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Simplical cohomology and n-term extensions of algebras

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Simplicial method is very useful in discussing (co)homology theory in a non-abelian category. M. André [1] and J. Beck [2] investigated the simplicial cohomology $H^*(A, M)$ of a commutative algebra A over a basic ring K with coefficients in an A-module M.

The purpose of the present paper is to interpret the cohomology $H^*(A, M)$. Our interpretation of $H^*(A, M)$ is an analogy of that of the functor Ext* by N. Yoneda [9].

It has been known that the 0-th cohomology group $H^0(A, M)$ is isomorphic to the module $\mathrm{Der}_K(A, M)$ of K-derivations of A into M, and the first cohomology group $H^1(A, M)$ is in $1-1$ correspondence with the set $\mathrm{Ex}^1(A, M)$ of isomorphic classes of 1-term extensions of A by M. N. Shimada and others [8] have shown that the second cohomology group is in $1-1$ correspondence with the set $\mathrm{Ex}^2(A, M)$ of equivalence classes of 2-term extensions of A by M in the sense of S. Lichtenberg and S. Schlessinger [7] (or in M. Gerstenhaber [5]).

We start with the definitions of quasi-simplicial algebras and the simplicial cohomology. Let $\mathcal{A}$ be the category of associative commutative algebras with unit over a basic ring K. Denote by $(\mathcal{A}, A)$ the category of morphisms in $\mathcal{A}$ with range A.

A *quasi-simplicial algebra* A_* over A is defined by a diagram in $(\mathcal{A}, A)$

$$\cdots \vdots A_n \underset{\varepsilon^n}{\overset{\varepsilon^0}{\vdots}} A_{n-1} \vdots \cdots \quad A_0 \xrightarrow{\varepsilon^0} A_{-1} = A \tag{0.1}$$

with $\varepsilon^i \varepsilon^j = \varepsilon^{j-1}\varepsilon^i$, $i<j$ (see §1 for details).

If B is an abelian group object in $(\mathcal{A}, A)$, then B is the idealization $A+M$ of some A-module M. For every object C in $(\mathcal{A}, A)$, A-module M is regarded as a C-module *via* the structure homomorphism $C \to A$, and we have an isomorphism of functors

$$\mathrm{Hom}_{(\mathcal{A}, A)}(C, B) \cong \mathrm{Der}_K(C, M).$$

A quasi-simplicial algebra (0.1) leads to a cochain complex

$$\cdots \xleftarrow{\partial^n} \mathrm{Hom}(A_n, B) \xleftarrow{\partial^{n-1}} \mathrm{Hom}(A_{n-1}, B) \leftarrow \cdots \leftarrow \mathrm{Hom}(A_0, B) \tag{0.2}$$

where $\partial^n = \sum_{i=0}^{n+1}(-1)^i \mathrm{Hom}(\varepsilon^i, B)$ and $\mathrm{Hom} = \mathrm{Hom}_{(\mathcal{A}, A)}$.

The derived group of the complex (0.2) does not depend on the choice of A_*, so far as A_* is "free" and "acyclic" (§1). It is called the simplicial cohomology group of A by M, and denoted by $H^*(A, M)$. Our cohomology is equivalent to that in M. André [1], Chap. II. In particular the "standard simplicial algebra" (§2) of A is free and acyclic, and hence our cohomology is also equivalent to the cotriple cohomology in the sense of J. Beck [2].

For a positive integer n, we define an *n-fold (quasi-)simplicial extension* of A by M to be a (quasi-)simplical algebra which satisfies certain conditions (§3). As every simplical module is determined by its "Moore complex" (§1), an n-fold simplial extension of A by M is as well determined by a sequence of K-modules

$$0 \to M \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} X_0 \xrightarrow{d_0} A \to 0 \text{ (exact)}$$

with certain conditions (§5). Such a sequence is called an *n-term extension* of A by M. The totality of n-fold (resp. quasi-) simplicial extensions of A by M is suitably classified into the set $\mathrm{Ex}^n(A, M)$ (resp. $\mathrm{Ex}^n_{q-s}(A, M)$) (§4).

The main theorem asserts that $H^n(A, M)$ is in $1-1$ correspondence with $\mathrm{Ex}^n(A, M)$ and simultaneously with $\mathrm{Ex}^n_{q-s}(A, M)$. The argument is functorial in substance, so it will be applicable to other algebraic systems: e.g. non-commutative algebras, algebras without unit and Lie algebras.

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§1. Simplicial objects

Let Φ be the category such that $\mathrm{ob}\Phi$ consists of the null set $[-1]$ and sets $[n]=\{0, 1, \cdots, n\}$ for non-negative integers n, $\mathrm{mor}\Phi$ consists of monotone non-decreasing maps. Let Ψ be the subcategory of Φ such that $\mathrm{mor}\Psi$ consists of all injections in Φ. Let Φ_0 (resp. Ψ_0) be the full subcategory of Φ (resp. Ψ) such that $\mathrm{ob}\Phi_0$ (resp. $\mathrm{ob}\Psi_0$) consists of $\mathrm{ob}\Phi$ (recp. $\mathrm{ob}\Psi$) except the null set.

For a monotone map α: $[p]\to[q]$ in Φ, p (resp. q) is called the *domain* (resp. the *range*) of α, and denoted by $d(\alpha)$ (resp. $r(\alpha)$). There exist the special monotone maps

$$\varepsilon^i=\varepsilon^i_n : [n-1]\to[n], \quad \delta^i=\delta^i_n : [n+1]\to[n]$$

with $0\leq i\leq n$ such that

$$(1.1)\qquad \varepsilon^i(j)=\begin{cases} j, & j<i, \\ j+1, & j\geq i, \end{cases} \quad \delta^i(j)=\begin{cases} j, & j\leq i, \\ j-1, & j>i. \end{cases}$$

Every monotone map α is represented by a composition of a surjection δ and an injection ε : $\alpha=\varepsilon\delta$. Every surjection (resp. injection) is represented by a composition of δ^i (resp. ε^i).

A contravariant functor of Φ, Φ_0, Ψ or Ψ_0 into a category C is called respectively an augmented simplicial object, a *simplicial object*, an augmented quasi-simplicial object, a *quasi-simplicial object* in C.

If X is one of them, then we write X_*, X_n, and $\bar{\alpha}$ instead of X,

$X([n])$ and $X(\alpha)$. The morphism $\bar{\varepsilon}_n^i : X_n \to X_{n-1}$ (resp. $\bar{\delta}_n^i : X_n \to X_{n+1}$) is called the face operator (resp. the degeneracy operator), and often denoted by $\varepsilon^i = \varepsilon_n^i$ (resp. $\delta^i = \delta_n^i$) for simplicity.

Let $\rho'_*, \rho_* : X_* \to Y_*$ be two morphisms of quasi-simplicial objects (a morphism ρ_* means a functor morphism). Consider a family $\omega^* = \{\omega_n^i\}$ of morphisms with $\omega^i = \omega_n^i : X_n \to Y_{n+1}$, $0 \leq i \leq n$, which satisfies the following conditions:

(1.2)
$$\begin{aligned}
&\varepsilon^0 \omega_n^0 = \rho'_n, \\
&\varepsilon^{n+1} \omega_n^n = \rho_n, \\
&\varepsilon^i \omega^j = \omega^{j-1} \varepsilon^i, \quad i < j, \\
&\varepsilon^{i+1} \omega^{i+1} = \varepsilon^{i+1} \omega^i, \\
&\varepsilon^i \omega^j = \omega^j \varepsilon^{i-1}, \quad i > j+1.
\end{aligned}$$

Then ω^* is called a *homotopy* between ρ'_* and ρ_*. ρ'_* is said *homotopic* to ρ_*, in notation $\rho'_* \sim \rho_*$. The relation $\sim$ is not an equivalence relation in general.

If $\rho'_*, \rho_* : X_* \to Y_*$ are two morphisms of simplicial objects, then a homotopy between ρ'_* and ρ_* is defined by a family ω^* which satisfies the above conditions (1.2) and the following conditions

(1.3)
$$\begin{aligned}
&\omega^j \delta^i = \delta^i \omega^{j-1}, \quad i < j, \\
&\omega^j \delta^i = \delta^{i+1} \omega^j, \quad i \geq j.
\end{aligned}$$

For an (augmented) (quasi-)simplicial object A_* in a category with zero object, if there exists

$$\widetilde{A}_n = \bigcap_{i=1}^{n} \operatorname{Ker} \varepsilon^i$$

for every n, then $\widetilde{A}_* = \{\widetilde{A}_n\}$ with $d_n = \varepsilon^0 | \widetilde{A}_n$ forms a chain complex. This chain complex is called the *Moore complex* of A_*.

A simplicial object in the category of sets is called a simplicial set. In the same sence we use terminologies "a simplicial module", etc.

An augmented (quasi-)simplicial set A_* is said *acyclic*, if, for every integer $n \geq 0$ and $n+1$ elements $a_0, a_1, \cdots, a_n \in A_{n-1}$ with $\varepsilon^i a_j = \varepsilon^{j-1} a_i$, $i < j$, there exists an element $a \in A_n$ such that $\varepsilon^i a = a_i$.

An augmented (quasi-)simplicial set A_* is said to satisfy the *Kan condition*, if, for every integer $n>0$ and n elements $a_0, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n \in A_{n-1}$ with $0 \leq k \leq n$ and $\varepsilon^i a_j = \varepsilon^{j-1} a_i$, $i<j$, there exists an elements $a \in A_n$ such that $\varepsilon^i a = a_i$ for $i \neq k$. If an augmented (quasi-) simplicial set A_* is acyclic, then it astifies the Kan codition.

A (quasi-)simplicial group, module or ring is said acyclic (resp. to satisfy the Kan condition), if the (quasi-)simplicial set consisting of its underlying set is acyclic (resp. satisfies the Kan condition). The Moore complex of an (augmented) (quasi-)simplicial ring is is defined to be the Moore complex of the underlying module. It is easily verified that a (quasi-)simplicial group (hence also module and ring) satisfying the Kan condition is acyclic if and only if its Moore complex is acyclic. It is well known that a simplicial group satisfies the Kan condition.

Proposition 1. (*Partition of unity*) *Let A_* be a simplicial object in a pre-additive category C with kernels. Then there exists the Moore complex $\widetilde{A}_*$ of A_*. And then for every integer $n \geq 0$ there exists one and only one family of morphisms*

$$\{\theta_\alpha : A_n \to A_r \mid \alpha : [n] \to [r] \text{ is a surjection}, 0 \leq r \leq n\}$$

satisfying the following conditions:

(1.4) $\quad \varepsilon^i \theta_\alpha = 0, \quad 0 < i \leq r,$

(1.5) $\quad id_{A_n} = \sum_\alpha \pi_\alpha, \quad \pi_\alpha = \bar{\alpha}\theta_\alpha,$
(α *runs over all surjections with donain* n).

Moreover we have

(1.6) $$\pi_\alpha \pi_\beta = \begin{cases} \pi_\alpha, & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{cases}$$

(1.7) $\quad A_n \cong \sum_\alpha \alpha A_{r(\alpha)}, \quad \bar{\alpha}\widetilde{A}_r = \mathrm{Im}(\bar{\alpha} \mid \widetilde{A}_r) \cong \widetilde{A}_r,$

where $\sum$ means biproduct (i.e. product and coproduct).

Before the proof we introduce the following notations. For each monotone map $\xi : [p] \to [q]$, define $\xi_+ : [p+1] \to [q+1]$ by

$$\xi_+(i)=\xi(i),\quad 0\leq i\leq p,$$
$$=q+1,\quad i=p+1.$$

Evidently $(\delta_n^i)_+=\delta_{n+1}^i$, $(\varepsilon_n^i)_+=\varepsilon_{n+1}^i$ and $(\xi\eta)_+=\xi_+\eta_+$. We extend formally this notation to $\bar{\xi}$ by $(\bar{\xi})_+=\bar{\xi}_+$ and also to linear combinations of these $\bar{\xi}$. If $\alpha:[n]\rightarrow[r]$ for $n>0$ is a surjection, then we have

case (1) $\alpha=\beta_+$ for some surjection β if $\alpha(n-1)<\alpha(n)$,

case (2) $\alpha=\beta\delta^{n-1}$ for some surjection β if $\alpha(n-1)=\alpha(n)$.

Proof of Proposition 1.1. For a surjection $\alpha:[n]\rightarrow[r]$ put

(1.8)
$$\theta_\alpha=\theta_\alpha^1\theta_\alpha^2\cdots\theta_\alpha^n,$$
$$\theta_\alpha^j=\varepsilon^j,\qquad \text{if } \alpha(j-1)=\alpha(j),$$
$$1-\delta^{j-1}\varepsilon^j,\quad \text{if } \alpha(j-1)<\alpha(j).$$

Then it follows (1.4). If a surjection α is in case 1, it follows $\theta_\alpha=(\theta_\beta)_+(1-\delta^{n-1}\varepsilon^n)$ and $\alpha\varepsilon^n=\varepsilon^{r(\alpha)}\beta$. If a surjection α is in case 2, it follows $\theta_\alpha=\theta_\beta\varepsilon^n$ and $\alpha\varepsilon^n=\beta$. Hence we have by induction on n

(1.9) for two surjections α, β with domain n.
$$\theta_\alpha\bar{\beta}\equiv 1,\quad \text{if } \alpha=\beta,$$
$$0,\quad \text{if } \alpha\neq\beta.$$

where the notation $\equiv$ implies the congruence modulo the submodule generated by $\bar{\gamma}\varepsilon_n^i$ with $0<i\leq n$, $\gamma\in\mathrm{Hom}_{\mathcal{C}}(A_{r(\beta)-1}, A_{r(\alpha)})$. If $\{\theta'_\alpha\}$ is another family of morphisms satisfying (1.4) and (1.5) then (1.9) implies $\theta_\alpha=\theta'_\alpha$. (1.6) follows from (1.4), (1.5) and (1.9). (1.5) is verified by induction on n as follows

$$\sum\nolimits_{d(\alpha)=n}\pi_\alpha=\sum\nolimits_{\alpha(n-1)=\alpha(n)}\pi_\alpha+\sum\nolimits_{\alpha(n-1)<\alpha(n)}\pi_\alpha$$
$$=\delta^{n-1}(\sum\nolimits_{d(\alpha_1)=n-1}\pi_{\alpha_1})\varepsilon^n+(\sum\nolimits_{d(\alpha_2)=n-1}\pi_{\alpha_2})_+(1-\delta^{n-1}\varepsilon^n)$$
$$=\delta^{n-1}\varepsilon^n+(1-\delta^{n-1}\varepsilon^n)$$
$$=1.$$

Hence A_n is isomorphic to the biproduct of all $\mathrm{Ker}(1-\pi_\alpha)=\mathrm{Im}(\pi_\alpha)$. Let $\widetilde{A}_n$ be the kernel of $1-\pi_\iota$ for the identity $\iota=\iota_n$ of $[n]$. Let $\hat{\iota}:\widetilde{A}_n\rightarrow A_n$ be the canonical injection. Then we have an exact sequence for each object B in $\mathcal{C}$;

$$0\to \mathrm{Hom}_C(B,\widetilde{A}_n)\xrightarrow{\mathrm{Hom}_C(\ ,\hat{\imath})}\mathrm{Hom}_C(B,A_n)$$

$$\xrightarrow{(\mathrm{Hom}_C(\ ,\varepsilon^i))}\prod_{i=1}^{n}\mathrm{Hom}_C(B,A_{n-1})\qquad\text{(exact)}$$

which leads to

$$\widetilde{A}_n=\bigcap_{i=1}^{n}\mathrm{Ker}\,\varepsilon^i.$$

For a surjection $\alpha:[n]\to[r]$

$$0\to \mathrm{Hom}_C(B,\widetilde{A}_r)\xrightarrow{\mathrm{Hom}_C(\ ,\bar{\alpha}\hat{\imath})}\mathrm{Hom}_C(B,A_n)$$

$$\xrightarrow{\mathrm{Hom}_C(\ ,1-\pi_\alpha)}\mathrm{Hom}_C(B,A_n)\qquad\text{(exact)}$$

which leads to

$$A_n\cong\textstyle\sum_\alpha \mathrm{Ker}(1-\pi_\alpha)\cong\sum_\alpha \widetilde{A}_{r(\alpha)}\,.$$

Proposition 1.2. (*Dold*) *If C is an additive category with kernels, then for a positive chain complex $\widetilde{A}_*$ in C there exists one and only one simplicial object A_* in C such that the Moore complex of A_* is $\widetilde{A}_*$. Therefore there exists an equivalence between the categories of simplicial objects and positive chain complexes in C.*

Proof. For a positive chain complex $\widetilde{A}_*$, let

$$A_n=\textstyle\sum_\alpha \widetilde{A}_{r(\alpha)},\quad n\geq 0$$

where α runs over all surjections with domain n. Denote by $\tilde{\alpha}:A_{r(\alpha)}\to A_{d(\alpha)}$ the canonical injection. For a surjection β in Φ_0, $\bar{\beta}:A_{r(\beta)}\to A_{d(\beta)}$ is defined by

$$\bar{\beta}\tilde{\alpha}=\widetilde{\alpha\beta},\ \text{for each surjection } \alpha \text{ with } d(\alpha)=r(\beta).$$

For an integer i with $0\leq i\leq n$, $\bar{\varepsilon}^i=\bar{\varepsilon}^i_n:A_n\to A_{n-1}$ is defined as follows.

$$\bar{\varepsilon}^i\tilde{\alpha}=0,\ i>0,\quad \alpha(i-1)<\alpha(i)<\alpha(i+1),\quad d(\alpha)=n,$$
$$\bar{\varepsilon}^i\widetilde{\alpha\delta^{i-1}}=\bar{\varepsilon}^i\widetilde{\alpha\delta^i}=\tilde{\alpha},\quad d(\alpha)=n-1,\ i>0,$$
$$\bar{\varepsilon}^0\widetilde{\alpha\delta^0}=\tilde{\alpha},\qquad d(\alpha)=n-1,$$
$$\bar{\varepsilon}^0\tilde{\alpha}^+=\tilde{\alpha}d_n\,,\qquad d(\alpha)=n-1,$$

where α^+ is defined by $\alpha^+(0)=0$, $\alpha^+\varepsilon^0=\varepsilon^0\alpha$ and $d_n : \widetilde{A}_n\to\widetilde{A}_{n-1}$ is a boundary morphism. Note that every map in Φ_0 is represented by $\alpha\delta^0$ or α^+. Straight forward calculations lead to $\overline{\beta}_1\overline{\beta}_2=\overline{\beta_2\beta_1}$ for two surjections β_1, β_2 with $r(\beta_1)=d(\beta_2)$, $\bar{\varepsilon}^i\bar{\varepsilon}^j=\bar{\varepsilon}^{j-1}\bar{\varepsilon}^i$, $\bar{\varepsilon}^i\bar{\delta}^j=\bar{\delta}^{j-1}\bar{\varepsilon}^i$, $\bar{\delta}^i\bar{\varepsilon}^j=\bar{\varepsilon}^{j+1}\bar{\delta}^i$ with $i<j$, $\bar{\varepsilon}^i\bar{\delta}^i=id$ and $\bar{\varepsilon}^{i+1}\bar{\delta}^i=0$.

Hence A_* is a simplicial object.

The remain of the proof follows from Proposition 1.1.

§2. Simplicial cohomology

Let K be an associative and commutative ring with unit. Let $\mathcal{A}$ be the category of associative and commutative K-algebra with unit. An object A in $\mathcal{A}$ is called simply a *K-algebra.* For a K-algebra A, denote by $(\mathcal{A}, A)$ the category of morphisms $\varepsilon=\varepsilon_B : B\to A$ in $\mathcal{A}$. An augmented (quasi-)simplicial object A_* in $\mathcal{A}$ is called a (quasi-)simplicial algebra *over* A_{-1}. By a morphism ρ_* of a (quasi-) simplicial algebra over A we mean a morphism of augmented (quasi-) simplicial algebras with $\rho_{-1}=id_A$.

Denoted by $\mathcal{S}$ the category of pointed sets. Let $U(A)$ be the underlying set of $A\in\mathrm{ob}\mathcal{A}$ with the base point $0\in U(A)$. Let $F(S)$ be the quotient algebra of the polynomial algebra generated by the set S identifying the base point with 0. Then we have an adjoint pair

$$(\varepsilon, \eta) : F\dashv U : (\mathcal{A}, \mathcal{S})$$

The pair $F\dashv U$ generates a cotriple $(G, \varepsilon, \eta)=(FU, \varepsilon, R\eta U)$. Functors $G_n=G^{n+1}$ and functor morphisms $\varepsilon_n^i=G^i\varepsilon G^{n-i}$, $\delta_n^i=G^i\delta G^{n-i}$ define a simplicial object in $\mathrm{Cat}(\mathcal{A}, \mathcal{A})$. For a K-algebra A, a family of $G_n(A)$ with $\varepsilon^i=\varepsilon^i(A)$, $\delta^i=\delta^i(A)$ defines a simplicial object called the *standard simplicial algebra over A.* An (augmented) quasi-simplicial algebra A_* is called *free,* if A_n is a polynomial algebra over K for $n\geq 0$. An (augmented) simplicial algebra A_* is called *free,* if there exists $S_n\in\mathrm{ob}\mathcal{S}$ for $n\geq 0$ such that $A_n=F(S_n)$ and $(U\delta^i)S_n\subset S_{n+1}$, $0\leq i\leq n$.

Proposition 2.1. *The standard simplicial algebra $G_*(A)$ is free and acyclic.*

Proof. Since $G_n(A)=FUG_{n-1}(A)$ and $(U\delta^i)(UG_n(A))\subset UG_{n+1}(A)$ for $0\leq i\leq n$, it follows that $G_*(A)$ is free. On the other hand, η induces a contracting homotopy of the Moore complex $\widetilde{G}_*(A)$.

Proposition 2.2. *Let F_* be a free (quasi-)simplicial algebra over a K-algebra A. Let A_* be an acyclic (quasi-)simplicial algebra over A. Then there exists a morphism $\rho_*: F_*\rightarrow A_*$ of (quasi-) simplicial algebra over A.*

Proof. We construct ρ_n for $n\geq -1$ such that $\rho_{-1}=id_A$, $\varepsilon^i\rho_n=\rho_{n-1}\varepsilon^i$, $0\leq i\leq n$, furthermore in the simplicial case $\rho_n\delta^i=\delta^i\rho_{n-1}$, $0\leq i\leq n$. Assume that such ρ_{-1}, ρ_0, $\cdots$, ρ_{n-1} are defined.

In the quasi-simplicial case, there exists a set $S\in\mathcal{S}$ such that $F_n=F(S)$. A quasi-simplicial set $\mathrm{Hom}_{\mathcal{A}}(F(S), A_*)=\mathrm{Hom}_{\mathcal{S}}(S, U(A_*))$ is acyclic by the assumption of the theorem. Hence there exists $\rho_n\in\mathrm{Hom}_{\mathcal{A}}(F(S), A_n)$ such that $\varepsilon^i\rho_n=\rho_{n-1}\varepsilon^i$.

In the simplicial case, there exist $S_n\in\mathcal{S}$ for $n\geq 0$ such that $F_n=F(S_n)$, $(U\delta^i)S_{n-1}\subset S_n$. Hence there exists a set map $\bar{\rho}_n: S_n\rightarrow UA$ such that

$$\bar{\rho}_n(x)=\bar{\delta}^i\bar{\rho}_{n-1}(y),\quad \text{for } x=\delta^i y \text{ for some } i \text{ and some } y\in S_{n-1},$$

$$\bar{\varepsilon}^i\bar{\rho}_n(x)=\bar{\rho}_{n-1}\bar{\varepsilon}^i(x),\ 0\leq i\leq n,\quad \text{for } x\in S_n-\bigcup_{i=0}^{n-1}\bar{\delta}^i S_{n-1},$$

where $\rho_{n-1}=U\rho_{n-1}$, $\bar{\delta}^i=U\delta^i$, $\varepsilon^i=U\varepsilon^i$, whence $\bar{\rho}_n$ determines a required morphism $\rho_n: F_n=F(S_n)\rightarrow A_n$.

Proposition 2.3. *Let F_* be a free (quasi-)simplicial algebra over a K-algebra A. Let A_* be an acyclic (quasi-)simplicial algebra over A. Let ρ'_*, ρ_* be two morphisms of F_* to A_*. Then ρ'_* is homotopic to ρ_*.*

Proof. We construct a homotopy ω^i_n, $0\leq i\leq n$, between ρ'_* and ρ_*. Since $\mathrm{Hom}_{\mathcal{S}}(S, U(A_*))$ is acyclic, it follows that there exists $\omega^0=\omega^0_0$ such that $\varepsilon^0\omega^0=\rho'_0$, $\varepsilon^1\omega^0=\rho_0$. For an integer $n>0$ assume that

ω_i^i, $0 \leq i \leq j < n$, are defined.

In the quasi-simplicial case, since the quasi-simplicial set $\mathrm{Hom}_{\mathcal{S}}(S, U(A_*))$ satisfies the Kan condition, it follows that there exists ω_n^0 such that $\varepsilon^0\omega_n^0 = \rho_n'$, $\varepsilon^i\omega_n^0 = \omega_{n-1}^0\varepsilon^{i-1}$, $0 < i \leq n+1$. In the same way we define inductively ω_n^j, $0 < j < n$, so that $\varepsilon^i\omega_n^j = \omega_{n-1}^{j-1}\varepsilon^i$ for $i < j$, $\varepsilon^j\omega_n^j = \varepsilon^j\omega_n^{j-1}$ and $\varepsilon^i\omega_n^j = \omega_{n-1}^j\varepsilon^{i-1}$ for $i > j+1$. Since $\mathrm{Hom}_{\mathcal{S}}(S, U(A_*))$ is acyclic, it follows that there exists ω_n^n such that $\varepsilon^i\omega_n^n = \omega_{n-1}^{n-1}\varepsilon^i$ for $i < n$, $\varepsilon^n\omega_n^n = \varepsilon^n\omega_n^{n-1}$, $\varepsilon^{n+1}\omega_n^n = \rho_n$.

In the simplicial case the proof can be obtained analogously, if we pay the same attention as in the proof of Theorem 1.

Let T be a contravariant functor of $\mathcal{A}$ (resp. $(\mathcal{A}, A)$) to an abelian category. If A_* is an augmented quasi-simplicial algebra, we have a chain complex

$$0 \to T(A_0) \xrightarrow{\partial^0} A(T_1) \xrightarrow{\partial^1} \cdots \to T(A_n) \xrightarrow{\partial^n} T(A_{n+1}) \to \cdots$$
$$\partial^n = \textstyle\sum_{0=i}^{n+1} (-1)^i T\varepsilon^i.$$

If ω^* is a homotopy between ρ_*' and ρ_* which are morphisms of augmented quasi-simplcial algebras of A_* into A_*', then $s^n = \sum_{i=0}^n (-1)^i T\omega_n^i$ for $n \geq 0$ form a chain homotopy between $T\rho_*'$ and $T\rho_*$ i.e.

$$s^{n+1}\partial^n + \partial^{n-1}s^n = T\rho_n' - T\rho_n$$

whence $T\rho_n'$ and $T\rho_n$ induce the same morphism of the derived objects $H^n(T(A_*')) \to H^n(T(A_*))$.

Let F_* be a free acyclic quasi-simplicial algebra over A' in $\mathcal{A}$ (resp. $(\mathcal{A}, A)$). Then we can consider cohomology $H^*(T(F_*))$, which does not depend on the choice of F_* by Proposition 2.2 and 2.3, and is denoted by $H^*(A', T)$. In the same way we can consider n-th homology $H_*(A', S) = H_*(S(F_*))$ for a covariant functor S of $\mathcal{A}$ (resp. $(\mathcal{A}, A)$) to an abelian category.

In particular for an abelian group object B in $(\mathcal{A}, A)$, we consider the cohomology group of A' in $(\mathcal{A}, A)$ by the functor $\mathrm{Hom}_{(\mathcal{A}, A)}(\ , B)$. B is represented as an idealization $A+M$ of an A-module M (J. Beck [2]).

We call the group $H^n(A, \mathrm{Hom}_{\mathcal{A}}(\ ,B))$ the *simplicial cohomology group of* A *by* M, in notation $H^n(A, M)$.

$\mathrm{Hom}_{(\mathcal{A}, A)}(A', B)$ for $A' \in \mathrm{ob}(\mathcal{A}, A)$ is isomorphic to the K-module $\mathrm{Der}_K(A', M)$ of K-derivations, where M is considered an A'-module *via* the structure homomorphism $\varepsilon : A' \to A$. Let A_* be a simplicial algebra over A. Put

$$\begin{aligned}\mathrm{Der}_K^{\sim}(A_n, M) &= \{f \in \mathrm{Der}_K(A_n, M) \mid f\delta^i = 0,\ 0 \leq i \leq n\}\\ &= \{f \in \mathrm{Der}_K(A_n, M) \mid f\pi_\iota = f\}.\end{aligned}$$

where we use the same π_ι as defined in Lemma 1 for identity $\iota = \iota_n$.

Proposition 2.4. *If A_* be a simplicial algebra over A, then we have the canonical isomorphism*

$$H^n(\mathrm{Der}_K^{\sim}(A_*, M)) \cong H^n(\mathrm{Der}_K(A_*, M)), \quad n \geq 0.$$

Proof. If $f \in \mathrm{Der}_K^{\sim}(A_n, M)$ then

$$(f\partial_{n+1})\pi_{\iota_{n+1}} = f\pi_{\iota_n}\partial_{n+1} = f\partial_{n+1}$$

whence $f\partial_{n+1} \in \mathrm{Der}_K^{\sim}(A_{n+1}, M)$, where $\partial_n = \sum_{i=0}^{n}(-1)^i \varepsilon^i$.

Hence $\mathrm{Der}^{\sim}(A_*, M)$ is a chain subcomplex.

Put

$$t_i = (1-\delta^0\varepsilon^1)(1-\delta^1\varepsilon^2)\cdots(1-\delta^{i-1}\varepsilon^i),$$
$$u^i = t_0\delta^0 - t_1\delta^1 + \cdots + (-1)^{i-1}t_{i-1}\delta^{i-1}.$$

It follows that

$$1 - t_n = \partial_{n+1}u_n + u_{n-1}\partial_n .$$

If f is an n-cocycle (i.e. $f\partial_{n+1} = 0$) in $\mathrm{Der}(A_*, M)$, then

$$f - f\pi_\iota = f(1-t_n) = (fu_{n-1})\partial_n ,$$

As u_{n-1} is represented by a linear combination of morphisms in $(\mathcal{A}, A)$, fu_{n-1} is a derivation. Hence f is cohomologous in $\mathrm{Der}(A_*, M)$ to a cocycle $f\pi_\iota$ in $\mathrm{Der}^{\sim}(A_*, M)$.

If $f \in \mathrm{Der}^{\sim}(A_n, M)$ is a coboundary in $\mathrm{Der}(A_n, M)$, there exists $g \in \mathrm{Der}(A_{n-1}, M)$ such that $f = g\partial_n$. Therefore

$$f=f\pi\iota_n=g\partial_n\pi\iota_n=(g\pi\iota_{n-1})\partial_n.$$

Hence f is a coboundary in $\mathrm{Der}^{\check{}}(A_n, M)$.

§3. Standard extensions

Let A_* be an (augmented) (quasi-)simplicial algebra. Let M_* be an (augmented) (quasi-)simplicial module. If M_n is an A_n-module for each n, and if the multiplications $A_n\otimes_K M_n\to M_n$ are compatible with the face operators ε^i, also with the degeneracy operators δ^i in the simplicial case, then we call M_* to be an *A_*-module.* The idealizations A_n+M_n form an (augmented) (quasi-)simplicial algebra, which we call the *idealization* of M_*, and denote it by A_*+M_*. A sub A_*-module I_* of A_* is called an *ideal* of A_*. A_n/I_n form an (augmented) (quasi-)simplicial algebra, which we denote by A_*/I_*. If I_* satisfies the Kan condition, then we have $(A_*/I_*)^{\check{}}=\widetilde{A}_*/\widetilde{I}_*$.

For a positive integer n and a module M over a K-algebra A, there exist one and only one simplicial A-module M_* such that $\widetilde{M}_n=\widetilde{M}_{n-1}=M$, $d_n=$identity and $\widetilde{M}_r=0$ for $r\neq n$, $n-1$. If A_* is a simplicial algebra over A, then M_* is canonically an A_*-module, the multiplication of which is given by $A_r\otimes M_r\xrightarrow{\varepsilon\otimes 1}A\otimes M_r\to M_r$. We can consider the idealization $B_*=A_*+M_*$. Let f be an n-cocycle in $\mathrm{Der}_K^{\check{}}(A_*, M)$. Define a subset I_r in B_r as follows

$$I_r=0,\quad r<n-1,$$
$$I_{n-1}=\varepsilon^0(\kappa_1-\kappa_2 f)(\widetilde{A}_n),$$
$$I_r=\{x\in B_r\,|\,\bar{\alpha}(x)\in I_{n-1} \text{ for every injection } \alpha:[n-1]\to[r]\},$$
$$r\geq n.$$

where $\kappa_1: A_n\to B_n$ and $\kappa_2: M\to B_n$ are the canonical injections.

If $\alpha:[n-1]\to[r]$ is not an injection, there exists an surjection $\beta:[n-1]\to[s]$, $s<n-1$ and an injection $\beta:[s]\to[r]$ such that $\alpha=\gamma\beta$. Hence $x\in I_r$ implies $\tilde{\gamma}(x)=0$, which means $\bar{\alpha}(x)=0$. Hence for every r, I_r is the set of all $x\in B_r$ such that $\tilde{\alpha}(x)\in I_{n-1}$ for every

monotone map $\alpha : [n-1] \to [r]$. Since I_{n-1} is an ideal of B_{n-1}, it follows that I_* is an ideal in B_*.

Denote by $E(f)$, ρ_* and τ, the simplicial algebra B_*/I_*, the canonical homomorphism $A_* \to E(f)$ and $M \to E(f)^{\sim}_n$ respectively.

Proposition 3.1. *Let A_* be an acyclic simplicial algebra over a K-algebra A. Let M be an A-module. If f is an n-cocycle in* $\mathrm{Der}^{\sim}_K(A_*, M)$, *then there exists one and only one acyclic simplicial algebra E_* over A satisfying the following conditions:* (1) *There exists a morphism $\rho_* : A_* \to E_*$ of simplicial algebras over A.* (2) *There exists an isomorphism $\tau : M \to \tilde{E}_n$ of A_n-modules such that $\rho_n = \tau f$.* (3) *$\tilde{E}_r = 0$ for $r > n$.* (4) *ρ_r is an isomorphism for* $0 \leq r \leq n-2$.

Proof. To prove the existence we may put $E_* = E(f)$ using the above notations. Then the condition (1), (2), (3) and (4) are satisfied. In the following commutative diagram the upper and middle raws and all the column are exact.

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & & & \\
\tilde{I}_{n+1} & \to & \tilde{I}_n & \longrightarrow & \tilde{I}_{n-1} & \longrightarrow & 0 & & & & & & \\
\| & & \downarrow & & \downarrow & & & & & & & & \\
\tilde{A}_{n+1} & \to & \tilde{A}_n + M & \to & \tilde{A}_{n-1} + M & \to & \tilde{A}_{n-2} & \to \cdots \to & \tilde{A}_0 & \to & A & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \| & & \| & & \\
0 & \longrightarrow & \tilde{E}_n & \longrightarrow & \tilde{E}_{n-1} & \longrightarrow & \tilde{E}_{n-2} & \to \cdots \to & \tilde{E}_0 & \to & A & \to & 0 \\
 & & \downarrow & & \downarrow & & & & & & & & \\
 & & 0 & & 0 & & & & & & & &
\end{array}$$

Hence the Moore complex $\tilde{E}_*$ is acyclic. It shows the acyclicity of E_*. The uniqueness of E_* follows from the Corollary 3.3 of Proposition 3.2.

Proposition 3.2. *With the same A_*, M and f as in Proposition* 3.1, *let f′ be another n-cocycle in* $\mathrm{Der}^{\sim}_K(A_*, M)$ *which is cohomologous to f. Let E'_* be an acyclic (quasi-)simplicial algebra over A satisfying the condition* (2) *and* (3) *in Proposition* 3.1

for a morphism $\rho'_* : A_* \to E'_*$ *of (quasi-)simplicial algebras over A and* $\tau' : M \to \widetilde{E}'_n$. *Then there exists a morphism* $\sigma_* : E_*(f) \to E'_*$ *of (quasi-)simplicial algebras over A such that* $\sigma_n \tau = \tau'$, $\sigma_r \rho_r = \rho'_r$ *for* $0 \leq r \leq n-2$.

Proof. There exists an $(n-1)$-cochain g such that $f - f' = g\partial_n$. Let $\sigma_r = \rho'_r : E_r = A_r \to E'_r$ for $0 \leq r \leq n-2$. Let $\bar{\sigma}_{n-1} : B_{n-1} = A_{n-1} + M \to E'_{n-1}$ be a homomorphism of K-algebras such that $\bar{\sigma}_{n-1} | A_{n-1} = \rho'_{n-1} + d_n \tau g d_n$ and $\bar{\sigma}_{n-1} | M = d_n \tau'$. $\bar{\sigma}_{n-1}$ induces a homomorphism of E_{n-1} into E'_{n-1}. For an integer $r \geq n$ assume that homomorphisms $\sigma_s : E_s \to E'_s$, $s < r$, $n \leq r$ are defined such that $\varepsilon^i \sigma_{r-1} = \sigma_{r-2} \varepsilon^i$, $0 \leq i \leq r-1$. By the acyclicity of E'_*, there exists an element $\sigma_r(x)$ for each $x \in E_r$ such that $\varepsilon^i(\sigma_r(x)) = \sigma_{r-1} \varepsilon^i(x)$, $0 \leq i \leq r$. Since $d_r : \widetilde{E}_r \to \widetilde{E}'_r$ is a monomorphism, it follows that $\sigma_r(x)$ is uniquely determined by x. Hence $\sigma_r : E_r \to E'_r$ is a homomorphism of K-algebras such that $\varepsilon^i \sigma_r = \sigma_{r-1} \varepsilon^i$. $d_n \tau' = \bar{\sigma}_{n-1} | M$ implies $d_n \tau' = \tilde{\sigma}_{n-1} d_n \tau = d_n \tilde{\sigma}_n \tau$. Hence $\tau' = \tilde{\sigma}_n \tau$. The lemma was shown in the quasi-simplicial case. In the simplicial case, $\sigma_r \delta^i = \delta^i \sigma_{r-1}$ for $0 \leq i \leq r < n$ follows from the definition. For $r \geq n$, $\sigma_r \delta^i = \delta^i \sigma_{r-1}$ follows from the uniquness of σ_r.

Corollary 3. 3. *With the same* A_* *and M as in Lemma, let f, f′ be two n-cocycles in* $\mathrm{Der}^{\sim}_K(A_*, M)$. *There exists an isomorphism* σ_* *between* $E_* = E(f)$ *and* $E'_* = E(f')$ *such that* $\tilde{\sigma}_n \tau = \tau'$, $\sigma_r \rho_r = \rho'_r$ *for* $0 \leq r \leq n-2$, *if and only if f and f′ are cohomologous.*

Proof. Assume that f and f' are cohomologous. By Proposition 3. 2, there exists a canonical morphism $\sigma_* : E_* \to E'_*$. σ_* gives a chain map $\tilde{\sigma}_* : \widetilde{E}_* \to \widetilde{E}'_*$, which is an isomorphism by the five lemma.

Conversely if $\sigma_* : E(f) \to E(f')$ is an isomorphism such that $\tilde{\sigma}_n \tau = \tau'$, $\sigma_r \rho_r = \rho'_r$ for $0 \leq r \leq n-2$, then $\varepsilon^i \sigma_{n-1} \rho_{n-1} = \sigma_{n-2} \rho_{n-2} \varepsilon^i = \rho'_{n-2} \varepsilon^i = \varepsilon^i \rho'_{n-1}$. There exists a homomorphism $\bar{g} : A_{n-1} \to E'_n$ of K-modules such that $d_n \bar{g} = \sigma_{n-1} \rho_{n-1} - \rho'_{n-1}$. Since $d_n \bar{g}(xy) = \sigma_{n-1} \rho_{n-1}(x) d_n \bar{g}(y) + \rho'_{n-1}(y) d_n \bar{g}(x)$, and d_n is a monomorphism, it follows that $\bar{g}(xy) = \delta^0 \sigma_{n-1} \rho_{n-1}(x) \bar{g}(y) + \delta^0 \rho'_{n-1}(y) \bar{g}(x)$. Let $g = \tau'^{-1} \bar{g} : A_{n-1} \to M$ then $g(xy) = xg(y) + yg(x)$. Since $d_n \bar{g} \delta^i = \delta^i(\sigma_{n-2} \rho_{n-2} - \rho'_{n-2}) = 0$, $0 \leq i \leq n$, it follows $g \in \mathrm{Der}^{\sim}_K(A_{n-1}, M)$.

Since $d_n\bar{g}\partial_n = d_n(\sigma_n g_n - g'_n) = d_n(\sigma_n \tau f - \tau' f') = d_n \tau'(f - f')$, it follows that $f - f' = g\partial_n$.

In particular if A_* is the standard simplicial algebra $G_*(A)$, and f is a cocycle in $\mathrm{Der}_K^{\sim}(G_*(A), M)$, then we call $E(f)$ the *standard n-fold extension.*

§4. (quasi-)simplicial extensions

Let M be a module over a K-algebra A. Let n be a positive integer. We define an *n-fold (quasi-)simplicial extension* E_* of A by M as follows:

(1) E_* is an acyclic (quasi-)simplicial algebra over A (so satisfying the Kan condition),

(2) $\widetilde{E}_r = 0$, $r > n$,

(3) $\widetilde{E}_n \cong M$ as E_n-modules,

(4) $\overline{E}_n^2 \cap \widetilde{E}_n = 0$, $\overline{E}_n = \mathrm{Ker}(\varepsilon : E_n \to A)$,

where $\overline{E}_n^2$ means the product $\overline{E}_n \cdot \overline{E}_n$ of ideals.

If E_* is a simplicial algebra, then the condition (4) is replaced by the following condition:

(4′) π_ι is a derivation.

In fact, it is easily verified that (4′) implies (4). Conversely assume that E_* satisfy (4). Then $\pi_\iota(xy) = 0$ for $x, y \in \overline{E}_n$. Denote by 0 the unique map $[n] \to [0]$, then π_0 is a homomorphism of algebras, and $\pi_\iota \pi_0 = 0$. Note that $\pi_\iota(\pi_0(x)y) = \pi_0(x)\pi_\iota(y)$. An equation for $x, y \in E_n$

$$xy = \pi_0(x)y + \pi_0(y)x - \pi_0(xy) + (1-\pi_0)(x) \cdot (1-\pi_0)(y)$$

leads to

$$\begin{aligned}\pi_\iota(xy) &= \pi_0(x)\pi_\iota(y) + \pi_0(y)\pi_\iota(x) \\ &= x\pi_\iota(y) + y\pi_\iota(x).\end{aligned}$$

This states that (4) implies (4′).

A *morphism* of such extensions is defined to be a morphism of an augmented (quasi-)simplicial algebras with $\rho_{-1} = id_A$, $\tilde{\rho}_n = id_M$.

If there exists a sequence of morphisms of extensions

$$E^0 \to E^1 \leftarrow E^2 \to \cdots \to E^{2r-1} \leftarrow E^{2r}$$

then E^0 and E^{2r} are called equivalent. The equivalent classes of extensions are called the *Yoneda classes*.

Proposition 4.1. *If E_* is an n-fold (quasi-)simplicial extension of a K-algebra A by an A-module M, then there exists an n-cocyle f in* $\mathrm{Der}_K^{\sim}(G_*(A), M)$ *and a morphism of the standard n-fold extension $E(f)$ into E_*.*

Proof. By Theorem 1, there exists a morphism ρ_* of $G_*(A)$ to E_* of (quasi-)simplicial algebras over A. $\rho_n\pi_\iota$ induces caonically a homomorphism $f: G_n(A) \to M$ of K-modules: $\tau' f = \rho_n\pi_\iota$.

It follows from (4) that

$$\rho_n\pi_\iota(xy) = \rho_n\pi_0(x)\rho_n\pi_\iota(y) + \rho_n\pi_0(y)\rho_n\pi_\iota(x),$$

which implies that f is a derivation. Since $\pi_\iota\delta^i = 0$, $0 \leq i \leq n$, it follows $f \in \mathrm{Der}_K^{\sim}(G_n(A), M)$.

By the fact $\rho_n\pi_\iota\partial_{n+1} = \varepsilon^0\rho_{n+1}\pi_\iota$ and the condition (2) for $r = n+1$, it follows $f\partial_{n+1} = 0$, which means f is a cocycle. It follows from Lemma 5 that there exists a morphism of $E(f)$ into E_*.

Proposition 4.2. *If $\rho_*: E(f) \to E_*$ and $\rho'_*: E(f') \to E_*$ are morphisms from the standard extension of n-fold (quasi-)simplicial extensions of a K-algebra A by an A-module M, then n-cocycles f and f′ are cohomogous.*

Proof. Morphisms ρ_* and ρ'_* are induced respectively from morphisms $\bar{\rho}_*$ and $\bar{\rho}'_*$ of the standard simplicial algebra $G_* = G_*(A)$ into E_*. By Theorem 2, there exists a homotopy ω^* of $\bar{\rho}'_*$ into $\bar{\rho}_*$. Put $\bar{g} = \sum_{i=0}^{n=1} (-1)^i (\omega^i_{n-1} - \rho_n\delta^i)\pi_\iota$. Then it follows

$$\varepsilon^0 \bar{g}\partial_n = \varepsilon^0(\rho'_n - \rho_n)\pi_\iota,$$

$$\varepsilon^i \bar{g} = 0, \quad 1 \leq i \leq n.$$

Hence $\bar{g}$ induces a homomorphism g of G_{n-1} into M of K-modules.

The acyclicity of E_* and the condition (2) for $r=n+1$ imply $(\bar{\rho}_n'-\bar{\rho}_n)\pi_\iota=\bar{g}\partial_n$, which means $f'-f=g\partial_n$.

If $x\in G_{n-1}$ and $y\in\bar{G}_{n-1}$ then

$$\begin{aligned}\bar{g}(\pi_0(x)y)&=\rho_n\delta^0\pi_0(x)\cdot\bar{g}(y)+\sum_{i=0}^{n-1}(-1)^i(\omega^i-\rho_n\delta^0)\pi_0(x)\cdot\omega^i\pi_\iota(y)\\&=\rho_n\delta^0\pi_0(x)\cdot\bar{g}(y).\end{aligned}$$

Hence it follows that g is an $(n-1)$-cochain in $\mathrm{Der}_K^{\sim}(G^*, M)$.

Hence f and f' are cohomologous.

By Corollary 3. 3, Proposition 4. 1 and Proposition 4. 2, we get the following theorem.

Theorem 4. 3. *Let n be a positive integer. Denote by* $\mathrm{Ex}^n(A, M)$ *(resp.* $\mathrm{Ex}^n_{q-s}(A, M)$*) the set of the Yoneda classes in the category of n-fold simplicial (resp. quasi-simplicial) extensions of a K-algebra A by an A-module M. Denote by* $H^n(A, M)$ *the n-th simplicial cohomology group of A by M. Then there exist canonical bijections between* $\mathrm{Ex}^n(A, M)$ *and* $\mathrm{Ex}^n_{q-s}(A, M)$ *and* $H^n(A, M)$.

§5. 3-fold extensions

Let n be a positive integer.

Proposition 5. 1. *Let* E_* *be an augmented acyclic (quasi-) simplicial K-module such that* E_r *is a K-algebra for* $-1\leq r<n$, *and* $\widetilde{E}_r=0$ *for* $r>n$. *Assume that the multiplications in* E_r *with* $-1\leq r<n$ *are compatible with the face operators, also with the degeneracy operators in the simplicial case. Then there uniquely exist multiplications in* E_r, $r>n$ *such that* E_* *becomes a (quasi-) simplicial algebra.*

Proof. For an integer $r\geq n$, assume that associative commutative multiplications $\varphi_s: E_s\times E_s\to E$, $s<r$ are defined and that $\varepsilon^i\varphi_s=\varphi_{s-1}(\varepsilon^i\times\varepsilon^i)$. Then there exists a set map φ_r such that $\varepsilon^i\varphi_r=\varphi_{r-1}(\varepsilon^i\times\varepsilon^i)$ by virtue of the acyclicity of E_*. Since $d_r:\widetilde{E}_r\to\widetilde{E}_{r-1}$ is an injection, φ_r is uniquely determined. Hence φ_r is associative

and commutative. In the simplicial case $\varphi_r(\delta^i\times\delta^i)=\delta^i\varphi_{r-1}$ is also satisfied by the uniqueness of φ_r.

In the following α, β, γ, δ, ξ and η imply monotone surjections. An exact sequence of K-modules

(1) $$0\longrightarrow M=\widetilde{E}_n\xrightarrow{d_n}\widetilde{E}_{n-1}\longrightarrow\cdots\xrightarrow{d_1}\widetilde{E}_0\xrightarrow{d_0}A\longrightarrow 0$$

with K-linear maps

$$\varphi^{\gamma}_{\alpha,\beta}:\widetilde{E}_{r(\alpha)}\otimes\widetilde{E}_{r(\beta)}\rightarrow\widetilde{E}_{r(\gamma)},\quad 0\leq d(\alpha)=d(\beta)=d(\gamma)\leq n,$$

is called an *n-term extension* of A by M, if the following conditions (2) to (9) are satisfied.

(2) $$\textstyle\sum_{\xi}\varphi^{\delta}_{\xi,\eta}(\varphi^{\xi}_{\alpha,\beta}\otimes 1)=\sum_{\eta}\varphi^{\delta}_{\alpha,\eta}(1\otimes\varphi^{\eta}_{\beta,\gamma}).$$

(3) $$\varphi^{\gamma}_{\alpha,\beta}=\varphi^{\gamma}_{\beta,\alpha}\tau,\qquad\text{where } \tau(x\otimes y)=y\otimes x.$$

(4) $$\varphi^{\gamma\delta}_{\alpha\delta,\beta\delta}=\varphi^{\gamma}_{\alpha,\beta}\,,$$

(5) $$\varphi^{\gamma}_{\alpha\delta^i\beta\delta^i}=0,\qquad\text{if } \gamma(i)\neq\gamma(i+1).$$

$$\varphi^{\gamma\delta^i}_{\alpha,\beta}=0,\quad\text{if } \alpha(i-1)<\alpha(i)<\alpha(i+1) \text{ and } \gamma(i-1)=\gamma(i).$$

$$\varphi^{\gamma\delta^{i-1}}_{\alpha,\beta}+\varphi^{\gamma\delta^i}_{\alpha,\beta}=0,\quad\text{if } \alpha(i-1)<\alpha(i)<\alpha(i+1).$$

(6) $$\varphi^{\gamma\delta^i}_{\alpha\delta^{i-1}\beta\delta^i}=\varphi^{\gamma}_{\alpha,\beta}\,,\quad\text{if } \gamma(i-1)=\gamma(i),$$

$$\varphi^{\gamma\delta^{i-1}}_{\alpha\delta^{i-1},\beta\delta^i}+\varphi^{\gamma\delta^i}_{\alpha\delta^{i-1},\beta\delta^i}=\varphi^{\gamma}_{\alpha,\beta}\,,\quad\text{if } \gamma(i-1)\neq\gamma(i).$$

(7) $$\varphi^{\gamma\delta^0}_{\alpha^+,\beta\delta^0}+d\varphi^{\gamma^+}_{\alpha^+,\beta\delta^0}=\varphi^{\gamma}_{\alpha,\beta}(d\otimes 1)$$

where α^+ is the surjection such that $\varepsilon^0\alpha=\alpha^+\varepsilon^0$.

(8) $\widetilde{E}_0$ is a K-algebra, and d_0 is a homorphism of K-algebras.

(9) $$\varphi^{\iota}_{\alpha,\beta}=0,\qquad\text{if } \iota=\iota_n \text{ and } r(\alpha)>0,$$

$$\varphi^{\iota}_{0,\iota}=\varphi_M(d_0\otimes 1),$$

where $\varphi_M:A\otimes M\rightarrow M$ is the multiplication of M.

A *morphism* of n-term extensions is a chain map ρ_* with $\rho_{-1}=id_A$ and $\rho_n=id_M$ which is compatible with $\varphi^{\gamma}_{\alpha,\beta}$.

Proposition 5.2. *If E_* is an n-fold simplicial extension of A by M, $\varphi=\varphi_m:E_m\otimes_K E_m\rightarrow E_m$ is the multiplication, then the Moore complex $\widetilde{E}_*$ of E_* with $\varphi^{\gamma}_{\alpha,\beta}$ is an n-term extension of A by M,* *where*

$$\varphi^{\gamma}_{\alpha,\beta} : \widetilde{E}_{r(\alpha)} \otimes \widetilde{E}_{r(\beta)} \to \widetilde{E}_{r(\gamma)}$$

is induced by $\theta_r\varphi(\bar{\alpha}\otimes\bar{\beta})$ *for three surjections* α, β, γ *with the same domain.*

The category of n-fold simplicial extensions of A by M is equivalent to the category n-term extensions of A by M.

Proof. The proof of the former part is streight-foreward. Given an n-term extension $\widetilde{E}_*$ of A by M. There exists one and only one simplicial K-module E_* over A such that its Moore complex is $\widetilde{E}_*$:

$$E_m = \sum_{d(\alpha)=m} \widetilde{E}_{r(\alpha)}, \quad m \geq 0.$$

Define $\varphi = \varphi_m : E_m \otimes E_m \to E_m$, $0 \leq m \leq n$ by

$$\varphi(\tilde{\alpha}\otimes\tilde{\beta}) = \sum_{r} \tilde{\gamma}\varphi^{\gamma}_{\alpha,\beta}.$$

(2) and (3) imply the associativity and commutativity of φ respectively. (4) implies the compatibility of φ with the degeneracy operators. (4), (5) and (6) follow the compatibility of φ with the face operators ε^i for $i > 0$. It follows that

$$\varepsilon^0\varphi(\widetilde{\alpha\delta^0}\otimes\widetilde{\beta\delta^0}) = \varphi(\tilde{\tilde{\alpha}}\otimes\tilde{\beta}),$$

$$\varepsilon^0\varphi(\tilde{\alpha}^+\otimes\widetilde{\beta\delta^0}) = \varphi(\tilde{\alpha}d\otimes\tilde{\beta}),$$

$$\varepsilon^0\varphi(\tilde{\alpha}^+\otimes\beta^+) = \varphi(\tilde{\alpha}d\otimes\tilde{\beta}d),$$

where the equation is obtained by the calculation

$$\begin{aligned}\varphi^{\gamma}_{\alpha,\beta}(d\otimes d) &= \varphi^{\gamma}_{\alpha,\beta}(d\otimes 1)(1\otimes d)\\ &= \varphi^{\gamma\delta^0\delta^1}_{\alpha\delta^0,\beta+\delta^1} + d\varphi^{\gamma+\delta^0}_{\alpha+\delta^0,\beta+\delta^1} + d\varphi^{\gamma+\delta^1}_{\alpha+\delta^0,\beta+\delta^1}\\ &= \varphi^{\gamma\delta^0}_{\alpha^+,\beta^+} + d\varphi^{\gamma^+}_{\alpha^+\beta^+}.\end{aligned}$$

Then E_m with $0 \leq m \leq n$ are K-algebras with the multiplication compatible with face and degeneracy operators, whence E_* is a simplicial algebra over A by Proposition 5. 1.

It follows from (8) that $\widetilde{E}^2_n \cap \widetilde{E}_n = 0$, Hence E_* is an n-fold simplicial extension.

Theorem 5.2. *Let A be a K-algebra and M be a A-module.*

(1) *a 1-term extension of A by M is given by an exact sequence*

$$0\longrightarrow M\xrightarrow{d_1}E_0\xrightarrow{d_0}A\longrightarrow 0$$

where d_0 is a homomorphism of K-algebras, d_1 is a homomorphism of E_0-modules.

(2) *a 2-term extension of A by M is given by an exact sequence*

$$0\longrightarrow M\xrightarrow{d_2}\widetilde{E}_1\xrightarrow{d_1}E_0\xrightarrow{d_0}A\longrightarrow 0$$

where d_0 is a homomorphism of K-algebras, d_1 and d_2 are homomorphisms of E_0-modules, and

$$d_1(x)y=d_1(y)x, \quad x, y\in\widetilde{E}_1\,.$$

(3) *a 3-term extension of A by M is given by an exact sequence*

$$0\longrightarrow M\xrightarrow{d_3}\widetilde{E}_2\xrightarrow{d_2}\widetilde{E}_1\xrightarrow{d_1}E_0\xrightarrow{d_0}A\longrightarrow 0$$

with a E_0-bilinear map

$$\langle\ ,\ \rangle:\widetilde{E}_1\otimes_{E_0}\widetilde{E}_1\to\widetilde{E}_2\,,$$

where d_0 is a homomorphism of K-algebras, d_1 is a homomorphism of E_0-modules with associative and commutative multiplications (i.e. E_0-algebras not necessary with unit), d_2 and d_3 are homomorphisms of E_0-modules, the map $\langle\ ,\ \rangle$ satisfies:

$$d_2\langle x_1, y_1\rangle=x_1y_1-d_1y_1x_1$$
$$\langle x_1, y_1z_1\rangle=\langle xy_1, z_1\rangle+d_1z_1\langle x_1, y_1\rangle$$
$$\langle d_2x_2, x_1\rangle=\langle x, d_2x_2\rangle-d_1x_1\cdot x_2\,.$$

Proof. The proof of (1) and (2) are seen in N. Shimada and others [8]. The proof of (3). For the 3-fold simplicial extension E_*, let $\widetilde{E}_*$ be the Moore complex and

$$\langle x_1, y_1\rangle=\delta^0x_1\cdot(\delta^0-\delta^1)y_1\,, \quad x_1, y_1\in\widetilde{E}_1\,.$$

Then

$$\varepsilon^0\langle x_1, y_1\rangle = x_1 y_1 - \delta^0\varepsilon^0 y_1\cdot x_1$$

$$\langle x_1 y, z_1\rangle - \langle x_1, y_1 z_1\rangle = \langle x_1, y_1\rangle\delta^1 z_1 .$$

Since $\overline{E}_3^2 \cap \widetilde{E}_3 = 0$, it follows that

$$\langle\varepsilon^0 x_2, x_1\rangle - \langle x_1, \varepsilon^0 x_2\rangle + \delta^0\delta^0\varepsilon^0 x_i\cdot x_2$$
$$= \varepsilon^0(\delta^1\delta^1 x_1\cdot\delta^0 x_2 - \delta^0\delta^1 x_1\cdot\delta^1 x_2 + \delta^0\delta^0 x_1\cdot\delta^2 x_2) = 0,$$
$$\delta^1 x_1\cdot x_2 - \delta^0\delta^0\varepsilon^0 x_2\cdot x_2 = \varepsilon^0((\delta^0 - \delta^1)\delta^1 x_1\cdot\delta^0 x_2) = 0,$$
$$x_2 y_2 - \langle\varepsilon^0 x_2, \varepsilon^0 y_2\rangle = \varepsilon^0(\delta^0 x_2\cdot\delta^0 y_2 - \delta^1 x_2\cdot\delta^1 y_2 + \delta^1 x_2\cdot\delta^2 y_2) = 0.$$

Conversely, let a sequence

$$0\to M = \widetilde{E}_3 \to \widetilde{E}_2 \to \widetilde{E}_1 \to \widetilde{E}_0 = E_0 \to A \to 0$$

satisfy the conditions in the theorem, E_* be a simplicial module such that its Moore complex is given by the above sequence. Denote by x_i, y_i, and z_i elements of $\widetilde{E}_i$.

Now we define multiplications in E_1, E_2, E_3 so that $\delta^0 x_1\cdot(\delta^0 - \delta^1)y_1 = \langle x_1, y_1\rangle$, and prove that E_* becomes a 3-fold simplicial extension. The required multiplications should be commutative and compatible with degeneracy operators. Therefore the multiplication is determined only by the following conditions.

In $E_1 = \delta^0 E_0 + \widetilde{E}_1$,

$$(\delta^0 x^0 + x_1)(\delta^0 y^0 + y_1) = \delta^0(x_0 y_0) + (x_0 y_1 + y_0 x_1 + x_1 y_1).$$

In $E_2 = \delta^0\delta^0 E_0 + \delta^0\widetilde{E}_1 + \delta^1\widetilde{E}_1 + \widetilde{E}_2$,

$$\delta^0\delta^0 x_0\cdot x_2 = x_0 x_2 ,$$
$$\delta^0 x_1\cdot\delta^1 y_1 = \delta^0(x_1 y_1) - \langle x_1, y_1\rangle,$$
$$\delta^0 x_1\cdot x_2 = \langle x_1, d_2 x_2\rangle,$$
$$\delta^1 x_1\cdot x_2 = d_1 x_1\cdot x_2$$
$$x_2\cdot y_2 = \langle d_2 x_2, d_2 y_2\rangle.$$

In $E_3 = \delta^0\delta^0\delta^0 E_0 + \delta^0\delta^0\widetilde{E}_1 + \delta^0\delta^1\widetilde{E}_1 + \delta^1\delta^1\widetilde{E}_1 + \delta^0\widetilde{E}_2 + \delta\widetilde{E}_2 + \delta^2\widetilde{E}_2 + \widetilde{E}_3$,

$$x\cdot x_3 = \varepsilon x\cdot x_3 , \quad x\in E_3 , \quad \varepsilon = \varepsilon_0^0\varepsilon_1^0\varepsilon_2^0\varepsilon_3^0 ,$$
$$\delta^0\delta^0 x_1\cdot\delta^2 x_2 = \delta^1\langle x_1, d_2 x_2\rangle - \delta^0\langle x_1, d_2 x_2\rangle,$$

$$\delta^0\delta^1 x_1\cdot\delta^1 x_2=\delta^1\langle x_1, d_2x_2\rangle+\delta^0(d_1x_1\cdot x_2-\langle x_1, d_2x_2\rangle),$$

$$\delta^1\delta^1 x_1\cdot\delta^0 x_2=\delta^0(d_1x_1\cdot x_2),$$

$$\delta^0 x^2\cdot\delta^1 y_2=\delta^0\langle d_2x_2, d_2y_2\rangle,$$

$$\delta^0 x_2\cdot\delta^2 y_2=0,$$

$$\delta^1 x_2\cdot\delta^2 y_2=\delta^1\langle d_2x_2, d_2y_2\rangle-\delta^0\langle d_2x_2, d_2y_2\rangle.$$

It is easily verified that the multiplications defined above are compatible with the face and degeneracy operators. The associativity of the multiplication in E_1 is seen imediately. It is not difficult but tedious to prove the associativity in E_2. By the definitiion of the multiplication in E_3, it follows that $\overline{E}_3^2\cap\widetilde{E}_3=0$. The associativity in E_3 follows from Proposition 5.1.

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