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The influence of small values of a holomorphic function on its maximum modulus

By

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1. Introduction

In a recent paper [5] we investigated the possible growth of the maximum modulus of a holomorphic function f defined in the unit disk D if the function tended to zero on certain sequences of Jordan arcs $\{\gamma_n\}$ in D. These sequences were distinguished by having

$$(1.0)\qquad \begin{aligned} &\text{i)}\quad \frac{1}{2}\leq r_n=\min_{z\in\gamma_n}|z|\to 1,\ n\to\infty;\\ &\text{ii)}\quad 0<\varliminf_{n\to\infty} HD(\gamma_n)\leq\varlimsup_{n\to\infty} HD(\gamma_n)<\infty;\end{aligned}$$

where $HD(\gamma_n)=\sup\rho(a,b)$, $a, b\in\gamma_n$, $\rho(a,b)$ denoting the hyperbolic distance between a and b. Such a sequence satisfying (1.0) is labeled a PHD sequence. If

$$R_n=\max|z|,\ z\in\gamma_n,\ n=1,2,\cdots,$$

then the closed circular sector of $|z|\leq R_n$ of minimum angle α_n containing γ_n is denoted by E_n. So E_n is of the form

$$0\leq|z|\leq R_n,\ \theta_n\leq\arg z\leq\theta_n+\alpha_n.$$

For convenience we suppose $0\leq\alpha_n\leq\pi$, all n. For a PHD sequence

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this is no restriction since necessarily $\alpha_n \to 0$, $n \to \infty$. For any α, $\alpha_n \leq \alpha < 2\pi$, $n=1, 2, \cdots$, put

$$(1.1) \qquad F_n^{(\alpha)} : 0 \leq |z| \leq R_n,\ \theta_n - \left(\frac{\alpha - \alpha_n}{2}\right) \leq \arg z \leq \theta_n + \left(\frac{\alpha + \alpha_n}{2}\right), \quad n=1, 2, \cdots.$$

Now $F_n^{(\alpha)}$ is a circular sector of fixed angle opening α containing E_n in a symmetric fashion. For any $S \subseteq D$ and holomorphic f let

$$\mathcal{M}(f, S) = \max(\sup_{z \in S} \log |f(z)|, 1).$$

For completeness we repeat Theorem 1 of [5] on which the present paper depends.

Theorem A. *Let f be holomorphic in D and satisfy for some PHD sequence $\{\gamma_n\}$, some finite value w_0, and some sequence $\{A_n\}$, $A_n > 0$,*

$$(1.2) \qquad |f(z) - w_0| \leq \exp\left(\frac{-A_n}{1-|z|}\right), \quad z \in \gamma_n, \ \text{all } n.$$

If there is a value α, $0 < \alpha < 2\pi$, for which

$$(1.3) \qquad \underline{\lim}_{n \to \infty} \frac{\mathcal{M}(f, F_n^{(\alpha)})}{A_n} = 0,$$

(where $F_n^{(\alpha)}$ is defined as in (1.1) relative to $\{\gamma_n\}$) then $f = w_0$.

2. Behavior of f away from its zeroes

In this note we exhibit a condition under which we can replace the *PHD* sequence of arcs by a sequence of points. If then (1.2) and (1.3) both hold for this sequence we still are able to conclude Theorem A. That is, we suppose there is sequence $\{z_n\}$ in D, with $\lim_{n \to \infty} |z_n| = 1$, such that for some positive sequence $\{A_n\}$, and some $|w_0| < \infty$,

$$(2.0) \qquad |f(z_n) - w_0| \leq \exp \frac{-A_n}{(1-|z_n|)}, \quad n=1, 2, \cdots.$$

It is obvious that such a inequality (2.0) is ineffective in influencing the growth of $|f|$ if either $f(z_n)=w_0$, or else z_n lies "near" a zero of $f-w_0$. In such circumstances (2.0) can be satisfied by a non-constant f for any sequence $\{A_n\}$ by suitably choosing $\{z_n\}$. So we must stay away from the zeroes of $f-w_0$ in the following sense. First set

$$Z(f)=\{z\in D|f(z)=0\};$$

and for any subset $S\subseteq D$, and any $a\in D$, let

$$\rho(a, S)=\inf \rho(a, s), \quad s\in S.$$

Then we may define for any $0<\delta<\infty$,

$$K_\delta(f)=\{z\in D|\rho(z, Z(f))\geq\delta\}.$$

To determine the sets over which we calculate the maximum modulus we proceed as follows. Let $\{z_n=|z_n|e^{i\theta_n}\}$ be a sequence with $z_n\in K_\delta(f-w_0)$, $n=1, 2, \cdots$, $|w_0|<\infty$, $0<\delta<\infty$, and $\lim_{n\to\infty}|z_n|=1$. Define a sequence of positive numbers $\{R_n\}$, $0<|z_n|<R_n<1$, by $\rho(|z_n|, R_n)=\delta$, $n=1, 2, \cdots$. For any $0<\alpha<2\pi$, set

$$G_n^{(\alpha)}: 0\leq|z|\leq R_n, \ \theta_n-\frac{\alpha}{2}\leq\arg z\leq\theta_n+\frac{\alpha}{2}, \quad n=1, 2, \cdots.$$

Note that the sequence of sets $\{G_n^{(\alpha)}\}$ depends on the sequence $\{z_n\}$, and the values δ and α. We will always view these sets in this context.

Theorem 1. *Let f be holomorphic and non-constant in D. For some finite value w_0, and some $0<\delta<\infty$, let $\{z_n\}$ be a sequence with $z_n\in K_\delta(f-w_0)$, all n. If*

$$(2.1) \qquad |f(z_n)-w_0|\leq\exp\left(\frac{-A_n}{1-|z_n|}\right), \quad A_n>0, \ n=1, 2, \cdots,$$

then for any choice of $0<\alpha<2\pi$,

$$(2.2) \qquad \varliminf_{n\to\infty}\frac{\mathcal{M}(f, G_n^{(\alpha)})}{A_n}>0.$$

Proof: We suppose (2.2) does not hold and so for some subsequence $\{n_k=j\}$, and some value $0<\alpha_0<2\pi$, we have

$$(2.3)\qquad \lim_{j\to\infty}\frac{\mathcal{M}(f, G_j^{(\alpha_0)})}{A_j}=0.$$

Now each of the non-Euclidean disks $N(z_j, \delta)=\{z\mid\rho(z_j, z)<\delta\}$ contains no zero of $f-w_0$. As a result there exists a *PHD* sequence $\{\gamma_j\}$, with $\gamma_j\subseteq N(z_j, \delta)$, all j, and on which

$$(2.4)\qquad |f(z)-w_0|\leq 2|f(z_j)-w_0|,\quad z\in\gamma_j,\quad all\quad j\cdots.$$

The existence of such a sequence can be verified by considering the image $N_j^*=f(N(z_j, \delta))$ on the Riemann surface R of f. Choose $0<\eta_j<\delta/2$ such that

$$|f(z)-w_0|\leq 2|f(z_j)-w_0|,\ z\in N(z_j, \eta_j).$$

Then select a $z_j^*\in N(z_j, \eta_j)$ for which $f'(z_j^*)\neq 0$. Let $f(z_j^*)=t_je^{i\varphi_j}$, and define L_j to be the line segment $w=te^{i\varphi_j}$, $0\leq t\leq t_j$. If there are no points $w\in L_j$ for which $f(z)=w$, $z\in N(z_j, \delta)$, and $f'(z)=0$, then consider the maximal segment of L_j which can be lifted into N_j^* with one endpoint at $f(z_j^*)\in N_j^*$. Call this lifted piece L_j^*. If there are (a finite number of) points on L_j for which f has a zero derivative at the corresponding $z\in N(z_j, \delta)$ we can alter L_j slightly to avoid these points and still maintain that the altered $L_j\subseteq\{|w|\leq t_j\}$. Consequently the curve γ_j in $N(z_j, \delta)$ corresponding to L_j^* is always a simple continuous curve starting at z_j^* and extending to the boundary of $N(z_j, \delta)$ for which (2.4) holds. It must extend to the boundary otherwise f would have a zero in $N(z_j, \delta)$. Consequently $\frac{\delta}{2}\leq HD(\gamma_j)\leq 2\delta$, and so $\{\gamma_j\}$ is the required *PHD* sequence.

One of the convenient inequalities in non-Euclidean geometry (which is known under various guises, see [4, Lemma 1] for a statement) says that for $z\in N(z_j, \Delta)$, $0<\Delta<1$

$$(2.5)\qquad (1-|z_j|)t_\Delta\leq 1-|z|\leq(1-|z_j|)t_\Delta^{-1},\quad 0<t_\Delta<\infty,\quad all\quad j\cdots.$$

By considering (2.1), (2.4) and (2.5) we have, for $z\in\gamma_j$,

$$(2.6)\qquad |f(z)-w_0|<\exp\left(-\frac{t_\delta A_j}{2(1-|z|)}\right).$$

We will be ready to apply Theorem A as soon as we notice that for j sufficiently large the set $F^{(\alpha_0/2)}$, defined by (1.1) relative to our just discovered *PHD* sequence $\{r_j\}$, is contained in $G_j^{(\alpha_0)}$. Thus (2.3) implies (1.3) while (1.2) holds because of (2.6). Hence $f=w_0$ contrary to hypothesis and so the theorem is proved.

Remark: If $\mathcal{M}(f, |z|<r)$ satisfies, for $0<r<1$, an inequality of the form

$$(2.7)\qquad \mathcal{M}(f, |z|<r)\leq\frac{A}{(1-r)^s},\quad A\geq 0,\ s\geq 0,$$

we can replace $\mathcal{M}(f, G_n^{(\alpha)})$ in (2.2) by $\mathcal{M}(f, H_n^{(\alpha)})$ where

$$H_n^{(\alpha)}:0\leq|z|\leq|z_n|,\ \theta_n-\frac{\alpha}{2}\leq\arg z\leq\theta_n+\frac{\alpha}{2},\quad n=1, 2, \cdots.$$

Simply observe that (2.5) and the fact that $\rho(|z_n|, R_n)=\delta$, $n=1, 2, \cdots$, guarantees that the maximum modulus on $G_n^{(\alpha)}$ has essentially the same order estimate as on $H_n^{(\alpha)}$.

By way of application we have

Corollary 1. *If f is a non-constant, normal, holomorphic function in D then for any finite value w_0, and any $0<\delta<\infty$,*

$$(2.8)\qquad (1-|z|)^2\log|f(z)-w_0|\geq C_\delta>-\infty,\quad z\in K_\delta(f-w_0);$$

while if f is bounded (2.8) can be improved to

$$(1-|z|)\log|f(z)-w_0|\geq C_\delta^*>-\infty,\quad z\in K_\delta(f-w_0).$$

Here C_δ and C_δ^* also depend on f and w_0.

Proof: If f is a normal holomorphic function in D, Hayman showed [2, p. 204] that

$$(2.9)\qquad \mathcal{M}(f, |z|<r)\leq\frac{Q_f}{1-r}.$$

For some $0<\delta<\infty$, if there was a sequence $\{z_n\}$, $z_n\in K_\delta(f-w_0)$, such that

$$(2.10)\qquad (1-|z_n|)^2\log|f(z_n)-w_0|=T_n\to-\infty,\quad n\to\infty,$$

then

$$(2.11)\qquad |f(z_n)-w_0| = \exp\left[\left(\frac{-1}{1-|z_n|}\right)\left(\frac{-T_n}{1-|z_n|}\right)\right].$$

So that, with $A_n=\dfrac{-T_n}{(1-|z_n|)}$, (2.9), (2.10) and (2.11) imply for any choice of $0<\alpha<2\pi$

$$\lim_{n\to\infty}\frac{\mathcal{M}(f, H_n^{(\alpha)})}{A_n}=0,$$

which, according to Theorem 1 and the remarks following, is impossible.

The proof in the case f is bounded is equally obvious.

The result for bounded f is reasonably sharp. Form the product f of the Blaschke product $B(z, \{a_n\})$, $a_n=1-\dfrac{1}{2^n}$, $n=1, 2, \cdots$, and $h(z)=\exp\dfrac{z+1}{z-1}$. If $\{x_n\}$ is a sequence in $K_\delta(f)$ with $0<x_n<x_{n+1}<1$, $x_n\to 1$, $n\to\infty$, we have

$$(1-x_n)\log|f(x_n)|\leq -1, \quad \text{all } n.$$

3. Interpolating sequences

If it happens that the set of zeroes of a Blaschke product $B(z, \{a_n\})$ in D form an interpolating sequence then Corollary 1 is not sharp. Cargo [1, Theorem 3.1] in a limited result, and Hoffman [3, Lemma 4.2] in more precise form showed that in such circumstances

$$(3.0)\qquad |B(z, \{a_n\})|\geq C_\delta>0, \quad z\in K_\delta(B).$$

We recall that $\{z_n\}$ is an interpolating sequence in D if for any bounded sequence of complex numbers $w=\{w_n\}$, there exists a bounded holomorphic function f_w in D with $f_w(z_n)=w_n$, all n. For a most penetrating analysis of the behavior of a Blaschke product away from its zeroes again see [3, p. 80ff.]. If we take $w_1\neq 0$, $w_n=0$, $n=2, 3, \cdots$, then an interpolating sequence must satisfy

$$(3.1)\qquad \sum_{n=1}^{\infty}(1-|z_n|)<\infty.$$

Corollary 1 can be rephrased with an interpolating sequence orientation to show that (3. 1) is still true for a sequence satisfying an (apparently) weaker interpolation problem.

Corollary 2. *Let* $\{z_n\}$ *be a sequence in* D *such that for any sequence of non-zero complex numbers* $w=\{w_n\}$, *with* $w_n\to 0$, $n\to\infty$, *there is a bounded holomorphic function* f_w *in* D *such that*

$$f_w(z_n)=w_n, \quad n=1, 2, \cdots. \tag{3.2}$$

If $w=\{w_n\}$ *is chosen so that*

$$\lim_{n\to\infty}(1-|z_n|)\log|w_n|=-\infty, \tag{3.3}$$

then the corresponding f_w *has a sequence of zeroes* $\{\zeta_n\}$ *satisfying*

$$\lim_{n\to\infty}\rho(\zeta_n, z_n)=0. \tag{3.4}$$

Consequently

$$\sum_{n=1}^{\infty}(1-|z_n|)<\infty. \tag{3.5}$$

Proof: According to Corollary 1 if $z_n\in K_\delta(f_w)$, some $\delta>0$, all n, it cannot happen that both (3. 2) and (3. 3) hold. Nor, in fact, can (3. 2) and (3. 3) hold for any subsequence $\{z_{n_k}\}$, satisfying $z_{n_k}\in K_\delta(f_w)$ so (3. 4) is verified. Since $\sum_{n=1}^{\infty}(1-|\zeta_n|)<\infty$, a glance at (2. 5) justifies our claim in (3. 5).

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