the sheaf associated with the quotient field $K$ of $A$, and $f: \mathcal{F} \to \mathcal{R}(X)$ corresponds to an $A$-homomorphism $\varphi: M \to K$. But, by tensoring $K$, $\varphi$ can be decomposed into $M^u \to \mathcal{M}_K$ and $\varphi$ is exactly the same to $f: \mathcal{F} \to \mathcal{O}_X$, hence, $f = \varphi = 0$ implies $\varphi = v. u = 0$.

2. Rational sections of vector- and projective fibres.

Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-Module of finite type and denote by $\mathcal{S}(\mathcal{E})$ the symmetric $\mathcal{O}_X$-Algebra of $\mathcal{E}$ (EGA. II. 1. 7. 4). And put $\mathcal{V}(\mathcal{E}) = \text{Spec} (\mathcal{S}(\mathcal{E}))$ (resp. $\mathcal{P}(\mathcal{E}) = \text{Proj} (\mathcal{S}(\mathcal{E}))$): $\mathcal{V}(\mathcal{E})$ (resp. $\mathcal{P}(\mathcal{E})$) is called the vector (resp. projective) fibre over $X$ defined by $\mathcal{E}$ (EGA. II. 1. 7. 8, 4. 1. 1).

**Proposition 2.** Let $X$ be a prescheme. For each quasi-coherent $\mathcal{O}_X$-Module $\mathcal{E}$ of finite type, (i) we have a canonical isomorphism

$$\Gamma_m(\mathcal{V}(\mathcal{E})/X) \cong \Gamma_m(X, \mathcal{E}),$$

and moreover (ii), if $X$ is integral, the canonical homomorphism $\iota: \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}_X \mathcal{R}(X)$ induces a canonical isomorphism

$$\iota: \Gamma_m(X, \mathcal{E}) \cong \Gamma(X, \mathcal{E} \otimes \mathcal{O}_X \mathcal{R}(X)).$$

**Proof**

(i) $\Gamma_m(\mathcal{V}(\mathcal{E})/X) = \lim \text{Hom}_m(U, \mathcal{V}(\mathcal{E})) \equiv \lim \text{Hom}_{\mathcal{O}_X|U}(\mathcal{E}|U, \mathcal{O}_X|U)$

$$= \Gamma_m(X, \mathcal{E})$$ (EGA. II. 1. 7. 8, 4. 1. 9). (ii) Assume $X$ to be integral. Since $\mathcal{E} \otimes \mathcal{O}_X \mathcal{R}(X)$ is a constant sheaf (Prop. 1. (v)), for any pair of open subsets $U$, $V$ of $X$ such that $U \supset V$, we get a commutative diagram:

$$\begin{array}{ccc}
\Gamma(U, \mathcal{E}) & \xrightarrow{\iota(U)} & \Gamma(U, \mathcal{E} \otimes \mathcal{R}(X)) \\
\Gamma(V, \mathcal{E}) & \xrightarrow{\iota(V)} & \Gamma(V, \mathcal{E} \otimes \mathcal{R}(X))
\end{array}$$

Passing to the direct limit, this defines our $\iota$. Consider the following canonical $\mathcal{R}(X)$-homomorphism

$$\beta: \mathcal{E} \otimes \mathcal{O}_X \mathcal{R}(X) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \otimes \mathcal{O}_X \mathcal{R}(X) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X)),$$

obtained by tensoring $\mathcal{R}(X)$ to the natural $\mathcal{O}_X$-homomorphism $\mathcal{E} = $
Rational sections and Chern classes of vector bundles

\( \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X)) \). Taking the global sections, we get an \( \mathcal{R}(X) \)-homomorphism

\[
\beta(X) : \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{R}(X)) \rightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X)).
\]

It is easy to see that \( \alpha = \beta \cdot \iota \), where \( \alpha \) is the canonical isomorphism defined in the section 1. In our case \( \alpha \) is an isomorphism (Cor. of Prop. 1), hence \( \iota \) is injective. Moreover \( \iota \) is surjective; in fact, for any \( s \in \Gamma(X, \mathcal{E} \otimes \mathcal{R}(X)) \), at the generic point \( x, s \in (\mathcal{E} \otimes \mathcal{R}(X))_x = \mathcal{E}_x \), hence there exist an open set \( U \) and an \( (\mathcal{O}_X|U) \)-homomorphism \( t : \mathcal{E}|U \rightarrow \mathcal{O}_X|U \) such that \( t \cdot \iota(U) = (s|U) \) in \( \Gamma(U, \mathcal{E} \otimes \mathcal{R}(X)) \). Q. E.D.

Remark. In the above proof, we may replace \( X \) by any open subset \( U \) of \( X \), hence \( \beta(U) : \Gamma(U, \mathcal{E} \otimes \mathcal{R}(X)) \rightarrow \Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X))) \) is a \( \Gamma(U, \mathcal{R}(X)) \)-isomorphism. Therefore we have the following

Corollary. For any quasi-coherent \( \mathcal{O}_X \)-Module \( \mathcal{E} \), of finite type, on a integral prescheme \( X \), we have a canonical \( \mathcal{R}(X) \)-isomorphism

\[
\beta : \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{R}(X) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X)).
\]

3. Now we shall give some fundamental notions and notations needed for our study. From now on, we shall assume the base prescheme \( X \) to be integral. Let \( \mathcal{E} \) be a quasi-coherent \( \mathcal{O}_X \)-Module of finite type; then, by Cor. of Prop. 1 and Prop. 2, we have canonical isomorphisms

\[
\Gamma_{\text{rat}}(\mathcal{V}(\mathcal{E})/X) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X)) \cong \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{R}(X)).
\]

For each rational section \( \omega \in \Gamma_{\text{rat}}(\mathcal{V}(\mathcal{E})/X) \), we denote by \( \omega^*_t \) and \( \omega^*_s \) the images of \( \omega \) under these isomorphisms in \( \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X)) \) and in \( \Gamma(X, \mathcal{E} \otimes \mathcal{R}(X)) \), respectively. Now fix a rational section \( \omega \in \Gamma_{\text{rat}}(\mathcal{V}(\mathcal{E})/X) \). Then, the \( \mathcal{O}_X \)-homomorphism of \( \mathcal{O}_X \)-Modules \( \omega^*_t : \mathcal{E} \rightarrow \mathcal{R}(X) \) can be uniquely extended to a homomorphism of graded \( \mathcal{O}_X \)-Algebras (of homogeneous degree 0)

\[
\omega^* : S(\mathcal{E}) \rightarrow \mathcal{R}(X)[T] = \mathcal{R}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[T].
\]
Put $I(\omega) = \text{Image of } \omega^*_1$ and $J(\omega) = \text{Kernel of } \omega^*$; then $I(\omega)$ is a quasi-coherent fractional Ideal of $\mathcal{R}(X)$, and \text{Image of } \omega^* = \bigoplus_{s \geq 0} I(\omega)^s$. $J(\omega) = \bigoplus_{s \geq 0} J_s(\omega)$ is a quasi-coherent homogeneous of $S(\mathcal{E})$, and it is generated by the component of degree 1: $J(\omega) = J_1(\omega) \cdot S(\mathcal{E})$.

Thus, we have exact sequences

(3) \hspace{1cm} 0 \rightarrow J(\omega) \rightarrow S(\mathcal{E}) \rightarrow \bigoplus_{s \geq 0} I(\omega)^s \rightarrow 0,

and

(3') \hspace{1cm} 0 \rightarrow J_1(\omega) \rightarrow \mathcal{E} \rightarrow I(\omega) \rightarrow 0.

Put $[\omega] = \text{Proj} \bigoplus_{s \geq 0} (I(\omega)^s)$; then $[\omega]$ is a closed subscheme of $P(\mathcal{E}) = \text{Proj}(S(\mathcal{E}))$; $[\omega] \mapsto P(\mathcal{E})$, and it is the $X$-prescheme obtained by blowing up the fractional Ideal $I(\omega)$ of $\mathcal{R}(X)$ (EGA. II. 8. 1. 3), and the canonical projection $[\omega] \mapsto P(\mathcal{E}) \rightarrow X$ is birational (EGA. II. 8. 1. 4). Note that $[\omega]$ is not empty if and only if $\omega^*_1 \neq 0$, i.e., $\omega \neq 0$.

If $\omega \neq 0$, there exists an open subset $U$ of $X$ such that $\omega^*$ induces a homomorphism $S(\mathcal{E}) \cdot U \rightarrow (\mathcal{O}_X|U)[T]$ (take a defining homomorphism $\mathcal{E}|U \rightarrow \mathcal{O}_X|U$ of $\omega^*_1$ (Cor. of Prop. 1) and extend it to $S(\mathcal{E}|U) = S(\mathcal{E})|U \rightarrow (\mathcal{O}_X|U)[T]$). This gives a rational map $U = \text{Proj}((\mathcal{O}_X|U)[T]) \rightarrow \text{Proj}(S(\mathcal{E})|U) \rightarrow P(\mathcal{E})$ (cf. EGA. II. 2. 8. 1), hence a rational map

$\bar{\omega} : X \rightarrow P(\mathcal{E})$.

By the definition of rational maps and the fact that $X$ is integral, this does not depend on the choice of $U$. And, since $\omega^*$ is an $\mathcal{O}_X$-homomorphism, the rational map $\bar{\omega}$ is a rational section of the projective fibre $P(\mathcal{E})/X$, and, obviously, it can be decomposed into $X \rightarrow [\omega] \mapsto P(\mathcal{E})$. $\bar{\omega}$ is called the \text{induced section} of $\omega$ to the projective fibre $P(\mathcal{E})/X$, and $[\omega]$ is called the \text{image} of $\bar{\omega}$ or the projective image of $\omega$. Hence we get a correspondence

$- : [\Gamma_{\text{rat}}(V(\mathcal{E})/X)] - \{0\} \rightarrow [\Gamma_{\text{rat}}(P(\mathcal{E})/X)]$.

\textbf{Proposition 3.} \textit{Let }$\bar{\omega}_0 : U \rightarrow P(\mathcal{E})$\textit{ be an }$X$\textit{-morphism which
Rational sections and Chern classes of vector bundles

represents \( \tilde{\omega} \), then the closure of the image \( \tilde{\omega}(U) \) in \( P(E) \) coincides with \([\omega]\).

Proof) Since \( \pi \cdot i: [\omega] \to X \) is birational, \( \phi: U \to [\omega] \) is also birational, hence the closure of \( \phi(U) \) in \( P(E) \) coincides with \([\omega] = [\omega] \).

Q. E. D.

To each open subset \( U \) of \( X \), associate a subset \( \mathcal{O}(U) \) of \( \mathcal{R}(U) \) consisting of the rational functions \( f \) such that \( f \cdot \mathcal{R}(U, I(\omega)) \subset \mathcal{R}(U, \mathcal{O}_X) \). This correspondence \( U \to \mathcal{O}(U) \) gives, with natural restrictions, a presheaf \( \mathcal{O} \) of sub-\( \mathcal{O}_X \)-modules of \( \mathcal{R}(X) \). The sheaf associated with \( \mathcal{O} \) is called the sheaf of \( \omega \) and denoted by \( \mathcal{O}_X(\omega) \).

Proposition 4. (i) The presheaf \( \mathcal{O} \) is a sheaf, i.e., \( \mathcal{O} = \mathcal{O}_X(\omega) \).

(ii) For each open subset \( U \) of \( X \), \( \mathcal{O}(U) = \mathcal{R}(U, \mathcal{O}_X(\omega)) \) coincides with the set of rational functions \( f \in \mathcal{R}(X) \) such that \( f \cdot \mathcal{R}(U, I(\omega)) \subset \mathcal{R}(U, \mathcal{O}_X) \).

Proof) (i) Easy. (ii) By the isomorphism \( \beta: \mathcal{E} \otimes \mathcal{R}(X) \cong \text{Hom}(\mathcal{E}, \mathcal{R}(X)) \) (Cor. of Prop. 2), \( f \cdot (\omega^*_1 | U) \) corresponds to \( f \cdot (\omega^*_1 | U) \), hence, by the commutative diagram

\[
\begin{array}{c}
\mathcal{E} \\
\downarrow \\
\mathcal{E} \otimes \mathcal{O}_X \mathcal{R}(X) \end{array} \xrightarrow{\beta} \begin{array}{c}
\text{Hom}(\mathcal{E}, \mathcal{O}_X) \\
\downarrow \\
\text{Hom}(\mathcal{E}, \mathcal{R}(X)) \end{array}
\]

it is easy to see that \( f(\omega^*_1 | U) \in \text{Im}[\mathcal{R}(U, \mathcal{E}) \to \mathcal{R}(U, \mathcal{E} \otimes \mathcal{R}(X))] \) if and only if \( \text{Im}[f \cdot (\omega^*_1 | U)] \subset \mathcal{O}_X | U \). On the other hand, \( \text{Im}[f \cdot (\omega^*_1 | U)] = f \cdot \text{Im}(\omega^*_1 | U) = f \cdot (I(\omega) | U) \). This proves (ii). Q. E. D.

Corollary. If, for each point \( x \) of \( X \), \( \mathcal{O}_X(\omega) \) is an unique factorization domain (in this case, we shall say that \( X \) is UFD), then \( \mathcal{O}_X(\omega) \) is an invertible sheaf on \( X \).

Proof) Let \( x \) be a point of \( X \). Then

\( \mathcal{O}_X(\omega)_x = \{ f \in \mathcal{R}(X) \mid f \cdot I(\omega), \subset \mathcal{O}_X, i \} \).

Let \( a_i \in \mathcal{R}(X) \) \( (i = 1, \ldots, r) \) be a set of generators of \( I(\omega) \), over \( \mathcal{O}_X \).
Since \( R(X) \) is the quotient field of \( \mathcal{O}_x \), and \( \mathcal{O}_x \) is an unique factorization domain, we may write \( a_i = gc_i \) such that \( g \in R(X), \ c_i \in \mathcal{O}_x, \) and \( c_i \)'s have no common factors in \( \mathcal{O}_x \). Then, for \( f \in R(X), \ f \) is in \( \mathcal{O}_x(\omega), \) if and only if \( f \cdot g \cdot c_i \) is in \( \mathcal{O}_x, \) for every \( i \). This proves that \( \mathcal{O}_x(\omega) = (1/g)\mathcal{O}_x \equiv \mathcal{O}_x. \) Hence, \( \mathcal{O}_x(\omega) \) is invertible.

Q. E. D.

Remark. The fractional invertible Ideal \( \mathcal{O}_x(\omega) \) of \( R(X) \) defines a Carier divisor \( (\omega) \) on \( X \), and \( g \) (of the above proof) is its local equation at \( x \) (cf. [7]). This \( (\omega) \) is called the divisor of the rational section \( \omega \).

From now on we shall assume that our integral prescheme \( X \) is UFD. By definition, \( \mathcal{O}_x(\omega)I(\omega) (\subset \mathcal{O}_x) \) is an quasi-coherent Ideal of \( \mathcal{O}_x \), we denote it \( \tilde{I}(\omega) = \mathcal{O}_x(\omega)I(\omega) \). Then, since \( \mathcal{O}_x(\omega) \) is invertible, we have a canonical isomorphism of \( X \)-preschemes (EGA. II. 3. 1. 8)

\[
g: [\omega] = \text{Proj}(\oplus \omega \cdot \tilde{I}(\omega)^*) \rightarrow [\omega] = \text{Proj}(\oplus \omega \cdot I(\omega)^*),
\]

and (EGA. II. 3. 2. 10)

\[
(4) \quad g_*(\mathcal{O}_{[\omega]}(n)) \cong \mathcal{O}_{[\omega]}(n) \otimes \mathcal{O}_x(\omega)^*.
\]

By means of \( g \), we shall identify \( [\omega]_1 \) and \( [\omega] \). Moreover, we shall denote by \( \langle \omega \rangle \) the closed sub-prescheme of \( X \) defined by the quasi-coherent Ideal \( \tilde{I}(\omega) \) of \( \mathcal{O}_x \) (\( \mathcal{O}_{\langle \omega \rangle} = \mathcal{O}_x/\tilde{I}(\omega) \)), then \( [\omega] = [\omega]_1 \) is the \( X \)-prescheme obtained by the blowing up centered at \( \langle \omega \rangle \).

4. Some results. We shall give here some relations among sheaves and preschemes defined in the above section.

Proposition 5. The underlying space of the closed sub-prescheme \( \langle \omega \rangle \) of \( X \) is the set of points of \( X \) at which the rational section \( \omega \) is not defined, i.e., \( X - \langle \omega \rangle \) is the domain of definition of \( \omega \).

Proof) Since the question is local, we may assume that \( X = \) 

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(1) We shall say that a rational map \( f: X \rightarrow Y \) is defined at \( x \in X \), if there exist an open nbd. \( U \) of \( x \) and a morphism \( f_U: U \rightarrow Y \) which represents \( f \), and the set of points of \( X \) at which \( f \) is defined is called the domain of definition of \( f \).
Spec($A$) is affine and that $\mathcal{E} = \tilde{E}$ is generated by its global section $E$ which is of finite type over $A$. Let $e_1, \ldots, e_n$ be a set of generators of $E$ over $A$ and put $\alpha_i = \omega_i^*(e_i) \in R(X)$ ($\omega_i^*: \mathcal{E} \to R(X)$). Let $x$ be a point of $X$ and $\mathfrak{p}$ the corresponding prime ideal of $A$; and write $\alpha_i = g \cdot a_i$ where $g \in R(X)$, $a_i \in A$ and $a_i$'s have no common divisors in $A_\mathfrak{p} = \mathcal{O}_{X, \mathfrak{p}}$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
E & \xrightarrow{\omega_i^*} & R(X) \\
\tau_i^* & \downarrow & \\
A & \xrightarrow{\text{multiplication by } g,} & \\
\end{array}
$$

where $\tau_i^*$ is the $A$-homomorphism defined by $\tau_i^*(e_i) = a_i$. It can be extended to the following commutative diagram of graded $A$-algebras:

$$
\begin{array}{ccc}
S_A(E) & \xrightarrow{\omega^*} & A[g \cdot T] \subset R(X)[T] \\
\tau^* & \downarrow & \\
A[T] & \xrightarrow{\mu^*} & \\
\end{array}
$$

and, passing to the associated projective fibres, we get the following:

$$
\begin{array}{ccc}
P(E) = \text{Proj}(S_A(E)) & \xleftarrow{\bar{\omega}} & \text{Proj}(A[g \cdot T]) \\
\tau & \downarrow & \\
\text{Proj}(A[T]) & \xleftarrow{\mu} & \\
\end{array}
$$

where $\bar{\omega}$, $\tau$ are rational maps. While $\text{Proj}(A[T])$ and $\text{Proj}(A[g \cdot T])$ can be canonically identified with $X$, and, by means of this identification, $\mu$ is the identity morphism of $X$ (EGA II. 3.1.7 and 3.1.8). Hence $\tau = \bar{\omega}$, in this sense. Now, since $I(\omega)_x = \mathcal{O}_X(\omega)_x = \sum a_i \mathcal{O}_X, \ldots$ (Cf. Proof of Cor. of Prop. 4), we see that

$$x \in \langle \omega \rangle \iff \tilde{I}(\omega)_x \neq \mathcal{O}_X \iff \text{all } a_i \text{'s in } \mathcal{O}_X \iff (\tau_i^*)^{-1}\mathcal{O}_X = E \iff \tau = \bar{\omega} \text{ is not defined at } x. \quad \text{Q. E. D.}$$

The prescheme structure of $\langle \omega \rangle$ (i.e., the sheaf $\mathcal{O}_{\langle \omega \rangle}$) may involve more detailed nature of the singular part of the rational section $\omega$ (or $\bar{\omega}$). The following two theorems will tell us some of these aspects.

**Theorem 1.** The Ideal $I(\omega) \cdot \mathcal{O}_{\langle \omega \rangle}$ of the closed sub-prescheme
$i^{-1}\pi^{-1}(o) = [o] \times _{x}[o]$ in $\mathcal{O}_{x}$ is isomorphic to the $\mathcal{O}_{x}$-Module $i^*(\mathcal{O}_{P(\mathcal{E})}(1) \otimes \mathcal{O}_{P(\mathcal{E})}^{\pi} \mathcal{O}_{x}(o)) = (\mathcal{O}_{P(\mathcal{E})}(1) \otimes \mathcal{O}_{x}(o)) | [o]$, i.e., we get an exact sequence of $\mathcal{O}_{P(\mathcal{E})}$-Modules

$$0 \rightarrow i^*(\mathcal{O}_{P(\mathcal{E})}(1) \otimes \mathcal{O}_{P(\mathcal{E})}^{\pi} \mathcal{O}_{x}(o)) \rightarrow \mathcal{O}_{[o]} \rightarrow i^*\pi^*\mathcal{O}_{<o>} \rightarrow 0.$$  

**Proof.** We have an exact sequence (EGA. II. 8. 1. 8)

$$0 \rightarrow \mathcal{O}_{[o]}(1) \rightarrow \mathcal{O}_{[o]} \rightarrow \mathcal{O}_{[o]} x_{<o>} \rightarrow 0.$$  

By this and the isomorphism (4), we get our assertion. Q. E. D.

**Proposition 6.** If $\mathcal{E}$ is locally free of rank 2, then $J_1(o)$ is an invertible $\mathcal{O}_x$-Module and $\overline{J(o)}$ (the Ideal of $[o]$ in $\mathcal{O}_{P(\mathcal{E})}$) is also an invertible $\mathcal{O}_{P(\mathcal{E})}$-Module.

**Proof.** At any point $x$ of $X$, we have an exact sequence

$$0 \rightarrow J_1(o) \rightarrow \mathcal{E} \rightarrow I(o) \rightarrow 0.$$  

Take a basis $(e_1, e_2)$ of $\mathcal{E}$ over $\mathcal{O}_{x}$, and put $\alpha_i = \omega_i^*(e_i)$ $(i = 1, 2)$; then, if we write $\alpha_i = g \cdot a_i$ as in the proof of Cor. of Prop. 4, we get the following commutative diagram

$$0 \rightarrow a_1 a_2 \mathcal{O} \rightarrow \mathcal{O}_x \otimes \mathcal{O}_x \rightarrow a_1 \mathcal{O} + a_2 \mathcal{O} \rightarrow 0$$

$$0 \rightarrow J_1(o) \rightarrow \mathcal{E} \rightarrow I(o) \rightarrow 0,$$

where $\lambda(a_1 a_2 c) = (a_1 c, -a_1 c), \mu(c, d) = a_1 c + a_1 d, f(c, d) = c \cdot e_1 + d \cdot e_2$ and $h(a_1 c + a_2 d) = g \cdot (a_1 c + a_2 d)$. And it is easy to see that the upper horizontal sequence is exact, hence $J_1(o) = a_1 a_2 \mathcal{O} \oplus \mathcal{O}_x$, i.e., $J_1(o)$ is invertible. Moreover, since $J(o) = J_1(o) \cdot S(\mathcal{E}), \overline{J(o)} = J_1(o) \mathcal{O}_{P(\mathcal{E})}$. This proves that $\overline{J(o)}$ is invertible. Q. E. D.

**Remark.** When $\mathcal{E}$ is locally free of rank 2, by the above proposition, we may regard $[o]$ as a Cartier divisor on $P(\mathcal{E})$, and $\overline{J(o)} = \mathcal{O}_{P(\mathcal{E})}(-[o])$, the invertible $\mathcal{O}_{P(\mathcal{E})}$-Module corresponding to the Cartier divisor $- [o]$.

**Theorem 2.** When $\mathcal{E}$ is locally free of rank 2,

$$J_1(o) \otimes \mathcal{O}_x A \mathcal{E} \equiv \mathcal{O}_x(o) \quad (o \neq 0).$$
Proof) Take an open covering \((U_a)_{a \in I}\) of \(X\) such that \(E|U_a \xrightarrow{\varphi_a} \mathcal{O}_X|U_a\) for each \(a \in I\). Let \(t^a = (t_{a1}^a, t_{a2}^a)\) be the basis of \(E|U_a\) over \(\mathcal{O}_X|U_a\), determined by \(\varphi_a\), and \(\tau^a = (\tau_{a1}^a, \tau_{a2}^a)\) the dual basis of \(\mathcal{E}|U_a\) of \(t^a\), then \(\tau_{a1}^a \land \tau_{a2}^a\) is a basis of \(\land^2 \mathcal{E}|U_a\). When that is so, the homomorphism \(\omega_i^*: \mathcal{E} \to \mathcal{R}(X)\) can be expressed, locally on \(U_a\), as

\[ \omega_i^* = A_{1i}^a \cdot \tau_{1i}^a + A_{2i}^a \cdot \tau_{2i}^a, \quad A_{1i}^a, A_{2i}^a \in \Gamma(U_a, \mathcal{R}(X)) = \mathcal{R}(X). \]

Note that, \(\omega \neq 0\) (i.e., \(\omega_i^* \neq 0\)) implies \(A_i^a \neq 0\) for \(i = 1\) or \(2\), and that, for \(e = \Sigma b_i^a \cdot t_i^a \in \Gamma(U_a, \mathcal{E})\), \(e \in \Gamma(U_a, J_i(\omega))\) if and only if \(\Sigma A_i^a \cdot b_i^a = 0\).

Consider the map

\[ (J_i(\omega) \otimes \mathcal{O}_X \land^2 \mathcal{E})|U_a \to \mathcal{R}(X)|U_a \]

given by the correspondence

\[ (b_{1i}^a \cdot t_{1i}^a + b_{2i}^a \cdot t_{2i}^a) \otimes c^a \tau_{1i}^a \land \tau_{2i}^a \mapsto -b_{1i}^a c^a / A_{1i}^a = -b_{2i}^a c^a / A_{2i}^a = k^a. \]

At any point \(x\) of \(U_a\), let \(A_i^a = g \cdot a_i, g \in \mathcal{R}(X)\), \(a_i \in \mathcal{O}_X\), such that \(a_1\) and \(a_2\) are relatively prime to each other in \(\mathcal{O}_X\). Then \(a_1, b_1 + a_2, b_2 = 0\), hence \(b_1 / a_2 = -b_2 / a_1\) is in \(\mathcal{O}_X\). Therefore \(k^a\) is an element of \((1/g) \cdot \mathcal{O}_X = \mathcal{O}_X(\omega)\), (cf. the proof of Cor. of Prop. 4). This means that the above map induces an \((\mathcal{O}_X|U_a)\)-homomorphism

\[ \phi_a : (J_i(\omega) \otimes \mathcal{O}_X \land^2 \mathcal{E})|U_a \to \mathcal{O}_X(\omega)|U_a, \]

and it is easy to see that this is an isomorphism. If \(G^{a\beta} = (G^{a\beta}_{ij})\) are the transition matrices of \(E\) (with respect to \(\varphi_a\)), then \((G^{a\beta})^{-1} = G^{\beta a}\) and \(\det(G^{a\beta})^{-1} = \det(G^{\beta a})\) are the transition matrices and functions of \(\mathcal{E}\) and \(\land^2 \mathcal{E}\), respectively. Hence,

\[ c^a = \det(G^{\beta a}) \cdot c^\beta, \quad b_{1i}^a = G_{11}^{\beta a} \cdot b_1 + G_{12}^{\beta a} \cdot b_2, \]

and

\[ A_{1i}^a = \det(G^{\beta a}) \cdot (-G_{11}^{\beta a} \cdot A_1^\beta + G_{12}^{\beta a} \cdot A_2^\beta), \]

therefore, by easy calculation, we get the identity \(k^a = k^\beta\). This shows that the \(\phi_a\)'s can be patched together and give a global isomorphism

\[ \phi : J_i(\omega) \otimes \land^2 \mathcal{E} \cong \mathcal{O}_X(\omega). \quad \text{Q. E. D.} \]

Corollary. Under the same assumptions in Th. 2,

\[ \mathcal{O} P(\mathcal{E})([\omega]) \equiv \mathcal{O} P(\mathcal{E})(1) \otimes \mathcal{O}_X \land^2 \mathcal{E} \otimes \mathcal{O}_X(\omega)^{-1} \]
**Proof**) Since \( J(\omega) = J_1(\omega) \cdot S(\mathcal{E}) \equiv J_1(\omega) \otimes _\mathcal{O} S(\mathcal{E})(-1) \), we get, by Th. 2, an isomorphism
\[
J(\omega) \otimes \mathcal{O}_x / \mathcal{E} \equiv S(\mathcal{E})(-1) \otimes \mathcal{O}_x \mathcal{O}_x(\omega).
\]
Hence, passing to the associated sheaves on \( P(\mathcal{E}) \), we get an isomorphism
\[
\overline{J(\omega)} \otimes \mathcal{O}_x / \mathcal{E} \equiv \mathcal{O}_P(\mathcal{E})(-1) \otimes \mathcal{O}_x \mathcal{O}_x(\omega).
\]
On the other hand \( \overline{J(\omega)} = \mathcal{O}_P(\mathcal{E})([\omega])^{-1} \), therefore, combining these two isomorphisms, we get our assertion. \( \quad \) Q. E. D.

5. **The case of algebraic schemes.** Let \( X \) be an algebraic scheme over an algebraically closed field \( k \). We denote, for each non-negative integer \( p \), by \( X^p \) the set of points \( x \) of \( X \) such that \( \text{codim}_x \mathcal{E} = p \), and by \( Z^*(X) \) the free abelian group generated by the irreducible closed subsets \( \{ x \} \) of \( X \), where \( x \) are in \( X^p \), and we shall say each element of \( Z^*(x) \) a cycle on \( X \) of codimension \( p \).

Let \( C^*(X) \) (\( p \geq 0 \)) be the abelian category of coherent \( \mathcal{O}_X \)-Modules whose supports are of codimension \( \geq p \), and
\[
\tau_s : C^*(X) \rightarrow K^*(X)
\]
the universal solution in the category of abelian groups satisfying the following axiom (i.e., the Grothendieck group of \( C^*(X) \)):

**Additivity** If \( 0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0 \) is exact in \( C^*(X) \), then \( \tau_s (\mathcal{T}) = \tau_s (\mathcal{T}') + \tau_s (\mathcal{T}'') \).

The immersion \( C^*(X) \rightarrow C^*(X) \) (for \( p \geq q \)) determines a canonical homomorphism \( K^q(X) \rightarrow K^s(X) \). By means of this homomorphism, we shall consider that every element of \( K^s(X) \) lies on \( K^s(X) \), especially on \( K^0(X) = K(X) \). Defining the product by
\[
\tau(\mathcal{T}) \cdot \tau(\mathcal{G}) = \Sigma_{p \geq 0} (-1)^p \tau(\mathcal{T} \otimes \mathcal{O}_p(\mathcal{E}, \mathcal{G})) \), \( \mathcal{T}, \mathcal{G} \in \text{Ob} C^0(X), \)
\( K^0(X) = K(X) \) has a ring structure (cf. Borel-Serre[1]).

For any \( \mathcal{T} \in \text{Ob} C^*(X) \), put
Rational sections and Chern classes of vector bundles

$$z_\nu(\mathcal{F}) = \Sigma_{i \in X} \text{length}_{\mathcal{O}_X} (\mathcal{F}_i) \cdot [x] \in Z^*(X),$$

and call it the cycle of codimension \( p \) associated to \( \mathcal{F} \) (cf. Serre [8]). Since the map \( z_\nu: C^*(X) \to Z^*(X) \) is clearly additive, it defines a group homomorphism \( z_\nu: K^*(X) \to Z^*(X) \) such that \( z_\nu(\gamma, (\mathcal{F})) = z_\nu(\mathcal{F}) \). We denote, for any closed subscheme \( Y \) of \( X \), \( z_\nu(\mathcal{O}_Y) = Y_\nu = Y \), i.e., the underlying space of \( Y \) with multiplicity 1. Moreover, if \( X \) is regular (i.e., nonsingular), the Cartier divisors on \( X \) are identified to the elements of \( Z^1(X) \) (i.e., the Weil divisors), hence we have a bijective canonical correspondence between \( Z^1(X) \) and the set of invertible sub-\( \mathcal{O}_X \)-Modules of \( \mathcal{R}(X) = (D \mapsto \mathcal{O}_X(D)) \), and it is easy to see that, for any positive divisor \( D \in Z^1(X) \), \( z_\nu(\mathcal{O}_D) = D_\nu = D \), where \( \mathcal{O}_D = \mathcal{O}_X(\mathcal{O}_X(-D)) \) (Cf. Mumford, [7]). The following theorem has been proved by Serre which is very useful for our study.

**Serre's Intersection Theory** (Serre [8], Prop. 1 of V, c.). Assume the algebraic scheme \( X \) to be regular. For elements \( \xi \in K^*(X) \) and \( \eta \in K^*(X) \) such that \( \xi, \eta \in K^{*+}(X) \), the cycles \( z_\nu(\xi) \) and \( z_\nu(\eta) \) intersect properly to each other and

$$z_{\nu+1}(\xi \cdot \eta) = X_\nu(\xi) \cdot z_\nu(\eta)$$

(the intersection product in usual sense).

**Lemma 1.** Assume \( X \) to be regular. For closed subscheme \( Y \) of \( X \) and any divisor \( D \) on \( X \), if we have an exact sequence of coherent \( \mathcal{O}_X \)-Modules

$$0 \to \mathcal{O}_X(-D) \otimes \mathcal{O}_Y \to \mathcal{O}_Y \to \mathcal{G} \to 0$$

then there exists a divisor \( D' \in Z^1(X) \), linearly equivalent to \( D \), such that the intersection product \( D' \cdot Y \), is defined and, for any \( p \leq \text{codim}_X Y \),

$$z_{p+1}(\mathcal{G}) = D' \cdot Y_\nu.$$

**Proof.** Take a \( D' \in Z^1(X) \) which is linearly equivalent to \( D \)
and intersects properly with $\text{Supp}(Y)$. Let $D' = E_1 - E_2$, $E_1 > 0$ and they have no common components. Then, since exact sequences

$$0 \to \mathcal{O}_X(-E_i) \to \mathcal{O}_X \to \mathcal{O}_{E_i} \to 0 \quad (i=1, 2)$$

are locally free resolution of $\mathcal{O}_{E_i}$, we get

$$0 \to \mathcal{O}_X\mathcal{O}_Y(\mathcal{O}_{E_i}, \mathcal{O}_Y) \to \mathcal{O}_X(-E_i) \otimes \mathcal{O}_Y(-E_1) \otimes \mathcal{O}_Y(-E_2) \to 0 \quad \text{(exact)}$$

and $\mathcal{O}_Y\mathcal{O}_X(\mathcal{O}_{E_i}, \mathcal{O}_Y) = 0$, if $p \geq 2$.

Since $E_i$ intersect properly with $\text{Supp}(Y)$ and $\text{Supp}(\mathcal{O}_{E_i_1} \mathcal{O}_Y) \subseteq \text{Supp}(\mathcal{O}_{E_i}) \cap \text{Supp}(\mathcal{O}_Y)$, we have codim. $\text{Supp}(\mathcal{O}_{E_i_1} \mathcal{O}_Y) = 1$. Hence, by Serre’s intersection theory, the intersection product

$$z_i(\mathcal{O}_{E_i}) \cdot z_p(\mathcal{O}_Y) = E_i \cdot Y_p$$

is defined and is equal to

$$z_{p+1}(\mathcal{O}_{E_i} \mathcal{O}_Y) - z_{p+1}(\mathcal{O}_X(-E_i) \mathcal{O}_Y).$$

Therefore

$$D' \cdot Y_p = E_1 \cdot Y_p - E_2 \cdot Y_p = z_{p+1}(\mathcal{O}_X(-E_1) \mathcal{O}_Y)$$

while $\mathcal{O}_X(-D) = \mathcal{O}_X(-D') = \mathcal{O}_X(-E_1) \mathcal{O}_X(E_2)$, hence, by tensoring $\mathcal{O}_X(-E_2)$ to the exact sequence (6), we get an exact sequence

$$0 \to \mathcal{O}_X(-E_1) \mathcal{O}_Y \to \mathcal{O}_X(-E_1) \mathcal{O}_Y \mathcal{O}_X(-E_2) \mathcal{O}_Y \to \mathcal{O}_X(-E_2) \mathcal{O}_Y \mathcal{O}_Y \to 0,$$ 

and, taking $z_{p+1}$,

$$z_{p+1}(\mathcal{O}_X(-E_2) \mathcal{O}_Y) - z_{p+1}(\mathcal{O}_X(-E_1) \mathcal{O}_Y)$$

$$= z_{p+1}(\mathcal{O}_X(-E_2) \mathcal{O}_Y) = z_{p+1}(\mathcal{O}_Y).$$

Thus we get the proof. Q. E. D.

Now we shall apply this result to Th. 1.

**Theorem 1'.** Let $X$ be a regular algebraic scheme, $\mathcal{E}$ a locally free $\mathcal{O}_X$-Module of rank $p+1$ and $H$ a divisor on $P(\mathcal{E})$ such that $\mathcal{O}_P(\mathcal{E})(H) = \mathcal{O}_P(\mathcal{E})(1)$. Then, for any non-zero rational section $\omega \in \Gamma_{\text{rat}}(V(\mathcal{E})/X)$, there exists a divisor $D$ on $P(\mathcal{E})$ such that it is linearly equivalent to $H + \pi^{-1}(\omega)$ and that the intersection product $D \cdot z_p(\mathcal{O}_\omega) = D \cdot [\omega]$ is defined and is equal to
Rational sections and Chern classes of vector bundles

\(-z_{p+1}(O,-i^{-1}\pi^{-1}\langle \omega \rangle) = -(i^{-1}\pi^{-1}\langle \omega \rangle)_{p+1}, \) \text{i.e., in the Chow ring} 
\(A(P(\mathcal{E})) \) of \(P(\mathcal{E})\) (if \(X\) is quasi-projective, cf. [2]), 
\((i^{-1}\pi^{-1}\langle \omega \rangle)_{p+1} = (-H-\pi^*(\omega)) \cdot [\omega].\)

Moreover, if \(\mathcal{E}\) is of rank 2,
\((\pi^{-1}\langle \omega \rangle)_{2} = -D \cdot [\omega], \) \text{i.e.,} \((\pi^{-1}\langle \omega \rangle)_{2} = (-H-\pi^{-1}(\omega)) \cdot [\omega] \) in \(A(P(\mathcal{E})).\)

**Proof:** Note that, \(P(\mathcal{E})\) is also a regular algebraic scheme and that the projection \(\pi: P(\mathcal{E}) \to X\) is flat; then it is easy to see that \(z_0(\pi^*O_x(\omega)) = \pi^{-1}(\omega).\) Then the first part is straightly obtained applying Lemma 1 to the exact sequence (5). The second part is an immediate consequence of the following lemma.

**Lemma 2.** Under the same assumption in Th. 1', if \(\mathcal{E}\) is of rank 2,

(i) \(\text{codim} \text{ Supp } (O_{<\omega}) \geq 2, \) and (ii) \(i^*\pi^*O_{<\omega} = \pi^*O_{<\omega}.\)

**Proof** (i) For any point \(x\) of \(X, \overline{I}(\omega),\) is generated by relatively prime two elements of \(O_x,\) hence \(\dim O_{<\omega} \leq \dim(O_x/\overline{I}(\omega)) \leq \dim O_x - 2.\) This proves (i).

(ii) Since \(i^*\pi^*O_{<\omega} = O_{P(\mathcal{E})}/\overline{I}(\omega) \cdot O_{P(\mathcal{E})} \otimes O_{\omega},\) in order to get our assertion, it is sufficient to prove that

\(J(\omega)(S(\mathcal{E})/\overline{I}(\omega) \cdot S(\mathcal{E})) = (J(\omega) + \overline{I}(\omega) \cdot S(\mathcal{E}))/\overline{I}(\omega) \cdot S(\mathcal{E}) = 0, \)
\text{i.e.,} \(J(\omega) \subset \overline{I}(\omega) \cdot S(\mathcal{E}).\)

At any point \(x\) of \(X,\) any element \(e \in J_1(\omega),\) is expressed as

\(e = b_1t_1 + b_2t_2, b_i \in O_x,\) such that \(b_1a_1 + b_2a_2 = 0\)
(with the notations used in the proof of Th. 2). Since \(\overline{I}(\omega) = a_1O_x + a_2O_x,\) and \(a_1\) and \(a_2\) are relatively prime to each other,

\(eS_{n-1}(\mathcal{E}) \subset b_1S_n(\mathcal{E}) + b_2S_n(\mathcal{E}) \subset a_1S_n(\mathcal{E}) + a_2S_n(\mathcal{E}) = \overline{I}(\omega)S_n(\mathcal{E}).\)

This proves \(J_1(\omega) \cdot S_{n-1}(\mathcal{E}) \subset \overline{I}(\omega)S_n(\mathcal{E}),\) i.e., \(I(\omega) = J_1(\omega)S(\mathcal{E}) \subset \overline{I}(\omega)S(\mathcal{E}).\)

Q. E. D.
In the algebraic scheme case, Cor. of Th. 2 also can be translated as follows.

**Theorem 2'.** Under the same assumptions in Th. 1', if $\mathcal{E}$ is of rank 2, the divisor $[\omega]$ is linearly equivalent to the divisor $H + \pi^{-1}K - \pi^{-1}(\omega)$, where $K$ is a divisor on $X$ such that $\mathcal{O}_X(K) \cong \mathcal{O}_X^2$.

**Corollary.** Under the same assumptions in Th. 2', if $X$ is quasi-projective, for any locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank 2, the first Chern class $c_1(\mathcal{E})$ of $\mathcal{E}$ is equal to $\langle \omega \rangle - c_1(\mathcal{E}) \cdot (\omega) - (\omega)^3$, where $\omega$ is non-zero rational section of the vector fibre $V(\mathcal{E})/X$ (The Chern classes are in the sense of Grothendieck, cf. [4], [5]).

**Proof.** Combine the results of Th. 1' and 2', we get an equality in the Chow-ring of $\mathcal{P}(\mathcal{E})$

$$H^2 + \pi^*K \cdot H + \pi^*(\langle \omega \rangle + (\omega) \cdot K - (\omega)^3) = 0.$$

This identity shows, by the definition (cf. [4], [5]), that

$$c_1(\mathcal{E}) = -c_1(\mathcal{E}) = -K = -\text{cl}_X(\wedge^2 \mathcal{E}) = \text{cl}_X(\wedge^2 \mathcal{E}),$$

and

$$c_1(\mathcal{E}) = c_1(\mathcal{E}) = \langle \omega \rangle + (\omega) \cdot K - (\omega)^3$$

$$= \langle \omega \rangle - (\omega) \cdot c_1(\mathcal{E}) - (\omega)^3.$$

**Q. E. D.**

**Remark.** We shall now apply the result to the case of surfaces. Let $X = F$ be a non-singular projective surface and $\mathcal{E} = \mathcal{F} = \mathcal{A} = \mathcal{A}_F \otimes \mathcal{O}_F$ the tangential sheaf on $F$. Then, for any linear differential form $\omega$ on $F$ (i.e., an element of $\Gamma(F, \mathcal{O}_F \otimes \mathcal{R}(F))$), we can express it, at any point $x$ of $F$, as $\omega = h(f \cdot dt_1 + g \cdot dt_2)$ ($t$'s are local parameters at $x$) where $h, f$ and $g$ are rational functions on $F$ such that $f$ and $g$ are regular at $x$ and are relatively prime in $\mathcal{O}_{F,x}$. Denote by $m_x$ the intersection multiplicity of the divisors $(f)$ and $(g)$ at $x$, and put $\langle \omega \rangle = \sum_m m_x \cdot x$; then the 0-cycle $\langle \omega \rangle$ is just the same thing of ours. And the second Chern class

$$c_2(\mathcal{F}) = c_2(F) = \langle \omega \rangle + (\omega) \cdot K - (\omega)^3.$$
Rational sections and Chern classes of vector bundles 311

\( (K = \text{cl}(\wedge^2 T_F) = \text{cl}(\mathcal{O}_F^2) = \text{the canonical divisor class on } F) \) is called the Severi-series which has been defined by F. Severi in [9], and used by J. Igusa, in [6], in order to prove the inequality \( B_2 \geq \rho \) where \( B_2 \) is the second Betti number of the surface \( F \) and \( \rho \) is the Picard number of \( F \).

Appendix

Let \( V \to \text{Spec}(k) \) be a non-singular projective algebraic variety of dimension \( n \) and \( \mathcal{O}_V = \wedge^i \mathcal{O}_V \) the sheaf of germs of holomorphic \( p \)-forms on \( V \). Then we get

\[
c_i(V) = \sum_{i=1}^{\chi} (-1)^i h^{p,q}, \quad h^{p,q} = \dim H^i(V, \mathcal{O}_V).
\]

In fact, let

\[
c_i(V) = \sum_{i=1}^{\chi} c_i t^i = \sum_{i=0}^{\chi} (-1)^i c_i(\mathcal{O}_V) t^i = \Pi^i(1 + \alpha, t)
\]

be the Chern polynomial of \( V \). Then we have

\[
c_i(\mathcal{O}_V) = \sum_{i=0}^{\chi} c_i(\mathcal{O}_V) t^i = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} (1 - (\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_p}))
\]

(cf. [5]). Hence

\[
\text{ch}(\mathcal{O}_V) = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \exp(-\alpha_{i_1} - \alpha_{i_2} - \cdots - \alpha_{i_p}).
\]

Applying this result to the theorem of Riemann-roch ([1]) we get

\[
x(V, \mathcal{O}_V) = \pi_*(\text{ch}(\mathcal{O}_V) \cdot T(V))
\]

\[
= \pi_*(\sum \exp(-\alpha_{i_1} - \alpha_{i_2} - \cdots - \alpha_{i_p}) \cdot \Pi(\alpha_i/1 - \exp(-\alpha_i)))
\]

(put) \( T^*(c_1, c_2, \cdots, c_n) \).

Therefore, the polynomial

\[
\sum_{i=0}^{\chi} x(V, \mathcal{O}_V) y^i = \sum_{i=0}^{\chi} T^*(c_1, \cdots, c_n) y^i (= T^*(c_1, \cdots, c_n))
\]

is the \( n \)-th term of the \( "m-Folge" \) belonging to the power series

\[
Q(y, x) = x(y + 1)/(1 - \exp(-x(y + 1))
\]

2) Igusa defined \( B_2 \) by the classical fact \( \Sigma(1)^i B_1 - c_2(F) \). On the other hand we can show \( c_2(F) = 2(-1)^{p+q} \mathcal{O}_F \) by means of the Riemann-Roch theorem of Grothendieck ([1]) (see Appendix).
Hiroshi Yamada

(cf. [10] p. 16, note that $\pi_* (\cdot) = \kappa [\cdot]$).

This proves that

$$c_* = \sum_{i=0}^r (-1)^i T^i_{\ast} (c_1, \cdots, c_r) = \sum (-1)^i \chi (V, \Theta_x^i)$$

$$= \sum_{j, \epsilon = 0} (-1)^j \epsilon h^{j, \epsilon}.$$

(cf. ibid. the formula (16) of Chap. 1, sect. 8, p. 17).

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