J. Math. Kyoto Univ.
7-1 (1967) 71-75

On complete homogeneous surfaces

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(Communicated by Prof. M. Nagata, April 24, 1967)

It is known that a complete nonsingular curve C, which is a homogeneous space for a connected algebraic group, is birationally isomorphic to either an abelian variety of dimension 1 or the projective line $\boldsymbol{P}^1$.

The purpose of this paper is to prove a similar result for the two-dimensional case. That is, we shall give the proof of the following

Theorem. *Let F be a complete nonsingular surface, which is a homogeneous space for a connected algebraic group G. Then F is birationally isomorphic to one of the following:*

1) *an abelian variety A of dimension 2,*

2) *a bijective rational image*[1] *of the direct product $A \times \boldsymbol{P}^1$ of an abelian variety A of dimension 1 and the projective line $\boldsymbol{P}^1$,*

3) *the projective space $\boldsymbol{P}^2$ of dimension 2,*

4) *the two-fold direct product $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ of the projective line $\boldsymbol{P}^1$.*

Of course, if the characteristic of the universal domain is 0, then 2) is same to

2') *the direct product $A \times \boldsymbol{P}^1$.*

We note that an algebraic homogeneous space can be embedded in some projective space (cf. [2]). Hence, in order to prove the theorem, we may assume that the complete homogeneous surface F is contained in a projective space.

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1) This means that F is (birationally isomorphic to) the image of $A \times \boldsymbol{P}$ by a bijective regular rational mapping.

1. First, we shall show that *a projective homogeneous surface F is a relatively minimal model* (cf. [8]).

We remark that if a projective nonsingular surface V is not a relatively minimal model then there exists an *irreducible* nonsingular exceptional curve of the first kind on V (cf. [8]). In fact, for such a surface V, there exist a relatively minimal model V_0 and an antiregular birational mapping of V_0 to V. Then, by the factorization theorem for antiregular birational transformations (cf. [8]), we have a sequence of quadratic transformations $V_0 \xrightarrow{\sigma_0} V_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-1}} V_n$, where V_i is a projective surface and V_n is birationally isomorphic to V. By the assumption, we have $n \geqq 1$ and so, for the center P_{n-1}' of σ_{n-1}, we see that $\sigma_{n-1}\{P_{n-1}\}$ is an irreducible nonsingular exceptional curve of the first kind on $V_n = V$.

Now let F be a projective homogeneous surface for a connected algebraic group G and (A, α) the Albanese variety of F. Then we have the inequality $0 \leqq q = \dim A \leqq \dim F = 2$ (cf. [3]).

(a) The case $q=2$. Then we have $F=A$, i.e. F is an abelian variety (cf. [3]). So, in this case, F is a minimal (and consequently a relatively minimal) model (cf. [7]).

(b) The case $q=1$. If, in this case, F is not a relatively minimal model, there exists an irreducible nonsingular exceptional curve E of the first kind on F. Then, as E is a rational curve, it is contained in one of the α-fibres. On the other hand, each α-fibre is a homogeneous space, of dimension $(\dim F) - q = 1$, for a connected linear algebraic group (cf. [3]) and so, of course, is an irreducible subvariety of dimension 1 on F. Hence we have $E = \alpha^{-1}(a)$ for some point a on A. Then the self-intersection number (E^2) of E must be equal to 0, which contradicts to the fact $(E^2) = -1$ (cf. [8]).

(c) The case $q=0$. Then F may be considered as a homogeneous space for a connected linear algebraic group L (cf. [3]). If F is not a relatively minimal model, there exists an irreducible nonsingular exceptional curve E of the first kind on F. Let k be an algebraically closed field of definition for F, L, E and the operation of L on F. Since L is linear and connected, we see that

$l(E)$ is linearly equivalent to E for any rational point l of L over k (cf. [5]). Hence we have $(l(E), E)=(E^2)=-1$. On the other hand, as E and $l(E)$ are irreducible, we see that if $l(E)\neq E$ then $(l(E), E)\geqq 0$. So we have $l(E)=E$ for any rational point l of L over k, i.e. $L_k E=E$ where L_k is the set of all the rational points of L over k. Let P_0 be a rational point of E over k. Then the mapping $\varphi: l\to lP_0$ of L to F is a surjective regular rational mapping defined over k. Since the set L_k is everywhere dense in L, we have, for any open subset $O\neq\phi$ of F, $\varphi^{-1}(O)\cap L_k\neq\phi$. Then, taking a point l_0 in $\varphi^{-1}(O)\cap L_k$, we see that $\varphi(l_0)=l_0P_0$ is in O and so L_kP_0 is everywhere dense in F. However, as L_kP_0 $(\subset L_kE)$ is contained in the proper closed subset E of F, we have a contradiction.

Therefore, in any cases, F is a relatively minimal model.

2. Next, we shall prove the following

Proposition.[2)] *Let V be a complete homogeneous space for a connected algebraic group G. Then there exist an abelian variety A and a connected linear algebraic group L such that V is the image of the product $A\times(L/H)$ by a bijective regular rational mapping, where H is a connected algebraic subgroup of L.* (Clearly we have $\dim A=$ the irregularity of V and L/H is a rational variety.) *In particular, if the characteristic of the universal domain is 0, V is birationally isomorphic to $A\times(L/H)$.*

Proof. We may assume that G operates effectively on V. Then G is generated by an abelian subvariety A and the maximal connected linear normal algebraic subgroup L of $G: G=A\cdot L$ (cf. [3]). Moreover the isotropy group of any point on V in G is connected and linear and so is contained in L (cf. [4]). Any element g in the intersection $A\cap L$ belongs to L and so has a fixed point P on the complete variety V (cf. [1]). On the other hand, as the operation of G on V is effective, the isotropy group of P in G has no common element other than the identity e with

2) Cf. A. Borel und R. Remmert, Über kompakte homogene Kählersche Mannigfaltigkeiten, Math. Ann. 145 (1962), 429-439.

the central subgroup A of G. Hence we have $A \cap L = \{e\}$ and so the canonical rational mapping π of $A \times L$ to $G = A \cdot L$ is bijective. Let P_0 be a point on V and H the isotropy group of P_0 in G which is connected and is contained in L. By means of π, $A \times L$ operates on V transitively and the isotropy group of P_0 in $A \times L$ is clearly $\{e\} \times H$. Then the rational mapping φ of $A \times L$ to V defined by $\varphi(a, l) = alP_0$ induces a bijective regular rational mapping of $A \times (L/H)$ onto V.

3. *Proof of Theorem.* If the irregularity q of F is 2, then F is (birationally isomorphic to) an abelian variety. If $q=1$, then, by Proposition, F is the image of $A \times (L/H)$ by a bijective regular rational mapping, where A is an abelian variety of dimension 1 and L is a connected linear algebraic group with an algebraic subgroup H. Then it is clear that L/H is an irreducible nonsingular rational projective curve and so is birationally isomorphic to the projective line $\boldsymbol{P}^1$. Now we consider the case $q=0$. It is known that a relatively minimal model of nonsingular rational projective surfaces is birationally isomorphic to one of 1) $\boldsymbol{P}^2$, 2) $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ and 3) F_n $(n=2, 3, \cdots)$ (cf. [6]). Clearly $\boldsymbol{P}^2$ and $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ are algebraic homogeneous spaces. On the other hand, on the surface F_n $(n=2, 3, \cdots)$, there exists an irreducible curve B_n such that $(B_n^2) = -n < 0$ (cf. [6]). Then, if F_n is a homogeneous space for a connected algebraic group and consequently for a connected linear algebraic group L, we see that $l(B_n)$ is linearly equivalent to B_n and so $(l(B_n), B_n) = (B_n^2) = -n < 0$ for any rational point l of L over k, where k is an algebraically closed field of definition for F_n, L, B_n and the operation of L on F_n. So, by a similar argument as in the case (c) in **1**, we have a contradiction.

Hence the proof of Theorem is completed.

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