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Branching Markov processes II

By

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The branching property of semi-groups and branching Markov processes were treated in Part I but the problem of construction was not discussed. We shall construct (X^0, π)-branching Markov processes in a probabilistic way. We shall first give a theorem on constructing a strong Markov process from a given Markov process by a piecing out procedure generalizing a method of Volkonsky [44], where a lemma on Markov time due to Courrège and Priouret [4] plays an important role. In chapter III, we shall apply the theorem to obtain (X^0, π)-branching Markov processes and give several examples.

The numbering continues that of the first part, pp. 237-278 of this journal. References such as [1] are to the list at the end of the first part.

II. Construction of a Markov process by piecing out

§2. 1. Construction

Let E be a locally compact Hausdorff space with a countable open base, $(W, \mathscr{B})$ be a measurable space on which a system $\{P_x, x\in E\}$ of probability measures is given, and $\mu(w, dy)$ be a stochastic kernal on $(W, \mathscr{B})\times(E, \mathscr{B}(E))$.[1] Let $\Omega = W\times E$, $\mathscr{F} = \mathscr{B}\otimes\mathscr{B}(E)$ and

1) We assume that, for every $B\in\mathscr{B}$, $P_x[B]$ is $\mathscr{B}(E)$-measurable in x. A *stochastic kernel* $\mu(w, dy)$ is a kernel such that for each w it is a probabillity in dy.

$\widetilde{\Omega}=\prod_{j=1}^{\infty}\Omega_j$ ($\Omega_j=\Omega$, $j=1, 2, \cdots$) with the product Borel field $\bigotimes_{j=1}^{\infty}\mathcal{F}_j$, ($\mathcal{F}_j=\mathcal{F}$, $j=1, 2, \cdots$). Further we define a stochastic kernel $Q(x, d\omega)$ on $(E, \mathcal{B}(E))\times(\Omega, \mathcal{F})$ by

$$Q(x, A)=\iint_A P_x[dw]\mu(w, dy), \qquad A\in\mathcal{F}, \tag{2.1}$$

where we denote $\omega=(w. y)$. The following theorem is a direct consequence of Ionescu-Tulcea's theorem (cf. [29] p. 137).

Theorem 2.1. *There exists a unique system* $\{\widetilde{P}_x, x\in E\}$ *of probability measures on* $(\widetilde{\Omega}, \bigotimes_{j=1}^{\infty}\mathcal{F}_j)$ *such that, for every measurable function* $F(\omega_1, \omega_2, \cdots, \omega_n)$ *on* $(\prod_{j=1}^{n}\Omega_j, \bigotimes_{j=1}^{n}\mathcal{F}_j)$ $(n=1, 2, \cdots)$,

$$\widetilde{E}_x[F(\omega_1, \omega_2, \cdots, \omega_n)]=\int\cdots\int_{\Omega\times\cdots\times\Omega} Q(x, d\omega_1)Q(x_1, d\omega_2)\cdots \times Q(x_{n-1}, d\omega_n)F(\omega_1, \omega_2, \cdots, \omega_n), \tag{2.2}$$

where $\omega_j=(w_j, x_j)$.

In the following we shall assume that we are given a right continuous strong Markov process $X^0=(W, \mathcal{B}_t, P_x, x\in E, x_t(w), \theta_t)$ on E such that $\mathcal{B}_t=\overline{\mathcal{B}}_{t+0}$. We assume also that X^0 has the terminal point $\Delta\in E$; the life time $\zeta(w)$ is defined by (0.7).

Definition 2.1. A stochastic kernel $\mu(w, dy)$ on $(W, \mathcal{N}_\infty)\times(E, \mathcal{B}(E))$ is called an *instantaneous distribution* if it satisfies

$$P_x[\mu(w, dy)=\mu(\theta_T w, dy),\ T<\zeta]=P_x[T<\zeta] \tag{2.3}$$

for every $\mathcal{B}_t$-Markov time T.

An instantaneous distribution gives a law which tells us how to piece out paths of the given Markov process x_t. We shall define a new process $X_t(\tilde{\omega})$, $\tilde{\omega}\in\widetilde{\Omega}$ as follows. First of all we put for $\omega=(w, y)\in\Omega=W\times E$,

$$\dot{x}_t(\omega)=\begin{cases} x_t(w), & t<\zeta(w), \\ y, & t\geq\zeta(w). \end{cases} \tag{2.4}$$

For $\tilde{\omega}=(\omega_1, \omega_2, \cdots)\in\tilde{\Omega}$, where $\omega_j=(w_j, y_j)$, putting

$$(2.5)\qquad N(\tilde{\omega})=\begin{cases}\inf\{j;\ \zeta(w_j)=0\}, \\ \infty, \quad \text{if} \quad \{\ \}=\phi,\end{cases}$$

we define $X_t(\tilde{\omega})$ on $(\tilde{\Omega}, \bigotimes_{j=1}^{\infty}\mathcal{F}_j)$ by

$$(2.6)\qquad X_t(\tilde{\omega})=\begin{cases}\dot{x}_t(\omega_1), & \text{if} \quad 0\leq t\leq\zeta(w_1), \\ \dot{x}_{t-\zeta(w_1)}(\omega_2), & \text{if} \quad \zeta(w_1)<t\leq\zeta(w_1)+\zeta(w_2), \\ \cdots\cdots & \\ \cdots\cdots & \\ \dot{x}_{t-(\zeta(w_1)+\zeta(w_2)+\cdots+\zeta(w_n))}(\omega_{n+1}), & \\ & \text{if} \quad \sum_{j=1}^{n}\zeta(w_j)<t\leq\sum_{j=1}^{n+1}\zeta(w_j), \\ \cdots\cdots & \\ \cdots\cdots & \\ \Delta, & \text{if} \quad t\geq\sum_{j=1}^{N(\tilde{\omega})}\zeta(w_j).\end{cases}$$

The life time $\tilde{\zeta}(\tilde{\omega})$ of $X_t(\tilde{\omega})$ is therefore defined by

$$(2.7)\qquad \tilde{\zeta}(\tilde{\omega})=\sum_{j=1}^{N(\tilde{\omega})}\zeta(w_j).$$

Further we shall introduce a sequence $\{\tau_n(\tilde{\omega}),\ n=0, 1, 2, \cdots\}$ of random times by

$$(2.8)\qquad \tau_0(\tilde{\omega})=0,\ \tau(\tilde{\omega})\equiv\tau_1(\tilde{\omega})=\zeta(w_1),\ \cdots$$
$$\tau_n(\tilde{\omega})=\sum_{j=1}^{n\wedge N(\tilde{\omega})}\zeta(w_j).$$

Remark 2.1. If $\mu(w, E-\{\Delta\})=1$, then clearly $\tilde{P}_x[\tau_n<\tilde{\zeta}$ for all $n=1, 2, \cdots]=1$, $x\in E-\{\Delta\}$, where $\tilde{P}_x$ is the probability measure constructed in Theorem 2.1.

Lemma 2.1. *Let $\tilde{P}_x$ be defined by Theorem* 2.1. *If we set*

$$\tilde{\Omega}_0=\{\tilde{\omega};\ X_t(\tilde{\omega}) \text{ is right continuous in } t\in[0, \infty)\},$$

then

$$\tilde{P}_x[\tilde{\Omega}_0]=1 \qquad \text{for every } x\in E.$$

Proof. If we put

$\tilde{\Omega}_1=\{\tilde{\omega};\ X_t(\tilde{\omega})$ is right continuous in $(\tau_n, \tau_{n+1}),\ n=1, 2, \cdots\}$,

$\tilde{\Omega}_2=\{\tilde{\omega};\ x_n=\lim_{t\downarrow 0} x_t(w_{n+1}),\ n=1, 2, \cdots\}$,

where $\tilde{\omega}=(\omega_1, \omega_2, \cdots)$ and $\omega_j=(w_j, x_j)$, then $\tilde{P}_x[\tilde{\Omega}_1]=1$ since $x_t(w)$ is right continuous. On the other hand, we have by the definition of the measure $\tilde{P}_x$ that

$$\tilde{P}_x[\tilde{\Omega}_2]=\lim_{n\to\infty}\int\cdots\int_{\Omega^{n+1}} Q(x, d\omega_1)Q(x_1, d\omega_2)\cdots Q(x_n, d\omega_{n+1})=1.$$

Hence we have $\tilde{P}_x[\tilde{\Omega}_0]=\tilde{P}_x[\tilde{\Omega}_1\cap\tilde{\Omega}_2]=1$.

By this lemma we can restrict every quantity defined on $\tilde{\Omega}$ to $\tilde{\Omega}_0$. Let φ_k be the projection of $\tilde{\Omega}$ to $\prod_{j=1}^{k}\Omega_j$ $(\Omega_j=\Omega)$ and define

$$\tilde{\mathcal{B}}'_{\tau_k}=\varphi_k^{-1}(\bigotimes_{j=1}^{k}\mathcal{F}')/\tilde{\Omega}_0,^{2)} \quad \text{where } \mathcal{F}'=\mathcal{N}_\infty\otimes\mathcal{B}(E), \tag{2.9}$$

$$\tilde{\mathcal{B}}=\bigvee_{k=1}^{\infty}\tilde{\mathcal{B}}'_{\tau_k}=\bigotimes_{j=1}^{\infty}\mathcal{F}'/\tilde{\Omega}_0, \quad \text{and}$$

$$\tilde{\mathcal{N}}_t=\sigma\{\tilde{\Omega}_0, \mathcal{B}(E);\ X_s(\tilde{\omega}), s\leq t\}.$$

In order to introduce new Borel fields we need

Definition 2.2. Let $T(\tilde{\omega})$ be a random time defined on $\tilde{\Omega}_0$ taking values in $[0, \infty]$. $\tilde{\omega}, \tilde{\omega}'\in\tilde{\Omega}_0$ are said to be *R_T-equivalent,* and denoted as

$$\tilde{\omega}\sim\tilde{\omega}'\ (R_T),$$

if

(a) $T(\tilde{\omega})=T(\tilde{\omega}')$,

(b) $X_s(\tilde{\omega})=X_s(\tilde{\omega}')$ for all $s\leq T(\tilde{\omega})$,

and

(c) if $\tau_k(\tilde{\omega})\leq T(\tilde{\omega})<\tau_{k+1}(\tilde{\omega})\leq\tilde{\zeta}(\tilde{\omega})$, then $\tau_k(\tilde{\omega}')\leq T(\tilde{\omega}')<\tau_{k+1}(\tilde{\omega}')\leq\tilde{\zeta}(\tilde{\omega}')$ and $\tau_j(\tilde{\omega})=\tau_j(\tilde{\omega}')$ for every $j\leq k$; while if $T(\tilde{\omega})\geq\tilde{\zeta}(\tilde{\omega})$, then $T(\tilde{\omega}')\geq\tilde{\zeta}(\tilde{\omega}')$ and $\tau_j(\tilde{\omega})=\tau_j(\tilde{\omega}')$ for every $j\geq 0$.

2) $\mathcal{B}/\tilde{\Omega}_0=\{E\cap\tilde{\Omega}_0;\ E\in\mathcal{B}\}$.

Definition 2. 3. We shall set

$$(2.10)\qquad \widetilde{\mathscr{B}}_T=\{A;\ \text{i) } A\in\widetilde{\mathscr{B}} \text{ and ii) if } \tilde{\omega}\in A \text{ and } \tilde{\omega}\sim\tilde{\omega}'\ (R_T),\ \text{then } \tilde{\omega}'\in A\}.$$

It is clear that $\widetilde{\mathscr{B}}_T$ is a Borel field on $\widetilde{\Omega}_0$. Several properties of $\widetilde{\mathscr{B}}_T$ are given in the following lemma.

Lemma 2. 2. (i) *$\{\widetilde{\mathscr{B}}_t;\ t\geq 0\}$*[3] *is an increasing family of Borel fields on $\widetilde{\Omega}_0$; $\widetilde{\mathscr{B}}_s\subset\widetilde{\mathscr{B}}_t$ if $s\leq t$. Also $\widetilde{\mathfrak{N}}_t\subset\widetilde{\mathscr{B}}_t$.*

(ii) *$\widetilde{\mathscr{B}}_{\tau_k}$ defined by* (2. 10) *for τ_k* (*defined by* (2. 8)) *coincides with $\widetilde{\mathscr{B}}'_{\tau_k}$ defined by* (2. 9).[4]

(iii) *τ_n is a $\widetilde{\mathscr{B}}_t$-Markov time for each n.*

(iv) *$T(\tilde{\omega})$ is a $\widetilde{\mathscr{B}}_t(\widetilde{\mathscr{B}}_{t+0})$-Markov time if and only if*

a) *$T(\tilde{\omega})$ is $\widetilde{\mathscr{B}}$-measurable and*

b) *if $T(\tilde{\omega})\leq t$* (resp. *$T(\tilde{\omega})<t$*) *and $\tilde{\omega}\sim\tilde{\omega}'(R_t)$, then $T(\tilde{\omega})=T(\tilde{\omega}')$.*

(v) *If T is $\widetilde{\mathscr{B}}_t$-Markov time, then*

$$\widetilde{\mathscr{B}}_T=\{B;\ B\in\widetilde{\mathscr{B}} \text{ such that } B\cap\{T\leq t\}\in\widetilde{\mathscr{B}}_t \text{ for all } t\geq 0\}.$$

Proof. (i) is clear. As for (ii), take $A\in\varphi_k^{-1}(\bigotimes_{j=1}^{k}\mathscr{F})/\widetilde{\Omega}_0$ and assume that $\tilde{\omega}\in A$ and $\tilde{\omega}\sim\tilde{\omega}'$ (R_{τ_k}). Then it is clear from the Definition 2. 2 that $\tilde{\omega}'\in A$. This proves $A\in\widetilde{\mathscr{B}}_{\tau_k}$. Conversely take $A\in\widetilde{\mathscr{B}}_{\tau_k}$. If $\tilde{\omega}\in A$ and $\varphi_k\tilde{\omega}=\varphi_k\tilde{\omega}'$, then clearly $\tilde{\omega}\sim\tilde{\omega}'$ (R_{τ_k}) and hence $\tilde{\omega}'\in A$. Therefore $\varphi_k^{-1}(\varphi_k(A))\cap\widetilde{\Omega}_0=A\in\varphi_k^{-1}(\bigotimes_{j=1}^{k}\mathscr{F}')/\widetilde{\Omega}_0=\widetilde{\mathscr{B}}'_{\tau_k}$.

Since (iii) follows from (iv), we shall prove (iv). Let $T(\tilde{\omega})$ be a $\widetilde{\mathscr{B}}_t$-Markov time and assume that $\tilde{\omega}\in\{T\leq t\}\in\widetilde{\mathscr{B}}_t$. If $\tilde{\omega}'\sim\tilde{\omega}$ (R_t) then by the definition of $\widetilde{\mathscr{B}}_t$ we have $\tilde{\omega}'\in\{T\leq t\}$, i.e., $T(\tilde{\omega}')\leq t$, and if we had $T(\tilde{\omega})\leq s<T(\tilde{\omega}')\leq t$ then this would imply $\tilde{\omega}\in\{T\leq s\}$ and $\tilde{\omega}\sim\tilde{\omega}'$ (R_s).[5] Hence $\tilde{\omega}'\in\{T\leq s\}$, i.e., $T(\tilde{\omega}')\leq s$ which is impossible. Therefore we have $T(\tilde{\omega})=T(\tilde{\omega}')$. Conversely if $T(\tilde{\omega})$

3) $\widetilde{\mathscr{B}}_t$ is defined by taking $T(\tilde{\omega})\equiv t$.

4) Therefore " ' " will be omitted in the sequel.

5) It is clear that $\omega\sim\omega'$ (R_t) implies $\omega\sim\omega'$ (R_s) for all $s\leq t$. (iv) is true for any system of equivalence relations (R_t) having this property.

satisfies a) and b), then clearly $\{T\leq t\}\in\widetilde{\mathscr{B}}$; and for $\tilde{\omega}\in\{T\leq t\}$, $\tilde{\omega}\sim\tilde{\omega}'$ (R_t) implies $\tilde{\omega}'\in\{T\leq t\}$. Thus $\{T\leq t\}\in\widetilde{\mathscr{B}}_t$ and hence T is a $\widetilde{\mathscr{B}}_t$-Markov time. This proves (iv).

Finally we shall prove (v). Let B be such that $B\cap\{T\leq t\}\in\widetilde{\mathscr{B}}_t$ for all $t\geq 0$. Take $\tilde{\omega}\in B$ and assume $\tilde{\omega}'\sim\tilde{\omega}$ (R_T). Then, if we put $t=T(\tilde{\omega})$, we have $\tilde{\omega}\in B\cap\{T=t\}\in\widetilde{\mathscr{B}}_t$ and $\tilde{\omega}'\sim\tilde{\omega}$ (R_t). Therefore $\tilde{\omega}'\in B\cap\{T=t\}$ which implies $\tilde{\omega}'\in B$ and hence $B\in\widetilde{\mathscr{B}}_T$. Conversely assume $B\in\widetilde{\mathscr{B}}_T$ and take $\tilde{\omega}\in B\cap\{T\leq t\}$ and $\tilde{\omega}'$ such that $\tilde{\omega}'\sim\tilde{\omega}(R_t)$. Since T is a $\widetilde{\mathscr{B}}_t$-Markov time, if $T(\tilde{\omega})\leq t$ and $\tilde{\omega}\sim\tilde{\omega}'$ (R_t), then $T(\tilde{\omega})=T(\tilde{\omega}')$ by (iv). Hence $\tilde{\omega}\sim\tilde{\omega}'$ (R_T) but this implies $\tilde{\omega}'\in B$ and hence $\tilde{\omega}'\in B\cap\{T\leq t\}$. Thus $B\cap\{T\leq t\}\in\widetilde{\mathscr{B}}_t$.

Now we shall define the shift operator $\tilde{\theta}_t:\widetilde{\Omega}_0\to\widetilde{\Omega}_0$ as follows: for $\tilde{\omega}\equiv(\omega_1,\omega_2,\omega_3,\cdots)$,

$$(2.11)\qquad \tilde{\theta}_t\tilde{\omega}=\begin{cases}((\theta_{t-\tau_k(\tilde{\omega})}w_{k+1},x_{k+1}),\omega_{k+2},\omega_{k+3},\cdots),\\ \qquad\qquad \text{if}\quad \tau_k(\tilde{\omega})\leq t<\tau_{k+1}(\tilde{\omega})\ \text{and}\ t<\tilde{\zeta}(\tilde{\omega}),\\ (\omega^k,\omega^{k+1},\cdots),\\ \qquad\qquad \text{if}\quad t\geq\tilde{\zeta}(\tilde{\omega})\ \text{and}\ k=\inf\{j;\ x_0(w_j)=\Delta\}.\end{cases}$$

By a straightforward calculation, it is easily checked that

$$(2.12)\qquad X_s(\tilde{\theta}_t\tilde{\omega})=X_{t+s}(\tilde{\omega})\qquad \text{for all } s,t\geq 0,\ \tilde{\omega}\in\widetilde{\Omega}_0.$$

On the basis of the above notation our theorems of construction read as follows:

Theorem 2.2. *Let $X^0=\{W,\mathscr{B}_t,P_x,x_t,\theta_t\}$ be a right continuous strong Markov process on E with $\Delta\in E$ as its terminal point such that $\overline{\mathscr{B}}_{t+0}=\mathscr{B}_t$ and let $\mu(w,dx)$ be an instantaneous distribution. Then the system $X=\{\widetilde{\Omega}_0,\widetilde{\mathscr{B}}_{t+0},\widetilde{P}_x,X_t,\tilde{\theta}_t,\tilde{\zeta}\}$ defined above is a right continuous strong Markov process on E with Δ as the terminal point such that*

(i) *the process $\{X_t,t<\tau,\widetilde{P}_x\}$ is equivalent to the process $\{x_t,t<\zeta,P_x\}$ and*

(ii) *for every $B\in\mathfrak{N}_\infty$ and $A\in\mathscr{B}(E)$*

$$\widetilde{P}_x[\{\tilde{\omega};\ w_1\in B\ \text{and}\ X_\tau(\tilde{\omega})\in A\}]=\int_B P_x[dw]\mu(w,A),$$

where we write $\tilde{\omega}=(\omega_1, \omega_2, \cdots)$ *and* $\omega_j=(w_j, x_j)$.

By Remark 0.1 (iii) we have

Corollary. $X=\{\widetilde{\Omega}_0, \mathcal{F}_t, \widetilde{P}_x, X_t, \tilde{\theta}_t, \tilde{\zeta}\}$ *is strong Markov, where we set* $\mathcal{F}_t=\overline{\overline{\mathcal{B}}}_{t+0}\equiv\bigcap_{\varepsilon>0}\overline{\overline{\mathcal{B}}}_{t+\varepsilon}$.

Theorem 2.3. *If* $X^0=(x_t, P_x)$ *satisfies* $P_x[x_{t-0}(w)$ *exists in* $t\in(0,\infty)]=1$ *for all* $x\in E$, *then* $X=(X_t, \widetilde{P}_x)$ *satisfies* $\widetilde{P}_x[X_{t-0}(\tilde{\omega})$ *exists in* $t\in(0, \tilde{\zeta}(\omega))]=1$ *for all* $x\in E$. *If further,* $\sup_{x\in E-\{\Delta\}} P_x[\zeta<\infty]=\alpha<1$, *then* $\widetilde{P}_x[X_{t-0}(\tilde{\omega})$ *exists in* $t\in(0,\infty)]=1$ *for all* $x\in E$.

Theorem 2.4. *If* $X^0=(x_t, \mathcal{B}_t, P_x)$ *is quasi-left continuous and* ζ *is totally inaccessible (cf. Meyer* [31]*), then* $X=(\widetilde{\Omega}_0, \mathcal{F}_t, X_t)$ *is quasi-left continuous before* $\tilde{\zeta}$, *i.e., if* T_n, $n=0, 1, 2, \cdots$ *and* T *are* $\mathcal{F}_t$*-Markov times such that* $T_n\uparrow T$, *then*

$$\widetilde{P}_x[\lim_{n\to\infty} X_{T_n}=X_T;\ T<\tilde{\zeta}]=\widetilde{P}_x[T<\tilde{\zeta}].$$

Theorem 2.5. 1) *Let* $X^0=(x_t, \mathcal{B}_t, P_x)$ *be a Hunt process and* ζ *be totally inaccessible. Further we assume*

(2.13) $$\widetilde{P}_x[\tilde{\zeta}=+\infty]=1 \qquad \textit{for all } x\in E-\{\Delta\},$$

then $X=(\widetilde{\Omega}^0, \widetilde{P}_x, \mathcal{F}_t, X_t)$ *is a Hunt process.*

2) *In order that the condition* (2.13) *be fulfilled, it is sufficient that* $\mu(w, E-\{\Delta\})=1$ *for all* w *such that* $+\infty>\zeta(w)>0$ *and that one of the following conditions be satisfied;*

(1) $$\sup_{x\in E-\{\Delta\}} P_x[\zeta(w)<\infty]=\alpha<1, \quad \textit{or}$$

(2) $$\textit{for some } \varepsilon>0.\ \inf_{x\in E-\{\Delta\}} P_x[\zeta(\omega)>\varepsilon]=\delta>0,$$

Proof of Theorems 2.2~2.5 will be given in the following. We shall give simple applications here but they will not be used in later sections.

Example 2.1. For a given strong Markov process $X^0=(W, \mathcal{B}_t, P_x, x_t, \theta_t, \zeta)$ on E having left limits with $\Delta\in E$ as its terminal point and for a given probability kernal $\tilde{\mu}(x, dy)$ on $(E-\{\Delta\})\times(E-\{\Delta\})$, define a kernal $\mu(w, dy)$ by

$$\mu(w, dy)=\begin{cases}\tilde{\mu}(x_{\zeta(w)-}(w), dy), & \text{if } 0<\zeta(w)<+\infty \text{ and } x_{\zeta-}\in E-\{\Delta\},\\ \delta_{\{\Delta\}}(dy),^{6)} & \text{if otherwise.}\end{cases}$$

It is easy to see that $\mu(w, dy)$ is an instantaneous distribution. The case of $\tilde{\mu}(x, dy)=\delta_{\{x\}}(dy)$ was considered by Volkonsky [44].

Example 2.2. Let $E'=E^0\cup\partial E$, where E' is compact and E^0 is a dense open set of E'. Let $E=E^0\cup\{\Delta\}$ be the one-point compactification of E^0 and $X^0=(W, \mathcal{B}_t, P_x, x_t, \theta_t, \zeta)$ be a strong Markov process on E with Δ as the terminal point. Suppose, for $x\in E^0$, $P_x[\lim_{t\uparrow\zeta} x_t(w)$ exists in ∂E in the topology of E', $\zeta(w)<+\infty]=P_x[\zeta(w)<\infty]$. If for a given probability kernel $\tilde{\mu}(\xi, dy)$ on $\partial E\times E^0$ we set

$$\mu(w, dy)=\begin{cases}\tilde{\mu}(x_{\zeta-}(w), dy), & \text{if } 0<\zeta(w)<\infty,\\ \delta_{\{\Delta\}}(dy), & \text{if otherwise,}\end{cases}$$

then we get an instantaneous distribution. The process constructed by Theorem 2.2 is called an instantaneous return process (cf. Feller [7], Kunita [26]).

§2.2. Proof of Theorems

We shall give here the proof of Theorems 2.2~2.5. It will consist of several lemmas.

Lemma 2.3. *$\tilde{\mathcal{B}}=\tilde{\mathcal{B}}_{\tau_k}\vee\tilde{\theta}_{\tau_k}^{-1}(\tilde{\mathcal{B}})$ for every $k=1, 2, \cdots$.*

Proof. Since $\tilde{\mathcal{B}}\supset\tilde{\mathcal{B}}_{\tau_k}\vee\tilde{\theta}_{\tau_k}^{-1}(\tilde{\mathcal{B}})$ is clear, we will prove $\tilde{\mathcal{B}}\subset\tilde{\mathcal{B}}_{\tau_k}\vee\tilde{\theta}_{\tau_k}^{-1}(\tilde{\mathcal{B}})$. For this it is sufficient to show $\{\tilde{\omega};\ \omega_j\in B\}\cap\tilde{\Omega}_0\in\tilde{\mathcal{B}}_{\tau_k}\vee\tilde{\theta}_{\tau_k}^{-1}(\tilde{\mathcal{B}})$ for every $B\in\mathcal{F}'$ where we write $\tilde{\omega}=(\omega_1, \omega_2, \cdots)$. This follows, however, from $\{\tilde{\omega};\ \omega_j\in B\}\cap\tilde{\Omega}_0=\{\tilde{\omega};\ (\tilde{\theta}_{\tau_k}\tilde{\omega})_{j-k}\in B\}\cap\tilde{\Omega}_0\in\tilde{\theta}_{\tau_k}^{-1}(\tilde{\mathcal{B}})$ if $j>k$ and $\{\tilde{\omega};\ \omega_j\in B\}\cap\tilde{\Omega}_0\in\tilde{\mathcal{B}}_{\tau_j}\subset\tilde{\mathcal{B}}_{\tau_k}$ if $j\leq k$.

6) $\delta_{\{\Delta\}}(dy)$ is the unit measure at Δ.

Lemma 2.4. *Let $T(\tilde{\omega})$ be a $\tilde{\mathscr{B}}_{t+0}$-Markov time (resp. $\tilde{\mathscr{B}}_t$-Markov time). Then for every non-negative integer k there exists $T_k(\tilde{\omega}, \tilde{\omega}')$ on $\tilde{\Omega}_0 \times \tilde{\Omega}_0$ satisfying*

(i) *$T_k(\tilde{\omega}, \tilde{\omega}')$ is $\tilde{\mathscr{B}}_{\tau_k} \otimes \tilde{\mathscr{B}}$-measurable,*

(ii) *for fixed $\tilde{\omega}$, $T_k(\tilde{\omega}, \cdot)$ is a $\tilde{\mathscr{B}}_{t+0}$-Markov time (resp. $\tilde{\mathscr{B}}_t$-Markov time), and*

(iii) $T(\tilde{\omega}) \vee \tau_k(\tilde{\omega}) = \tau_k(\tilde{\omega}) + T_k(\tilde{\omega}, \tilde{\theta}_{\tau_k}\tilde{\omega})$.

Proof. Let $T(\tilde{\omega})$ be a $\tilde{\mathscr{B}}_{t+0}$-Markov time and set

$$T'_k(\tilde{\omega}) = T(\tilde{\omega}) \vee \tau_k(\tilde{\omega}) - \tau_k(\tilde{\omega});$$

then by the previous lemma there exists a $\tilde{\mathscr{B}}_{\tau_k} \otimes \tilde{\mathscr{B}}$-measurable function $T'_k(\tilde{\omega}, \tilde{\omega}')$ such that

$$T'_k(\tilde{\omega}) = T'_k(\tilde{\omega}, \tilde{\theta}_{\tau_k}\tilde{\omega}).$$

We modify T'_k and put

$$T_k(\tilde{\omega}, \tilde{\omega}') = \begin{cases} T'_k(\tilde{\omega}, \tilde{\omega}'), & \text{if } X_{\tau_k}(\tilde{\omega}) = X_0(\tilde{\omega}'), \\ \infty, & \text{if } X_{\tau_k}(\tilde{\omega}) \neq X_0(\tilde{\omega}'). \end{cases}$$

Clearly $T_k(\tilde{\omega}, \tilde{\omega}')$ is also $\tilde{\mathscr{B}}_{\tau_k} \otimes \tilde{\mathscr{B}}$-measurable. It is only necessary to prove (ii). For this it is sufficient to show by virtue of (iv) of Lemma 2.2 that if $T_k(\tilde{\omega}, \tilde{\omega}_1) < t$ and $\tilde{\omega}_1 \sim \tilde{\omega}_2$ (R_t), then $T_k(\tilde{\omega}, \tilde{\omega}_1) = T_k(\tilde{\omega}, \tilde{\omega}_2)$. Put $\tau_k(\tilde{\omega}) = s$ and write $\tilde{\omega} = (\omega_1, \omega_2, \omega_3, \cdots)$, $\tilde{\omega}_1 = (\omega_1^1, \omega_2^1, \omega_3^1, \cdots)$ and $\tilde{\omega}_2 = (\omega_1^2, \omega_2^2, \omega_3^2, \cdots)$. Then from $T_k(\tilde{\omega}, \tilde{\omega}_1) < t$ and $\tilde{\omega}_1 \sim \tilde{\omega}_2$ (R_t), we have $X_{\tau_k}(\tilde{\omega}) = X_0(\tilde{\omega}_1) = X_0(\tilde{\omega}_2)$. Therefore if we set

$$\tilde{\omega}'_1 = (\omega_1, \omega_2, \cdots, \omega_k, \omega_1^1, \omega_2^1, \omega_3^1, \cdots)$$
$$\tilde{\omega}'_2 = (\omega_1, \omega_2, \cdots, \omega_k, \omega_1^2, \omega_2^2, \omega_3^2, \cdots)$$

we have, noting $\tau_k(\tilde{\omega}'_1) = \tau_k(\tilde{\omega}'_2) = \tau_k(\tilde{\omega}) = s$,

$$\tilde{\omega} \sim \tilde{\omega}'_1 \sim \tilde{\omega}'_2 \ (R_{\tau_k}). \tag{2.14}$$

Moreover, we have

$$\tilde{\theta}_{\tau_k}\tilde{\omega}'_i = \tilde{\omega}_i, \quad (i = 1, 2) \text{ and} \tag{2.15}$$

$$\tilde{\omega}'_1 \sim \tilde{\omega}'_2 \ (R_{t+s}). \tag{2.16}$$

Therefore, from (2.14) and (2.15)

$$(2.17)\qquad \begin{aligned} T_k(\tilde{\omega}, \tilde{\omega}_i) &= T_k(\tilde{\omega}_i', \tilde{\theta}_{\tau_k}\tilde{\omega}_i') \\ &= \tau_k(\tilde{\omega}_i') \vee T(\tilde{\omega}_i') - \tau_k(\tilde{\omega}_i') \\ &= \tau_k(\tilde{\omega}_i') \vee T(\tilde{\omega}_i') - s, \qquad (i=1, 2) \end{aligned}$$

and also

$$(2.18)\qquad \tau_k(\tilde{\omega}_1') \vee T(\tilde{\omega}_1') = \tau_k(\tilde{\omega}_1') + T_k(\tilde{\omega}, \tilde{\omega}_1) < s+t.$$

By virtue of (iv) of Lemma 2.2, (2.18) and (2.16) imply

$$(2.19)\qquad \tau_k(\tilde{\omega}_1') \vee T(\tilde{\omega}_1') = \tau_k(\tilde{\omega}_2') \vee T(\tilde{\omega}_2').$$

Then by (2.17) we have $T_k(\tilde{\omega}, \tilde{\omega}_1) = T_k(\tilde{\omega}, \tilde{\omega}_2)$.

The proof when T is a $\tilde{\mathscr{B}}_t$-Markov time is quite similar.

Lemma 2.5. (i) *For any* $B \in \tilde{\mathscr{B}}$ *and* $A \in \tilde{\mathscr{B}}_{\tau_k}$,

$$(2.20)\qquad \tilde{P}_x[A, \tilde{\theta}_{\tau_k}\tilde{\omega} \in B] = \tilde{E}_x[\tilde{P}_{X_{\tau_k}}[B]; A].$$

(ii) *Let* $g(\tilde{\omega}, t)$ *be a bounded* $\tilde{\mathscr{B}} \otimes \mathscr{B}[0, \infty]$*-measurable function on* $\tilde{\Omega}_0 \times [0, \infty]$. *If* $\sigma(\tilde{\omega}) \geq 0$ *is* $\tilde{\mathscr{B}}_{\tau_k}$*-measurable and* $A \in \tilde{\mathscr{B}}_{\tau_k}$,

$$(2.21)\qquad \tilde{E}_x[g(\tilde{\theta}_{\tau_k}\tilde{\omega}, \sigma); A] = \tilde{E}_x[\tilde{E}_{X_{\tau_k}}[g(\cdot, s)]\big|_{s=\sigma}; A].$$

(iii) *Let* $g(\tilde{\omega}, \tilde{\omega}')$ *be a bounded* $\tilde{\mathscr{B}}_{\tau_k} \otimes \tilde{\mathscr{B}}$*-measurable function on* $\tilde{\Omega}_0 \times \tilde{\Omega}_0$. *Then for every* $A \in \tilde{\mathscr{B}}_{\tau_k}$,

$$(2.22)\qquad \tilde{E}_x[g(\tilde{\omega}, \tilde{\theta}_{\tau_k}\tilde{\omega}); A] = \tilde{E}_x[\tilde{E}_{X_{\tau_k}}[g(u, \cdot)]\big|_{u=\tilde{\omega}}; A].$$

Proof. For the proof of (i), taking $A_j \in \mathscr{F}'$, $j=1, 2, \cdots, n$, we have from the definition of $\tilde{P}_x$,

$$\begin{aligned} &\tilde{P}_x[\{\tilde{\omega};\ \omega_1 \in A_1, \omega_2 \in A_2, \cdots, \omega_n \in A_n\}] \\ &= \int_{A_1} \cdots \int_{A_k} Q(x, d\omega_1) Q(X_{\tau_1}(\tilde{\omega}), d\omega_2) \cdots Q(X_{\tau_{k-1}}(\tilde{\omega}), d\omega_k) \int_{A_{k+1}} \cdots \int_{A_n} \\ &\qquad\qquad \cdot Q(X_{\tau_k}, d\omega_{k+1}) \cdots Q(X_{\tau_{n-1}}, d\omega_n) \\ &= \int_{A_1} \cdots \int_{A_k} Q(x, d\omega_1) Q(X_{\tau_1}, d\omega_2) \cdots Q(X_{\tau_{k-1}}, d\omega_k) \\ &\qquad\qquad \cdot \tilde{P}_{X_{\tau_k}}[\{\tilde{\omega};\ \omega_1 \in A_{k+1}, \cdots, \omega_{n-k} \in A_n\}] \\ &= \tilde{E}_x[\tilde{P}_{X_{\tau_k}}[\{\tilde{\omega};\ \omega_1 \in A_{k+1}, \cdots, \omega_{n-k} \in A_n\}];\ \{\tilde{\omega};\ \omega_1 \in A_1, \cdots, \omega_k \in A_k\}] \end{aligned}$$

This proves (2.20) for $A=\{\tilde{\omega};\ \omega_1\in A_1,\ \cdots,\ \omega_k\in A_k\}$ and $B=\{\tilde{\omega};\ \omega_1\in A_{k+1},\ \cdots,\ \omega_{n-k}\in A_n\}$. By a standard argument we have (2.20) for any $A\in\tilde{\mathscr{B}}_{\tau_k}$ and $B\in\tilde{\mathscr{B}}$. (ii) follows from (i) by a standard argument. To prove (iii), we first assume $g(\tilde{\omega},\tilde{\omega}')=g_1(\tilde{\omega})g_2(\tilde{\omega}')$, where g_1 is bounded $\tilde{\mathscr{B}}_{\tau_k}$-measurable and g_2 is bounded $\tilde{\mathscr{B}}$-measurable; then it follows at once from (i). By a standard argument (2.22) holds for every bounded $\tilde{\mathscr{B}}_{\tau_k}\otimes\tilde{\mathscr{B}}$-measurable function $g(\tilde{\omega},\tilde{\omega}')$.

Lemma 2.6. *Let T be a $\tilde{\mathscr{B}}_{t+0}$-Markov time (resp. $\tilde{\mathscr{B}}_t$-Markov time); then there exists an $\mathcal{N}_{t+0}$-Markov time (resp. $\mathcal{N}_t$-Markov time) $T'(w)$ defined on W such that*

$$(2.23)\qquad T'(w)=T(\tilde{\omega})\quad for\ \tilde{\omega}\in\{\tilde{\omega};\ T(\tilde{\omega})<\tau(\tilde{\omega}),\ w_1=w\},$$

where we write $\tilde{\omega}=((w_1,x_1),\omega_2,\omega_3,\cdots)$.

Proof. For a fixed $w\in W$, put $A_w=\{\tilde{\omega};\ T(\tilde{\omega})<\tau(\tilde{\omega})$ and $w_1=w\}$, *where* $\tilde{\omega}=((w_1,y),\omega_2,\cdots)$. First of all, note that if $\tilde{\omega}$ and $\tilde{\omega}'$ belong to A_w, then $T(\tilde{\omega})=T(\tilde{\omega}')$. In fact, if $T(\tilde{\omega})<t<\tau(\tilde{\omega})$, then we have $\tilde{\omega}\sim\tilde{\omega}'$ (R_t) since $x_s(w_1)=x_s(w_1')$ for $s\leq t$. This implies $T(\tilde{\omega})=T(\tilde{\omega}')$ by (iv) of Lemma 2.2.

Now set

$$(2.24)\qquad T'(w)=\begin{cases}T(\tilde{\omega}),\ \tilde{\omega}\in A_w & \text{if } A_w\neq\phi,\\ \infty, & \text{if } A_w=\phi.\end{cases}$$

We shall prove $T'(w)$ is $\mathcal{N}_{t+0}$-Markov time ($\mathcal{N}_t$-Markov time). In fact, if we assume $T'(w)<t$ and $x_s(w)=x_s(w')$ for all $s\leq t$, then $\tilde{\omega}\sim\tilde{\omega}'(R_{t\wedge\tau(\tilde{\omega})})$, where we set $\tilde{\omega}=((w,x),\omega_2,\omega_3,\cdots)$ and $\tilde{\omega}'=((w',x'),\omega_2',\omega_3',\cdots)$. Therefore $T'(w)=T(\tilde{\omega})=T(\tilde{\omega}')=T'(w')$. This implies that $T'(w)$ is an $\mathcal{N}_{t+0}$-Markov time by Lemma 2.2 (iv) (cf. Footnote 5 of §2.1).

Lemma 2.7. *Let f be a bounded measurable function on E, $g(x,t)$ be a bounded measurable function on $E\times[0,\infty]$ and T be a $\tilde{\mathscr{B}}_{t+0}$-Markov time. Then*

(2.25) $$\widetilde{E}_x[f(X_T)g(X_\tau, \tau-T);\ T<\tau] = \widetilde{E}_x[f(X_T)\widetilde{E}_{X_T}[g(X_\tau, \tau)];\ T<\tau].$$

Proof. It is sufficient to prove (2.25) for g of the form $g(x, t)=g_1(x)g_2(t)$. In this case we have by Lemma 2.6

$$\begin{aligned}&\widetilde{E}_x[f(X_T)g_1(X_\tau)g_2(\tau-T);\ T<\tau]\\ &=\widetilde{E}_x[f(X_T)I_{\{T'(w)<\zeta(w)\}}g_1(X_\tau)g_2(\zeta(\theta_{T'}w))]\\ &=\int_\Omega P_x[dw]\mu(w, dy)f(x_{T'(w)}(w))I_{\{T'<\zeta\}}g_1(y)g_2(\zeta(\theta_{T'}w)).\end{aligned}$$

This is equal to, since μ is an instantaneous distribution,

$$\int_W P_x[dw]f(x_{T'})I_{\{T'<\zeta\}}g_2(\zeta(\theta_{T'}w))\int_E \mu(\theta_{T'}w, dy)g_1(y).$$

Then using the strong Markov property of $X^0=\{x_t, P_x\}$, this is equal to

$$\begin{aligned}&E_x[f(x_{T'})I_{\{T'<\zeta\}}E_{x_{T'}}[g_2(\zeta)\int_E \mu(w, dy)g_1(y)]]\\ &=E_x[f(x_{T'})I_{\{T'<\zeta\}}\widetilde{E}_{x_{T'}}[g_1(X_\tau)g_2(\tau)]]\\ &=\widetilde{E}_x[f(X_T)I_{\{T<\zeta\}}\widetilde{E}_{X_T}[g_1(X_\tau)g_2(\tau)]]\end{aligned}$$

and the proof is complete.

Lemma 2.8. *Let $g(x, t)$ be a bounded measurable function on $E\times[0, \infty]$, T be a $\widetilde{\mathcal{B}}_{t+0}$-Markov time and $A\in\widetilde{\mathcal{B}}_{T+0}$. Then*

(2.26) $$\widetilde{E}_x[g(X_{\tau(\tilde\theta_T\tilde\omega)}(\tilde\theta_T\tilde\omega), \tau(\tilde\theta_T\tilde\omega));\ A] = \widetilde{E}_x[\widetilde{E}_{X_T}[g(X_\tau, \tau)];\ A].$$

Proof.

$$\begin{aligned}&\widetilde{E}_x[I_{\{\tau_k\le T<\tau_{k+1}\}}\widetilde{E}_{X_T}[g(X_\tau, \tau)];\ A]\\ &=\widetilde{E}_x[I_{\{\tau_k\le T\}}\cdot I_{\{0\le T-\tau_k<\tau(\tilde\theta_{\tau_k}\tilde\omega)\}}\widetilde{E}_{X_{T-\tau_k}(\tilde\theta_{\tau_k}\tilde\omega)}[g(X_\tau, \tau)];\ A].\end{aligned}$$

By Lemma 2.4 this is equal to

$$\widetilde{E}_x[I_{\{\tau_k\le T\}}I_{\{0\le T_k(\tilde\omega, \tilde\theta_{\tau_k}\tilde\omega)<\tau(\tilde\theta_{\tau_k}\tilde\omega)\}}\widetilde{E}_a[g(X_\tau, \tau)];\ A],$$

where $a=X_{T_k(\tilde\omega, \tilde\theta_{\tau_k}\tilde\omega)}(\tilde\theta_{\tau_k}\tilde\omega)$, and by Lemma 2.5 this is equal to

$$\widetilde{E}_x[I_{\{\tau_k\leq T\}}\widetilde{E}_{X_{\tau_k}}[I_{\{0\leq T_k(u,\cdot)<\tau\}}\widetilde{E}_{X_{T_k(u,\cdot)}}[g(X_\tau, \tau)]]\Big|_{u=\tilde{\omega}}; A].$$

Applying Lemma 2.7 on $T_k(u, \cdot)$ and by Lemma 2.5 this is equal to

$$\widetilde{E}_x[I_{\{\tau_k\leq T\}}\widetilde{E}_{X_{\tau_k}}[I_{\{0\leq T_k(u,\cdot)<\tau\}}g(X_\tau, \tau-T_k(u, \cdot))]\Big|_{u=\tilde{\omega}}; A]$$

$$=\widetilde{E}_x[I_{\{\tau_k\leq T\}}I_{\{0\leq T_k(\tilde{\omega},\tilde{\theta}_{\tau_k}\tilde{\omega})<\tau(\tilde{\theta}_{\tau_k}\tilde{\omega})\}}$$

$$\cdot g(X_{\tau(\tilde{\theta}_{\tau_k}\tilde{\omega})}(\tilde{\theta}_{\tau_k}\tilde{\omega}), \tau(\tilde{\theta}_{\tau_k}\tilde{\omega})-T_k(\tilde{\omega}, \tilde{\theta}_{\tau_k}\tilde{\omega})); A]$$

$$=\widetilde{E}_x[I_{\{\tau_k\leq T\}}I_{\{0\leq T-\tau_k<\tau_{k+1}-\tau_k\}}g(X_{\tau_{k+1}}, \tau_{k+1}-T); A]$$

$$=\widetilde{E}_x[I_{\{\tau_k\leq T<\tau_{k+1}\}}g(X_{\tau(\tilde{\theta}_T\tilde{\omega})}(\tilde{\theta}_T\tilde{\omega}), \tau(\tilde{\theta}_T\tilde{\omega})); A].$$

Now summing up the first and the last expressions over k, we obtain (2.26).

Proof of Theorem 2.2. We have only to prove the strong Markov property of $X=(X_t, \widetilde{P}_x, \widetilde{\mathscr{B}}_{t+0})$. Let f be a bounded measurable function on E such that $f(\Delta)=0$, T be a $\widetilde{\mathscr{B}}_{t+0}$-Markov time and $A\in\widetilde{\mathscr{B}}_{T+0}$. We shall prove

(2.27) $$\widetilde{E}_x[f(X_{T+t}); A]=\widetilde{E}_x[\widetilde{E}_{X_T}[f(X_t)]; A].^{7)}$$

Set

$$I=\widetilde{E}_x[f(X_{T+t}); A\cap\{T<\tau_k\leq T+t, \text{ for some } k\}]$$

and

$$II=\widetilde{E}_x[f(X_{T+t}); A\cap\{\tau_k\leq T, T+t<\tau_{k+1} \text{ for some } k\}].$$

Then clearly the left hand side of (2.27) is equal to $I+II$. Now

$$\widetilde{E}_x[f(X_{T+t}); \tau_k\leq T, T+t<\tau_{k+1}, A]$$

$$=\widetilde{E}_x[f(X_{T-\tau_k+t}(\tilde{\theta}_{\tau_k}\tilde{\omega}); 0\leq T-\tau_k<\tau(\tilde{\theta}_{\tau_k}\tilde{\omega}),$$

$$0\leq T-\tau_k+t<\tau(\tilde{\theta}_{\tau_k}\tilde{\omega}); A].$$

By Lemma 2.4 this is equal to

$$\widetilde{E}_x[f(X_{T_k(\tilde{\omega},\tilde{\theta}_{\tau_k}\tilde{\omega})+t}(\tilde{\theta}_{\tau_k}\tilde{\omega})); 0\leq T_k(\tilde{\omega}, \tilde{\theta}_{\tau_k}\tilde{\omega})<\tau(\tilde{\theta}_{\tau_k}\tilde{\omega}),$$

$$0\leq T_k(\tilde{\omega}, \tilde{\theta}_{\tau_k}\tilde{\omega})+t<\tau(\tilde{\theta}_{\tau_k}\tilde{\omega}); A]$$

---

7) For convenience, we set $X_\infty(\tilde{\omega})\equiv\Delta$.

and by Lemma 2.5 (iii) this is equal to

$$\widetilde{E}_x[I_{\{\tau_k\leq T\}\cap A}\widetilde{E}_{X_{\tau_k}}[f(X_{T_k(u,\cdot)+t});\ 0\leq T_k(u,\cdot)<\tau,\ 0\leq T_k(u,\cdot)+t<\tau]\,\big|_{u=\tilde{\omega}}].$$

If we apply Lemma 2.6 to $T_k(u,\cdot)$ we get an $\mathcal{N}_{t+0}$-Markov time $T_k'(u,w)$ on W. Therefore by the strong Markov property of $\{x_t, P_x, \mathcal{N}_{t+0}\}$,[8] the last expression is equal to

$$\begin{aligned}&\widetilde{E}_x[I_{\{\tau_k\leq T\}\cap A}\cdot E_{X_{\tau_k}}[E_{x_{T_k'(u,\cdot)}}[f(x_t);\ 0\leq t<\zeta];\ 0\leq T_k'(u,\cdot)<\zeta]\,\big|_{u=\tilde{\omega}}]\\ =&\widetilde{E}_x[I_{\{\tau_k\leq T\}\cap A}\cdot \widetilde{E}_{X_{\tau_k}}[\widetilde{E}_{X_{T_k(u,\cdot)}}[f(X_t);\ 0\leq t<\tau];\ 0\leq T_k(u,\cdot)<\tau]\,\big|_{u=\tilde{\omega}}].\end{aligned}$$

By Lemma 2.5 (iii) this is equal to

$$\begin{aligned}&\widetilde{E}_x[I_{\{\tau_k\leq T\}\cap A}\cdot \widetilde{E}_{X_{T_k(\tilde{\omega},\tilde{\theta}_{\tau_k}\tilde{\omega})}(\tilde{\theta}_{\tau_k}\tilde{\omega})}[f(X_t);\ 0\leq t<\tau];\\ &\qquad 0<T_k(\tilde{\omega},\tilde{\theta}_{\tau_k}\tilde{\omega})<\tau(\tilde{\theta}_{\tau_k}\tilde{\omega})]\\ =&\widetilde{E}_x[I_{\{\tau_k\leq T<\tau_{k+1}\}\cap A}\widetilde{E}_{X_T}[f(X_t);\ 0\leq t<\tau]].\end{aligned}$$

Thus we have

$$\begin{aligned}II&=\sum_{k=0}^{\infty}\widetilde{E}_x[f(X_{T+t});\ \tau_k\leq T,\ T+t<\tau_{k+1};\ A]\\ &=\widetilde{E}_x[\widetilde{E}_{X_T}[f(X_t);\ 0\leq t<\tau];\ A].\end{aligned}$$

Hence

$$\widetilde{E}_x[\widetilde{E}_{X_T}[f(X_t)];\ A]-II=\widetilde{E}_x[\widetilde{E}_{X_T}[f(X_t);\ \tau\leq t];\ A]. \tag{2.28}$$

It remains therefore to prove

$$I=\widetilde{E}_x[\widetilde{E}_{X_T}[f(X_t);\ \tau\leq t];\ A], \tag{2.29}$$

and this can be verified as follows:

$$\begin{aligned}&\widetilde{E}_x[\widetilde{E}_{X_T}[f(X_t);\ \tau\leq t];\ A]\\ =&\widetilde{E}_x[\widetilde{E}_{X_T}[f(X_{t-\tau}(\tilde{\theta}_\tau\tilde{\omega}));\ \tau\leq t];\ A].\end{aligned}$$

By Lemma 2.5 this is equal to

$$\widetilde{E}_x[\widetilde{E}_{X_T}[\widetilde{E}_{X_\tau}[f(X_{t-u})]\,\big|_{u=\tau};\ \tau\leq t];\ A]$$

---

8) By assumption $\{x_t, \mathcal{B}_t\}$ is strong Markov and $\mathcal{N}_{t+0}\subset\overline{\mathcal{B}}_{t+0}=\mathcal{B}_t$; therefore $\{x_t, \mathcal{N}_{t+0}\}$ is strong Markov.

and by Lemma 2.8 this equals

$$\widetilde{E}_x[\widetilde{E}_{X_{\tau(\tilde{\theta}_T\tilde{\omega})}(\tilde{\theta}_T\tilde{\omega})}[f(X_{t-u})]\mid_{u=\tau(\tilde{\theta}_T\tilde{\omega})};\ \tau(\tilde{\theta}_T\tilde{\omega})\leq t;\ A].$$

Because of $\tau(\tilde{\theta}_T\tilde{\omega})=\tau_{k+1}-T$ on $\{\tau_k\leq T<\tau_{k+1}\}$, the above expression becomes

$$\sum_{k=0}^{\infty}\widetilde{E}_x[I_{\{\tau_k\leq T<\tau_{k+1}\}}\widetilde{E}_{X_{\tau_{k+1}}}[f(X_{t-u})]\mid_{u=\tau_{k+1}-T};\ \tau_{k+1}-T\leq t;\ A].$$

Since $\{\tau_{k+1}-T\leq t\}\cap\{\tau_k\leq T<\tau_{k+1}\}\cap A$ is $\widetilde{\mathscr{B}}_{\tau_{k+1}}$-measurable, by Lemma 2.5 this is equal to

$$\begin{aligned}
&\sum_{k=0}^{\infty}\widetilde{E}_x[I_{\{\tau_k\leq T<\tau_{k+1}\}\cap\{\tau_{k+1}-T\leq t\}\cap A}\cdot f(X_{t+T-\tau_{k+1}}(\tilde{\theta}_{\tau_{k+1}}\tilde{\omega}))]\\
&=\sum_{k=0}^{\infty}\widetilde{E}_x[I_{\{\tau_k\leq T<\tau_{k+1}<T+t\}\cap A}\cdot f(X_{t+T})]\\
&=\widetilde{E}_x[f(X_{T+t});\ A\cap\{T<\tau_k\leq T+t \text{ for some } k\}]\\
&=I.
\end{aligned}$$

This completes the proof.

Proof of Theorem 2.3. The first assertion is almost clear from the definition. Assume

$$\sup_{x\in E-\{\Delta\}}P_x[\zeta<\infty]=\alpha<1;$$

then

$$\begin{aligned}
&\widetilde{P}_x[\tau_n(\tilde{\omega})<\infty,\ N(\tilde{\omega})=\infty]\\
&=\widetilde{E}_x[\widetilde{P}_{X_\tau}[\tau_{n-1}<\infty,\ N=+\infty];\ X_\tau\in E-\{\Delta\},\ \tau<\infty]\\
&\leq\alpha\sup_{x\in E-\{\Delta\}}\widetilde{P}_x[\tau_{n-1}<\infty,\ N=+\infty].
\end{aligned}$$

Thus we have

$$\sup_{x\in E-\{\Delta\}}\widetilde{P}_x[\tau_n(\tilde{\omega})<\infty,\ N(\tilde{\omega})=\infty]\leq\alpha\sup_{x\in E-\{\Delta\}}\widetilde{P}_x[\tau_{n-1}(\tilde{\omega})<\infty,\ N(\tilde{\omega})=\infty]$$

and hence

$$\sup_{x\in E-\{\Delta\}}\widetilde{P}_x[\tau_n<\infty,\ N=\infty]\leq\alpha^n.$$

This proves that for every $x\in E-\{\Delta\}$

$$\widetilde{P}_x[\tau_\infty<\infty,\ N=\infty]\leq\lim_{n\to\infty}\widetilde{P}_x[\tau_n<\infty,\ N=\infty]=0,$$

that is,

$$\tilde{P}_x[\tau_\infty(\tilde{\omega})=\infty \text{ or } N(\tilde{\omega})<\infty]=1.$$

Now the second assertion is clear from this and the way of the construction.

Proof of Theorem 2. 4.

$$\tilde{P}_x[\lim_{n\to\infty} X_{T_n}=X_T;\ T<\tilde{\zeta}]$$
$$=\sum_{k=0}^{\infty}\tilde{P}_x[\lim_{n\to\infty} X_{T_n}=X_T;\ \tau_k<T\leq\tau_{k+1}].$$

Applying Lemma 2. 4 for T_n and T, we have

$$\tilde{P}_x[\lim_{n\to\infty} X_{T_n}=X_T;\ \tau_k<T\leq\tau_{k+1}]$$
$$=\tilde{P}_x[\lim_{n\to\infty} X_{T_n^k(\tilde{\omega},\tilde{\theta}_{\tau_k}\tilde{\omega})}(\tilde{\theta}_{\tau_k}\tilde{\omega})=X_{T^k(\tilde{\omega},\tilde{\theta}_{\tau_k}\tilde{\omega})}(\tilde{\theta}_{\tau_k}\tilde{\omega});$$
$$\tau_k<T,\ T^k(\tilde{\omega},\tilde{\theta}_{\tau_k}\tilde{\omega})\leq\tau(\tilde{\theta}_k\tilde{\omega})]$$
$$=\tilde{E}_x[\tilde{P}_{X_{\tau_k}}[\lim_{n\to\infty} X_{T_n^k(u,\cdot)}=X_{T^k(u,\cdot)};\ 0<T^k(u,\cdot)\leq\tau]\,|_{u=\tilde{\omega}};\ \tau_k<T].$$

Noticing that x_t is quasi-left continuous and ζ is totally inaccessible, the last expression is equal to

$$=\tilde{E}_x[\tilde{P}_{X_{\tau_k}}[0<T^k(u,\cdot)\leq\tau]\,|_{u=\tilde{\omega}};\ \tau_k<T]$$
$$=\tilde{P}_x[\tau_k<T\leq\tau_{k+1}].$$

Thus we have

$$\tilde{P}_x[\lim_{n\to\infty} X_{T_n}=X_T;\ T<\tilde{\zeta}]=\sum_{k=0}^{\infty}\tilde{P}_x[\tau_k<T\leq\tau_{k+1}]$$
$$=\tilde{P}_x[T<\tilde{\zeta}].$$

Proof of Theorem 2. 5. The first assertion follows from Theorem 2. 3 and Theorem 2. 4. Now suppose

(2. 30) $\mu(w, E-\{\Delta\})=1$ for all w such that $\zeta(w)>0$,

and $X^0=(x_t, P_x)$ satisfies

(i) $\sup_{x\in E-\{\Delta\}} P_x[\zeta<\infty]=\alpha<1.$

We have noticed in the proof of Theorem 2. 3 that (i) implies

$$\widetilde{P}_x[\tau_\infty(\omega)=+\infty \text{ or } N(\tilde{\omega})<\infty]=1;$$

but by (2. 30),

$$\widetilde{P}_x[N(\tilde{\omega})=+\infty]=1 \quad \text{for } x\in E-\{\varDelta\}.$$

Hence,

$$\widetilde{P}_x[\tau_\infty(\tilde{\omega})=+\infty \text{ or } N(\tilde{\omega})<\infty]=\widetilde{P}_x[\tau_\infty(\tilde{\omega})=+\infty]$$

$$=\widetilde{P}_x[\tilde{\zeta}(\tilde{\omega})=+\infty]=1 \quad \text{for } x\in E-\{\varDelta\}.$$

Next we assume (2. 30) and

(ii) $$\inf_{x\in E-\{\varDelta\}} P_x[\zeta(w)>\varepsilon]=\delta>0 \quad \text{for some } \varepsilon>0.$$

Since $\{\zeta(\tilde{\omega})<\infty\}\subset\bigcup_{n=1}^{\infty}\bigcap_{k=n+1}^{\infty}\{\zeta(w_k)<\varepsilon\}$,[9] we have for $x\in E-\{\varDelta\}$

(2. 31) $$\widetilde{P}_x[\tilde{\zeta}(\tilde{\omega})<\infty]\leq\lim_{n\to\infty}\widetilde{P}_x[\bigcap_{k=n+1}^{\infty}\{\zeta(w_k)<\varepsilon\}\cap\{\tau_n<\infty\}]$$

$$=\lim_{n\to\infty}\widetilde{P}_x[\widetilde{P}_{X_{\tau_n}}[\bigcap_{k=1}^{\infty}\{\zeta(w_k)<\varepsilon\}];\ \tau_n<\infty].$$

On the other hand, for $x\in E-\{\varDelta\}$

$$\widetilde{P}_x[\bigcap_{k=1}^{\infty}\{\zeta(w_k)<\varepsilon\}]\leq\widetilde{E}_x[\widetilde{P}_{X_\tau}[\bigcap_{k=1}^{\infty}\{\zeta(w_k)<\varepsilon\}];\ \zeta(w_1)<\varepsilon]$$

$$\leq\sup_{y\in E-\{\varDelta\}}\widetilde{P}_y[\bigcap_{k=1}^{\infty}\{\zeta(w^k)<\varepsilon\}]\cdot\widetilde{P}_x[\zeta(w)<\varepsilon],$$ [10]

and hence

$$\sup_{x\in E-\{\varDelta\}}\widetilde{P}_x[\bigcap_{k=1}^{\infty}\{\zeta(w_k)<\varepsilon\}]\leq(1-\delta)\sup_{x\in E-\{\varDelta\}}\widetilde{P}_x[\bigcap_{k=1}^{\infty}\{\zeta(w_k)<\varepsilon\}].$$

This indicates that we should have

$$\sup_{x\in E-\{\varDelta\}}\widetilde{P}_x[\bigcap_{k=1}^{\infty}\{\zeta(w_k)<\varepsilon\}]=0,$$

and hence by (2. 31)[11]

$$\widetilde{P}_x[\tilde{\zeta}(\tilde{\omega})<\infty]=0 \quad \text{for every } x\in E-\{\varDelta\}.$$

9) $\tilde{\omega}=(\omega_1, \omega_2, \cdots)$, $\omega_j=(w_j, x_j)$.

10) By (2. 30), $\widetilde{P}_x[X_\tau\in E-\{\varDelta\}, \tau<\infty]=\widetilde{P}_x[\tau<\infty]$ if $x\in E-\{\varDelta\}$.

11) By (2. 30), $\widetilde{P}_x[X_{\tau_n}\in E-\{\varDelta\}, \tau_n<\infty]=\widetilde{P}_x[\tau_n<\infty]$ if $x\in E-\{\varDelta\}$.

III. Construction of branching Markov processes

In this chapter we will construct an (X^0, π)-branching Markov process (cf. Definition (1. 6)) in a probabilistic way. Given a Markov process X^0 on $S\cup\{\Delta\}$ with Δ as the terminal point, we will first of all construct the n-fold direct product X_n^* of X^0 and the n-fold symmectric direct product $\widetilde{X}_n$ of X^0, which are Markov processes on $S^{(n)}\cup\{\Delta\}$ and $S^n\cup\{\Delta\}$, respectively, with Δ as the terminal point. Then we shall construct the direct sum $\widetilde{X}$ of $\widetilde{X}_n$, which is a Markov process on $\widehat{S}=\bigcup_{n=0}^{\infty}S^n\cup\{\Delta\}$ with Δ as the terminal point. We will next construct from X^0 and a branching law π an instantaneous distribution μ (cf. Definition (2. 1)) for the process $\widetilde{X}$. Then we will piece out the path functions of $\widetilde{X}$ by μ according to the previous chapter to get a strong Markov process $\boldsymbol{X}$ on $\widehat{S}$, which will certainly be the (X^0, π)-branching Markov process. The other analytic ways of construction will be discussed in Chapter IV.

§3. 1. Direct products and symmetric direct products of a Markov process

Let S be a compact Hausdorff space with a countable open base; and let $S^{(n)}$, S^n, $\mathbf{S}=\bigcup_{n=0}^{\infty}S^n$ and $\widehat{\mathbf{S}}=\mathbf{S}\cup\{\Delta\}$ be defined as in §0. 2. Let $X^0=\{W, x_t^0(w), \mathcal{B}_t^0, P_x^0, x\in S\cup\{\Delta\}, \theta_t^0, \zeta^0\}$ be a right continuous strong Markov process on $S\cup\{\Delta\}$[1] with Δ as its terminal point such that $\mathcal{B}_t^0=\overline{\mathcal{B}}_{t+0}^0$.

Definition 3. 1. (i) For each $n=1, 2, \cdots$, a Markov process $X_n^*=\{x_t^*, \mathcal{B}_t^*, P_{\boldsymbol{x}}^{*(n)}\}$ on $S^{(n)}\cup\{\Delta\}$ with Δ as the terminal point is called the *n-fold direct product of* X^0 if it satisfies

$$(3. 1)\qquad E_{\boldsymbol{x}}^{*(n)}[f_1\otimes f_2\otimes\cdots\otimes f_n(x_t^*)]=\prod_{i=1}^{n}E_{x_i}^0[f_i(x_t^0)]$$

for every $\boldsymbol{x}=(x_1, x_2, \cdots, x_n)$, $f_i\in\boldsymbol{C}(S)$, $i=1, 2, \cdots, n$, and $t\geq 0$.[2]

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1) Δ is attached to S as an isolated point. ζ^0 is the life time.

2) $f_1\otimes\cdots\otimes f_n$ is a continuous function on $S^{(n)}$ defined by $f_1\otimes\cdots\otimes f_n(x_1, x_2, \cdots, x_n)=\prod_{i=1}^{n}f_i(x_i)$. We set $f(\Delta)=0$ for every function f.

(ii) For each $n=1, 2, \cdots$, a Markov process $\widetilde{X}_n=\{\tilde{x}_t, \widetilde{\mathcal{B}}_t, \widetilde{P}_{x}^{(n)}\}$ on $S^n\cup\{\varDelta\}$ with $\varDelta$ as the terminal point is called the *n-fold symmetric direct product of* X^0 if it satisfies

$$(3.2)\qquad \widetilde{E}_{x}^{(n)}[\widehat{f}(\tilde{x}_t)]=\prod_{i=1}^{n}E_{x_i}^0[f(x_t^0)]$$

for every $x=[x_1, x_2, \cdots, x_n]$, $f\in C^*(S)$, and $t\geq 0$.

The direct product and the symmetric direct product of X^0 are uniquely determined from X^0 up to equivalence because of the denseness of the linear hull of $\{f_1\otimes f_2\otimes\cdots\otimes f_n;\ f_i\in C(S)\}$ in $C(S^{(n)})$ and the linear hull of $\{\widehat{f}|_{S^n};\ f\in C^*(S)\}$ in $C(S^n)$.[3)]

Now we shall construct a version of the direct product and the symmetric direct product of X^0 in the following way. Let $W^{(n)}$ be the n-fold product of W, whose elements will be denoted as $\overline{w}=(w_1, w_2, \cdots, w_n)$, where $w_j\in W$, and put

$$(3.3)\qquad \overline{\zeta}(\overline{w})=\min_{1\leq k\leq n}\{\zeta(w_k)\}$$

$$(3.4)\qquad x_t^*(\overline{w})=\begin{cases}(x_t^0(w_1), \cdots, x_t^0(w_n)), & \text{if } t<\overline{\zeta}(\overline{w}),\\ \varDelta, & \text{if } t\geqq\overline{\zeta}(\overline{w}),\end{cases}$$

$$(3.5)\qquad \overline{\theta}_t\overline{w}=(\theta_t^0w_1, \theta_t^0w_2, \cdots, \theta_t^0w_n),$$

$$(3.6)\qquad \mathcal{N}_t^{*(n)}=\sigma(W^{(n)}, \mathcal{B}(S^{(n)}\cup\{\varDelta\});\ x_s^*(\overline{w}),\ s\leq t),\ \mathcal{N}_\infty^{*(n)}=\bigvee_{t>0}\mathcal{N}_t^{*(n)},$$

$$(3.7)\qquad P_x^{*(n)}[A]=\begin{cases}P_{x_1}^0\times\cdots\times P_{x_n}^0[A], & \text{if } x=(x_1, \cdots, x_n)\in S^{(n)},\\ P_\varDelta^0\times\cdots\times P_\varDelta^0[A], & \text{if } x=\varDelta,\end{cases}$$

for $A\in\mathcal{N}_\infty^{*(n)}$.

By Theorem 3.1 given below, one can see that the process

$$X_n^*=\{W^{(n)}, x_t^*(\overline{w}), \mathcal{B}_t^{*(n)}=\overline{\mathcal{N}}_{t+0}^{*(n)}, P_x^{*(n)}, x\in S^{(n)}\cup\{\varDelta\}, \overline{\theta}_t, \overline{\zeta}\}$$

defined above is a strong Markov process and it satisfies clearly (3.1). Hence, it is a version of the n-fold direct product of X^0. We will call this X_n^* *the canonical realization of the n-fold direct product of* X^0. Now let ρ be the natural mapping $S^{(n)}\to S^n$ and set

3) Cf. Lemma 0.2.

(3.8) $$\tilde{x}_t(\overline{w})=\rho[x_t^*(\overline{w})],$$ [4]

(3.9) $$\widetilde{\mathcal{N}}_t^{(n)}=\sigma(W^{(n)}, \mathcal{B}(S^n\cup\{\Delta\}); \tilde{x}_s(\overline{w}), s\leq t), \widetilde{\mathcal{N}}_\infty^{(n)}=\bigvee_{t>0}\widetilde{\mathcal{N}}_t^{(n)}$$

and define $\{\widetilde{P}_x^{(n)}\}$, $x\in S^n\cup\{\Delta\}$ on $\widetilde{\mathcal{N}}_\infty^{(n)}$ by

(3.10) $$\widetilde{P}_x^{(n)}[A]=\begin{cases}P_{x_1}^0\times\cdots\times P_{x_n}^0[A], & \text{if } x=[x_1, x_2, \cdots, x_n]\in S^n,\\ P_\Delta^0\times\cdots\times P_\Delta^0[A], & \text{if } x=\Delta.\end{cases}$$

$\widetilde{P}_x^{(n)}$ is well defined just as in Lemma 1.1. We shall define the process $\widetilde{X}_n$ by $\widetilde{X}_n=\{W^{(n)}, \tilde{x}_t(\overline{w}), \widetilde{\mathcal{B}}_t^{(n)}=\widetilde{\mathcal{N}}_{t+0}^{(n)}, \widetilde{P}_x^{(n)}, x\in S^n\cup\{\Delta\}, \overline{\theta}_t, \overline{\zeta}(\overline{w})\}$. X_n is the process induced from X_n^* by the mapping ρ in the sense of Dynkin ([6] Theorem 10.13, p. 325), i.e., $\widetilde{X}_n=\rho(X_n^*)$. The process $\widetilde{X}_n$ is certainly a version of the n-fold symmetric direct product of X^0. We will call this $\widetilde{X}_n$ the *canonical realization of the n-fold symmetric direct product of* X^0.

Theorem. 3.1. *The canonical realization of the n-fold direct product X_n^* and the canonical realization of the n-fold symmetric direct product $\widetilde{X}_n$ are right continuous strong Markov processes on* $S^{(n)}\cup\{\Delta\}$ and $S^n\cup\{\Delta\}$, *respectively. If X^0 has left limits, then X_n^* and $\widetilde{X}_n$ have left limits.*

Proof. We shall prove this theorem only for X_n^*: the proof for $\widetilde{X}_n$ follows then from the Theorem 10.13 of Dynkin [6]. First we shall prove the following

Lemma 3.1. (i) *Let $A\in\mathcal{N}_t^{*(n)}$ and $A_{[\overline{w}(j)]}$ be the j-section of A defined by $A_{[\overline{w}(j)]}=\{w_j;\ \overline{w}=(w_1, \cdots, w_n)\in A\}$ for fixed $\overline{w}(j)=(w_1, \cdots, w_{j-1}, w_{j+1}, \cdots, w_n)$. Then for each $\overline{w}(j)$, $A_{[\overline{w}(j)]}$ belongs to* $\mathcal{N}_t^0$.[5]

(ii) *Let $T(\overline{w})$ be an $\mathcal{N}_{t+0}^{*(n)}$-Markov time; then for each fixed $\overline{w}(j)$, the j-section $T_{[\overline{w}(j)]}$ of T defined by $T_{[\overline{w}(j)]}(w_j)=T(\overline{w})$ is an $\mathcal{N}_{t+0}^0$-Markov time.*

(iii) *Let T be an $\mathcal{N}_{t+0}^{*(n)}$-Markov time and $A\in\mathcal{N}_{T+0}^{*(n)}$; then for fixed*

4) We extend ρ as the mapping $S^{(n)}\cup\{\Delta\}\to S^n\cup\{\Delta\}$ by setting $\rho\{\Delta\}=\Delta$.

5) $\mathcal{N}_t^0=\sigma(W, \mathcal{B}(S\cup\{\Delta\}); x_s^0(w), s\leq t)$ and hence $\mathcal{N}_t^0\subset\mathcal{B}_t^0$.

$\overline{w}(j)$, $A_{[\overline{w}(j)]}$ belongs to $\mathcal{N}^0_{T[\overline{w}(j)]+0}$.[6]

Proof. (i) We assume $n \geqq 2$, the case of $n=1$ being clear. Fixing $\overline{w}(j)$, set $\mathcal{B}=\{A \in \mathcal{N}_t^{*(n)};\ A_{[\overline{w}(j)]} \in \mathcal{N}_t^0\}$. Then clearly $\mathcal{B}$ is a sub-Borel field of $\mathcal{N}_t^{*(n)}$ over $W^{(n)}$. For $\Gamma \in \mathcal{B}(S^{(n)})$ and $s \leqq t$,

$$\{\overline{w};\ x_s^*(\overline{w}) \in \Gamma\} = \{\overline{w}=(w_1, \cdots, w_n);\ (x_s^0(w_1), \cdots, x_s^0(w_n)) \in \Gamma,\ s<\zeta^0(w_i),\ i=1, 2, \cdots, n\},$$

and hence its j-section is given by

$$\{x_s^*(\overline{w}) \in \Gamma\}_{[\overline{w}(j)]} = \begin{cases} \{w_j;\ x_s^0(w_j) \in \Gamma_{[x_s^0(w_1), \cdots, x_s^0(w_{j-1}), x_s^0(w_{j+1}), \cdots, x_s^0(w_n)]}, \\ \quad s<\zeta^0(w_j)\}^{7)}, & \text{if } s<\zeta^0(w_i) \text{ for all } i \neq j, \\ \phi, & \text{if otherwise.} \end{cases}$$

Thus $\{x_s^*(\overline{w}) \in \Gamma\} \in \mathcal{B}$. Also we have for $s \leqq t$,

$$\{x_s^*(\overline{w}) = \Delta\}_{[\overline{w}(j)]} = \begin{cases} W, & \text{if for some } k \neq j,\ \zeta^0(w_k) \leqq s, \\ \{w_j;\ \zeta^0(w_j) \leqq s\}, & \text{if otherwise,} \end{cases}$$

and hence $\{x_s^*(\overline{w}) = \Delta\} \in \mathcal{B}$. This proves $\{x_s^* \in \Gamma\} \in \mathcal{B}$ for all $s \leqq t$ and $\Gamma \in \mathcal{B}(S^{(n)} \cup \{\Delta\})$; therefore $\mathcal{N}_t^{*(n)} = \mathcal{B}$.

The proof of (ii) and (iii) is clear from (i) since

$$\{w_j;\ T_{[\overline{w}(j)]}<t\} = \{T<t\}_{[\overline{w}(j)]}$$

and

$$A_{[\overline{w}(j)]} \cap \{w_j;\ T_{[\overline{w}(j)]}<t\} = \{A \cap \{T<t\}\}_{[\overline{w}(j)]}.$$

Now we return to the proof of the theorem. We shall prove only the strong Markov property of X_n^*, the other part of the theorem being trivial. For this it is sufficient to prove[8]

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6) $$\mathcal{N}_{T+0}^{*(n)} = \{A \in \mathcal{N}_\infty^{*(n)};\ A \cap \{T<t\} \in \mathcal{N}_t^{*(n)} \text{ for every } t \geqq 0\}$$
$$= \{A \in \mathcal{N}_\infty^{*(n)};\ A\{\cap T \leqq t\} \in \mathcal{N}_{t+0}^{*(n)} \text{ for every } t \geqq 0\}.$$
$\mathcal{N}_{S+0}^0$ is defined similarly.

7) For $\Gamma \in \mathcal{B}(S^{(n)})$ and for a fixed $x(j)=(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$, $\Gamma_{[x(j)]}$ is the j-section of Γ: $\Gamma_{[x(j)]} = \{x_j;\ (x_1, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_n) \in \Gamma\}$.

8) For convenience, we set $x_\infty^* = \Delta$ and we extend every function f defined on $S^{(n)}$ as a function defined on $S^{(n)} \cup \{\Delta\}$ by setting $f(\Delta)=0$.

$$E_{x}^{*(n)}[f(x_{t+T}^{*});\ A]=E_{x}^{*(n)}[E_{x_{T}^{*}}^{*(n)}[f(x_{t}^{*})];\ A]$$

for every $f\in C(S^{(n)})$, an $\mathcal{N}_{t+0}^{*(n)}$-Markov time T and $A\in\mathcal{N}_{T+0}^{*(n)}$. We may assume $f=g_{1}\otimes g_{2}\otimes\cdots\otimes g_{n}$, $g_{i}\in C(S)$ since the linear hull of such functions is dense in $C(S^{(n)})$. Then,[9] if $x=(x_{1}, x_{2}, \cdots, x_{n})$,

$$\begin{aligned}&E_{x}^{*(n)}[f(x_{t+T}^{*});\ A]\\&=E_{x_{1}}^{0}\times\cdots\times E_{x_{n}}^{0}[\prod_{i=1}^{n}g_{i}(x_{t+T}^{0}(w_{i}));\ A]\\&=\int_{W\times\cdots\times W}\cdots\int P_{x_{1}}^{0}(dw_{1})P_{x_{2}}^{0}(dw_{2})\cdots P_{x_{n-1}}^{0}(dw_{n-1})\\&\quad\left\{\int_{W}P_{x_{n}}^{0}(dw_{n})\prod_{i=1}^{n-1}g_{i}(x_{t+T[\bar{w}(n)]}^{0}(w_{i}))g_{n}(x_{t}^{0}(\theta_{T[\bar{w}(n)]}^{0}w_{n}))\cdot I_{A[\bar{w}(n)]}(w_{n})\right\}.\end{aligned}$$

Note that for fixed $\bar{w}(n)$, $\prod_{i=1}^{n-1}g_{i}(x_{t+T[\bar{w}(n)]}^{0}(w_{i}))$ is $\mathcal{N}_{T[\bar{w}(n)]+0}^{0}$ measurable in w_{n}, then by Lemma 3.1 and the strong Markov property of X^{0} the above integral is equal to

$$\begin{aligned}&\int_{W\times\cdots\times W}\cdots\int P_{x_{1}}^{0}(dw_{1})P_{x_{2}}^{0}(dw_{2})\cdots P_{x_{n-1}}^{0}(dw_{n-1})\\&\cdot\left\{\int_{W}P_{x_{n}}^{0}(dw_{n})\prod_{i=1}^{n-1}g_{i}(x_{t+T[\bar{w}(n)]}^{0}(w_{i}))I_{A[\bar{w}(n)]}(w_{n})\cdot E_{x_{T[\bar{w}(n)]}^{0}(w_{n})}^{0}[g_{n}(x_{t}^{0})]\right\}\\&=E_{x_{1}}^{0}\times\cdots\times E_{x_{n}}^{0}[\prod_{i=1}^{n-1}g_{i}(x_{t+T}^{0}(w_{i}))E_{x_{T}^{0}(w_{n})}^{0}[g_{n}(x_{t}^{0})];\ A].\end{aligned}$$

Repeating this, we have

$$\begin{aligned}&E_{x}^{*(n)}[f(x_{t+T}^{*});\ A]\\&=E_{x}^{*(n)}[\prod_{i=1}^{n}E_{x_{T}^{0}(w_{i})}^{0}[g_{i}(x_{t}^{0})];\ A]\\&=E_{x}^{*(n)}[E_{x_{T}^{*}}^{*(n)}[f(x_{t}^{*})];\ A].\end{aligned}$$

Theorem 3.2. (i) *If* $X^{0}=\{W, x_{t}^{0}, \mathcal{B}_{t}^{0}, P_{x}^{0}, x\in S\cup\{\Delta\}, \theta_{t}^{0}, \zeta^{0}\}$ *is quasi-left continuous before* ζ^{0}, i.e.,

$$P_{x}^{0}[\lim_{m\to\infty}x_{T_{m}}^{0}=x_{T}^{0};\ T<\zeta^{0}]=P_{x}^{0}[T<\zeta^{0}]$$

for every $x\in S$, *and for every increasing sequence* $\{T_{n}\}$ *of* $\mathcal{B}_{t}^{0}$-

9) We extend each g_{i} as a function defined on $S\cup\{\Delta\}$ by setting $g_{i}(\Delta)=0$.

Markov times such that $T_n \uparrow T$, *then* X_n^* *and* $\widetilde{X}_n$ *are also quasi-left continuous before* $\bar{\zeta}$.
(ii) *If* X^0 *is a Hunt process and* ζ^0 *is totally inaccessible* (cf. *Meyer* [31] p. 130), *then* X_n^* *and* $\widetilde{X}_n$ *are Hunt processes.*

Proof. It is clearly sufficient to consider the case of X_n^*. Let $T_n \uparrow T$ be an increasing sequence of $\mathcal{N}_{t+0}^{*(n)}$-Markov times; then by Lemma 3.1,

$$P_{\boldsymbol{x}}^{*(n)}[\lim_{m\to\infty} x_{T_m}^* = x_T^*,\ T<\zeta]$$
$$= P_{x_1}^0 \times P_{x_2}^0 \times \cdots \times P_{x_n}^0 [\bigcap_{i=1}^{n} \{\lim x_{T_m}^0(w_i) = x_T^0(w_i)\} \cap \{T(\overline{w}) < \bar{\zeta}(\overline{w})\}]$$
$$= \int \cdots \int_{W\times\cdots\times W} P_{x_1}^0(dw_1)\cdots P_{x_{n-1}}^0(dw_{n-1})$$
$$\{P_{x_n}^0 [\bigcap_{i=1}^{n-1} \{\overline{w};\ \lim x_{T_m}^0(w_i) = x_T^0(w_i),\ T(\overline{w}) < \zeta^0(w_i)\}_{[\overline{w}(n)]}$$
$$\cap \{w_n;\ \lim x_{T_m[\overline{w}(n)]}^0(w_n) = x_{T[\overline{w}(n)]}^0(w_n),\ T_{[\overline{w}(n)]} < \zeta(w_n)\}]\}$$
$$= \int \cdots \int_{W\times\cdots\times W} P_{x_1}^0(dw_1)\cdots P_{x_{n-1}}^0(dw_{n-1}) \{P_{x_n}^0 [\bigcap_{i=1}^{n-1} \{\overline{w};\ \lim x_{T_m}^0(w_i)$$
$$= x_T^0(w_i),\ T(\overline{w}) < \zeta^0(w_i)\}_{[\overline{w}(n)]} \cap \{w_n;\ T_{[\overline{w}(n)]} < \zeta(w_n)\}]\}$$
$$= P_{\boldsymbol{x}}^{*(n)} [\bigcap_{i=1}^{n-1} \{\lim x_{T_m}^0(w_i) = x_T^0(w_i)\} \cap \{T(\overline{w}) < \bar{\zeta}(\overline{w})\}].$$

Repeating this we have

$$P_{\boldsymbol{x}}^{*(n)}[\lim_{m\to\infty} x_{T_m}^*(\overline{w}) = x_T^*(\overline{w}),\ T<\bar{\zeta}] = P_{\boldsymbol{x}}^{*(n)}[T<\bar{\zeta}].$$

(ii) can be proved quite similarly if we note that if ζ^0 is totally inaccessible and $\{T_m\}$ is an increasing sequence of $\mathcal{B}_t^0$-Markov times such that $T_m \uparrow T$, then

$$\{\{T<\zeta^0\} \cup \bigcup_{n=1}^{\infty} \{T_n \wedge \zeta^0 = \zeta^0\}\} \cap \{T<\infty\} \underset{\text{a.s}}{=} \{T<\infty\}.$$

§3.2. Direct sum of X_n^* and $\widetilde{X}_n$

Given a right continuous strong Markov process X^0 on $S\cup\{\Delta\}$ with Δ as the terminal point such that $\overline{\mathcal{B}}_{t+0} = \mathcal{B}_t$, let X_n^* and $\widetilde{X}_n$ $(n=1, 2, \cdots)$ be the canonical realizations of the n-fold direct product and the n-fold symmetric direct product of X^0, respectively, defined in the previous section. Let $\widehat{S}^* = \bigcup_{n=0}^{\infty} S^{(n)}$ and $S^* = S^* \cup \{\varDelta\}$ be the topological sum of $S^{(n)}$ and its one-point compactification, respectively; then the natural mapping ρ from $S^{(n)}$ to S^n can be extended from $\widehat{S}^*$ to $\widehat{S}$, where we set $\rho(\partial) = \partial$ and $\rho(\varDelta) = \varDelta$.

Now put

$$(3.11) \qquad W^{(0)} = \{w_\partial\},^{10)} \quad \overline{W} = \bigcup_{n=0}^{\infty} W^{(n)},$$

$$(3.12) \qquad x_t^*(\overline{w}) = \begin{cases} x_t^*(\overline{w}) \text{ defined by (3.4)}, & \text{if } \overline{w} \in \bigcup_{n=1}^{\infty} W^{(n)}, \\ \partial, & \text{if } \overline{w} = w_\partial \in W^{(0)}, \end{cases}$$

$$(3.13) \qquad \overline{\zeta}(\overline{w}) = \begin{cases} \overline{\zeta}(\overline{w}) \text{ defined by (3.3)}, & \text{if } \overline{w} \in \bigcup_{n=1}^{\infty} W^{(n)}, \\ +\infty, & \text{if } \overline{w} = w_\partial \in W^{(0)}, \end{cases}$$

$$(3.14) \qquad \overline{\theta}_t \overline{w} = \begin{cases} \overline{\theta}_t \overline{w} \text{ defined by (3.5)}, & \text{if } \overline{w} \in \bigcup_{n=1}^{\infty} W^{(n)}, \\ w_\partial, & \text{if } \overline{w} = w_\partial \in W^{(0)}, \end{cases}$$

$$(3.15) \qquad \mathfrak{N}_t^* = \sigma(\overline{W}, \mathscr{B}(\widehat{S}^*);\ x_s^*(\overline{w}),\ s \leq t), \quad \mathfrak{N}_\infty^* = \bigvee_{t>0} \mathfrak{N}_t^*,$$

$$(3.16) \qquad P_x^*[A] = P_x^{*(n)}[A \cap W^{(n)}],\ x \in S^{(n)},\ A \in \mathfrak{N}_\infty^*,^{11)}$$
$$P_\partial^*[A] = \delta_{\{w_\partial\}}(A),\ A \in \mathfrak{N}_\infty^*,$$

and $P_\varDelta^*$ is any probability measure on $(\overline{W}, \mathfrak{N}_\infty^*)$ such that

$$P_\varDelta^*[x_t^*(\overline{w}) \equiv \varDelta \text{ for all } t \geq 0] = 1.$$

Definition 3.2. The stochastic process

$$X^* = \{\overline{W},\ x_t^*(\overline{w}),\ \mathscr{B}_t^* = \mathfrak{N}_{t+0}^*,\ P_x^*,\ x \in \widehat{S}^*,\ \overline{\theta}_t,\ \overline{\zeta}\}$$

on $\widehat{S}^*$ defined above is called the *direct sum of* X_n^*.[12]

Now let

$$(3.17) \qquad \tilde{x}_t(\overline{w}) = \rho(x_t^*(\overline{w})),\ \overline{w} \in \overline{W},$$

10) w_∂ is an extra point.

11) Note that if $A \in \mathfrak{N}_\infty^*$, then $A \cap W^{(n)} \in \mathfrak{N}_\infty^{*(n)}$.

12) We consider $\varDelta$ as the terminal point of X^*, and hence $\overline{\zeta}$ is the life time.

and define $\widetilde{\mathcal{N}}_t$, $\widetilde{\mathcal{N}}_\infty$ and $\widetilde{P}_{\boldsymbol{x}}$, $\boldsymbol{x}\in\widehat{\boldsymbol{S}}$, for $\tilde{x}_t(\overline{w})$ in a similar way as (3.15) and (3.16).

Definition 3.3. The stochastic process

$$\widetilde{X}=\{\overline{W},\ \tilde{x}_t(\overline{w}),\ \widetilde{\mathcal{B}}_t=\overline{\widetilde{\mathcal{N}}}_{t+0},\ \widetilde{P}_{\boldsymbol{x}},\ \boldsymbol{x}\in\widehat{\boldsymbol{S}},\ \overline{\theta}_t,\ \overline{\zeta}\}$$

on $\widehat{\boldsymbol{S}}$ is called the *direct sum of* $\widetilde{X}_n$.[13]

Clearly $\widetilde{X}$ is the process induced from X^* by the mapping ρ, i.e., $\widetilde{X}=\rho(X^*)$.

The following theorem is a direct consequence of Theorem 3.1 and Theorem 3.2.

Theorem 3.3. *X^* and $\widetilde{X}$ are right continuous strong Markov processes on $\widehat{\boldsymbol{S}}^*$ and $\widehat{\boldsymbol{S}}$, respectively, with ∂ and Δ as traps. If X^0 has left limits (is quasi-left continuous before ζ^0, is a Hunt process and ζ^0 is totally inaccessible), then X^* and $\widetilde{X}$ have left limits (resp., are quasi-left continuous before $\overline{\zeta}$, are Hunt processes).*

§3.3. Construction of an instantaneous distribution

Let $X^0=\{W,\ x_t^0(w),\ \mathcal{B}_t^0,\ P_x^0,\ x\in S\cup\{\Delta\},\ \theta_t^0,\ \zeta^0\}$ be a right continuous strong Markov process on $S\cup\{\Delta\}$ with Δ as the terminal point such that $\mathcal{B}_t^0=\overline{\mathcal{B}}_{t+0}^0$. Further we shall assume

(3.18) $\quad P_x^0[\zeta^0=t]=0$ for every $t\geqq 0$ and $x\in S$

and

(3.19) $\quad P_x^0[x_{\zeta^0-}^0\text{-exists},\ \zeta^0<\infty]=P_x^0[\zeta^0<\infty]$ for every $x\in S$.

Let $\widetilde{X}^{(n)}(n=1,2,\cdots)$ be the canonical realization of the n-fold symmetric direct product of X^0, and $\widetilde{X}$ be the direct sum of $\widetilde{X}^{(n)}$ (cf. Definition 3.3).

Now let $\pi(x, dy)$ be a stochastic kernel on $S\times\widehat{\boldsymbol{S}}$[14] such that

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13) We consider Δ as the terminal point of $\widetilde{X}$, and hence $\overline{\zeta}$ is the life time.

14) i.e., it is a kernel on $(S, \mathcal{B}(S))\times(\widehat{\boldsymbol{S}}, \mathcal{B}(\widehat{\boldsymbol{S}}))$ such that for each fixed $x\in S$ it is a probability measure on $(\widehat{\boldsymbol{S}}, \mathcal{B}(\widehat{\boldsymbol{S}}))$.

$$\pi(x, S) \equiv 0 \quad \text{for every } x \in S. \tag{3.20}$$

If we restrict this kernel on $S \times S$, then it is a substochastic kernel with the property (3.20), and conversely, a given substochastic kernel π on $S \times S$ with the property (3.20) defines a stochastic kernel on $S \times \hat{S}$ with the property (3.20) by setting

$$\pi(x, \{\Delta\}) = 1 - \pi(x, S), \quad x \in S. \tag{3.21}$$

Hence it is equivalent to give a stochastic kernel on $S \times \hat{S}$ with the property (3.20) and to give a substochastic kernel on $S \times S$ with the property (3.20). It is also equivalent to give a system $\{q_n(x), \pi_n(x, dy)\}$, where $q_n(x)$, $n = 0, 2, 3, \cdots$ are non-negative $\mathcal{B}(S)$-measurable functions such that

$$\sum_{n=0}^{\infty} q_n(x) \leq 1,$$

and $\pi_n(x, dy)$, $n = 0, 2, 3, \cdots$ are stochastic kernels on $S \times S^n$, by the relation

$$\pi(x, E) = \sum_{n=0}^{\infty} q_n(x) \pi_n(x, E \cap S^n), \quad E \in \mathcal{B}(S), \ x \in S, \tag{3.22}$$

$$q_n(x) = \pi(x, S^n), \quad \pi_n(x, E) = \frac{1}{q_n(x)} \pi(x, E),^{15)} \quad E \in \mathcal{B}(S^n). \tag{3.23}$$

Given a stochastic kernel on $S \times \hat{S}$ with the property (3.20), we shall define a kernel μ' on $(W^{(n)}, \widetilde{\mathfrak{N}}_\infty^{(n)}) \times (\hat{\mathbf{S}}^{(n)}, \mathcal{B}(\hat{\mathbf{S}}^{(n)}))$[16] by

$$\begin{aligned} &\mu'(\overline{w}, d\boldsymbol{x}_1, d\boldsymbol{x}_2, \cdots, d\boldsymbol{x}_n) \\ &= \begin{cases} \displaystyle\sum_{i=1}^{n} I_{\{\bar{\zeta}(\overline{w}) = \zeta^0(w_i)\}}(\overline{w}) \cdot \pi(x^0_{\zeta^0(w_i)-}(w_i), d\boldsymbol{x}_i) \prod_{j \neq i} \delta_{\{x^0_{\zeta^0(w_i)}(w_j)\}}(d\boldsymbol{x}_j), \\ \qquad\qquad\qquad\qquad\qquad \text{if } 0 < \bar{\zeta}(\overline{w}) < \infty, \\ \delta_{\{\underbrace{\Delta, \cdots, \Delta}_{n}\}}(d\boldsymbol{x}_1, d\boldsymbol{x}_2, \cdots, d\boldsymbol{x}_n), \quad \text{if } \bar{\zeta}(\overline{w}) = 0 \text{ or } \bar{\zeta}(\overline{w}) = \infty, \end{cases} \end{aligned} \tag{3.24}$$

where $\overline{w} = (w_1, w_2, \cdots, w_n)$.

---

15) Let $\pi_n(dy)$ be a probability measure on S^n and set $\pi_n(x, dy) = \pi_n(dy)$ *if* $q_n(x) = 0$.

16) $\hat{\mathbf{S}}^{(n)} = \underbrace{\hat{S} \times \hat{S} \times \cdots \times \hat{S}}_{n}$.

Let γ be the mapping defined by (0.19) and define a kernel μ on $(W^{(n)}, \tilde{\mathfrak{N}}_\infty^{(n)}) \times (\hat{S}, \mathcal{B}(\hat{S}))$ by

$$\mu(\bar{w}, d\boldsymbol{x}) = \mu'(\bar{w}, \gamma^{-1}(d\boldsymbol{x})). \tag{3.25}$$

We have in this way a stochastic kernel on $(\bigcup_{n=1}^{\infty} W^{(n)}, \tilde{\mathfrak{N}}_\infty) \times (\hat{S}, \mathcal{B}(\hat{S}))$. We set further

$$\mu(w_\partial, d\boldsymbol{x}) = \delta_{\{\partial\}}(d\boldsymbol{x}). \tag{3.26}$$

Thus we have defined a stochastic kernel on $(\bar{W}, \tilde{\mathfrak{N}}_\infty) \times (\hat{S}, \mathcal{B}(\hat{S}))$, and the following theorem is clear from the definition.

Theorem 3.4. *$\mu(\bar{w}, dx)$ is an instantaneous distribution for the process $\tilde{X}$.*

§3.4. Construction of an (X^0, π)-branching Markov process

For a given X^0 satisfying (3.18) and (3.19), and a given stochastic kernel $\pi(x, dy)$ on $S \times \hat{S}$ satisfying (3.20), we construct the direct sum $\tilde{X}$ of the canonical realizations of the symmetric direct products of X^0 and the instantaneous distribution μ of $\tilde{X}$ as in the previous sections. Now we apply Theorem 2.2; we have a right continuous strong Markov process $\boldsymbol{X} = \{\tilde{\Omega}, \boldsymbol{X}_t(\tilde{\omega}), \boldsymbol{P}_{\boldsymbol{x}}, \boldsymbol{x} \in \hat{S}, \mathcal{F}_t, \theta_t, \zeta\}$ on $\hat{S}$ with ∂ and Δ as traps such that $\overline{\mathcal{F}}_{t+0} = \mathcal{F}_t$. We will show that $\boldsymbol{X}$ is the (X^0, π)-branching Markov process (cf. Definition (1.6)). First, it is easy to see that $\tau(\tilde{\omega})$ defined by (2.8) coincides with that defined by (1.7). Also it is clear that $\boldsymbol{X}$ satisfies the conditions (C.1) and (C.2) by the way of the construction and by (3.18). Next, we shall prove that $\boldsymbol{X}$ has the property B.III. In fact, if $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]$, we have by Theorem 2.2 (i) and (ii) that, for $f \in \boldsymbol{B}^*(S)$,

$$\begin{aligned} \boldsymbol{E}_{\boldsymbol{x}}[\hat{f}(\boldsymbol{X}_t); t < \tau] &= \tilde{E}_{\boldsymbol{x}}[\hat{f}(\tilde{x}_t); t < \bar{\zeta}] \\ &= \int \cdots \int_{W \times \cdots \times W} P^0_{x_1}(dw_n) \cdots P^0_{x_n}(dw_n) \{\prod_{i=1}^n (f(x^0_t(w_i)) I_{\{t < \zeta^0(w_i)\}})\} \\ &= \prod_{i=1}^n E^0_{x_i}[f(x^0_t(w)); t < \zeta^0] \\ &= \prod_{i=1}^n \boldsymbol{E}_{x_i}[f(\boldsymbol{X}_t); t < \tau], \end{aligned} \tag{3.27}$$

and for $f\in \boldsymbol{B}^*([0,\infty)\times S)$,

$$
\begin{aligned}
(3.28)\quad & \boldsymbol{E}_x[\widehat{f}(\tau, \boldsymbol{X}_\tau);\ \tau\leq t] \\
&=\widetilde{E}_x\Big[\int_{\widehat{S}}\mu(\bar{w}, d\boldsymbol{y})\widehat{f}(\bar{\zeta}(\bar{w}), \boldsymbol{y});\ \bar{\zeta}(\bar{w})\leq t\Big] \\
&=\widetilde{E}_x\Big[\int_{\widehat{S}}\cdots\int_{\widehat{S}}\sum_{i=1}^n I_{\{\bar{\zeta}(\bar{w})=\zeta^0(w_i)\leq t\}}(\bar{w})\cdot\pi(x^0_{\zeta^0(w_i)-}(w_i), d\boldsymbol{x}_i) \\
&\qquad\cdot\prod_{j\neq i}\delta_{\{x^0_{\zeta(w_i)}(w_j)\}}(d\boldsymbol{x}_j)\cdot\prod_{j=1}^n\widehat{f}(\zeta^0(w_i), \boldsymbol{x}_j)\Big] \\
&=\sum_{i=1}^n\int_W P^0_{x_i}(dw_i)\Big[I_{\{\zeta^0(w_i)\leq t\}}\cdot\int_{\widehat{S}}\pi(x^0_{\zeta^0(w_i)-}, d\boldsymbol{x}_i)\widehat{f}(\zeta^0(w_i), \boldsymbol{x}_i) \\
&\quad\cdot\Big\{\int\cdots\int_{W\times\cdots\times W}P^0_{x_1}\times\cdots\times P^0_{x_{i-1}}\times P^0_{x_{i+1}}\times\cdots\times P^0_{x_n}[dw_1, \cdots, \\
&\qquad\qquad dw_{i-1}, dw_{i+1}, \cdots, dw_n] \\
&\qquad\cdot\prod_{j\neq i}[\widehat{f}(\zeta^0(w_i), x_{\zeta^0(w_i)}(w_j))\cdot I_{\{\zeta^0(w_i)<\zeta^0(w_j)\}}]\Big\}\Big] \\
&=\sum_{i=1}^n\int_W P^0_{x_i}[dw_i]\Big(I_{\{\zeta^0(w_i)\leq t\}}\int_{\widehat{S}}\mu(w_i, d\boldsymbol{x})\widehat{f}(\zeta^0(w_i), \boldsymbol{x}) \\
&\quad\cdot\Big\{\int\cdots\int_{W\times\cdots\times W}P^0_{x_1}\times\cdots\times P^0_{x_{i-1}}\times P^0_{x_{i+1}}\times\cdots\times P^0_{x_n}[dw_1, \cdots, \\
&\qquad\qquad dw_{i-1}, dw_{i+1}, \cdots, dw_n] \\
&\qquad\cdot\prod_{j\neq i}[\widehat{f}(\zeta^0(w_i), x_{\zeta^0(w_i)}(w_j))\cdot I_{\{\zeta^0(w_i)<\zeta^0(w_j)\}}]\Big\}\Big) \\
&=\sum_{i=1}^n\int_0^t\int_{\widehat{S}}\boldsymbol{P}_{x_i}[\tau\in ds, \boldsymbol{X}_\tau\in d\boldsymbol{y}]\{\widehat{f}(s, \boldsymbol{y})\cdot\prod_{j\neq i}\boldsymbol{E}_{x_j}[\widehat{f}(s, \boldsymbol{X}_s);\ s<\tau]\}.
\end{aligned}
$$

Therefore, by Theorem 1.2 (d), $\boldsymbol{X}$ is a branching Markov process. Finally we shall show that $\boldsymbol{X}$ is the (X^0, π)-branching Markov process. In fact, $\{\boldsymbol{X}_t, t<\tau, \boldsymbol{P}_x\}$ and X^0 are equivalent and hence the non-branching part of $\boldsymbol{X}$ coincides with X^0. Next we have, for $x\in S$, $f\in\boldsymbol{B}^*(S)$, $g\in\boldsymbol{B}(S)$ and $\lambda>0$ that

$$
\begin{aligned}
(3.29)\quad & \boldsymbol{E}_x[e^{-\lambda\tau}\widehat{f}(\boldsymbol{X}_\tau)g(\boldsymbol{X}_{\tau-})] \\
&=E^0_x\Big[e^{-\lambda\zeta^0}g(x^0_{\zeta^0-})\int_{\boldsymbol{S}}\mu(w, d\boldsymbol{y})\widehat{f}(\boldsymbol{y})\Big] \\
&=E^0_x\Big[e^{-\lambda\zeta^0}g(x^0_{\zeta^0-})\int_{\boldsymbol{S}}\pi(x^0_{\zeta^0-}, d\boldsymbol{y})\widehat{f}(\boldsymbol{y})\Big] \\
&=\boldsymbol{E}_x\Big[e^{-\lambda\tau}g(\boldsymbol{X}_{\tau-})\int_{\boldsymbol{S}}\pi(\boldsymbol{X}_{\tau-}, d\boldsymbol{y})\widehat{f}(\boldsymbol{y})\Big]
\end{aligned}
$$

and therefore π is the branching law of the process $\boldsymbol{X}$.

Summarizing the above arguments, we have the following

Theorem 3.5. *For a given right continuous strong Markov process $X^0=(x_t^0, \mathscr{B}_t^0)$ on $S\cup\{\Delta\}$ with Δ as its terminal point satisfying* (3.18), (3.19) *and $\overline{\mathscr{B}}_{t+0}^0=\mathscr{B}_t^0$, and a given stochastic kernel $\pi(x, dy)$ on $S\times\widehat{S}$ satisfying* (3.20), *we construct the direct sum $\widetilde{X}$ of the canonical realizations of the symmetric direct products of X^0 and an instantaneous distribution μ as in* §3.2 *and* §3.3. *Next, applying Theorem* 2.2 *for $\widetilde{X}$ and μ, we construct a right continuous strong Markov process $\boldsymbol{X}=(\boldsymbol{X}_t, \mathscr{F}_t)$ on $\widehat{S}$ such that $\overline{\mathscr{F}}_{t+0}=\mathscr{F}_t$. Then $\boldsymbol{X}$ is the (X^0, π)-branching Markov process. Further if X^0 has left limits, then $\boldsymbol{X}$ has left limits for $t<\tau_\infty$, and if X^0 is quasi-left continuous and ζ^0 is totally inaccessible, then $\boldsymbol{X}$ is quasi-left continuous before τ_∞.*

The last assertion of the theorem follows immediately from Theorem 3.3, Theorem 2.3 and Theorem 2.4.

§3.5. Examples

Example 3.1. Branching process with a single type

Consider the simplest case when $S=\{a\}$ then S can be identified with $\boldsymbol{Z}^+=\{0, 1, 2, \cdots\}$ and $\widehat{S}$ with $\widehat{\boldsymbol{Z}}^+=\boldsymbol{Z}^+\cup\{+\infty\}$.[17] Therefore a branching Markov process on $\widehat{S}$ is a Markov chain on $\widehat{\boldsymbol{Z}}^+$ such that its system of transition matrices $\{P_{ij}(t), t\geq 0, i, j\in\widehat{\boldsymbol{Z}}^+\}$ satisfies

$$\begin{cases}\sum_{j=0}^{\infty}P_{ij}(t)f^j=\left(\sum_{j=0}^{\infty}P_{1j}(t)f^j\right)^i, \quad 0<f<1,\ i=0, 1, 2, \cdots, \\ P_{+\infty, +\infty}(t)=1.\end{cases}$$

It is easy to see that $P_{ij}(t)$ defines a strongly continuous semi-group on $\boldsymbol{C}_0(\boldsymbol{Z}^+)$, and hence $\boldsymbol{X}$ is a Hunt process. This implies that $\boldsymbol{X}$ is a minimal Markov chain. If we set

17) Cf. Example 1.3.

$$b_i = E_i[\tau]^{-1}, \quad \pi_{ij} = P_i[X_\tau = j]$$

where τ is the first jumping time, the property B. III. of §1. 2 is equivalent to

(3. 30) $\quad b_i = ib_1$ and $\pi_{ij} = \pi_{1, j-i+1}$, $i = 0, 1, 2, \cdots$.

Thus a Markov chain on $\hat{Z}^+$ *is a branching Markov process if and only if it is a* (b_i, π_{ij})*-minimal chain with the property* (3. 30).

Fundamental equations which will be treated in Chapter IV are given as follows: if we set, for $0 \leq f < 1$

$$u(t, i; f) = T_t \hat{f}(i) = \sum_{j=0}^{\infty} P_{ij}(t) f^j, \quad i = 1, 2, \cdots,$$

$$u(t; f) = u(t, 1; f)$$

and

$$F(f) = \sum_{j=0}^{\infty} \pi_{1,j} f^j,$$

then

(3. 31) $\quad u(t; f) = f \cdot e^{-b_1 t} + b_1 \int_0^t F(u(t-s; f)) e^{-b_1 s} ds$, ($S$-equation),

(3. 32) $\quad \dfrac{\partial u(t; f)}{\partial t} = b_1 \{F(u(t; f)) - u(t; f)\}$,

$\quad\quad u(0+, f) = f$, $\quad$ (backward equation)

and

(3. 33) $\quad \dfrac{\partial u(t, i; f)}{\partial t} = b_1 (F(f) - f) \dfrac{\partial u(t, i; f)}{\partial f}$,

$\quad\quad u(0+, i; f) = f^i$, $i = 0, 1, 2, \cdots$, (forward equation).

Now assume

$$\pi_{1,0} = P_1[X_\tau = \partial] = 0$$

and

$$\pi_{1,\infty} = P_1[X_\tau = \Delta] = 0.$$

We shall prove an intimate relation between the uniqueness of the solution of S-equation (3. 31) and the occurence of no explosion in a Corollary of Theorem 4. 7, i.e., $P_i[e_\Delta = +\infty] = 1$ if and only if $u(t) \equiv 1$ is the unique solution of (3. 31) with the initial value $f = 1$.

As is well known (and it can be proved easily) $u(t)\equiv 1$ is the unique solution of (3.31) or (3.32) if and only if

$$\int^{1-0}\frac{df}{f-F(f)}=+\infty,$$

(cf. Harris [8]). Here we shall give another probabilistic proof of this fact. The proof is based on the following

Lemma 3.2. $\boldsymbol{E}_1[e_\Delta]=\infty$ *if and only if* $\boldsymbol{P}_1[e_\Delta=\infty]=1$.

Proof. "If" part is trivial and hence we shall prove "only if" part. Assume $\boldsymbol{P}_1[e_\Delta=\infty]<1$. Then $\boldsymbol{P}_1[e_\Delta>t]\equiv \boldsymbol{T}_t\hat{1}(1)<1$ for every $t>0$. In fact, if for some t, $\boldsymbol{T}_t\hat{1}(1)=1$, then $\boldsymbol{T}_{nt}\hat{1}(1)=\boldsymbol{T}_{(n-1)t}(\boldsymbol{T}_t\hat{1})(1)$ $=\boldsymbol{T}_{(r-1)t}\hat{1}(1)=\cdots=\boldsymbol{T}_t\hat{1}(1)=1$ and hence $\lim_{n\to\infty}\boldsymbol{T}_{nt}\hat{1}(1)=\boldsymbol{P}_1[e_\Delta=\infty]=1$. But this is a contradiction. Therefore $\boldsymbol{T}_t\hat{1}(1)<1$ for every $t>0$. Next we shall show that for fixed $t_0>0$

$$\boldsymbol{T}_{nt_0}\hat{1}(1)\leq(\boldsymbol{T}_{t_0}\hat{1}(1))^n.$$

In fact, since $\boldsymbol{T}_t\hat{1}(i)=(\boldsymbol{T}_t\hat{1}(1))^i\leq\boldsymbol{T}_t\hat{1}(1)$, $i=1, 2, \cdots$,

$$\boldsymbol{T}_{nt_0}\hat{1}(1)=\boldsymbol{T}_{t_0}(\boldsymbol{T}_{(n-1)t_0}\hat{1})(1)\leq\boldsymbol{T}_{t_0}\hat{1}(1)\,\boldsymbol{T}_{(n-1)t_0}\hat{1}(1)\leq\cdots\leq(\boldsymbol{T}_{t_0}\hat{1}(1))^n.$$

Hence $\boldsymbol{T}_t\hat{1}(1)\leq e^{-Kt}$ for some constant $K>0$. Therefore

$$\boldsymbol{E}_1[e_\Delta]=\int_0^\infty \boldsymbol{T}_t\hat{1}(1)dt<\infty.$$

Now it is clear that $e_\Delta=\tau_\infty$ a.s. under the above assumptions. Hence $\boldsymbol{E}_1[e_\Delta]=\boldsymbol{E}_1[\tau_\infty]$. Since

$$\tau_\infty=\sum_{k=1}^\infty(\tau_k-\tau_{k-1})=\sum_{k=1}^\infty\tau(\theta_{\tau_{k-1}}\omega),$$

$$\boldsymbol{E}_1[\tau_\infty]=\sum_{k=1}^\infty\boldsymbol{E}_1[\boldsymbol{E}_{X_{\tau_{k-1}}}[\tau]].$$

On the other hand

$$\boldsymbol{E}_1[\boldsymbol{E}_{X_{\tau_{k-1}}}[\tau]]=\sum_{n_1=1}^\infty\cdots\sum_{n_k=1}^\infty\pi_{1,n_1+1}\cdots\pi_{1,n_k+1}\frac{1}{n_1+n_2+\cdots+n_k+1}\frac{1}{b_1},$$

and noting that, for $0<\epsilon<1$,

$$\sum_{k=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\pi_{1,n_1+1}\cdots\pi_{1,n_k+1}\frac{(1-\epsilon)^{n_1+n_2+\cdots+n_k+1}}{n_1+n_2+\cdots+n_k+1}<\infty,$$

we see that $E_1[\tau_\infty]=\infty$ is equivalent to

$$\int_{1-\epsilon}^{1}\sum_{k=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\pi_{1,n_1+1}\cdots\pi_{1,n_k+1}\xi^{n_1+n_2+\cdots+n_k}d\xi$$

$$=\int_{1-\epsilon}^{1}\sum_{k=1}^{\infty}\left(\frac{F(\xi)}{\xi}\right)^k d\xi=\int_{1-\epsilon}^{1}\frac{\xi}{\xi-F(\xi)}d\xi=+\infty.$$

Therefore by the above Lemma, $P_1[e_J=+\infty]=1$ if and only if $\int^{1-0}\frac{1}{\xi-F(\xi)}d\xi=+\infty$. The conclusion is still valid when $\pi_{1,0}>0$: the proof is reduced to the case $\pi_{1,0}=0$ by the transformation of §5. 5.

Example 3. 2. Branching process with finite number of types

Let $S=\{a_1, a_2, \cdots, a_k\}$; then S can be identified with

$$(\boldsymbol{Z}^+)^k=\overbrace{\boldsymbol{Z}^+\times\boldsymbol{Z}^+\times\cdots\times\boldsymbol{Z}^+}^{k}=\{\boldsymbol{i}=\{i_1, i_2, \cdots, i_k\}; i_l\in\boldsymbol{Z}^+\}$$

and $\hat{S}$ with $\widehat{(\boldsymbol{Z}^+)^k}\equiv(\boldsymbol{Z}^+)^k\cup\{+\infty\}$. Therefore a branching Markov process on $\hat{S}$ is a right-continuous Markov chain on $\widehat{\boldsymbol{Z}^{+(k)}}$ such that its system of transition matrices $\{P_{\boldsymbol{i},\boldsymbol{j}}(t), t\geq 0, \boldsymbol{i}, \boldsymbol{j}\in\widehat{\boldsymbol{Z}^{+(k)}}\}$ satisfies

$$\begin{cases}\sum_{\boldsymbol{j}}P_{\boldsymbol{i},\boldsymbol{j}}(t)\hat{f}(\boldsymbol{j})=\prod_{l=1}^{k}\left(\sum_{\boldsymbol{j}}P_{\boldsymbol{e}_l,\boldsymbol{j}}(t)\cdot\hat{f}(\boldsymbol{j})\right)^{i_l}\text{[18]},\\ P_{+\infty,+\infty}(t)\equiv 1,\end{cases}$$

where $\boldsymbol{f}=(f_1, f_2, \cdots, f_k)$, $0\leq f_l<1$, $\boldsymbol{i}=(i_1, i_2, \cdots, i_k)$ and $\boldsymbol{e}_l=(0, \cdots, \overset{l\text{-th}}{1}, \cdots, 0)$. From this it is easy to see that $P_{\boldsymbol{i},\boldsymbol{j}}(t)$ defines a strongly continuous semi-group on $\boldsymbol{C}_0(\widehat{\boldsymbol{Z}^{+(k)}})$, and hence $\boldsymbol{X}$ is a Hunt process. This implies $\boldsymbol{X}$ is a minimal Markov chain. By Theorem 1. 4, it is given as an (X^0, π)-branching Markov process. In this way every branching Markov process on $\hat{S}$ is determined by a Markov chain X^0 on $S\cup\{\Delta\}$, with $\{\Delta\}$ as its terminal point, and a substochastic kernel $\pi(\boldsymbol{e}_l, d\boldsymbol{y})$ on $S\times\boldsymbol{S}$ such that $\pi(\boldsymbol{e}_l, S)=0$, $l=1, 2, \cdots, k$. But every such X^0 is given in the following way: given $0\leq\pi_{ij}\leq 1$, $\pi_{ii}=0$,

18) $\hat{f}(\boldsymbol{i})=f_1^{i_1}f_2^{i_2}\cdots f_k^{i_k}$.

$\sum_j \pi_{ij}=1,\ i, j=1, 2, \cdots, k$ and $0\leq b_i<+\infty,\ 0\leq c_i<+\infty\ \ i=1, 2, \cdots, k$, X^0 is the $e^{-\int_0^t c(x_s)ds}$-subprocess[19] of (π_{ij}, b_i)-Markov chain x_t on $S=(\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_k)$.[20] *Thus there is a one-to-one correspondence between the set of all branching Markov process on* $\hat{S}$ *and the set of all systems* $\{b_i, c_i, \pi_{ij}, \pi(\boldsymbol{e}_i, d\boldsymbol{y})\}$ $i, j=1, 2, \cdots, k$ *satisfying the above conditions.*

Given such a system $\{b_i, c_i, \pi_{ij}, \pi(\boldsymbol{e}_i, d\boldsymbol{y})\}$, define a sub-stochastic kernel $\pi'(\boldsymbol{e}_i, d\boldsymbol{y})$ on $S\times \boldsymbol{S}$ and b'_i, $i=1, 2, \cdots, k$, by

$$\begin{cases} \pi'(\boldsymbol{e}_i, \{\boldsymbol{e}_j\})=\dfrac{b_i}{b_i+c_i}\pi_{ij}, & i, j=1, 2, \cdots, k, \\ \pi'(\boldsymbol{e}_i, \{\boldsymbol{y}\})=\dfrac{c_i}{b_i+c_i}\pi(\boldsymbol{e}_i, \{\boldsymbol{y}\}), & i=1, 2, \cdots, k,\ \boldsymbol{y}\in \boldsymbol{S}-S, \end{cases}$$

and

$$b'_i=b_i+c_i, \quad i=1, 2, \cdots, k,$$

Set

$$F_i(\boldsymbol{f})=\sum_{\boldsymbol{y}}\pi'(\boldsymbol{e}_i, \{\boldsymbol{y}\})\hat{\boldsymbol{f}}(\boldsymbol{y}),$$

then the fundamental equations which will be discussed in Chapter IV are now given as follows: if we set, for $\boldsymbol{f}=(f_1, \cdots, f_k)$, $0\leq f_i<1$,

$$\boldsymbol{u}(t, \boldsymbol{i}; \boldsymbol{f})=\sum_{\boldsymbol{j}} P_{\boldsymbol{i},\boldsymbol{j}}(t)\hat{\boldsymbol{f}}(\boldsymbol{j}),$$
$$\boldsymbol{u}(t; \boldsymbol{f})=(u_1(t; \boldsymbol{f}), u_2(t; \boldsymbol{f}), \cdots, u_k(t; \boldsymbol{f})),$$

where

$$u_i(t; \boldsymbol{f})=\boldsymbol{u}(t, \boldsymbol{e}_i; \boldsymbol{f}),$$

then

$$(3.34)\qquad u_i(t; \boldsymbol{f})=f_i e^{-b'_i t}+b'_i\int_0^t F_i(\boldsymbol{u}(t-s; \boldsymbol{f}))e^{-b'_i s}ds,$$
$$i=1, 2, \cdots, k, \quad (S\text{-equation})$$

$$(3.35)\qquad \frac{\partial u_i}{\partial t}(t; \boldsymbol{f})=b'_i\{F_i(\boldsymbol{u}(t; \boldsymbol{f}))-u_i(t; \boldsymbol{f})\},$$
$$u_i(0+, \boldsymbol{f})=f_i, \quad i=1, 2, \cdots, k, \quad (\text{backward equation})$$

---

19) $\boldsymbol{c}$ is a function on S defined by $\boldsymbol{c}(e_i)=c_i$, $i=1, 2, \cdots, k$.

20) That is, x_t is a Markov chain on S such that $E_{e_i}(\sigma)=b_i^{-1}$ and $P_{e_i}[x_\sigma=e_j]=\pi_{ij}$, where σ is the first jumping time.

and

(3.36)
$$\frac{\partial u(t, \boldsymbol{i}; f)}{\partial t}=\sum_{l=1}^{k} b_l'\{F_l(f)-f_l\}\frac{\partial u(t, \boldsymbol{i}; f)}{\partial f_l},$$
$$u(0+, \boldsymbol{i}; f)=\widehat{f}(\boldsymbol{i}), \quad \boldsymbol{i}\in \boldsymbol{S}\equiv\widehat{\boldsymbol{Z}^{+(k)}}, \quad \text{(forward equation)}.$$

Example 3.3. Age dependent branching process

Let $S=[0, \infty]$, $k(x)$ be a non-negative locally integrable function on $[0, \infty)$ and $\{q_n(x)\}_{n=0}^{\infty}$ be a sequence of non-negative measurable functions on $[0, \infty)$ such that $\sum_{n=0}^{\infty} q_n(x)\equiv 1$ and $q_1(x)\equiv 0$.[21] Define a probability kernel $\pi(x, dy)$ on $S\times S$ by

(3.37)
$$F(x; f)\equiv\int_S \widehat{f}(y)\pi(x, dy)$$
$$=\begin{cases}\sum q_n(x)f^n(0), & x\in[0, \infty),\\ f(\infty), & x=+\infty.\end{cases}$$

Let X^0 be the $e^{-\int_0^t k(x_s)ds}$-subprocess of the uniform motion x_t on S. By Theorem 3.5 we have the (X^0, π)-branching Markov process $\boldsymbol{X}$, and we shall call it an *age dependent branching process.* The fundamental system of $\boldsymbol{X}$ is given as (T_t^0, K, π), where

$$T_t^0 f(x)=e^{-\int_x^{x+t}k(s)ds}f(x+t), \quad x\in[0, \infty),$$
$$=f(\infty), \quad x=\infty,$$
$$\int_0^\infty K(x; dsdy)f(y)=T_s^0(k\cdot f)(x)ds$$

and π is defined by (3.37). Hence $u(t, x)=\boldsymbol{T}_t\widehat{f}(x)=\boldsymbol{E}_x[\widehat{f}(\boldsymbol{X}_t)]$, $f\in\overline{\boldsymbol{B}^*[0, \infty]^+}$, satisfies the S-equation:

(3.38)
$$u(t, x)=f(x+t)e^{-\int_x^{x+t}k(s)ds}$$
$$+\int_0^t k(x+r)e^{-\int_x^{x+r}k(s)ds}\sum_{n=0}^{\infty}q_n(x+r)u^n(t-r, 0)dr.$$

Now let

$$H=\{f\in\boldsymbol{B}(S);\ f|_{[0,\infty)}\in\boldsymbol{C}[0, \infty)\}.$$

Then for the semi-group T_t of the uniform motion,[22] H_0 and $\widetilde{H}_0$ are

21) We extend $k(x)$ and $q_n(x)$ as functions on $[0, \infty]$ by setting them 0 at $x=\infty$.

22) i.e., the semi-group T_t defined by $T_t f(x)=\begin{cases}f(x+t), & x\in[0, \infty),\\ f(\infty), & x=\infty.\end{cases}$

given by

$$H_0=\{f\in \boldsymbol{B}(S);\ f|_{[0,\infty)} \text{ is uniformly continuous on } [0,\infty)\}$$

(cf. Chapter IV) and

$$\widetilde{H}_0=H.$$

In the following we shall use the results which will be developed in Chapter IV. It is easy to see that the fundamental system is H-regular (weakly H-regular) if k and q_n are in H_0 (resp. in H). The infinitesimal generator A_H and the weak infinitesimal generator $\widetilde{A}_H$ are given by

$$A_H f(x)=\widetilde{A}_H f(x)=f'(x)$$

with domains

$$D(A_H)=\{f\in H_0;\ f' \text{ exists and } f'\in H_0\}$$

and

$$D(\widetilde{A}_H)=\{f\in H;\ f' \text{ exists and } f'\in H\}.$$

By a corollary of Theorem 4.10, we see that

(i) if k and q_n are in H_0 and $f\in \boldsymbol{B}^*(S)\cap D(A_H)$, then $u(t,x) = \boldsymbol{T}_t\widehat{f}(x)=\boldsymbol{E}_x[\widehat{f}(X_t)]$ is in $D(A_H)$ for all $t\geq 0$, strongly differentiable in t and satisfies

$$\text{(3.39)}\qquad \begin{cases} \dfrac{\partial u(t,x)}{\partial t}=\dfrac{\partial u(t,x)}{\partial x}+k(x)\left\{\sum_{n=0}^{\infty} q_n(x)u^n(t,0)-u(t,x)\right\} \\ u(0+,x)=f(x), \end{cases}$$

(ii) if k and q_n are in H and $f\in \boldsymbol{B}^*(S)\cap D(\widetilde{A}_H)$, then $u(t,x)$ is in $D(\widetilde{A}_H)$ for all $t\geq 0$, has right-hand derivatives $D_t^+u(t,x)$ in t and satisfies

$$\text{(3.40)}\qquad \begin{cases} D_t^+u(t,x)=\dfrac{\partial u(t,x)}{\partial x}+k(x)\left\{\sum_{n=0}^{\infty} q_n(x)u^n(t,0)-u(t,x)\right\}, \\ u(0+,x)=f(x). \end{cases}$$

Next set

$$G(x;\ f)=\sum_{n=0}^{\infty} nq_n(x)\cdot f(0).$$

Then $v(t, x)=M_t f(x)=\boldsymbol{E}_x[\check{f}(\boldsymbol{X}_t)]$ satisfies

$$v(t, x)=f(x+t)e^{-\int_x^{x+t}k(s)ds} + \int_0^t k(x+r)e^{-\int_x^{x+r}k(s)ds}G(x+r)v(t-r, 0)dr, \tag{3.41}$$

where

$$G(x)=\sum_{n=0}^{\infty} nq_n(x).$$

Further if $G(x)\in H_0$ $(G(x)\in H)$ and $f\in D(A_H)$ (resp. $D(\widetilde{A}_H)$), then $v(t, x)$ is in $D(A_H)$ for all $t\geq 0$, strongly differentiable in t and satisfies

$$\begin{cases} \dfrac{\partial v(t, x)}{\partial t}=\dfrac{\partial v(t, x)}{\partial x}+k(x)[G(x)v(t, 0)-v(t, x)], \\ v(0+, x)=f(x), \end{cases} \tag{3.42}$$

(resp. $v(t, x)$ is in $D(\widetilde{A}_H)$ for all $t\geq 0$, has right-hand derivatives in t and satisfies (3.42), where $\dfrac{\partial v}{\partial t}$ is now replaced by the right-hand derivative).

Example 3.4. Branching diffusion processes

By a *branching diffusion process* we mean a branching Markov process whose non-branching part X^0 is given as an e^{-A_t}-subprocess of a conservative diffusion process $X=\{x_t, P_x\}$ on a manifold S, where A_t is a non-negative continuous additive functional of x_t. In the following we shall consider some of typical examples.

(A) Branching Brownian motions

Let $S=\widehat{R^N}=R^N\cup\{\infty\}$ be one-point compactification of N-dimensional Euclidean space R^N and $X=\{x_t, P_x\}$ be a standard Brownian motion on S.[23] Let $k\in \boldsymbol{C}(S)^+$ and define A_t by

$$A_t=\int_0^t k(x_s)ds.$$

Let X^0 be the e^{-A_t}-subprocess of X. Let $q_n\in \boldsymbol{C}(S)^+$, $n=0, 2, \cdots$, such that $\sum_{n=0}^{\infty} q_n(x)\equiv 1$ and define $\pi(x, dy)$ by

23) ∞ is attached to R^N as a trap: $P_\infty[x_t=\infty$, for all $t\geq 0]=1$.

$$\pi(x, dy)=\sum_{n=0}^{\infty}q_n(x)\delta_{[\underbrace{x,\cdots,x}_{n}]}(dy).\text{[24]} \tag{3.43}$$

Then we have the (X^0, π)-branching Markov process $\boldsymbol{X}$, and we shall call it a *branching Brownian motion*.[25] The fundamental system (T_t^0, K, π) of $\boldsymbol{X}$ is given by

$$T_t^0 f(x)=\int_{R^N}P^0(t, x, y)f(y)dy, \qquad x\in R^N,$$

$$K(x;\ dsdy)=P^0(s, x, y)k(y)dyds, \quad x\in R^N,$$

where $P^0(s, x, y)$ is the fundamental solution of

$$\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u-k\cdot u.$$

It is easy to see that the fundamental system is regular. Hence we can apply all the results in Chapter IV, and we see that

$$u(t, x)=\boldsymbol{T}_t\widehat{f}(x)=\boldsymbol{E}_x[\widehat{f}(\boldsymbol{X}_t)],\ f\in \boldsymbol{C}^*(S)^+,\ x\in R^N,$$

satisfies S-equation;

$$u(t, x)=T_t^0\widehat{f}(x)+\int_0^t T_s^0(kF(\cdot\,;\ u(t-s, \cdot)))ds, \tag{3.44}$$

where

$$F(x;\ f)=\sum_{n=0}^{\infty}q_n(x)f^n(x). \tag{3.45}$$

If further $f\in D(\bar{\Delta})\cap \boldsymbol{C}^*(S)$,[26] then $u(t, x)$ belongs to $D(\bar{\Delta})\cap \boldsymbol{C}^*(S)$, is strongly differentiable in t and satisfies

$$\begin{cases}\dfrac{\partial u}{\partial t}=\dfrac{\bar{\Delta}}{2}u+k\cdot\{F(\cdot\,;\ u)-u\},\\ \|u(t, \cdot)-f\|\to 0, \quad (t\downarrow 0).\end{cases} \tag{3.46}$$

---

24) $\delta_{[\overbrace{x,\cdots,x}^{n}]}(dy)$ is a unit measure on $[\overbrace{x, \cdots, x}^{n}]\in S^n$.

25) It is clear that if $\boldsymbol{x}=[x_1, \cdots, x_n]$, $x_i\in R^N$ for all i then with $\boldsymbol{P}_{\boldsymbol{x}}$-probabillity one $\boldsymbol{X}_t\in \bigcup_{n=0}^{\infty}(\overbrace{R^N\times R^N\times\cdots\times R^N}^{n})/\sim \cup\{\Delta\}$. We are interested in the part of process $\boldsymbol{X}$ on this space.

26) $D(\bar{\Delta})=\{f\in \widehat{\boldsymbol{C}}(R^N), \Delta f\in \widehat{\boldsymbol{C}}(R^N)\}$, where $\widehat{\boldsymbol{C}}(R^N)=\{f\in \boldsymbol{C}(R^N);\ \lim_{|x|\to\infty}f(x)$ exists$\}$. Thus $\widehat{\boldsymbol{C}}(R^N)$ and $\boldsymbol{C}(S)$ are essentially the same space. $D(\bar{\Delta})$ coincides with the domain (in Hille-Yosida sense) of the infinitesimal generator $A\left(=\frac{1}{2}\bar{\Delta}\right)$ of the semi-group of the standard Brownian motion x_t on $\widehat{\boldsymbol{C}}(R^N)$.

If $G(x)\in\widehat{C}(R^N)^+$, where

$$G(x)=\sum_{n=0}^{\infty}nq_n(x),$$

then $v(t, x)=M_tf(x)=\boldsymbol{E}_x[\check{f}(\boldsymbol{X}_t)]$, $x\in R^N$ defines a strongly continuous semi-group on $\boldsymbol{C}(R^N)$ with the infinitesimal generator L given by

$$Lu=\frac{\bar{\Delta}}{2}u+k(x)(G(x)-1)\cdot u, \tag{3.47}$$

$$D(L)=D(\bar{\Delta}). \tag{3.48}$$

Hence we see that M_t is represented as

$$M_tf(x)=E_x[e^{\int_0^t k(G-1)(x_s)ds}f(x_t)]$$

in terms of the standard Brownian motion x_t.

If, in particular, $a(x)\in\widehat{C}(R^N)$ and we define k and q_n by $k(x)=|a|(x)$, $q_0(x)=I_{\{a^-(x)>0\}}$, $q_2(x)=I_{\{a^-(x)=0\}}$[27] and $q_n(x)=0$ $(n=3, 4, \cdots)$, then M_t is the semi-group corresponding to the infinitesimal generator $\frac{\bar{\Delta}}{2}+a$, or

$$M_tf(x)=E_x[e^{\int_0^t a(x_s)ds}f(x_t)].$$

Many arguments can be carried over to the case of unbounded k: we can construct the (X^0, π)-branching Markov process $\boldsymbol{X}$ by Theorem 3.5 and if, e.g., $\pi(x, dy)=\delta_{[x,x]}(dy)$, then $u(t, x)=\boldsymbol{E}_x[\widehat{f}(\boldsymbol{X}_t)]$ is a solution in a weak sense of the equation

$$\frac{\partial u}{\partial t}=\frac{\Delta}{2}u+k(u^2-u),\ u(0+,\cdot)=f.$$

The case of $k(x)=|x|^\gamma$ was considered in Ito-McKean [19].

(B) Branching A-diffusion processes

Let D be a bounded domain in R^N with sufficiently smooth boundary ∂D and $a^{ij}(x)$, $b^j(x)$ $(i, j=1, 2, \cdots, N)$ be sufficiently smooth functions on $\bar{D}=D\cup\partial D$ such that $\sum_{i,j=1}^{N}a^{ij}(x)\xi^i\xi^j\geqq\varepsilon|\xi|^2$ for

27) $I_{\{\ \}}$ is the indicator function of the set $\{\ \}$. $a^-=(-a)\vee 0$.

every $\xi=(\xi_1, \xi_2, \cdots, \xi_N)$.[28] Set

$$Au(x)=\sum_{i,j=1}^{N}\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left(a^{ij}(x)\sqrt{a(x)}\frac{\partial u}{\partial x^j}\right)+\sum_{j=1}^{N}b^j(x)\frac{\partial u}{\partial x^j}(x),$$

where $a(x)=[\det(a^{ij}(x))]^{-1}$. It is known that for given $c\in C(\overline{D})$ and $\beta\in C(\partial\overline{D})$ such that $c\geq 0$ and $\beta\geq 0$ there exists a unique diffusion process $X^0=(x_t^0, P_x^0)$ on $\overline{D}\cup\{\Delta\}$ with Δ as the terminal point such that if f is sufficiently smooth, $u(t, x)=E_x^0[f(x_t^0)]$ defines the solution of

$$(3.49)\qquad \begin{cases} \dfrac{\partial u}{\partial t}=Au-c\cdot u, \\ \left(\dfrac{\partial u}{\partial n}-\beta\cdot u\right)\Big|_{\partial D}=0.^{29)} \end{cases}$$

If $c(x)=\beta(x)\equiv 0$, the corresponding process is conservative: we shall denote it by $X=(x_t, P_x)$ and call it the reflecting A-diffusion process on $S\equiv\overline{D}$. Then X^0 is the e^{-A_t}-subprocess of X, where $A_t=\int_0^t c(x_s)ds+\int_0^t \beta(x_s)d\varphi_s$.[30] Let $q_n(x)\in \boldsymbol{C}(S)^+$, $q_1(x)\equiv 0$ and $\sum_{n=0}^{\infty}q_n(x)\equiv 1$, and define $\pi(x, dy)$ by (3·43). We shall call the (X^0, π)-branching Markov process $\boldsymbol{X}$ a *branching A-diffusion process.*

The fundamental system (T_t^0, K, π) is given by

$$T_t^0 f(x)=\int_{\overline{D}}P^0(t, x, y)f(y)m(dy),$$

$$K(x; dsdy)=P^0(s, x, y)c(y)m(dy)ds+P^0(s, x, y)\beta(y)\widetilde{m}(dy)ds,^{31)}$$

where $P^0(s, x, y)$ is the fundamental solution of (3.49) (cf. Nagasawa-Sato [37], Ikeda-Nagasawa-Sato [17]). In this case T_t^0 maps $\boldsymbol{B}(S)$ into $\boldsymbol{C}(S)$, and from this we see that the semi-group $\boldsymbol{T}_t$ of $\boldsymbol{X}$

28) $|\xi|=\sqrt{\sum_{i=1}^{N}\xi_i^2}$.

29) $\dfrac{\partial}{\partial n}$ is the derivative in the direction of the inner normal at ∂D determined by the metric tensor $a^{ij}(x)$.

30) φ_t is the local time on ∂D of x_t: the precise definition and the above facts we refer to Sato-Ueno [39].

31) $m(dx)=\sqrt{a(x)}\,dx^1dx^2\cdots dx^n$, and $\widetilde{m}(dx)$ is the surface element on ∂D.

maps $C_0(S)$ into $C_0(S)$ and is strongly continuous. Hence X is a Hunt process. $u(t,x)=T_t\widehat{f}(x)$, $f\in C^*(S)$, $x\in S$, satisfies

$$u(t,x)=T_t^0f(x)+\int_0^t\int_S K(x;\,dsdy)F(y;\,u(t-s,\cdot)), \tag{3.50}$$

(*S*-equation)

where $F(x;f)$ is given by (3.45). Hence $u(t,x)$ can be regarded as a solution (in a weak sense) of

$$\begin{cases}\dfrac{\partial u}{\partial t}=Au+c(F(\cdot\,;\,u)-u),\\[2mm] \dfrac{\partial u}{\partial n}+\beta\{F(\cdot\,;\,u)-u\}\,|_{\partial D}=0,\\[2mm] u(0+,\cdot)=f. \qquad \text{(backward equation).}\end{cases} \tag{3.51}$$

Remark 3.1. If $c\equiv 0$, (3.51) is a parabolic differential equation with a non-linear boundary condition.

Now assume $\sum_{n=0}^{\infty}nq_n(x)\equiv\alpha(x)\in C(\overline{D})$; then $v(t,x)=M_tf(x)\equiv E_x[\check{f}(X_t)]$, $f\in C(\overline{D})$ satisfies

$$v(t,x)=T_t^0f(x)+\int_0^t\int_{\overline{D}} K(x;\,dsdy)\alpha(y)v(t-s,y), \tag{3.52}$$

and hence $v(t,x)$ can be regarded as a solution in a weak sense of

$$\begin{cases}\dfrac{\partial v}{\partial t}=Av+c(\alpha-1)v,\\[2mm] \dfrac{\partial v}{\partial n}+\beta(\alpha-1)v\,|_{\partial D}=0,\\[2mm] v(0+,\cdot)=f.\end{cases} \tag{3.53}$$

The expectation semi-group M_t can be represented in terms of the reflecting A-diffusion $X=(x_t,P_x)$ as

$$M_tf(x)=E_x[e^{\int_0^t(\alpha-1)(x_s)dA_s}],$$

where

$$A_t=\int_0^t c(x_s)ds+\int_0^t\beta(x_s)d\varphi_s\,.$$

(C) Branching A-diffusion processes with absorbing boundaries

Let (x_t,P_x) be an absorbing barrier A-diffusion process, i.e. a diffusion process on $S=D\cup\{\delta\}$[32] with δ as a trap such that $v(t,x)=E_x[f(x_t)]$, for sufficiently regular $f\in \boldsymbol{C}_0(D)$,[33] is a solution of

$$\frac{\partial u}{\partial t}=Au,\quad \lim_{x\to\delta}u(t,x)=0,$$

where A is the same differential operator as in (B). For given $c(x)\in \boldsymbol{C}(S)^+$ and $q_n(x)\in \boldsymbol{C}(S)^+$ such that $q_1(x)\equiv 0$ and $\sum_{n=0}^{\infty}q_n(x)\equiv 1$, let $X^0=\{x_t^0, P_x\}$ be the $e^{-\int_0^t c(x_s)ds}$-subprocess of X and π be defined by (3. 43). We shall call the (X^0,π)-branching Markov process $\boldsymbol{X}$ a *branching A-diffusion process with absorbing boundary.* In this case it is easy to see that if we set $T=\{\partial, \delta, [\delta,\delta], [\delta,\delta,\delta], \cdots\}$, then, with probability one for all $\boldsymbol{P}_{\boldsymbol{x}}$, $\boldsymbol{X}_t\in T$ implies $\boldsymbol{X}_s\in T$ for all $s\geq t$. It is natural to set

$$\xi_t=\check{I}_D(\boldsymbol{X}_t) \tag{3. 54}$$

and call it the *number of particles,* that is, we are interested in only those particles which are in D. Then the extinction time and the explosion time are defined respectively by

$$e_\partial=\inf\{t;\ \xi_t=0\}=\inf\{t;\ \boldsymbol{X}_t\in T\}$$

and

$$e_\Delta=\lim_{n\to\infty}e_n,\quad \text{where } e_n=\inf\{t;\ \xi_t\geq n\}.$$

The case when $A=\frac{1}{2}\Delta$ and $c(x)\equiv c$ (constant) was studied by Sevast'yanov [41] and Watanabe [46].

(D) One-dimensional branching diffusion processes

Let $X=(x_t, P_x)$ be a regular conservative one-dimensional diffusion process on $S=[r_1, r_2]$ with appropriate boundary conditions. Suppose the local infinitesimal generator of X is given as

$$Au(x)=\frac{u^+(dx)}{m(dx)}.$$[34]

32) D is a domain in R^N with sufficiently smooth boundary and $D\cup\{\delta\}$ is its one-point compactification.

33) $\boldsymbol{C}_0(D)=\{f;$ continuous on D and $\lim_{x\to\delta}f(x)=0\}$.

34) $u^+(dx)$ is the Stieltjes measure of $u^+(x)=\frac{d^+u}{dx}$ (if u^+ is of bounded variation). Cf. Ito-McKean [19].

Let $k(dx)$ be a non-negative Radon measure on S and A_t be the corresponding additive functional.[35] Given $q_n(x)\in \boldsymbol{C}(S)^+$ such that $q_1(x)\equiv 0$ and $\sum_{n=0}^{\infty} q_n(x)\equiv 1$, define $\pi(x, dy)$ by (3. 43). Let $X^0=\{x_t^0, P_x^0\}$ be the e^{-A_t}-subprocess of X. We shall call the (X^0, π)-branching Markov process $\boldsymbol{X}$ a *one-dimensional branching diffusion process.* If $P^0(t, x, y)m(dy)$ is the transition probability of x_t^0, then the kernel $K(x; dsdy)$ is given by

$$K(x; dsdy)=P^0(s, x, y)k(dy)ds,$$

and hence $u(t, x)=\boldsymbol{T}_t\widehat{f}(x)=\boldsymbol{E}_x[\widehat{f}(\boldsymbol{X}_t)]$, $x\in S$, satisfies

$$\begin{aligned}(3.55)\qquad u(t,x)=&\int_S P^0(t,x,y)f(y)m(dy)\\&+\int_0^t ds\int_S P^0(s,x,y)F(y,u(t-s,\cdot))k(dy)\end{aligned}$$

(S-equation)

where $F(x; f)$ is given by (3. 45). If r_j $(j=1$ or $2)$ is regular and the boundary condition of x_t^0 is given by

$$p_j^{(1)}u(r_j)+(-1)^j p_j^{(2)}\frac{\partial u}{\partial x}(r_j)+p_j^{(3)}\lim_{x\to r_j}A^0u(x)=0,$$ [36]

$$(p_j^{(i)}\geq 0,\ i=1, 2, 3),$$

$u(t, x)$ can be regarded as a solution in a weak sense of

$$(3.56)\qquad \begin{cases}\dfrac{\partial u}{\partial t}=\dfrac{u^+(dx)+k(dx)(F(x;u)-u)}{m(dx)},\\ p_j^{(1)}[u(r_j)-F(r_j;u)]+(-1)^j p_j^{(2)}\dfrac{\partial u}{\partial x}(r_j)\\ \qquad\qquad +p_j^{(3)}\lim_{x\to r_j}A^0u(x)=0,\\ u(0+,\cdot)=f.\end{cases}$$

If $\alpha(x)=\sum_{n=0}^{\infty} nq_n(x)\in \boldsymbol{C}(S)$, then $v(t, x)=M_tf(x)=\boldsymbol{E}_x[\check{f}(\boldsymbol{X}_t)]$ satisfies

35) $A_t=\int_S \varphi(t, x)k(dx)$ where $\varphi(t, x)$ is the local time at $x\in S$. cf. [19].

36) $A^0u(x)=\dfrac{u^+(dx)-u(x)k(dx)}{m(dx)}$.

$$(3.57)\qquad v(t,x)=\int_S P^0(t,x,y)f(y)m(dy)+\int_0^t ds\int_S P^0(s,x,y)\alpha(y)v(t-s,y)k(dy),$$

and hence $v(t,x)$ can be regarded as a solution in a weak sense of

$$(3.58)\qquad \begin{cases} \dfrac{\partial v}{\partial t}=\dfrac{v^+(dx)+(\alpha-1)v(x)k(dx)}{m(dx)}, \\ p_j^{(1)}(1-\alpha(r_j))v(r_j)+(-1)^j p_j^{(2)}\dfrac{\partial v}{\partial x}(r_j)+p_j^{(2)}\lim\limits_{x\to r_j}A^0 v(x)=0, \\ v(0+,\cdot)=f(x). \end{cases}$$

$M_t f(x)$ is expressed in terms of the original diffusion process $X=(x_t, P_x)$ as

$$M_t f(x)=E_x[e^{\int_0^t(\alpha-1)(x_s)dA_s}f(x_t)].$$

Example 3.5. Electron-Photon cascades

These branching processes are discussed in detail in Harris [8] Chapter VII. Unfortunately a cascade process with infinite cross section can not be put into our formulation and so we shall formulate only a cascade process with finite cross section.

Let $S=[0,\infty]\times\{1,2,3\}$ and T_t^0 and K be defined by

$$(3.59)\qquad T_t^0 f(a,j)=f(a,j)e^{-c_j t},$$

$$(3.60)\qquad \int_S K((a,j);dsdy)f(y)=c_j f(a,j)e^{-c_j s}ds,$$
$$0<c_j<\infty,\quad j=1,2,3,\quad a\in[0,\infty).$$

Let $\pi(x,dy)$ be a substochastic kernel on $S\times S$ such that $\pi(x,S)\equiv 0$ and satisfies the following conditions:

$$(3.61)\qquad \pi((a,1),\{y=[(au,2),(a(1-u),3)]\in S^2;\ 0\leq u\leq 1\})=1,$$

$$(3.62)\qquad \pi((a,k),\{y=[(au,1),(a(1-u),k)]\in S^2;\ 0\leq u\leq 1\})=1,\quad k=2,3.$$

Let X^0 be a Markov process on $S\cup\{\Delta\}$ with $\{\Delta\}$ as its terminal point such that its semi-group is given by (3.59). We shall call the (X^0, π)-branching Markov process $\boldsymbol{X}$ an *electron-photon cascade process with finite cross section.* Physical meanings are the following; the number a in $(a,1)\in[0,\infty]\times\{1\}$, $(a,2)\in[0,\infty]\times\{2\}$ and $(a,3)\in[0,\infty]\times\{3\}$ represent the energy of a photon, of a positive electron and of a negative electron, respectively. (3.61) describes the law of pair production of positive and negative electrons, and so on.

We set further the following assumptions;

(3.63) $$c_2=c_3,$$

(3.64) there exist measurable functions $k_1(u)$, $k_2(u)$ on $[0,1]$ such that $k_1(u)=k_1(1-u)$ and for every $E\in\mathscr{B}[0,1]$,

$$\pi((a,1),\{\boldsymbol{y}=[(au,2),(a(1-u),3)]\,;\,u\in E\})=\int_E k_1(u)du,$$

$$\pi((a,k),\{\boldsymbol{y}=[(au,1),(a(1-u),k)]\,;\,u\in E\})=\int_E k_2(u)du,$$
$$k=2,\,3.$$ [37]

In the sequel we do not distinguish positive and negative electrons and therefore *consider only such* $f\in\boldsymbol{C}^*(S)$ that $f(a,2)=f(a,3)$. It is clear from (3.63) and (3.64) that $\boldsymbol{E}_{(a,2)}[\widehat{f}(\boldsymbol{X}_t)]=\boldsymbol{E}_{(a,3)}[\widehat{f}(\boldsymbol{X}_t)]$ for every $f\in\boldsymbol{C}^*(S)$ with $f(a,2)=f(a,3)$.

Now $u_j(t,a)=\boldsymbol{E}_{(a,j)}[\widehat{f}(\boldsymbol{X}_t)]$, $(j=1,2)$ satisfy

(3.65)
$$\begin{cases} u_1(t,a)=f(a,1)e^{-c_1t}+c_1\displaystyle\int_0^t\left\{\int_0^1 u_2(t-s,au)u_2(t-s,a(1-u))\,k_1(u)du\right\}e^{-c_1s}ds \\ u_2(t,a)=f(a,2)e^{-c_2t}+c_2\displaystyle\int_0^t\left\{\int_0^1 u_1(t-s,au)u_2(t-s,a(1-u))\,k_2(u)du\right\}e^{-c_2s}ds, \end{cases}$$

(S-equation),

and hence they satisfy the backward equations:

37) By (3.61) and (3.62) it follows that $\int_0^1 k_i(u)du=1$, $i=1,2$.

$$(3.66)\quad \begin{cases} \dfrac{\partial u_1}{\partial t}(t,a)=-c_1u_1(t,a)+c_1\displaystyle\int_0^1 u_2(t,au)u_2(t,a(1-u))k_1(u)du, \\ \dfrac{\partial u_2}{\partial t}(t,a)=-c_2u_2(t,a)+c_2\displaystyle\int_0^1 u_1(t,au)u_2(t,a(1-u))k_2(u)du. \end{cases}$$

$v_j(t,a)=M_tf(a,j)=\boldsymbol{E}_{(a,j)}[\check{f}(\boldsymbol{X}_t)]$, $(j=1,2)$, satisfy

$$(3.67)\quad \begin{cases} v_1(t,a)=f(a,1)e^{-c_1t}+c_2\displaystyle\int_0^t\left\{\int_0^1[v_2(t-s,au) \right. \\ \qquad\qquad \left. +v_2(t-s,a(1-u))]k_1(u)du\right\}e^{-c_1s}ds \\ \qquad =f(a,1)e^{-c_1t}+2c_1\displaystyle\int_0^t\left\{\int_0^1 v_2(t-s,au)k_1(u)du\right\}e^{-c_1s}ds, \\ v_2(t,a)=f(a,2)e^{-c_2t}+c_2\displaystyle\int_0^t\left\{\int_0^1[v_1(t-s,au) \right. \\ \qquad\qquad \left. +v_2(t-s,a(1-u))]k_2(u)du\right\}e^{-c_2s}ds, \end{cases}$$

and hence they satisfy

$$(3.68)\quad \begin{cases} \dfrac{\partial v_1}{\partial t}(t,a)=-c_1v_1(t,a)+2c_1\displaystyle\int_0^1 v_2(t,au)k_1(u)du, \\ \dfrac{\partial v_2}{\partial t}(t,a)=-c_2v_2(t,a)+c_2\displaystyle\int_0^1[v_1(t,au) \\ \qquad\qquad +v_2(t,a(1-u))]k_2(u)du. \end{cases}$$

Consider, for instance, the number $N_t(E)$ of electrons at time t whose energy is greater than E. If we set

$$g_E(x)=\begin{cases}1, & x=(a,2) \text{ and } a\geq E \text{ or } x=(a,3) \text{ and } a\geq E, \\ 0, & \text{otherwise,}\end{cases}$$

then clearly $N_t(E)=\check{g}_E(\boldsymbol{X}_t)$. For $0\leq\lambda<1$, set $f_E(x)=\lambda^{g_E(x)}$; then $\boldsymbol{E}_x[\hat{f}_E(X_t)]=\boldsymbol{E}_x[\lambda^{N_t(E)}]$, $x\in S$. It is easy to verify that

$$(3.69)\qquad \boldsymbol{E}_{(a,j)}[\hat{f}_E(\boldsymbol{X}_t)]=\boldsymbol{E}_{(1,j)}[\hat{f}_{E/a}(\boldsymbol{X}_t)],$$

and hence if we set

$$(3.70)\qquad \varphi_j(t,E)=\boldsymbol{E}_{(1,j)}[\lambda^{N_t(E)}] \qquad (j=1,2),$$

we have from (3.66) that

$$(3.71)\quad \begin{cases} \dfrac{\partial\varphi_1}{\partial t}(t,E)=-c_1\varphi_1(t,E)+c_1\displaystyle\int_0^1\varphi_2\Big(t,\frac{E}{u}\Big)\varphi_2\Big(t,\frac{E}{1-u}\Big)k_1(u)du, \\ \dfrac{\partial\varphi_2}{\partial t}(t,E)=-c_2\varphi_2(t,E)+c_2\displaystyle\int_0^1\varphi_1\Big(t,\frac{E}{u}\Big)\varphi_2\Big(t,\frac{E}{1-u}\Big)k_2(u)du. \end{cases}$$

Similarly if we set $m_j(t,E)=\boldsymbol{E}_{(1,j)}[N_t(E)]$, $(j=1,2)$, then we have from (3.68) that

$$(3.72)\quad \begin{cases} \dfrac{\partial m_1}{\partial t}(t,E)=-c_1m_1(t,E)+2c_1\displaystyle\int_0^1 m_2\Big(t,\frac{E}{u}\Big)k_1(u)du, \\ \dfrac{\partial m_2}{\partial t}(t,E)=-c_2m_2(t,E)+c_2\displaystyle\int_0^1\Big\{m_1\Big(t,\frac{E}{u}\Big) \\ \qquad\qquad +m_2\Big(t,\frac{E}{1-u}\Big)\Big\}k_2(u)du. \end{cases}$$

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