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Regular operators and spaces of harmonic functions with finite Dirichlet integral on open Riemann surfaces

By

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Introduction Let R be an open Riemann surface and let W be an open subset of R consisting of a finite number of regularly imbedded regions on R such that $R-W$ is connected and compact. We denote by $C^{\omega}(\partial W)$ the family of real analytic functions on ∂W and by $H(\overline{W})$ the family of harmonic functions on $\overline{W}$. L. Sario introduced (see [1]) the notion of a normal operator L: $C^{\omega}(\partial W)\to H(\overline{W})$ which is defined by the following conditions:

(1) $Lf=f$ on ∂W, (2) $L(c_1f_1+c_2f_2)=c_1Lf_1+c_2Lf_2$,

(3) $L1=1$, (4) $Lf\geqq 0$ if $f\geqq 0$, (5) $\int_{\partial W}(dLf)^*=0$.

One of Sario's important results is the following existence theorem for principal functions: *Let a harmonic function s be given on* $\overline{W}$. *Then there exists a harmonic function* p *on* R *satisfying* $p-s=L(p-s)$ *on* W *if and only if* $\int_{\partial W}(ds)^*=0$. *The function* p *is uniquely determined up to an additive constant.*

He constructed two normal operators L_0 and $(P)L_1$. Using the above existence theorem for these operators he gave elegant proofs to some classical theorems and obtained some results which have been applied to the theory of conformal mapping by many authors ([6], [9], [10], etc). However neither Dirichlet operator H^W ([3]

or [4]) nor Neumann operator N^W (§11) is normal in general, and so the existence of the Green function or the Neumann function is not derived by a direct application of the above existence theorem. On the other hand, B. Rodin [11] showed that L_0- or $(P)L_1$-principal functions give the reproducing kernels for some subspaces of Γ_h (=the space of harmonic differentials on R with finite Dirichlet norm), while he remarked that the reproducing kernel for the subspace Γ_S of Schottky differentials do not seem to be obtained in terms of these principal functions.

Thus, in this paper, we modify conditions (1)-(5) and introduce an operator L: $C^\omega(\partial W)\to H(\overline{W})$ which is defined by the following conditions

(1*) $Lf=f$ on ∂W,

(2*) $D_W(Lf)<\infty$,

(3*) $D_W(Lf, Lg)=\int_{\partial W} f(dLg)^*$ for all $f, g\in C^\omega(\partial W)$.

Here $D_W(Lf)$ (resp. $D_W(Lf, Lg)$) denotes the Dirichlet integral (resp. mixed Dirichlet integral) over W. We shall call such an operator L *regular* (§2). Although normal operators are not always regular, H^W and N^W as well as L_0 and $(P)L_1$ are regular operators.

A large part of this paper is devoted to the establishment of correspondence between regular operators and subspaces of the Banach space HD, the space of Dirichlet finite harmonic functions u on R with the norm $|||u|||_R=\sqrt{D_R(u)}+|u(a_0)|$. Using this correspondence, the existence theorem of principal functions for a regular operator is proved by a method of orthogonal decomposition (§7). Also a condition that a regular operator be normal is given by a property of the corresponding subspace (§§4, 8). From these results, we shall see that the reproducing kernels for Γ_S cannot be always expressed in terms of normal operators but they are expressed in terms of a regular operator (§§9, 11).

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§1. Preliminaries

Given an open Riemann surface R, we denote by $\mathfrak{W}$ the collection of open sets W of R such that for each W, $R-W$ is connected and compact and W and its exterior have the same non-empty relative boundary in common which consists of a finite number of mutually disjoint simple analytic closed curves. For each $W\in\mathfrak{W}$, we denote by ∂W the relative boundary of W and set $\overline{W}=W\cup\partial W$. We suppose that the orientation of ∂W is positive with respect to W. We assume that functions defined on subsets of R are always real-valued. As for the differentials we shall use the notations and terminology used in Chapter V of L. Ahlfors and L. Sario [1], but we restrict ourselves to real differentials. The classes Γ, Γ_h, Γ_e, Γ_{he}, $\cdots$ are Hilbert spaces with the inner product $(\omega_1, \omega_2)_R=\int_R \omega_1\omega_2^*$. Also we refer to C. Constantinescu and A. Cornea [4] for the Dirichlet functions and Dirichlet potentials and use the same notations e.g., D, HD, D_0, C_0^∞, dD, dHD, $\cdots$. The following propositions are well-known:

Proposition 1 (See [1], V, $10A$) *Let Γ_1 be a closed linear subspace of Γ_h. Then $\Gamma=\Gamma_1+\Gamma_{e0}+\Gamma_1^\perp+\Gamma_{e0}^*$ where $\Gamma_1^\perp$ is the orthogonal complement of Γ_1 with respect to Γ_h.*

Proposition 2 (Royden decomposition) (See [4], Satz 7.6) *If R is hyperbolic, then $D=HD+D_0$ on R. Moreover, let $u_1, u_2\in HD$ and $f_{01}, f_{02}\in D_0$ such that $u_1+f_{01}=u_2+f_{02}$ quasi-everywhere on some $W\in\mathfrak{W}$. Then $u_1=u_2$ on R.*

Proposition 3 (See [1], II, $13B$) *Let $\{u_n\}$ be a sequence in HD satisfying $\lim_{m,n\to\infty}\|du_n-du_m\|_R=0$ and let $\{u_n(a)\}$ converge at least* *at one point a in R. Then there exists a function u in HD such that* $\lim_{n\to\infty}\|du_n - du\|_R = 0$ *and* $\{u_n\}$ *converges locally uniformly to u on R.*

Proposition 4 (See [4], Hilfssatz 7.8) *Suppose that R is hyperbolic. Let* $\{f_{0n}\}$ *be a sequence in* D_0 *such that* $\|df_{0n+1} - df_{0n}\|_R < 1/2^n$. *Then* $\{f_{0n}\}$ *converges to an* $f_0 \in D_0$ *quasi-everywhere on R and* $\lim_{n\to\infty}\|df_{0n} - df_0\|_R = 0$.

Finally we explain the meaning of notation $\int_\beta \omega$. Let ω be a differential of class C^1. Then, if for any exhaustion $\{\Omega_n\}$ of R $\lim_{n\to\infty}\int_{\partial\Omega_n}\omega$ exists then we write $\int_\beta \omega = \lim_{n\to\infty}\int_{\partial\Omega_n}\omega$ where β stands for the ideal boundary of R and Ω_n is a relatively compact region in R whose boundary consists of a finite number of analytic curves. On using Green's formula we obtain the following fact: *Let* $f \in C^1(\overline{W}) \cap D(W)$ *and* $\omega^* \in \Gamma_c^1(\overline{W})$. *Then* $\int_\beta f\omega^*$ *exists and* $\int_\beta f\omega^* = (df, \omega)_W - \int_{\partial W} f\omega^*$.

§2. Regular operators

Let $W \in \mathfrak{W}$. Let $C^\omega(\partial W)$ be the family of real-analytic functions f on ∂W. We denote by $H(\overline{W})$ the family of restrictions u to $\overline{W}$ of harmonic functions on open sets containing $\overline{W}$.

Definition We say that an operator $L\colon C^\omega(\partial W) \to H(\overline{W})$ is *regular* (with respect to W), if it satisfies the following conditions

(1) $Lf = f$ on ∂W,

(2) $\|dLf\|_W < \infty$,

(3) $(dLf, dLg)_W = \int_{\partial W} f(dLg)^*$ for any $f, g \in C^\omega(\partial W)$.

On account of the equality obtained at the end of §1, condition (3) is equivalent to

(3′) $\int_\beta (Lf)(dLg)^* = 0$ for any $f, g \in C^\omega(\partial W)$.

We see from Green's formula that a regular operator is linear:

$$L(c_1 f_1 + c_2 f_2) = c_1 L f_1 + c_2 L f_2 .$$

Definition A regular operator is called *canonical*, if $L1=1$ on W. A regular operator is called *positive*, if $f \geqq 0$ on ∂W implies $Lf \geqq 0$ on W.

Condition (3) implies that $\|dL1\|_W^2 = \int_{\partial W} (dL1)^*$ and $(dL1, dLf)_W = -\int_\beta (dLf)^*$ for any $f \in C^\omega(\partial W)$. It follows that L is canonical if any only if $\int_\beta (dL1)^* = 0$. Moreover in this case $\int_\beta (dLf)^* = 0$ for any $f \in C^\omega(\partial W)$.

Remark In case L is not canonical, let $L' = L(f - c_f) + c_f$ where $c_f = \int_\beta (dLf)^* / \int_\beta (dL1)^*$. Then L' is canonical.

Uniqueness theorem *Let L be a regular operator with respect to W. Let $u \in HD$ satisfy the equation $u = Lu$ on W. Then u is a constant. If, in addition, L is not canonical, then the constant must be zero.*

Proof Since $R - W$ is compact, we have by Green's formula $\|du\|_{R-W}^2 = \int_{\partial(R-W)} u(du)^* = -\int_{\partial W} u(du)^*$. It follows from $u = Lu$ on $\overline{W}$ that $\|du\|_R^2 = \|du\|_{R-W}^2 + \|dLu\|_W^2 = -\int_{\partial W} u(du)^* + \int_{\partial W} u(dLu)^* = 0$. Hence u is a constant c on R. If L is not canonical, $c = Lc$ implies $c = 0$.

§3. Subspaces A of HD, regular operators L_W^A and consistent sysems $\mathcal{L}^A$

If a given Riemann surface R is parabolic, conditions (1), (2) in §2 imply condition (3), and moreover, for each $W \in \mathfrak{W}$, all the regular operators become identical. Hence hereafter we always assume that R is hyperbolic. Let $a_0 \in R$ be fixed. The space HD is a Banach space with respect to the norm $|||u|||_R = \|du\|_R + |u(a_0)|$.

Let a subspace A[1] of Banach space HD be given, that is, A is a closed linear subset of HD. Let $W \in \mathfrak{W}$. For any $f \in C^{\omega}(\partial W)$, let

$$M_f^A = M_f = \{g \in A + D_0;\ g = f \text{ quasi-everywhere on } \partial W\}.$$

Denote by $\Lambda_f^A = \Lambda_f$ the set of restrictions of the differentials $dg(\in dM_f)$ to W. Then, Λ_f is a non-empty subset of the Hilbert space $dD(W) = \Gamma_e(W)$.

Lemma 1 *Λ_f is convex and complete.*

Proof The convexity is clear. To show the completenss, let $\{\omega_n\}$ be any sequence in Λ_f which satisfies $\|\omega_n - \omega_m\|_W \to 0$ as $m, n \to \infty$. We can find $u_n \in A$, $f_{0n} \in D_0$ such that $u_n + f_{0n} \in M_f$, $d(u_n + f_{0n}) = \omega_n$ on W. Moreover we can require $u_n + f_{0n} = H_f^{R-\overline{W}}$ on $R - W$ for all n, where $H_f^{R-\overline{W}}$ is the harmonic function on $R - W$ with the boundary values f on $\partial(R-W)$ $(= -\partial W)$. For, when f_{0n} does not satisfy $u_n + f_{0n} = H_f^{R-\overline{W}}$ on $R - W$, we consider the function g_{0n} which is f_{0n} on $\overline{W}$ and is $H_{f-u_n}^{R-\overline{W}}$ on $R - W$. Then $g_{0n} \in D_0$, $u_n + g_{0n} \in M_f$, $d(u_n + g_{0n}) = \omega_n$ and $u_n + g_{0n} = H_f^{R-\overline{W}}$. Since $\lim_{m,n\to\infty} \|d(u_m + f_{0m}) - d(u_n + f_{0n})\|_R = \lim_{m,n\to\infty} \|\omega_m - \omega_n\|_W = 0$, we can choose a subsequence $\{u_{n_k} + f_{n_k}\}$ which satisfies $\|d(u_{n_{k+1}} + f_{0n_{k+1}}) - d(u_{n_k} + f_{0n_k})\|_R < 1/2^k$. Hence we have $\|du_{n_{k+1}} - du_{n_k}\|_R < 1/2^k$ and $\|df_{0n_{k+1}} - df_{0n_k}\|_R < 1/2^k$, because $dHD \perp dD_0$. It follows from Proposition 4 that $\{f_{0n_k}\}$ converges quasi-everywhere on R to a Dirichlet potential f_0 and $\lim_{k\to\infty} \|df_{0n_k} - df_0\|_R = 0$. Since $u_{n_k} + f_{0n_k} = H_f^{R-\overline{W}}$ on $R - W$ for all k, $\{u_{n_k}\}$ converges at least at one point in $R - W$. It follows from Proposition 3 that $\{u_{n_k}\}$ converges locally uniformly on R to an HD-function u. Hence $\lim_{k\to\infty} \||u_{n_k} - u\||_R = 0$ and $u + f_0 = f$ quasi-everywhere on ∂W. By the closedness of A we have $u \in A$. We conclude that $u + f_0 \in M_f$ and $\lim_{n\to\infty} \|d(u + f_0) - \omega_n\|_W = 0$. Consequently Λ_f is complete.

Under these preliminaries we shall construct a regular operator

1) If $A = HD$, then most of the proofs in this section coincide with those in Chapter 15 (pp. 154–166) of C. Constantinescu and A. Cornea [4]. Also see M. Ohtsuka [8].

L_W^A for a given subspace A and $W \in \mathfrak{W}$. By a well-known theorem in the theory of Hilbert space, Lemma 1 implies that there exists a unique element $\omega_f \in \Lambda_f$ such that $\|\omega_f\|_W = \inf\{\|\omega\|_W;\ \omega \in \Lambda_f\}$. Moreover ω_f must be a harmonic differential on W and $(\omega_f, \omega)_W = 0$ for all $\omega \in \Lambda_0$ and therefore $\|\omega - \omega_f\|_W^2 = \|\omega\|_W^2 - \|\omega_f\|_W^2$ for all $\omega \in \Lambda_f$. Since $\omega_f \in \Lambda_f$ and ω_f is harmonic on W, there exists a function F in M_f which satisfies $dF = \omega_f$ and is harmonic on W. Since $F = f$ quasi-everywhere on ∂W, the restriction F_1 of F to W is uniquely determined. Moreover F_1 assumes continuously f on ∂W,[2] i.e., for any $\zeta \in \partial W$, $\lim_{\substack{z \to \zeta \\ z \in W}} F_1(z) = f(\zeta)$. Therefore if we denote by $L_W^A f (= Lf)$ the function which is F_1 on W and f on ∂W, then $Lf \in H(\overline{W}) \cap HD(W)$. We have thus an operator L_W^A: $C^\omega(\partial W) \to H(\overline{W})$ which satisfies conditions (1), (2) in §2. It furthermore satisfies condition (3) in §2. In fact, let $f, g \in C^\omega(\partial W)$ and consider a function $f_0 \in C_0^\infty(R) \cap M_f$. It follows from $d(Lf - f_0) \in \Lambda_0$ that $(d(Lf - f_0), dLg)_W = 0$. By Green's formula we have $0 = \int_{\beta + \partial W} (Lf - f_0)(dLg)^* = \int_\beta (Lf)(dLg)^*$. Consequently, L_W^A is a regular operator with respect to W. We shall call L_W^A the *regular operator induced by A for W*. The system $\mathcal{L}^A = \{L_W^A\}_{W \in \mathfrak{M}}$ has the following property:

Proposition 5 *If $W_1, W_2 \in \mathfrak{W}$ such that $W_1 \supset W_2$, then for any $f \in C^\omega(\partial W_1)$, $L_{W_2}^A(L_{W_1}^A f) = L_{W_1}^A f$ on W_2.*

Proof We write simply $L_{W_1}^A = L_1$ and $L_{W_2}^A = L_2$. Consider the function g on W_1 which is $L_2(L_1 f)$ on W_2 and is $L_1 f$ on $\overline{W}_1 - W_2$. Then $dg \in \Lambda_f(W_1)$, and hence $\|dg\|_{W_1}^2 \geqq \|dL_1 f\|_{W_1}^2$ or $\|dL_2(L_1 f)\|_{W_2}^2 \geqq \|dL_1 f\|_{W_2}$. It follows from $dL_1 f \in \Lambda_{L_1 f}(W_2)$ that $L_1 f = L_2(L_1 f)$ on W_2.

Definition For each $W \in \mathfrak{W}$ suppose a regular operator L_W with respect to W is given. Then the system $\mathcal{L} = \{L_W\}_{W \in \mathfrak{M}}$ is said to be *consistent* if, for any $W_1, W_2 \in \mathfrak{W}$ such that $W_1 \supset W_2$, $L_{W_1} f$

2) This fact can be proved by making use of Lemma 3 of M. Ohtsuka [8].

$=L_{W_2}(L_{W_1}f)$ on W_2 for any $f\in C^{\omega}(\partial W_1)$.

Proposition 5 shows that the system $\mathcal{L}^A$ induced by A is consistent.

Definition Suppose a consistent system $\mathcal{L}=\{L_W\}_{W\in\mathfrak{W}}$ is given. Let u be a harmonic function defined near the ideal boundary β. Precisely speaking, there is a compact subset K of R such that u is defined and is harmonic on $R-K$. Then we say that u has *$\mathcal{L}$-behavior on β*, if $u=L_W u$ on W for any $W\in\mathfrak{W}$ such that $\overline{W}\subset R-K$.

We note that u has $\mathcal{L}$-behavior on β, if $u=L_W u$ for some $W\in\mathfrak{W}$ such that $\overline{W}\subset R\text{-}K$. In fact, let W_1 be any set in $\mathfrak{W}$ such that $\overline{W}_1\subset R\text{-}K$. Choose $W_2\in\mathfrak{W}$ sach that $W\cap W_1\supset W_2$. Then the consistency of $\mathcal{L}$ implies that, on W_2, $u=L_W u=L_{W_2}(L_W u)=L_{W_2}u$ and $L_{W_1}u=L_{W_2}(L_{W_1}u)$. By making use of condition (3) in §2 we have $\|d(u-L_{W_1}u)\|^2_{W_1}=\int_{\beta+\partial W_1}(u-L_{W_1}u)(d(u-L_{W_1}u))^*=\int_{\beta}L_{W_2}(u-L_{W_1}u)(dL_{W_2}(u-L_{W_1}u))^*=0$. It follows that $u=L_{W_1}u$ on W_1.

§4. Canonical and positive operators

We shall give a necessary and sufficient condition in order that the regular operator L_W^A induced by A should be canonical or positive. Let $u\in HD$ and write $u\vee 0=\inf\{v\in HP;\ v\geqq\max(u,0)$ on $R\}$. It is well-known that $u\vee 0\in HD$, $\|d(u\vee 0)\|_R\leqq\|du\|_R$ and $u\vee 0$ is the harmonic part of the Royden decomposition of $\max(u,0)$. Moreover, let $\{\Omega_n\}$ be an exhaustion of R such that Ω_n is a relatively compact and open subset in R and $\partial\Omega_n$ consists of a finite number of analytic curves. Then $\{H^{\Omega_n}_{\max(u,0)}\}$ converges locally uniformly on R to $u\vee 0$ and $\lim_{n\to\infty}\|dH^{\Omega_n}_{\max(u,0)}-d(u\vee 0)\|_{\Omega_n}=0$ (see [4], p. 61). We say that a subspace A of HD forms a *vector lattice*, if $u\in A$ implies $u\vee 0\in A$ (see [4], p. 16).

Theorem 1 *If a subspace A contains* 1, *then L_W^A is canonical for any $W\in\mathfrak{W}$. Conversely if L_W^A is canonical for some $W\in\mathfrak{W}$, then $A\ni 1$. If a subspace A forms a vector lattice, then L_W^A is* *positive for any* $W \in \mathfrak{W}$[3]. *If a subspace* A *does not form a vector lattice, then there exists* $W_0 \in \mathfrak{W}$ *such that* L_W^A *is not positive for any* $W \in \mathfrak{W}$ *which is contained in* W_0.

Proof If A contains 1, then the construction of L_W^A implies $L_W^A 1 = 1$ on W. Conversely, for some $W \in \mathfrak{W}$ suppose that $L_W^A 1 = 1$ on W. Then we find $u_1 \in A$, $f_0 \in D_0$ such that $u_1 + f_0 = 1$ on W. It follows from Proposition 2 that $u_1 = 1$ on R, proving $A \ni 1$.

Suppose that A forms a vector lattice. Let $W \in \mathfrak{W}$ and let $f \in C^\omega(\partial W)$ such that $f \geqq 0$. We can choose $u_f \in A$ and $f_0 \in D_0$ such that $L_W^A f = u_f + f_0$ on W. Consider the function g_0 which is equal to $H^W_{f-(u_f \vee 0)}$ on W and $H^{R-W}_{f-(u_f \vee 0)}$ on $R-W$. Here $H^W_{f-(u_f \vee 0)}$ denotes the Dirichlet solution on W with the boundary values $f-(u \vee 0)$ on ∂W and 0 on the ideal boundary β of R. Then $g_0 \in D_0$. Hence by our assumption we have $u_f \vee 0 + g_0 \in M_f^A(W)$. On the other hand, the function $u_f \vee 0 + g_0$ on W is equal to $(u_f + f_0) \cup 0 = \inf\{v \in HP(W);\ v \geqq \max(0, u_f + f_0) \text{ on } W\}$. In fact, let $\{\Omega_n\}$ be an exhaustion of R. Then, because of the fact that $f \geqq 0$ on ∂W, we can prove that both $u_f \vee 0 + g_0$ and $(u_f + f_0) \cup 0$ are equal to the limit of the Dirichlet solutions in $W \cap \Omega_n$ for the boundary function equal to f on ∂W and to $\max(u_f, 0)$ on $\partial \Omega_n$ (cf. [8]). It follows from $\|d(u_f + f_0)\|_W \geqq \|d((u_f + f_0) \cup 0)\|_W$ that $\|d(u_f \vee 0 + g_0)\|_W \leqq \|d(u_f + f_0)\|_W = \|dL_W^A f\|_W$. By minimum property of $L_W^A f$ we infer that $L_W^A f = u_f \vee 0 + g_0$, or $L_W^A f = (u_f + f_0) \cup 0$ on W. Hence $L_W^A f \geqq 0$ on W, proving that L_W^A is positive for any $W \in \mathfrak{W}$ under the hypothesis that A forms a vector lattice.

If the last assertion were not true, there would exist a sequence $\{W_n\}$ in $\mathfrak{W}$ such that $W_n \supset \overline{W}_{n+1}$, $\bigcap_{n=1}^{\infty} W_n = \phi$ and $L_{W_n}^A$ is positive. If we write $\Omega_n = R - \overline{W}_n$, then $\{\Omega_n\}$ is an exhaustion of R such that $\overline{\Omega}_n$ is compact and $\partial \Omega_n = -\partial W_n$. Let u be any function of A and let $u^+ = \max(u, 0)$ on R. For the sake of convenience we write

3) This condition for the positiveness was suggested by Professor F-Y. Maeda.

$L^A_{W_n}=L_n$. First we shall show $\|du\|_{W_n}\geqq\|dL_nu^+\|_{W_n}$ for all n. Since L_nu^+ is non-negative on $\overline{W}$ and vanishes on $\partial W_n\cap\{u<0\}$, we have $(dL_nu^+)^*=(\partial L_nu^+/\partial n)ds\leqq 0$ on $\partial W_n\cap\{u<0\}$ where $\partial/\partial n$ denotes differentiation in the direction of the exterior normal with respect to W_n. It follows from condition (3) in §2 that

$$(dL_nu^+,\ dL_nu)_{W_n}=\int_{\partial W_n}u(dL_nu^+)^*=\int_{W_n\cap\{u\geqq 0\}}u(dL_nu^+)^*+\int_{\partial W_n\cap\{u<0\}}u(dL_nu^+)^*$$
$$\geqq\int_{\partial W_n\cap\{u\geqq 0\}}u(dL_nu^+)^*=\int_{\partial W_n}u^+(dL_nu^+)^*=\|dL_nu^+\|^2_{W_n}.$$

By making use of Schwarz's inequality we have $\|dL_nu^+\|_{W_n}\|dL_nu\|_{W_n}\geqq|(dL_nu^+,\ dL_nu)_{W_n}|\geqq\|dL_nu^+\|^2_{W_n}$, or $\|dL_nu\|_{W_n}\geqq\|dL_nu^+\|_{W_n}$. Since $u\in M^A_u(W_n)$, the definition of L_nu implies $\|du\|_{W_n}\geqq\|dL_nu\|_{W_n}$. Hence we see that $\|du\|_{W_n}\geqq\|dL_nu^+\|_{W_n}$ for all n. Next we find $u_n\in A$ and $f_{0n}\in D_0$ such that $L_nu^+=u_n+f_{0n}$ on W_n and $u_n+f_{0n}=H^{\Omega_n}_{u^+}$ on $\overline{\Omega}_n$ (see the proof of Lemma 1). For $m>n$, with the help of triangle inequality, we obtain

$$\begin{aligned}&\|d(u_m+f_{0m})-d(u_n+f_{0n})\|^2_R\\&\quad=\|dH^{\Omega_m}_{u^+}-dH^{\Omega_n}_{u^+}\|^2_{\Omega_n}+\|dH^{\Omega_m}_{u^+}-dL_nu^+\|^2_{\Omega_m-\Omega_n}+\|dL_mu^+-dL_nu^+\|^2_{W_m}\\&\quad\leqq\|dH^{\Omega_m}_{u^+}-dH^{\Omega_n}_{u^+}\|^2_{\Omega_n}+(\|dH^{\Omega_m}_{u^+}\|_{\Omega_m-\Omega_n}+\|dL_nu^+\|_{\Omega_m-\Omega_n})^2\\&\qquad\qquad+(\|dL_mu^+\|_{W_m}+\|dL_nu^+\|_{W_m})^2\\&\quad\leqq\|dH^{\Omega_m}_{u^+}-dH^{\Omega_n}_{u^+}\|^2_{\Omega_n}+(\|dH^{\Omega_m}_{u^+}\|_{\Omega_m-\Omega_n}+\|du\|_{W_n})^2+4\|du\|^2_{W_n}.\end{aligned}$$

On the other hand, since $\{H^{\Omega_n}_{u^+}\}$ converges locally uniformly on R to $u\vee 0$, $\lim_{n\to\infty}\|dH^{\Omega_n}_{u^+}-d(u\vee 0)\|_{\Omega_n}=0$ and $u_n+f_{0n}=H^{\Omega_n}_{u^+}$ on Ω_n, it follows that $\lim_{n\to\infty}\|d(u\vee 0)-d(u_n+f_{0n})\|_R=0$. Hence $\lim_{n\to\infty}\|d(u\vee 0)-du_n\|_R=\lim_{n\to\infty}\|df_{0n}\|_R=0$, because $dHD\perp dD_0$. It follows from Proposition 4 that there exists a subsequence $\{f_{0n_k}\}$ in D_0 which converges to zero quasi-everywhere on R. This, together with $\lim_{k\to\infty}(u_{n_k}+f_{0n_k})(z)=(u\vee 0)(z)$ on R, implies that $\{u_{n_k}\}$ converges to $u\vee 0$ quasi-everywhere on R. Since $\lim_{k\to\infty}\|d(u\vee 0)-du_{n_k}\|_R=0$, we have by Proposition 3 $\lim_{k\to\infty}|||u\vee 0-u_{n_k}|||_R=0$. Hence $u\vee 0\in A$. In other words, the subspace A must form a vector lattice, which is a contradiction to our hypothesis. The theorem is completely proved.

§5. Lemmas

Let $\bar{\Omega}$ be a compact bordered Riemann surface with contours $\beta(\Omega)$: $\bar{\Omega}=\Omega\cup\beta(\Omega)$. We orient $\beta(\Omega)$ positively with respect to Ω, and let $\beta(\Omega)$ consist of q contours $\beta_1, \cdots, \beta_q$. Consider $\omega\in\Gamma_c^1(S)$ where S is an open set (with respect to $\bar{\Omega}$) which contains q closed ring domains $\{S_i\}_{i=1}^q$ such that $\partial S_i\supset\beta_i$ and $\bar{S}_i\cap\bar{S}_j=\phi$ for $i\neq j$. We denote by α_i the other contour of S_i and orient it negatively with respect to S_i: $\partial S_i=\beta_i-\alpha_i$. Using this notation we have the following elementary lemma:

Lemma 2 *There exists an* $\hat{\omega}\in\Gamma_c^1(\Omega)$ *such that* $\hat{\omega}=\omega$ *on* $\bigcup_{i=1}^q S_i$ *if and only if* $\int_{\beta(\Omega)}\omega=0$.

Proof For the proof it is essential that Ω is connected. The "only if" part is clear from the closedness of $\hat{\omega}$. To prove the "if" part, suppose that $\int_{\beta(\Omega)}\omega=0$. We write $a_i=\int_{\beta_i}\hat{\omega}$. Then $\sum_{i=1}^n a_i=0$. We fix a closed disk $\bar{\Delta}$ in $\Omega-\bar{S}$, and orient its contour $\delta=\partial\Delta$ positively with respect to Δ. For each i, we can easily construct a closed of class C^1 differential ω_i on $\bar{\Omega}-\Delta$ such that $\int_{\beta_i}\omega_i=\int_\delta\omega_i=1$ and $\int_{\beta_j}\omega_i=0$ for any $j\neq i$. If we consider the closed differential $\sigma=\sum_{i=1}^q a_i\omega_i$ on $\bar{\Omega}-\Delta$, then $\int_\delta\sigma=\sum_{i=1}^q a_i\int_\delta\omega_i=\sum_{i=1}^q a_i=0$. Hence there exists a function g of class C^2 on $\bar{\Delta}_1-\Delta$ such that $\sigma=dg$ on $\bar{\Delta}_1-\Delta$, where Δ_1 is an open disk containing $\bar{\Delta}$. We are thus able to extend g to $\bar{\Delta}_1$ being kept of class C^2. If we set $\hat{\sigma}=dg$ on $\bar{\Delta}_1$ and σ on $\Omega-\Delta$, then $\hat{\sigma}\in\Gamma_c^1(\bar{\Omega})$ and $\int_{\beta_i}\hat{\sigma}=a_i$ for each i. Because of the fact that $\int_{\beta_i}(\omega-\hat{\sigma})=0$, we find a function f_i of class C^2 on S_i such that df_i

$=\omega-\widehat{\sigma}$ on S_i. Obviously, there exists a function f on $\overline{\Omega}$ of class C^2 such that $f=f_i$ on each S_i. If we set $\widehat{\omega}=df+\widehat{\sigma}$ on $\overline{\Omega}$, then $\widehat{\omega}$ is one of the required differentials.

We return to a hyperbolic Riemann surface R. Let A be a *linear subset* of HD. Then we have

Lemma 3 *If A is closed in HD, then dA is closed in Γ_{he}. Conversely, suppose that dA is closed in Γ_{he}. Then, if either $A \ni 1$ or $\overline{A} \not\ni 1$, then A is closed in HD.*[4)]

Proof It is clear from Proposition 3 that, if $A \ni 1$, then A is closed if and only if dA is closed. Suppose that A is closed and $A \not\ni 1$. First we shall show that, if a is an arbitrarily fixed point in R, then there exists a positive number $\lambda_a^A=\lambda_a$ such that $|u(a)| \leq \lambda_a \|du\|_R$ for any $u \in A$. If such a λ_a did not exist, then we find a sequence $\{u_n\}$ in A such that $u_n(a)=1$ and $\lim_{n\to\infty}\|du_n\|_R=0$. It follows from Proposition 3 that $\lim_{n\to\infty}|||u_n-1|||_R=0$, and hence $A \ni 1$. This is a contradiction. Secondly, let $u_n \in A$ and $\{du_n\}$ form a Cauchy sequence. Then the above inequality implies that $\lim_{m,n\to\infty}|||u_n-u_m|||_R=\lim_{m,n\to\infty}(|u_n(a_0)-u_m(a_0)|+\|du_n-du_m\|_R) \leq \lim_{m,n\to\infty}$

4) There exists a linear non-closed subset A such that dA is closed. In fact, let R be a Riemann surface such that the dimension of Γ_{he} is infinite. First, take a sequence $\{u_n\}$ in HD which satisfies $u_n(a_0)=0$ for all n and $(du_n, du_m)_R=0$ if $m \neq n$ and $=1$ if $m=n$. Consider the space $l^2=\{\xi(=\{\xi_i\});\ \xi_i$ is real and $\sum_{i=1}^{\infty}\xi_i^2<\infty\}$ and write $\|\xi\|_{l^2}=(\Sigma\xi_i^2)^{1/2}$. Then $\{\Sigma\xi_i du_i:\ \xi=\{\xi_i\}\in l^2\}$ is a closed subspace in Γ_{he} and Proposition 3 guarantees that, for any $\xi=\{\xi_i\}\in l^2$, $\Sigma\xi_i u_i(z)$ certainly is a harmonic function on R. Next, consider a *non-continuous* linear functional f on l^2 and set $A=\{f(\xi)+\Sigma\xi_i u_i(z):\ \xi\in l^2\}$ $(\subset HD)$. Then it is clear that A is linear and dA is closed. If we suppose $A \ni 1$, then there exists $\xi \neq 0$ in l^2 such that $1=f(\xi)+\Sigma\xi_i u_i(z)$ on R. Considering Dirichlet integrals, we have $0=\|d(\Sigma\xi_i u_i)\|_R^2=\|\xi\|_{l^2}^2$, or $\xi=0$, which is a contradiction. Hence $A \not\ni 1$. On the other hand, since f is not continuous at 0, there exists a sequence $\{\xi^{(n)}\}$ in l^2 such that $\lim_{n\to\infty}\|\xi^{(n)}\|_{l^2}=0$ and $\lim_{n\to\infty} f(\xi^{(n)})=1$. Then we have $\lim_{n\to\infty}|||f(\xi^{(n)})+\Sigma\xi_i^{(n)}u_i(z)-1|||_R=\lim_{n\to\infty}\|\xi^{(n)}\|_{l^2}=0$. Hence closure of A in HD contains 1. Consequently, A is not closed.

$(\lambda_{a_0}+1)\|du_n-du_m\|_R=0$. On account of the closedness of A, we find u in A such that $\lim_{n\to\infty}|||u_n-u|||_R=0$, and hence $\lim_{n\to\infty}\|du_n-du\|_R=0$. Thus dA is closed in Γ_{hc}.

Finally suppose that dA is closed and $\bar{A}\not\ni 1$. If we denote by C the family of constant functions, then $d(A+C)$ $(=dA)$ is closed and $A+C\ni 1$. The fact mentioned above implies $A+C$ is closed, and hence $A+C\supset\bar{A}$. We have thus $dA=d(A+C)\supset d\bar{A}\supset dA$, or $dA=d\bar{A}$. It follows from $\bar{A}\not\ni 1$ that $A=\bar{A}$, i.e., A is closed.

Corollary 1 *Let A be a subspace of HD which does not contain* 1 *and write $\langle u_1, u_2\rangle=(du_1, du_2)_R$ for any u_1, $u_2\in A$. Then A becomes a Hilbert space with respect to the inner product $\langle\ ,\ \rangle$. Moreover the linear functional T_a: $u\to u(a)$ is continuous.*

Corollary 2 *Suppose that A and B are subspaces of HD such that dA is orthogonal to dB. Then A+B is closed.*

Proof Since dA is othogonal to dB, $d(A+B)$ is closed (see for example [1], *V*, *7G*). First suppose that A or B contains 1. Then the above lemma implies that $A+B$ is closed. Next suppose that neither A nor B contains 1. Then $\overline{A+B}$ does not contain 1. For, if $\overline{A+B}$ contained 1, then there would exist $u_n\in A$ and $v_n\in B$ such that $\lim_{n\to\infty}|||u_n+v_n-1|||_R=0$. We have thus $\lim_{n\to\infty}(u_n(a_0)+v_n(a_0))=1$ and $0=\lim_{n\to\infty}\|du_n+dv_n\|_R^2=\lim_{n\to\infty}(\|du_n\|_R^2+\|dv_n\|_R^2)$, because $dA\perp dB$. Since $A\not\ni 1$ and $B\not\ni 1$, it follows from $\lambda_{a_0}^A\|du_n\|_R\geqq|u_n(a_0)|$, $\lambda_{a_0}^B\|dv\|_R\geqq|v_n(a_0)|$ that $\lim_{u\to\infty}u_n(a_0)=\lim_{n\to\infty}v_n(a_0)=0$, which contradicts $\lim_{n\to\infty}(u_n(a_0)+u_n(a_0))=1$. Consequently $\overline{A+B}\not\ni 1$. We see from the above lemma that $A+B$ is closed in *HD*.

§6. A characterization of $L_W^A f$

Proposition 6 *Let $W\in\mathfrak{W}$. Let $f_0\in C^1(R)\cap D_0$ and $\omega\in\Gamma_c^1(W)$. Then $\int_\beta f_0\omega=0$.*

Proof Consider a function F in $C^1(R)$ which is 1 outside of a compact set in R and is 0 on a (relatively compact) open set Ω which contains $R-W$ and whose boundary consists of a finite number of closed analytic curves. Then $Ff_0 \in D_0 \cap C^1(R)$ and $Ff_0 = 0$ on $\partial\Omega$. Therefore the restriction of Ff_0 to $R-\Omega$ is a Dirichlet potential on $R-\bar{\Omega}$ (see for example [13], Lemma 1). It follows from $dD_0(R-\Omega) \perp \Gamma_c^*(R-\Omega)$ that $(d(Ff_0), \omega^*)_{R-\Omega}=0$. By making use of Green's formula we have thus $0=\int_{\beta-\partial\Omega}(Ff_0)\omega$ $=\int_\beta (Ff_0)\omega = \int_\beta f_0\omega$. q.e.d.

Let A be a subspace of HD. Since dA is a subspace of Γ_h (Lemma 3), we have the following decompsition: $\Gamma_h = dA + (dA)^\perp$. For each $W \in \mathfrak{W}$, we consider the following subset of $\Gamma_c^1(W)$:

$$\Sigma_A(W) = \{\omega \in \Gamma_c^1(\bar{W});\ \int_\beta u\omega = 0 \text{ for any } u \in A\}.$$

Since $(dA)^\perp + \Gamma_{e0}^*$ is orthogonal to dA (Proposition 1), $\omega^* \in (dA)^\perp + \Gamma_{e0}^* \cap \Gamma^1$ implies that $0=(du, \omega^*)_R = -\int_\beta u\omega$ for any $u \in A$, and hence $\omega \in \Sigma_A(W)$. That is, $\{\omega | W;\ \omega \in (dA)^{\perp *} + \Gamma_{e0} \cap \Gamma^1\} \subset \Sigma_A(W)$, where $\omega | W$ denotes the restriction of ω to W. If A contains 1, the opposite inclusion relation is valid. In fact, let ω be any differential in $\Sigma_A(W)$. Since $A \ni 1$, we have $\int_{\partial W}\omega = -\int_\beta 1\omega = 0$. It follows from Lemma 2 that there exists $\hat{\omega} \in \Gamma_c^1(R)$ such that $\hat{\omega} = \omega$ on $\bar{W}$. For any $u \in A$, we obtain $(du, \hat{\omega}^*)_R = -\int_\beta u\hat{\omega} = -\int_\beta u\omega = 0$. Since $\Gamma_c^* = \Gamma_h + \Gamma_{e0}^* = (dA) + (dA)^\perp + \Gamma_{e0}^*$, it follows that $\hat{\omega}^* \in ((dA)^\perp + \Gamma_{e0}^*) \cap \Gamma^1$, and hence $\hat{\omega} \in (dA)^{\perp *} + \Gamma_{e0} \cap \Gamma^1$. We state this fact as

Remark If $A \ni 1$, then $\Sigma_A(W) = \{\omega | W;\ \omega \in (dA)^{\perp *} + \Gamma_{e0} \cap \Gamma^1\}$.

The following characterization of $L_W^A f$ will be used frequently.

Theorem 2[5] *Let A be a subspace of HD. Let $W \in \mathfrak{W}$ and*

5) This formulation is due to Professor M. Yoshida (see [14]). The author's original one was much more complicated, though they are essentially the same. On account of this formulation the author could make the following argument simpler.

$f\in C^{\omega}(\partial W)$. The function $u=L_W^A f$ satisfies the following conditions:

(a) $u=f$ *on* ∂W,

(b) $u=v+f_0$ *on* W *for some* $v\in A$ *and* $f_0\in D_0$,

(c) $(du)^*\in\Sigma_A(W)$.

Conversely, a function u with properties (a), (b), (c) *must be equal to* $L_W^A f$.

Proof In this proof we write simply $L_W^A=L$. From the construction of Lf we see that $u=Lf$ satisfies (a), (b). It further satisfies (c). In fact, it is clear that $(dLf)^*\in\Gamma_c^1(\overline{W})$. Let w be any function in A. Take $g_0\in C_0^{\infty}(R)$ such that $g_0=w$ on ∂W. Then $w-g_0\in M_0^A$. Since dLf is orthogonal to Λ_0^A, we have thus $0=(d(w-g_0), dLf)_W=\int_{\beta+\partial W}(w-g_0)(dLf)^*=\int_{\beta}w(dLf)^*$. Hence $(dLf)^*\in\Sigma_A(W)$. In order to prove the converse, suppose that u satisfies (a), (b), (c). Conditions (a), (b) imply $u(=v+f_0)\in\mathfrak{W}_f^A$. Hence it is enough to show that $\|dLf\|_W\geqq\|du\|_W$. There exists $v_1\in A$, $f_{01}\in D_0\cap C^1(R)$ such that $L_W^A f=v_1+f_{01}$ on W. By condition (c) we have $\int_{\beta}v(du)^*=\int_{\beta}v_1(du)^*=0$. This, with Proposition 6, implies that $\int_{\beta}u(du)^*=\int_{\beta}(Lf)(du)^*=0$. Hence $(dLf, du)_W=\int_{\beta+\partial W}(Lf)(du)^*=\int_{\partial W}f(du)^*=\int_{\beta+\partial W}u(du)^*=\|du\|_W^2$. We conclude from Schwarz's inequality that $\|dLf\|_W\geqq\|du\|_W$.

Corollary *Suppose that A and B are subspaces of HD such that $dA\perp dB$. Let $W\in\mathfrak{W}$ and $f\in C^{\omega}(\partial W)$. Then there exist $f_A\in C^{\omega}(\partial W)$ and $u_B\in B$ such that $L_W^{A+B}f=L_W^A f_A+u_B$ on W.*

Proof Corollary 2 to Lemma 3 guarantees that $A+B$ induces the regular operator L_W^{A+B}. We find $u_A\in A$, $u_B\in B$ and $f_0\in D_0$ such that $L_W^{A+B}f=u_A+u_B+f_0$ on W. Let u be any function in A. Because of $dA\perp dB$, we have $0=(du, du_B)_R=\int_{\beta}u(du_B)^*$. This, together with

(c) in Theorem 2, implies $0=\int_\beta u(d(u_A+f_0))^*$, that is, $(d(u_A+f_0))^* \in \Sigma_A(W)$. It follows from Theorem 2 that $L_W^A(u_A+f_0)=u_A+f_0$ on W. If we set $f_A=u_A+f_0$ on ∂W, then $f_A \in C^\omega(\partial W)$ and $L_W^{A+B}f= L_W^A f_A+u_B$.

Let L be regular operator with respect to $W\in\mathfrak{W}$. Now we are going to find a subspace A of HD which induces L for W, i.e., $L_W^A=L$. Let $f\in C^\omega(\partial W)$. We extend Lf onto $R-W$ to be a Dirichlet function F on R and denote by u_f^L the harmonic part of F in the Royden decomposition, which is uniquely determined by Lf on account of Proposition 2. Consider the following subfamily of HD:

$$\mathfrak{A}(L)=\{u_f^L;\ f\in C^\omega(\partial W)\}$$

and denote by A_L the closure of $\mathfrak{A}(L)$ in HD. With these preparations we prove

Lamma 4 *Let L be a regular operator with respect to $W\in\mathfrak{W}$. Then the subspace A_L induces L for W.*

Proof Let $f\in C^\omega(\partial W)$. We shall prove $L_{A_L}^W f=Lf$ by applying Theorem 2 to $A=A_L$. It is clear that Lf satisfies (a), (b). Let $u_g^L(=w)$ be any function in $\mathfrak{A}(L)$. Then we find from the definition of w a Dirichlet potential g_0 such that $w+g_0=Lg$ on W. Condition (3′) in §2, together with Proposition 6, implies that $\int_\beta w(dLf)^*=\int_\beta (Lg)(dLf)^*-\int_\beta g_0(dLf)^*=0$, and hence that $(dw, dLf)_W=-\int_{\partial W} w(dLf)^*$ for any $w\in\mathfrak{A}(L)$. Next let w be any function in A_L. Then there exists a sequence $\{w_n\}$ in $\mathfrak{A}(L)$ such that $\lim_{n\to\infty}|||w_n-w|||_R=0$. It follows that $\lim_{n\to\infty}\|dw_n-dw\|_W\leqq\lim_{n\to\infty}\|dw_n-dw\|_R=0$ and $\{w_n\}$ converges to w uniformly on ∂W. We have thus $(dw, dLf)_W=\lim_{n\to\infty}(dw_n, dLf)_W=-\lim_{n\to\infty}\int_{\partial W_n} w_n(dLf)^*=-\int_{\partial W} w(dLf)^*$, and hence $\int_\beta w(dLf)^*=0$, proving $(dLf)^*\in\Sigma_{A_L}(W)$.

§7. Existence theorem for principal functions

Let L be a regular operator with respect to $W \in \mathfrak{W}$. Let s be a harmonic function on $\bar{W}$ except for isolated singularities not accumulating to the boundary ∂W. We investigate the existence and the uniqueness of function p harmonic on R except for the singularities of s which satisfies the following equation:

$$p-s=L(p-s) \quad \text{on} \quad W.$$

Theorem 3[6] (I) *If L is canonical, the necessary and sufficient condition for the existence of p is that $\int_{\partial W}(ds)^*=0$. The function p is uniquely determined up to an additive constant.*

(II) *If L is not canonical, then for any s the function p exists and is uniquely determined.*

Proof We denote by $\{a_i\}$ the set of singular points of s. The necessity in (I) is immediately proved. To prove the uniqueness in (I) or (II), suppose that both p_1 and p_2 satisfy $p-s=L(p-s)$ on W. Then $p_1-p_2 \in HD(R)$ and $p_1-p_2=L(p_1-p_2)$ on W. By virtue of uniqueness theorem in §2, we see that $p_1=p_2+\text{const.}$ if L is canonical, and $p_1=p_2$ if L is not canonical.

The sufficiency in (I): Suppose a given s satisfies the condition $\int_{\partial W}(ds)^*=0$. We extend s on $R-W$ so that we obtain $\hat{s} \in C^2(R-W)$. Lemma 2, together with $\int_{\partial W}(ds)^*=0$ and the fact that $R-W$ is connected, implies that $(ds)^*$ is also extendible to a closed differential σ on $R-W$, that is, $\sigma \in \Gamma_c^1(R-W)$ and $\sigma=(ds)^*$ on $\bar{W}-\{a_i\}$. Then since $\widehat{ds}+\sigma^*$ is identically zero on $W-\{a_i\}$, it is square integrable on R. Namely,

6) The existence and the uniqueness of p for L_0-operator are well-known (see [1], III, 3A). Using this fact, we can easily prove Theorem 3 (see §11). Here we shall prove Theorem 3 by a method of orthogonal decomposition, which is different from the one used in M. Nakai and L. Sario [7]. In this connection confer with B. Rodin and L. Sario [12].

$\widehat{ds}+\sigma^*\in\Gamma$. Now we take the subspace $A_L=A$ inducing L for W (Lemma 4) and consider the orthogonal decomposition:

$$\Gamma=d(A+D_0)+((dA)^{\perp}+\Gamma_{e0}^*).$$

We use this to obtain

$$\widehat{ds}+\sigma^*=dF+\omega \quad \text{on} \quad R,$$

where $F\in A+D_0$ and $\omega\in(dA)^{\perp}+\Gamma_{e0}^*$. On rewriting the equation in the form

$$\widehat{ds}-dF=-\sigma^*+\omega \quad \text{on} \quad R-\{a_i\},$$

we find that the differential on the left is exact (and hence closed) and the differential on the right is coclosed on any region which does not contain any a_i. Therefore the above differential is harmonic on $R-\{a_i\}$ (Weyl's lemma). In particular, we may assume that F is of class C^2 on R and ω is of class C^1 on R. If we set $p=\hat{s}-F$ on $R-\{a_i\}$, the function p is harmonic on $R-\{a_i\}$. Let us prove $L(p-s)=p-s$ on W. Since $\hat{s}=s$ on $\overline{W}$, it is enough to prove $LF=F$ on W. It is clear that F satisfies conditions (a), (b) in Theorem 2 for $f=F$. Since $\sigma=(ds)^*$ on W, we have, on W, $(dF)^*=(\widehat{ds}+\sigma^*-\omega)^*=(ds)^*-\sigma-\omega^*=-\omega^*$. It follows from $\omega^*\in(dA)^{\perp *}+\Gamma_{e0}\cap\Gamma^1$ and the remark in §6 that $(dF)^*|W\in\Sigma_A(W)$. Consequently Theorem 2 shows $L_W^A F=F$ on W, or $LF=F$ on W. Hence p is one of the required functions.

The existence in (II): For given L consider the canonical operator L' which is defined in the remark in §2. Also for a given s consider the function $s'=s-cL1$ on W where $c=\int_{\partial W}(ds)^*/\int_{\partial W}(dL1)^*$. Then we can apply (I) to these L' and s', and have a harmonic function p' on R such that $p'-s'=L'(p'-s')$ on W. Precisely speaking, $p'-(s-Lc)=L(p'-(s-c)-c_1)+c_1$ on W where $c_1=\int_{\partial W}(dL(p'-s'))^*/\int_{\partial W}(dL1)^*$. If we set $p=p'-c_1$ on R, then p is harmonic on $R-\{a_i\}$ and $p-s=L(p-s)$ on W.

p will be called the *principal function* associated with s for L.

§8. Uniqueness for A which induces a given L

Lemma 5 *Let A be any subspace of HD and W be any set in $\mathfrak{W}$. Then $\overline{\mathfrak{A}(L_W^A)}=A$.*

Proof First we shall show that, for arbitrary two points a and b in $R-\overline{W}$, there exists a function $u_{a,b}^A=u_{a,b}$ in $\mathfrak{A}(L_W^A)(\subset A)$ such that $(du, du_{a,b})_R=u(a)-u(b)$ for any $u\in A$. Let $\bar{\Delta}_a$ (resp. Δ_b) be a closed disk in $R-\overline{W}$ with center at a (resp. b) and let $\bar{\Delta}_a\cap\bar{\Delta}_b=\phi$. We set $W_1=W\cup\Delta_a\cup\Delta_b$. We apply Theorem 3 to $W=W_1$, $L=L_{W_1}^A$ and $s=\log 1/|z-a|$ on Δ_a, $=-\log 1/|z-b|$ on Δ_b and $=0$ on W. Then $\int_{\partial W_1}(ds)^*=0$, and hence we can solve the equation $p-s=L(p-s)$ on W_1. Thus there exists a harmonic function $G_{a,b}(z;A)=G^A(z)=G(z)$ on $R-\{a,b\}$ such that $G(z)-\log 1/|z-a|$ and $G(z)+\log 1/|z-b|$ are harmonic on Δ_a and Δ_b respectively and that $L_W G=G$ on W.[7] Hence we find $v\in\mathfrak{A}(L_W^A)$ and $f_0\in D_0$ such that $G=v+f_0$ on W. Now we consider a special case: $A=\{0\}$ and write $G_{a,b}(z;\{0\})=G^0(z)$. Then $G^0=0+g_0=g_0$ on W where g_0 is a certain Dirichlet potential on R. Set $u_{a,b}=1/2\pi(G-G^0)$. Since $u_{a,b}\in HD(R)$ and $u_{a,b}=1/2\pi(((v+f_0)-g_0)$ on W, it follows from Proposition 2 that $u_{a,b}=(1/2\pi)v$, proving $u_{a,b}\in\mathfrak{A}(L_W^A)$. Let u be any function in A. By (c) in Theorem 2 we have $\int_\beta u(dG)^*=\int_\beta u(dL_W^A G)^*=0$. According to Proposition 6 we also have $\int_\beta G^0(du)^*=\int_\beta g_0(du)^*=0$. Computing Cauchy's principal values, we have thus $(du, dG)_R=2\pi(u(a)-u(b))$ and $(du, dG^0)_R=0$. Consequently, $(du, du_{a,b})_R=u(a)-u(b)$, proving the assertion.

Next we shall show $\overline{d[\mathfrak{A}(L_W^A)]}=dA$. We write simply $\overline{\mathfrak{A}(L_W^A)}$

7) It is an immediate consequence of the defintion of regular operators that $L_{W_1}^A$ is composed of H^{Δ_a}. H^{Δ_b} and L_W^A on Δ_a, Δ_b and W respectively where $H^{\Delta_a}f$, for instance, is defined by $H_f^{\Delta_a}$. Hence $H^{\Delta_a}{}_{G-\log 1/|\zeta-a|}(z)=G(z)-\log 1/|z-a|$ and $H^{\Delta_b}{}_{G+\log 1/|\zeta-b|}(z)=G(z)+\log 1/|z-b|$ on Δ_a and Δ_b.

$=A'$. Let u be any function of A such that du is orthogonal to dA'. Then since $du_{a,b}\in d\mathfrak{A}(L_W^A)\subset dA'$ for any $a, b\in R-\overline{W}$, we have $0=(du, du_{a,b})_R=u(a)-u(b)$. Hence u is a constant on a non-empty open set $R-\overline{W}$, or on R. Namely $du=0$. We have thus $(dA')^{\perp}\cap(dA)=\{0\}$. On the other hand, $dA\supset dA'$, because $A\supset\mathfrak{A}(L_W^A)$. Since dA and dA' are closed (Lemma 3), it follows that $dA=dA'+(dA')^{\perp}\cap(dA)=dA'$. Finally let u be any function in A. Then $dA=dA'$ implies that there is a function $v\in A'$ such that $dv=du$, that is, $v=u+c$ where c is a constant. If $A\ni 1$, then L_W^A is canonical (Theorem 1,) and hence $\mathfrak{A}(L_W^A)\ni 1$. Consequently $A'\ni u$. If $A\not\ni 1$, then $c=u-v(\in A)$ must be 0. Hence $A'\ni v=u$. Therefore $A=A'$.

Remark In the above proof we do not assume $A\ni 1$ or $\not\ni 1$. Now suppose that $A\not\ni 1$. Fix a point a in R and let $W\in\mathfrak{W}$ such that $\overline{W}\not\ni a$. Since L_W^A is not canonical, we use (II) in Theorem 3, by a similar method to that in the construction of $G_{a,b}(z; A)$, to obtain a harmonic function $G_a(z; A)=G_a(z)$ on $R-\{a\}$ which has the singularity $\log 1/|z-a|$ at a and has $L_W^A G_a=G_a$ on W. The same reasoning as in the above proof implies that $u_a^A=u_a=G_a-G^0 \in\mathfrak{A}(L_W^A)$ and, for any $u\in A$, $(du, du_a)_R=u(a)$. Using the terminology in Corollary 2 in §5, we have thus $T_a u=\langle u, u_a\rangle$ for any $u\in A$.

Theorem 4 *Let L be a regular operator with respect to $W\in\mathfrak{W}$. Then there exists a unique subspace of HD which induces L for W.*

Proof The existence was shown in Lemma 4. In order to prove the uniqueness, let A be any subspace such that $L_W^A=L$. Then, it follows from Lemma 5 that $A=\overline{\mathfrak{A}(L_W^A)}=\overline{\mathfrak{A}(L)}=A_L$, which is defined in §6. Hence A is uniquely determined by L and coincides with A_L.

For a given regular operator L with respect to W, we denote in the sequel by $A(L)$ the subspace inducing L for W, i.e., $L_W^{A(L)}=L$.

Consider the following subset of HD:

$$A^L=\{u\in HD;\ \int_\beta u(dLf)^*=0 \text{ for all } f\in C^\omega(\partial W)\}.$$

With these notations we have

Corollary 1 $A^L=A_L=A(L)$.

Proof It is clear that A^L is closed in HD. Hence it is sufficient to prove that A^L induces L for W. For the sake of convenience we write $L_W^{A^L}=L_1$ and $L_W^{A_L}=L_2$. Let u be any function of A_L. Then we see from $L_2=L$ (Lemma 4) and (c) in Theorem 2 that, for any $u\in A_L$, $\int_\beta u(dLf)^*=\int_\beta u(dL_2f)^*=0$ for all $f\in C^\omega(\partial W)$. Hence $A^L\supset A_L$. If f is any function in $C^\omega(\partial W)$, we have thus $\|dL_1f\|_W\leqq\|dL_2f\|_W=\|dLf\|_W$. On the other hand, we find $u\in A^L$ and $f_0\in D_0$ such that $L_1f=u+f_0$ on W. $\int_\beta u(dLf)^*=0$, together with Proposition 6, implies $\int_\beta (L_1f)(dLf)^*=0$. It follows that $(dL_1f, dLf)_W=\int_{\partial W} f(dLf)^*=\|dLf\|_W^2$, and hence $\|dL_1f\|_W\geqq\|dLf\|_W$. Consequently $L_1f=Lf$ on W.

In §3 we have seen that a subspace A of HD induces a consistent system $\mathcal{L}^A=\{L_W^A\}_{W\in\mathfrak{M}}$. Here we shall state the converse as

Corollary 2 *Let* $\mathcal{L}=\{L_W\}_{W\in\mathfrak{M}}$ *be any consistent system. Then there exists a unique subspace A such that* $\mathcal{L}^A=\mathcal{L}$.

Proof The uniqueness is clear from Theorem 4. Let us prove the existence. For each $W\in\mathfrak{W}$, consider the subspace $A(L_W)$. Let W_1, W_2 be any set in $\mathfrak{W}$ such that $W_1\supset W_2$. We write simply $L_{W_1}=L_1$ and $L_{W_2}=L_2$. Then, since $L_2(L_1f)=L_1f$ on W_2 for any $f\in C^\omega(\partial W_1)$, we easily obtain $A^{L_1}\supset A^{L_2}\supset\mathfrak{A}(L_2)\supset\mathfrak{A}(L_1)$. Because of $A^{L_1}=A_{L_1}=\overline{\mathfrak{A}(L_1)}$ (Corollary 1), we have $A(L_1)=A^{L_1}=A^{L_2}$

$=A(L_2)$. It follows that all the $A(L_W)$ coincide and hence we denote it by A. Then, for any $W\in\mathfrak{W}$, $L_W^A=L_W^{A(L_W)}=L$. That is, $\mathcal{L}^A=\mathcal{L}$.

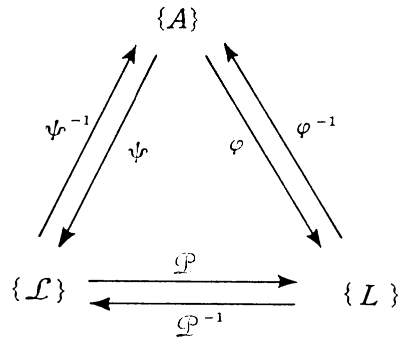
Fix W_0 in $\mathfrak{W}$. Consider the following families:

$\{L\}$ = the family of regular operators with respect to W_0,

$\{A\}$ = the family of subspaces of HD,

$\{\mathcal{L}\}$ = the family of consistent systems of regular operators.

On account of Theorem 4 we have an onto and one-to-one mapping φ: $\{A\}\to\{L\}$ such that $\varphi(A)=L_{W_0}^A$. By Corollary 2 we also obtain an onto and one-to-one mapping ψ: $\{A\}\to\{\mathcal{L}\}$ such that $\psi(A)=\mathcal{L}^A$. Denote by $\mathcal{P}$ the projection: $\{\mathcal{L}\}\to\{L\}$, i.e., $\mathcal{P}(\mathcal{L})=L_{W_0}$ where $\mathcal{L}=\{L_W\}_{W\in\mathfrak{M}}$. Then we obtain $\mathcal{P}=\varphi\circ\psi^{-1}$, and hence $\mathcal{P}$ is onto and one to one. Otherwise stated, *for a given regular operator L with respect to W_0 there exists a unique consistent system $\{L_W\}_{W\in\mathfrak{M}}$ such that $L_{W_0}=L$.* Summarizing the result, we have the following commutative diagram:



Let L be a regular operator with respect to W and let a be an arbitrary point in W. Then we have the following

Proposition 7 *There exists a signed measure $\mu_a^L=\mu_a$ which satisfies $\int_{\partial W} f d\mu_a=Lf(a)$ for all $f\in C^\omega(\partial W)$. Hence we can extend the domain of L from $C^\omega(\partial W)$ to $C(\partial W)$, where $C(\partial W)$ is the* *family of continuous functions on* ∂W.

Proof We choose a point b in $R-\overline{W}$ and consider the subspace $A(L)$. In the proof of Lemma 5 we constructed a harmonic function $G_{a,b}(z; A(L))=G(z)$ on $R-\{a,b\}$ which has positive and negative logarithmic singularities at a and b respectively and has $\mathcal{L}^{A(L)}$-behavior on β (see §3). If we set $\widetilde{G}_a(z; L, W)=\widetilde{G}(z)=G(z)-LG(z)$ on W, then the function $\widetilde{G}$ is a harmonic function $W-\{a\}$ which has a positive logarithmic singularity at a, assumes continuously 0 on ∂W and $\mathcal{L}^{A(L)}$-behavior on β. Therefore, computing $(dLf, d\widetilde{G})_W$ as Cauchy's principal value, we have $Lf(a)=-(1/2\pi)\int_{\partial W} f(d\widetilde{G})^*$. Hence the required measure μ_a exists and $d\mu_a=-(1/2\pi)(\partial\widetilde{G}/\partial n)ds$.

§9. Reproducing kernels

Let Γ_x be any subspace of $\Gamma_h=\Gamma_h(R)$ and γ be a 1-chain of R. Then there exists a unique element $\omega_\gamma^x=\omega_\gamma$ in Γ_x such that $(\omega, \omega_\gamma)_R=\int_\gamma \omega$ for all $\omega\in\Gamma_x$. The differential ω_γ is called the *reproducing kernel* associated with γ for Γ_x (see [1], [2], [11] or [14]). B. Rodin expressed the kernels for several known subspaces in terms of principal functions for L_0 and $(P)L_1$. But the situation is different for the kernel for Γ_S as he could not treat. Here we shall express the kernel for any subspace Γ_x such that $\Gamma_x\subset\Gamma_{he}$ or $\Gamma_x\supset\Gamma_{h0}$ in terms of principal functions for regular operators. The proof can be achieved, under our existence theorem, by a method similar to B. Rodin [11]. Since $\Gamma_S\supset\Gamma_{h0}$, this will give us an answer to Rodin's question ([11], p. 989, Remark). We may assume that γ is 1-simplex contained in a parametric disk $\Delta=\{|z|<1\}$ and write $\partial\gamma=a-b$.

The case where $\Gamma_x\subset\Gamma_{he}$: Let $A^x=A=\{u\in HD; du\in\Gamma_x\}$. Then A is a subspace of HD containing 1. Thus A induces a consistent system $\mathcal{L}^A$ of canonical operators. By exactly the same reasoning as in the proof of Lemma 5 we have

$$\omega_\gamma = (1/2\pi)(dG - dG^0)$$

where G (resp. G_0) *is a harmonic function on* $R-\{a, b\}$ *which has logarithmic singularities with coefficients* $+1$, -1 *at* a, b *respectively and has* $\mathcal{L}^A$-(resp. $\mathcal{L}^{\{0\}}$-) *behavior on* β.

The case where $\Gamma_x \supset \Gamma_{h0}$: Consider the orthogonal complement $\Gamma_x^\perp$ of Γ_x. $\Gamma_x \supset \Gamma_{h0}$ implies $\Gamma_x^{\perp *} \subset \Gamma_{h0}^{\perp *} = \Gamma_{he}$. If we write $B = \{u \in HD;\ du \in \Gamma_x^{\perp *}\}$, then B also induces the canonical system $\mathcal{L}^B = \{L_W^B\}_{W \in \mathfrak{M}}$. We set $s = \arg(z-a)/(z-b)$ on $\Delta - \gamma$ and $=0$ on $W \in \mathfrak{W}$ such that $\overline{W} \cap \overline{\Delta} = \phi$ and $R - \overline{W}$ is a disk. Taking $R-\gamma$ to be a given Riemann surface, we chocse the regular operator L with respect to $W \cup (\Delta - \gamma)$ such that for any $f \in C^\omega((\partial W) \cup (\partial \Delta))$, Lf is $L_W^A f$ on W and is the restriction to $\Delta - \gamma$ of the Dirichlet solution H_f^Δ. Applying (I) in Theorem 3, we have a harmonic function p on $R-\gamma$ satisfying $p - s = L(p-s)$ on $W \cup (\Delta - \gamma)$. That is, $p = L_W^B p$ on W and $p(z) = \arg(z-a)/(z-b) + u(z)$ on $\Delta - \gamma$ where u is a harmonic function on Δ. Since $\Gamma_x = (dB)^{\perp *}$, condition (b) in Theorem 2, together with the remark in §6, implies $0 = \int_\beta (L_B^W p)\omega = \int_\beta p\omega$ for all $\omega \in \Gamma_x$, and hence $(\omega, (dp)^*)_R = (\omega, (dp)^*)_{R-\gamma} = \int_{(\gamma)} (\arg(z-a)/(z-b))\omega = 2\pi \int_\gamma \omega$, where $\int_{(\gamma)}$ indicates the integration carried arround γ. Since $G^0 = g_0$ on W for some $g_0 \in D_0 \cap C^1(R)$, it follows from Proposition 6 that $(\omega, dG^0)_R = 0$ as a Cauchy's principal value. Hence $(\omega, 1/2\pi((dp)^* + dG^0))_R = \int_\gamma \omega$. On the other hand, $(dp)^* + dG^0 \in \Gamma_x$. In fact, obviously $(dp)^* + dG^0 \in \Gamma_h$. We see from the remark in §6 and (c) in Theorem 2 that there exists $\omega_x \in \Gamma_x$ and $\omega_{e0} \in \Gamma_{e0} \cap \Gamma^1$ such that $(dp)^* = (dL_W^B)^* = \omega_x + \omega_{e0}$ on W. We have thus $(dp)^* + dG^0 = \omega_x + \omega_{e0} + dg_0$ on W. On rewriting the equation in the form

$$(dp)^* + dG^0 - \omega_x = \omega_{e0} + dg_0 \text{ on } W,$$

we find that the differential on R on the left belongs to Γ_{he}, because

$R-W$ is simply connected. Since the differential on R on the right belongs to $dD_0=\Gamma_{e0}$, it follows from Proposition 2 that $(dp)^*+dG^0-\omega_x=0$ on R, proving $(dp)^*+dG^0\in\Gamma_x$. Hence

$$\omega_\gamma=1/2\pi((dp)^*+dG^0),$$

where dp is a harmonic differential on $R-\{a, b\}$ such that p is single valued on $R-\gamma$, has $\mathcal{L}^B$-behavior on β and has the following form near γ: $p=\arg(z-a)/(z-b)+u$ where u is a harmonic function on Δ.

§10. Convergence theorem

Let S be a subset of HD. We denote by $[\overline{S}]$ the smallest subspace of HD containing S.

Theorem 5 (I) *Let $\{A_n\}$ be a sequence of subspaces of HD such that $\bigcap_{n=1}^{\infty}[\overline{\bigcup_{k=n}^{\infty}A_k}]=\overline{\bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty}A_k)}$, which we denote by A. Then, for any $W\in\mathfrak{W}$ and $f\in C^\omega(\partial W)$, we have $\lim_{n\to\infty}\|dL_W^{A_n}f-dL_W^{A}f\|_W=0$.*

(II) *Let $\{\Omega_n\}$ be a sequence of regions in R such that $\Omega_{n+1}\supset\Omega_n$ and $\bigcup_{n=1}^{\infty}\Omega_n=R$. Suppose that A_n* (resp. A) *is a subspace of HD (Ω_n)* (resp. $HD(R)$) *which satisfies the following conditions*: (α) *For each $u\in A$, there exists $u_n\in A_n$ such that $\lim_{n\to\infty}|||u_n-u|||_{\Omega_n}=0$.* ($\beta$) *If $\{u_{n_j}\}$, $u_{n_j}\in A_{n_j}$ is a sequence such that $\sup\|du_{n_j}\|_{\Omega_{n_j}}<\infty$ and $\{u_{n_j}\}$ converges to u locally uniformly on R, then u belongs to A. Then, for any $W\in\mathfrak{W}$ and $f\in C^\omega(\partial W)$, we have $\lim_{n\to\infty}\|dL_{W_n}^{A_n}f-dL_W^{A}f\|_{W_n}=0$ where $W_n=W\cap\Omega_n$.*

Proof (I) If we set $B_n=[\overline{\bigcup_{k=n}^{\infty}A_k}]$, then $B_n\supset B_{n+1}$ and $\bigcap_{n=1}^{\infty}B_n=A$. We write simly $L_W^{B_n}=L_n$ and $L_W^A=L$. We can find $u_n\in B_n$ and $f_{0n}\in D_0$ such that $L_nf=u_n+f_{0n}$ on W and $u_n+f_{0n}=H_f^{R-W}$ on $R-W$. Since $B_n\supset B_{n+1}\supset A$, we have $\|dL_nf-dL_{n+1}f\|_W^2=\|dL_{n+1}f\|_W^2-\|dL_nf\|_W^2\geqq0$ and $\infty>\|dLf\|_W\geqq\|dL_nf\|_W$. Hence $0=\lim_{n\to\infty}\|dL_nf-dL_{n+1}f\|_W^2$

$=\lim_{n\to\infty}\|d(u_n+f_{0n})-d(u_{n+1}+f_{0n+1})\|_R^2=\lim_{n\to\infty}(\|du_n-du_{n+1}\|_R^2+\|df_{0n}-df_{0n+1}\|_R^2)$. Therefore using Propositions 3, 4 by the same method as in the proof of Lemma 1, we see that there exist subsequences $\{u_{n_k}\}$, $\{f_{0n_k}\}$ and $u\in HD$, such that $\lim_{k\to\infty}|||u_{n_k}-u|||_R=\lim_{k\to\infty}\|df_{0n_k}-df_0\|_R=0$ and $\{f_{0n_k}\}$ converges to f_0 quasi-everywhere on R. It follows that $u\in B_{n_k}$ for all k, i.e., $u\in A$. Consequently, $u+f_0\in M_f^A\subset M_f^{B_{n_k}}$. This implies that

$$\lim_{k\to\infty}\|dLf-dL_{n_k}f\|_W^2=\lim_{k\to\infty}(\|dLf\|_W^2-\|dL_{n_k}f\|_W^2\leq\lim_{k\to\infty}(\|d(u+f_0)\|_W^2-\|dL_{n_k}f\|_W^2)=\lim_{k\to\infty}\{(\|du\|_R^2-\|du_{n_k}\|_R^2)+(\|df_0\|_R^2-\|df_{0n_k}\|_R^2)\}=0.$$

Since $\{B_n\}$ decreases, we have thus $\lim_{n\to\infty}\|dL_W^Af-dL_W^{B_n}f\|_W^2=0$. On the other hand, if we set $C_n=\bigcap_{k=n}^{\infty}A_k$, we have similarly $\lim_{n\to\infty}\|dL_W^Af-dL_W^{C_n}f\|_W=0$. It follows from $C_n\subset A_n\subset B_n$ that $\lim_{n\to\infty}\|dL_W^Af-dL_W^{A_n}f\|_W=0$.

(II) We write simply $L_{W_n}^{A_n}=L_n$, $L_W^A=L$ and $R-\overline{W}=G$. We can find $u_n\in A_n$, $f_{0n}\in D_0(\Omega_n)$ such that $u_n+f_{0n}=L_nf$ on W_n and $=H_f^G$ on G. Observe that $\|dL_nf\|_{W_n}\leq\|dL_{W_n}^{\{0\}}f\|_{W_n}=\|dH_f^{W_n}\|_{W_n}\leq\|dH_f^{W_1}\|_{W_1}(=M_1)<\infty$ where $H_f^{W_n}$ denotes the Dirichlet solution on W_n whose boundary values are f on ∂W and 0 on the ideal boundary $\beta(\Omega_n)$ of Ω_n. We have $\|du_n\|_{\Omega_n}^2+\|df_{0n}\|_{\Omega_n}^2=\|du_n+df_{0n}\|_{\Omega_n}^2=\|dL_nf\|_{W_n}^2+\|dH_f^G\|_G^2\leq M_1^2+\|dH_f^G\|_G^2$ $(=M_2)<\infty$. Hence we see that $\{\|df_{0n}\|_{\Omega_n}\}$ is bounded. This implies that a subsequence $\{f_{0n_k}\}$ converges locally uniformly on W. In fact, because of the fact that $f_{0n}=H_{f_{0n}}^{W_n}$ on W_n and that $\|df_{0n}\|_{W_n}\leq M_2$ it is enough to show that, for a fixed point a in W, $\{f_{0n}(a)\}$ is bounded. Since the harmonic measure $\omega_a^{W_n}$ (for Ω_n) has finite energy, $f_{0n}\in D_0(\Omega_n)$ yields (see [4], p. 79) $(df_{0n}, dp^{\omega_a^{W_n}})_{\Omega_n}=2\pi\int_{\partial G}f_{0n}d\omega_a^{W_n}=2\pi H_{f_{0n}}^{W_n}(a)=2\pi f_{0n}(a)$. $\{\|dp^{\omega_a^{W_n}}\|_{\Omega_n}\}$ is bounded, because $\lim_{n\to\infty}\|dp^{\omega_a^{W_n}}\|_{\Omega_n}^2=2\pi\lim_{n\to\infty}\int_{\partial G}p^{\omega_a^{W_n}}d\omega_a^{W_n}=2\pi\lim_{n\to\infty}H_{g_a^{\Omega_n}}^{W_n}(a)=2\pi H_{g_a^R}^{W}(a)=\|dp^{\omega_a^W}\|_R^2<\infty$ where $g_a^{\Omega_n}$ and g_a^R are Green's functions with pole at a on Ω_n and R respectively. It follows from Schwarz's inequality that $\{f_{0n}(a)\}$ is bounded.

Now, since $\|dL_{n_k}f\|_{W_{n_k}} \leqq M_1$ and $L_{n_k}f=f$ on ∂W for all k, we can find a subsequence $\{L_{n_{k_j}}f\}$ (which we denote by $\{L_\nu f\}$) which converges locally uniformly on an open set containing $\overline{W}$. Since $u_\nu = L_\nu f - f_{0\nu}$ on $\overline{W}_\nu$, we conclude that $\{u_\nu\}$ converges locally uniformly to u on R. In particular, $\{f_{0\nu}\} (=\{L_\nu f - u_\nu\})$ converges to $f_0 \in C^\omega(\partial W)$ uniformly on ∂G, and hence from $f_{0\nu} = H^{W_n}_{f_{0\nu}}$ on W_n it converges to $H^W_{f_0}$ uniformly on $\overline{W}$. Because of $\|du_\nu\|_{\Omega_\nu} \leqq M_2$, condition ($\beta$) implies $u \in A$. If we extend $H^W_{f_0}$ to G by $H^G_{f_0}$, which we denote by g_0, then $u+g_0 \in M^A_f$. Hence we have $\|d(u+g_0)-dLf\|^2_W = \|d(u+g_0)\|^2_W - \|dLf\|^2_W \geqq 0$. Since $\{L_\nu f\}$ converges to $u+g_0$ uniformly on every compact set K on W, it follows from Fatou's lemma that $\|d(u+g_0)\|_W \leqq \varliminf_{\nu\to\infty} \|dL_\nu f\|_{W_\nu}$. Moreover $\lim_{\nu\to\infty}(dL_\nu f, d(u+g_0))_{W_\nu} = \|d(u+g_0)\|^2_W$. In fact,

$$
\begin{aligned}
&|(dL_\nu f, d(u+g_0))_{W_\nu} - \|d(u+g_0)\|^2_W| \leqq |(dL_\nu f, d(u+g_0))_{W_\nu - K}| \\
&+ \|d(u+g_0)\|^2_{W-K} + |(dL_\nu f - d(u+g_0), d(u+g_0))_K| \\
&\leqq \|dL_\nu f\|_{W_\nu} \|d(u+g_0)\|_{W-K} + \|d(u+g_0)\|_W \|d(u+g_0)\|_{W-K} + \|dL_\nu f \\
&- d(u+g_0)\|_K \|d(u+g_0)\|_W \leqq M_1(2\|d(u+g_0)\|_{W-K} + \|dL_\nu f - d(u+g_0)\|_K).
\end{aligned}
$$

Since $\sup_{K\subset W} \|d(u+g_0)\|_K = \|d(u+g_0)\|_W$ and $\lim_{\nu\to\infty}\|dL_\nu f - d(u+g_0)\|_K = 0$, this inequality yields the above relation.

On the other hand, it is clear from condition (α) that $\varlimsup_{n\to\infty}\|dL_n f\|_{W_n} \leqq \|dLf\|_W$. We have thus $\|d(u+g_0)\|_W = \lim_{\nu\to\infty}\|dL_\nu f\|_{W_\nu}$ $(=\|dLf\|_W)$. Consequently, $\lim_{\nu\to\infty}\|dL_\nu f - d(u+g_0)\|_{W_\nu} = 0$ and $u+f_0$ is equal to Lf on W, which is independent of the choice of subsequence $\{L_\nu f\}$. It follows that $\lim_{n\to\infty}\|dL_n f - dLf\|_{W_n} = 0$.

§11. Examples

Example 1 (L_0- and $(P)L_1$-operators) If we take HD (resp. $(P)HM$) for A in §3, then it contains 1 and forms a vector lattice. It follows from Theorem 1 that HD (resp. $(P)HM$) induces a canonical and positive operator L^{HD}_W (resp. $L^{(P)HM}_W$) for each $W \in \mathfrak{W}$.

By virtue of Oikawa's characterization for L_{0W} in [9], we have

$$L_W^{HD}=L_{0W} \quad \text{and} \quad L_W^{(P)HM}=(P)L_{1W}.$$

For the definition of L_{0W} and $(P)L_{1W}$ refer to Chapter III of L. Ahlfors and L. Sario [1]. In case P is the canonical partition ([1], *I*, 38*A*) also see Y. Kusunoki [5].

Remark Let us prove (I) in Theorem 3 by making use of the fact that there exists p' on R which satisfies $p'-s=L_{0W}(p'-s)$ (see footnote 6 on page 185). Consider the subspace $A=A_L$ (Lemm 4) and set $B=\{u\in HD;\ du\in(dA)^{\perp}\cap\Gamma_{he}\}$. Then $A+B=HD$ and $dA\perp dB$. It follows from the corollary to Theorem 2 and $L_W^{HD}=L_{0W}$ that there exist $g\in C^{\omega}(\partial W)$ and $u\in B$ such that $p'-s=L_W^A g+u$ on W. Therefore $p=p'-u$ is one of the required functions.

Example 2 ($L_W^{HD\cap HD^*}$-operator) We consider the following subspace: $HD\cap HD^*=\{u\in HD;\ u$ has a single valued conjugate harmonic function on $R\}$. Then $HD\cap HD^*$ contains 1 but there exist surfaces for which $HD\cap HD^*$ does not form a vector lattice. On such surfaces, $L_W^{HD\cap HD^*}$ is canonical for each $W\in\mathfrak{W}$, but for sufficiently small $W\in\mathfrak{W}$ it is not positive (Theorem 1). Since $\Gamma_S^{*\perp}=d(HD\cap HD^*)$, we see from the case where $\Gamma_x\supset\Gamma_{h0}$ in §9 that the reproducing kernel for Γ_S can be expressed in terms of $L_W^{HD\cap HD^*}$-principal functions.

Example 3 (Dirichlet operator H^W) We take $\{0\}$ for A. Then we have positive but not canonical operator $L_W^{\{0\}}$. Obviously each $L_W^{\{0\}}$ is equal to the Dirichlet operator H^W (see [3]).

Example 4 (Neumann operator N^W) We assume that a given Riemann surface R is a region in the plane whose boundary β consists of a finte number of closed analytic Jordan curves. Consider the following subset of HD:

$$\mathfrak{N}=\{u\in H(R\cup\beta);\ \int_{\beta}uds=0\}$$

and write its closure in HD by N. Then N is a subspace of HD which neither contains 1 nor forms a vector lattice. Hence L_W^N is not canonical or positive. We say that L_W^N is the *Neumann operator* with respect to W, and denote it by N^W. $N_f^W(=u)$ is characterized by the following properties: *u is a harmonic function on $\overline{W}\cup\beta$ such that $u=f$ on ∂W, $\int_\beta uds=0$ and $\partial u/\partial u=$*const. on β. Now, let $a\in R$ and let Δ be a disk with center at a and $\Delta^{1/2}$ be the concentric disk with radius one-half of Δ. We write $W=R-(\overline{\Delta}-\Delta^{1/2})$ and $W_1=W-\overline{\Delta}$, and set $s=\log 1/|z-a|$ on $\Delta^{1/2}$, $=0$ on W_1. On applying (II) in Theorem 5 with $L=N^W$, we have the harmonic function $N(a, z)$ on $R-\{a\}$ such that $N(a, z)=\log 1/|z-a|+u(z)$ on $\Delta^{1/2}-\{a\}$ where u is a harmonic function on $\Delta^{1/2}$, $\int_\beta N(a, \zeta)ds_\zeta=0$ and $\partial N(a, \zeta)/\partial n_\zeta=$const. on β. Consequently, $N(a, z)$ is what is called the *Neumann function* with pole at a (see [2]).

Example 5 Fix a point a in R and consider the following subspace of HD:

$$N_a=\{u\in HD;\ u(a)=0\}.$$

Similarly we see that $L_W^{N_a}$ is not canonical or positive. In particular, let $R=\{z;\ |z|<1\}$ and $a=0$. Then, since $\int_{|\zeta|=1} u(\zeta)ds_\zeta=2\pi u(0)$ for any $u\in H\{|z|\leqq 1\}$, we can prove $N_a=N$. Hence in this case $L_W^{N_a}$ coincides with the Neumann operator.

Let $\{\Omega_n\}$ be a canonical exhaustion of R (see [1], *I*, 29*A*). If we set $A_n=HD(\Omega_n)$ (resp. $(P)HM(\Omega_n)$, $(HD\cap HD^*)(\Omega_n)$, $\{0\}$ and $N_a(\Omega_n)$) and $A=HD$ (resp. $(P)HM$, $HD\cap HD^*$, $\{0\}$ and N_a), then it is clear that they satisfy conditions (α), (β) of (II) in Theorem 5.

Suppose that R is a hyperbolic Riemann surface. Fix $W_0\in\mathfrak{W}$. On applying Corollary 1 in §8 to $L=H^{W_0}$ and $L=L_{0W_0}$, we have the following two simple facts: i) If $u\in HD$ and $\int_\beta u(dH_f^{W_0})^*=0$ for all $f\in C^\omega(\partial W)$, then $u\equiv 0$. ii) The space HD is identical with the closure of the set {the harmonic part of the Royden decomposition of $L_{0W}f$; $f \in C^{\omega}(\partial W_0)$}.

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