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Derivations in Azumaya algebras

By

A. Roy and R. Sridharan

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§ 1. Introduction

Let A be an Azumaya algebra, i. e. a central separable algebra, over a commutative ring R. A theorem of Jacobson-Hochschild states that if R is a field, then any derivation of R can be extended to a derivation of A ([6], Theorem 2); the proof consists in reducing the problem to crossed products and using a cohomological argument. We prove here that the theorem is valid, more generally, for any semi-local ring R; even for the case of a field is "functorial". The method of proof is as follows: in § 2 we show that for any commutative ring R, the canonical homomorphism $R[X]/(X^2) \to R$ induces an isomorphism of the corresponding Brauer groups; this comes out as a corollary to Theorem 2. 1, which seems to be of independent interest. In § 3 we prove the Skolem-Noether theorem over semi-local ring (Theorem 3. 1) and deduce a "cancellation law" for Azumaya algebras over such rings. We use these facts in § 4 to prove the main theorem.

For standard concepts and results regarding Azumaya algebras and Brauer groups over commutative rings, we refer to Auslander-Goldman [3] and Bass [4].

In what follows, R will denote a commutative noetherian ring and $\otimes$ will stand for $\otimes_R$.

§ 2. A theorem on Brauer groups

Theorem 2. 1. *Let R be a commutative noetherian ring and let $\mathfrak{A}$ be an ideal of R contained in the radical such that R is complete* *with the $\mathfrak{A}$-adic topology. Then the canonical mapping $R \to R/\mathfrak{A}$ induces a monomorphism $Br(R) \to Br(R/\mathfrak{A})$.*

To prove the theorem we need a few lemmas.

Lemma 2.2. *Let R and $\mathfrak{A}$ be as in theorem 2.1, and let A be an R-algebra which is of finite type as an R-module. Then*

(i) *any idempotent of $A/\mathfrak{A}A$ can be lifted to an idempotent of A;*

(ii) *if $\bar{P}$ is a finitely generated projective module over $A/\mathfrak{A}A$, then there exists a finitely generated projective A-module P such that $\bar{P} \approx A/\mathfrak{A}A \otimes_A P$.*

Proof. (i) The classical method of lifting idempotents works in our case, since A is complete under the $\mathfrak{A}$-adic topology ([1], Lemma 9.8E).

(ii) There exists a finitely generated free $A/\mathfrak{A}A$-module $\bar{F}$ and an idempotent endomorphism $\bar{e}$ of $\bar{F}$ such that $\bar{P}$ is isomorphic to the cokernel of $\bar{e}$. Let F be a finitely generated free A-module such that $\bar{F} = A/\mathfrak{A}A \otimes_A F$. We have $End_{A/\mathfrak{A}A} \bar{F} = (End_A F)/(\mathfrak{A}\, End_A F)$, and $End_A F$ is finitely generated as an R-module. Hence by (i), there exists an idempotent endomorphism e of F which "lifts" $\bar{e}$. The cokernel of e is a projective A-module, say P, and obviously $\bar{P} \approx A/\mathfrak{A}A \otimes_A P$.

Lemma 2.3. *Let R be a commutative ring and let $\mathfrak{A}$ be an ideal of R contained in the radical. If P is a finitely generated projective R-module such that $P/\mathfrak{A}P$ is faithful over $R/\mathfrak{A}$, then P is faithful.*

Proof. Clearly ann $P \subset \mathfrak{A} \subset \operatorname{rad} R$. For any maximal ideal $\mathfrak{m}$ of R, ann $P_{\mathfrak{m}} = (\operatorname{ann} P)_{\mathfrak{m}} \subset \mathfrak{m}A_{\mathfrak{m}} \neq A_{\mathfrak{m}}$. Thus $P_{\mathfrak{m}}$ is a non-zero free $A_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of R, whence $(\operatorname{ann} P)_{\mathfrak{m}} = 0$ for all $\mathfrak{m}$, i.e. ann $P = 0$.

Lemma 2.4. *Let A be an Azumaya algebra over a commutative ring R. If there exists an A-module P which is finitely generated, faithful and projective as an R-module and $rk_R A = (rk_R P)^2$, then A is isomorphic to $End_R P$.*

(For a projective module P over R, $rk_R A$ denotes the function

Spec $R\to \boldsymbol{Z}$ defined by $(rk_R A)(\mathfrak{P})=$ the rank of the free $R_{\mathfrak{P}}$-module $P_{\mathfrak{P}}$).

Proof. We have a homomorphism

$$\varphi : A\to End_R P$$

of R-algebras and both of these algebras are Azumaya over R. Therefore φ is a monomorphism and $End_R P=\varphi(A)\otimes C$, where C is the commutant of $\varphi(A)$ in $End_R P$. Now $rk_R(End_R P)=(rk_R P)^2 =rk_R A$, so that $rk_R C=1$. But an Azumaya R-algebra of rank 1 is R itself. Thus $C=R$ and φ is an isomorphism.

Proof of Theorem 2.1. Let A be an Azumaya R-algebra such that $A/\mathfrak{A}A$ represents the trivial element in $Br(R/\mathfrak{A})$. Then there exists a finitely generated faithful projective $R/\mathfrak{A}$-module $\bar{P}$ such that $A/\mathfrak{A}A\approx End_{R/\mathfrak{A}}\bar{P}$. By Proposition A. 3 of [2], $\bar{P}$ is a finitely generated projective $A/\mathfrak{A}A$-module. Let P be a finitely generated projective A-module such that $\bar{P}\approx A/\mathfrak{A}A\otimes_A P$ (Lemma 2.2 (ii)). As an R-module, P is finitely generated, projective and faithful (Lemma 2.3). Also, since $rk_{R/\mathfrak{A}}A/\mathfrak{A}A=(rk_{R/\mathfrak{A}}\bar{P})^2$ and $\mathfrak{A}\subset rad\, R$, it follows that $rk_R A=(rk_R P)^2$. Thus, by Lemma 2.4, $A\approx End_R P$, i. e. A represents the trivial element in $Br(R)$. This completes the proof of Theorem 2.1.

Corollary 2.5. *For any commutative noetherian ring R, and n a positive integer, the canonical homomorphism $R[X]/(X^n)\to R$ induces an isomorphism $Br(R[X]/(X^n))\to Br(R)$.*

Proof. We have only to show that the map $Br([X]/(X^n)) \to Br(R)$ is surjective. But this follows from the fact that the composite of canonical homomorphisms $R\to R[X]/(X^n)\to R$ is the identity mapping of R so that the composite of the induced maps $Br(R)\to Br(R[X]/(X^n))\to Br(R)$ is the identity mapping of $Br(R)$.

§3. Skolem-Noether Theorem

Theorem 3.1. *Let R be a commutative noetherian semi-local ring. Let A and B be Azumaya R-algebras and let $f, g: B\to A$ be R-monomorphisms. Then there exists an inner automorphism θ*

of A such that $g=\theta\circ f$.

Proof. We shall first prove the theorem assuming that R is a finite product of fields. Let in fact $R=R_1\times\cdots\times R_n$, with R_i a field for each i. We can write $A=A_1\times\cdots\times A_n$ and $B=B_1\times\cdots\times B_n$, where A_i and B_i are central simple R_i-algebras ([4] Prop. 2. 20 (b)). Further, f and g induce monomorphisms $f_i, g_i: B_i\to A_i$. By the Skolem-Noether theorem over fields, we can find inner automorphisms θ_i of A_i such that $g_i=\theta_i\circ f_i$ for $1\leqslant i\leqslant n$. Then $\theta=\theta_1\times\cdots\times\theta_n$ is an inner automorphism of A satisfying $g=\theta\circ f$.

Let now R be any semi-local ring and let $\mathfrak{w}$ denote the radical of R. Let $\bar{f}, \bar{g}: B/\mathfrak{w}B\to A/\mathfrak{w}A$ be the induced $R/\mathfrak{w}$-monomorphisms. Since $R/\mathfrak{w}$ is a finite product of fields, by the case considered above, there exists an inner automorphism $\bar{\theta}$ of $A/\mathfrak{w}A$ such that $\bar{g}=\bar{\theta}\circ\bar{f}$. Since A is an R-module of finite type, $\mathfrak{w}A$ is contained in the radical of A and hence $\bar{\theta}$ can be lifted to an inner automorphism θ of A. The monomorphisms g and $\theta\circ f$ from B into A induce the same map modulo $\mathfrak{w}$. Thus we can assume that $\bar{f}=\bar{g}$. Now, since B is separable, there exists an element $e\in B\otimes B$ such that $be=eb$ for all $b\in B$ and the image of e under the canonical map $m_B: B\otimes B\to B$ is 1 ([5], Chap. IX, Prop. 7. 7). Let u denote the image of e in A under the map $B\otimes B \xrightarrow{f\otimes g} A\otimes A \xrightarrow{m_A} A$. Since $\bar{f}=\bar{g}$, it follows that $u\equiv 1 \pmod{\mathfrak{w}A}$, and hence u is a unit. Since $be=eb$ (i. e. $(b\otimes 1)e=e(1\otimes b)$) for every $b\in B$, we have $m_A\circ(f\otimes g)((b\otimes 1)e)=m_A\circ(f\otimes g)(e(1\otimes b))$, whence $f(b)u=ug(b)$. Thus $g=\theta\circ f$, where θ denotes the inner automorphism of A given by u^{-1}.

Proposition 3. 2. *Let R be a semi-local ring and let A, B and C be Azumaya R-algebras such that $A\otimes C\approx B\otimes C$. Then $A\approx B$.*

Proof. $h: A\otimes C\to B\otimes C$ be an isomorphism. We have two monomorphisms of C into $B\otimes C$ namely, $f: C\to 1\otimes C\subset B\otimes C$ and $g: C\to 1\otimes C\subset A\otimes C \xrightarrow{h} B\otimes C$. By the theorem above, there exists an automorphism θ of $B\otimes C$ such that $g=\theta\circ f$. Hence the commutants of $f(C)$ and $g(C)$ in $B\otimes C$ are isomorphic. But these commutants are isomorphic to B and A respectively, which proves the proposition.

Corollary 3. 3. *Let R be semi-local and let A and B be similar Azumaya R-algebras such that $rk_R A = rk_R B$. Then $A \approx B$.*

Proof. Let P_1 and P_2 be finitely generated faithful projective R-modules such that $A \otimes End_R P_1 \approx B \otimes End_R P_2$. Since $rk_R A = rk_R B$ and $rk_R(End_R P_i) = (rk_R P_i)^2$, it follows that $rk_R P_1 = rk_R P_2$. Hence $P_1 \approx P_2$ (it is enough to show that P_1 and P_2 are isomorphic modulo the radical of R and this is achieved by the Chinese Remainder Theorem). Thus $End_R P_1 \approx End_R P_2$. The proposition above now shows that $A \approx B$.

§ 4. Extension of derivations to Azumaya algebras

The aim of this section is to prove the following

Theorem 4. 1. *Let R be a commutative noetherian semi-local ring and let A be an Azumaya R-algebra. Then any derivation of R into itself can be extended to a derivation of A.*

Before embarking on the proof of the theorem, we recall a few things about derivations. Let Λ be any ring and let $\Lambda[x]$ be the ring of dual numbers $\Lambda \cdot 1 + \Lambda \cdot x$ with $x^2 = 0$. There is a homomorphism $\eta_\Lambda : \Lambda[x] \to \Lambda$ given by $\eta_\Lambda(\lambda + \mu x) = \lambda$. If $d : \Lambda \to \Lambda$ is a derivation, the mapping $t_d : \Lambda \to \Lambda[x]$ defined by $t_d(\lambda) = \lambda + d\lambda x$ is a *section* of η_Λ, i. e., it is a homomorphism of rings and satisfies the condition $\eta_\Lambda \circ t_d =$ identity. Conversely, if t is a section of η_Λ, write $t(\lambda) = \lambda + d\lambda x$. Clearly d is a derivation of Λ and $t = t_d$.

Proof of the theorem. For any derivation d of R, we shall denote by $R[x]_d$ the ring $R[x]$ considered as an R-algebra through the homomorphism t_d; in particular, corresponding to the zero derivation, $R[x]_0$ is the usual R-algebra structure on $R[x]$.

Since $\eta_R \circ t_d = \eta_R \circ t_0 =$ identity, we have $Br(\eta_R) \circ Br(t_d) = Br(\eta_R) \circ Br(t_0) =$ identity. However $Br(\eta_R) : Br(R[x]) \to Br(R)$ is an isomorphism by Corollary 2. 5. Thus, it follows that $Br(t_d) = Br(t_0)$. This means that for any Azumaya R-algebra A, the $R[x]$-algebras

$R[x]_d \otimes A$ and $R[x]_0 \otimes A$ are similar. The ranks of these algebras over $R[x]$ are the same. Therefore, by corollary 3. 3, there exists an isomorphism $f: R[x]_d \otimes A \to R[x]_0 \otimes A$ of $R[x]$-algebras. For any $a \in A$, let us write

$$f(1 \otimes a) = 1 \otimes g(a) + x \otimes h(a) .$$

We assert that g is an R-automorphism of A. In fact, for $r \in R$ and $a \in A$,

$$f(1 \otimes ra) = 1 \otimes g(ra) + x \otimes h(ra)$$

and also

$$\begin{aligned} f(1 \otimes ra) &= f((r + drx) \otimes a) \\ &= (r + drx) f(1 \otimes a) \quad \text{by } R[x]\text{-linearity of } f, \\ &= 1 \otimes rg(a) + x \otimes (drg(a) + rh(a)) . \end{aligned}$$

So

$$g(ra) = rg(a)$$

and

$$h(ra) = dr\, g(a) + rh(a) \qquad (*)$$

Thus g is R-linear and hence it is an R-endomorphism of A. Since A is Azumaya over R, g is actually an automorphism ([3], Corollary 3. 4).

Set $f_1 = (1 \otimes g)^{-1} \circ f$. Clearly $f_1: R[x]_d \otimes A \to R[x]_0 \otimes A$ is an $R[x]$-isomorphism such that

$$f_1(1 \otimes a) = 1 \otimes a + x \otimes d_1 a$$

for some mapping $d_1: A \to A$. It is immediate that d_1 is a derivation of A. We shall show that $d_1(r) = d(r)$ for $r \in R$. In fact, this follows from (*) by putting $g = 1$, $h = d_1$ and $a = 1$, since $d_1(1) = 0$. This completes the proof of the theorem.

Remark. If Λ is any ring, any higher derivation of rank n of Λ into itself can be identified with a section of the canonical map $\Lambda[X]/(X^{n+1}) \to \Lambda$ (see [7]). Using the same kind of argument as in the proof of the theorem above, one can show that, for any semi-local ring R and an Azumaya R-algebra A, any higher derivation of rank n of R can be extended to a higher derivation of rank n of A. This was proved in [7] in case R is a field.

Tata Institute of Fundamental Research, Bombay
and
Centre for Advanced Training and Research
in Mathematics, University of Bombay.

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