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General boundary conditions for multi-dimensional diffusion processes

By

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Let E be a bounded open set in n-dimensional Euclidean space. The real L^p-spaces with bounded Borel measure m over E are denoted by $L^p(E, m)$. $L^p_{\mathrm{loc}}(E, m)$ are defined as usual. Suppose we are given an elliptic operator L defined on E as

$$(0.1)\qquad L=\sum\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)+\sum b_i\frac{\partial}{\partial x_i}.$$

Here, a_{ij} is bounded measurable, symmetric in i and j, uniformly positive definite and $b_i\in L^{p_0}(E, dx)$ with some $p_0>n$, where dx is the Lebesgue measure. A function u on E is called a (weak) solution of $Lu=f$ for a given $f\in L^1_{\mathrm{loc}}(E, dx)$ if weak derivatives $\frac{\partial u}{\partial x_i}$, $i=1, 2, \cdots, n$ belongs to $L^{q_0}_{\mathrm{loc}}(E, dx)$ (q_0 is the conjugate of p_0) and

$$(0.2)\qquad \int_E\sum a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx-\int_E v\left(\sum b_i\frac{\partial u}{\partial x_i}\right)dx=-\int vf\,dx$$

is fulfilled for all $v\in C_0^\infty(E)=\{$the space of infinitely differentiable functions with compact supports in $E\}$. The formal adjoint of (0.1) is defined as

$$(0.3)\qquad L^*=\sum\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)-\sum\frac{\partial}{\partial x_i}(b_i\cdot),$$

namely, v is a weak solution of $L^*v=f$, if (0.2) is fulfilled for all $u\in C_0^\infty(E)$, replacing $\int vf\,dx$ by $\int uf\,dx$.

Now let R_α, $\alpha>0$ be a family of linear transformations from $L^\infty(E, dx)$ to $C(E)=\{$the space of continuous functions in $E\}$, satisfying the resolvent equation: $R_\alpha-R_\beta+(\alpha-\beta)R_\alpha R_\beta=0$ for α, $\beta>0$ and sub-Markov property: $0\leqq\alpha R_\alpha f\leqq 1$ if $0\leqq f\leqq 1$. If $u=R_\alpha f$ satisfies

$$(0.4)\qquad (\alpha-L)u=f \qquad \text{for all } f\in L^\infty(E, dx),$$

R_α is called an L-diffusion resolvent. Set $D(A)=\{R_\alpha f\,|\,f\in L^\infty(E, dx)\}$ and define $Au=(\alpha-R_\alpha^{-1})u(=Lu)$. A with the domain $D(A)$ is called the generator of the resolvent.

What we will concern in this paper is to determine all L-diffusion resolvents satisfying (R.1) and (R.2) (stated in §1), or equivalently, to derive the most general boundary condition which prescribes the function family $D(A)$. The L-diffusion resolvent we are going to investigate are very general in the two points; neither the smoothness (or ordinal differentiability) of $u\in D(A)$ nor the smoothness of the boundary is assumed. So it is not our aim that we derive the boundary condition in an explicit form such as Wentzell [22]; our expression of the boundary condition is more like the one for Markov chain such as Feller's and Dynkin's [4].

The problem of boundary condition for diffusion process (or resolvent) has been proposed by W. Feller and has been studied in full details by Feller, Dynkin and Ito-Mckean, in case of one dimensional diffusion process. For the multi-dimensional diffusion process in a bounded domain with smooth boundary, Wentzell [22] has obtained the boundary condition of the function $R_\alpha f$ belonging to $C^2(E)$. Then Ueno has introduced the notion of the *Markov process on the boundary* associated with Wentzell's boundary condition, and Sato-Ueno [19], Courrege and others have investigated the existence of the diffusion process satisfying Wentzell's boundary condition, making use of Ueno's Markov process on the boundary.

All of these works are based, at least analytically, on the semigroup operating in the Banach space consisting of continuous functions. However, in our situation, it is natural and powerful to discuss the problem in L^2-setting. Actually, Fukushima [6] has determined all symmetric Brownian processes over an arbitrary bounded domain, making use of the theory of the Dirichlet space introduced by Beurling-Deny [1] and Doob's representation of the Dirichlet integral of harmonic functions [2].

In this article, the symmetric assumption which Fukushima's work is based on is removed, and his results [6] are extended to arbitrary L-diffusion processes. Especially, our emphasis lies in characterizing the generator Q of the Ueno's Markovian semigroup on the boundary associated with a given L-diffusion resolvent.

The approach of this article would be applied to the Markov chain, too. It is interesting to compare this with Dynkin [4] and Shiga-Watanabe [20].

§1. Outlines and main results.

1.1. Among L-diffusion resolvents, one of the fundamental is so called the minimal L-diffusion resolvent. Let $H_0^1(E)$ be the completion of $C_0^\infty(E)$ by the norm $\|u\|_{H^1(E)}=\left[\int\left[\sum\left(\frac{\partial u}{\partial x_i}\right)^2+u^2\right]dx\right]^{\frac{1}{2}}$. A diffusion resolvent is called minimal if the domain $D(A)$ is included in $H_0^1(E)$. The existence and the uniqueness of the minimal L-diffusion resolvent (except possibly the sub-Markov property) are due to Stampacchia [21]. Let us denote the minimal L-diffusion resolvent by G_α. Then the adjoint G_α^* of G_α in $L^2(E, dx)$ maps $L^\infty(E, dx)$ into $C(E)$ ([21]). We have further,

Theorem 1[1]. *There exists a unique standard diffusion process*[2] (x_t, ζ, P_x), $x\in E$ *such that*

$$G_\alpha f(x)=E_x\left(\int_0^\zeta e^{-\alpha t}f(x_t)\,dt\right) \qquad \text{for all } f\in L^\infty(E, dx). \tag{1.1}$$

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1) In the case $b_i=0$ and ∂E is regular, this theorem has been proved by Kanda [9].

2) For the definition of the standard process, see Dynkin's book or [11]. We use the same notation as [11]. But the σ-field $\mathfrak{F}$ associated with the standard process is omitted. It should be noted that the minimal L-diffusion process satisfies Meyer's Hypothesis (L), by the property (i).

Furthermore,

(i) $G_\alpha f(x)$, $\alpha \geqq 0$ (defined by the right hand of (1.1) if $\alpha=0$) *and its adjoint* G_α^*, $\alpha \geqq 0$ *maps* $L^\infty(E, dx)$ *into* $C(E)$.

(ii) *Both of* $\alpha G_\alpha f$ *and* $\alpha G_\alpha^* f$ *converge to* f *as* $\alpha \to \infty$, *if* $f \in C(E)$.

The above (x_t, ζ, P_x) is called the *minimal L-diffusion process.*

Theorem 1 will be proved in Section 2.

Now, since our expression of the boundary condition is made in the Martin boundary of the minimal L-diffusion process, we will discuss several problems concerned with it. Here we introduce the definition following [14] to fix the notation. Let $g(x, y)$ be Green's function of the minimal L-diffusion such that $G_0 f(x) = \int g(x, y) f(y)\, dy$. We fix a finite measure γ so that both of $\gamma g(y) = \int \gamma(dx)\, g(x, y)$ and $g\gamma(x) = \int g(x, y)\, \gamma(dy)$ are continuous and strictly positive on E. The *Martin exit and entrance kernels* $K(x, y)$ and $K^*(x, y)$ are defined by

$$K(x, y) = \frac{g(x, y)}{\gamma g(y)}, \qquad K^*(x, y) = \frac{g(x, y)}{g\gamma(x)}.$$

Set $fK(y) = \int f(x) K(x, y)\, dx$ and $K^* f(x) = \int K^*(x, y) f(y)\, dy$. The completion of E relative to the weakest uniform topology in which the function family $\{fK \mid f \in C_0(E)\}$ (or $\{K^* f \mid f \in C_0(E)\}$) are all uniformly continuous, is denoted by M(or M^*). Here $C_0(E)$ denotes the function family consisting of $f \in C(E)$ with compact supports in E. The sets $\partial M = M - E$ and $\partial M^* = M^* - E$ are called the *Martin exit boundary* and the *Martin entrance boundary*, respectively, Let $f\tilde{K}$ and $\tilde{K}^* f$ be continuous extensions of fK and $K^* f$ to M and M^*, respectively. Then there exist kernels $K(x, \eta)$ and $K^*(\xi, x)$ such that $\int f(x) K(x, \eta)\, dx = f\tilde{K}(\eta)$ and $\int K^*(\xi, x) f(x)\, dx = \tilde{K}^* f(\xi)$ hold for all $f \in C_0(E)$. They are *Martin exit* and *entrance kernels*. Then we have the following Martin representation: Let u be a nonnegative function such that $\int \gamma(dx)\, u(x) < +\infty$, $\frac{\partial u}{\partial x_i} \in L^2_{loc}(E, dx)$ and $Lu = 0$ (or $L^* u = 0$) hold. Then u has a unique representation

$$(1.2)\quad u(x)=\int_{\partial M}K(x,\eta)\mu(d\eta),\ (\text{or } u(x)=\int_{\partial M^*}K^*(\xi,x)\mu^*(d\xi))$$

making use of the measure μ (or μ^*) concentrated in the extremal points of ∂M (or ∂M^*). We shall call μ (or μ^*) as the canonical measure of u.

1.2. On the Martin spaces M and M^*, let us define new kernels. For $\alpha>0$, set

$$(1.3)\qquad K_\alpha(x,\eta)=K(x,\eta)-\alpha\int G_\alpha(x,dz)K(z,\eta),$$

$$K_\alpha^*(\xi,x)=K^*(\xi,x)-\alpha\int K^*(\xi,z)G_\alpha^*(x,dz).$$

Since $K(\cdot,\eta)$ and $K^*(\xi,\cdot)$ are excessive and co-excessive functions respectively, $K_\alpha(x,\eta)$ and $K_\alpha^*(\xi,x)$ are nonnegative. Furthermore, it can be proved that the sets $M_{en}=\{\xi\in M^*|K_\alpha^*(\xi,\cdot)\not\equiv 0\}$ and $M_{ex}=\{\eta\in M|K_\alpha(\cdot,\eta)\not\equiv 0\}$ are independent of $\alpha>0$ and include E.

Let us now define linear transformations S_α and S_α^* on M_{en} and M_{ex} as follows;

$$(1.4)\qquad S_\alpha f(\xi)=\int_E K_\alpha^*(\xi,x)\,g\gamma(x)f(x)\,dx,$$

$$S_\alpha^* f(\eta)=\int_E \gamma g(x)f(x)K_\alpha(x,\eta)\,dx.$$

Then it turns out that both of S_α and S_α^* are sub-Markov resolvents. The weakest topology in which all S_α-excessive (or S_α^*-excessive) functions are continuous is called the fine topology of the space M_{en} (or M_{ex}).

Now if ξ is a point of E, $S_\alpha f(\xi)$ coincides with $g\gamma(\xi)^{-1}G_\alpha[g\gamma f](\xi)$, so that the resolvent S_α restricted to the set E is the so called $g\gamma$-transform of the original resolvent G_α. Thus if u is an excessive function of G_α, $u/g\gamma$ is an excessive function of the resolvent S_α restricted to E. This fact permits us to define the finely continuous extension of $u/g\gamma$ to the space M_{en}. In particular if u is a potential, it is natural to regard the fine limit of $u/g\gamma$ to the boundary ∂M_{en}

$=M_{en}-E$ as the normal derivative of u, which we denote as $\frac{\partial u}{\partial g}$.

A similar observation enable us to define Naim's Θ-kernel. For $x, y\in E$, we define

$$\Theta(x, y)=\frac{g(x, y)}{g\gamma(x)\gamma g(y)}. \tag{1.5}$$

Then this kernel Θ has a uniquely fine continuous extension to $M_{en}\times M_{ex}$.

These facts will be proved in §4.2 and §4.3.

1.3. In order to state the boundary condition, it is necessary to identify the exit and extrance boundary points. Let $\eta\in\partial M_{ex}\equiv M_{ex}-E$ and $\xi\in\partial M_{en}\equiv M_{en}-E$. If the restriction of each fine neighbourhood of η to the set E is the restriction of a suitable fine neighbourhood of ξ, and vice versa, the two points η and ξ are identified. Then every $\eta\in\partial M_{ex}$ is identified at most with one point $\xi\in\partial M_{en}$, and vice versa.

Now let h_0 be a bounded and uniformly positive function such that $L^*h_0=0$ and let μ_0^* (concentrated in ∂M^*) be the canonical measure of h_0. The canonical measure of the constant function 1 is denoted by μ_0 (concentrated in ∂M). It can be proved that $\mu_0(\partial M-\partial M_{ex})=0$ and $\mu_0^*(\partial M-\partial M_{en})=0$. Moreover we have

Theorem 2. *μ_0-almost all points of ∂M_{ex} are identified with μ_0^*-almost all points of ∂M_{en}. Furthermore, μ_0 and μ_0^* are mutually absolutely continuous after this identification.*

The above identified set is denoted by ∂E. Then if u and h are represented as

$$u(x)=\int_{\partial E}K(x, \eta)\bar{u}(\eta)\mu_0(d\eta), \qquad h(x)=\int_{\partial E}K^*(\xi, x)\bar{h}(\xi)\mu_0^*(d\xi), \tag{1.6}$$

u and $\frac{h}{h_0}$ have the (fine) limits $\bar{u}(\xi)$ and $\bar{h}(\xi)$ at μ_0-almost all point as x tends to ξ along with the fine topologies of M_{en} and M_{ex}. We shall call this $\bar{u}$ (or $\bar{h}$) as the boundary value of $u\left(\text{or } \frac{h}{h_0}\right)$.

By virtue of Theorem 2, we are able to generalize the Doob representation of the Dirichlet integral of harmonic functions. Let h be a strictly positive function represented as (1.6). We define the bilinear form $\tilde{D}^h$ as

$$\tilde{D}^h(\varphi, \psi) = \frac{1}{2}\iint \Theta(\xi, \eta)(\psi(\xi)-\psi(\eta))(\varphi(\xi)-\varphi(\eta))\,\bar{h}(\xi)\,\mu_0^*(d\xi)\,\mu_0(d\eta). \tag{1.7}$$

The following theorem is a generalization of Doob's [2].

Theorem 3. *Let h be a strictly positive function represented as (1.6) with boundary value $\bar{h}$. Then every bounded function u such that $Lu=0$ and $\int \sum a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, h\, dx<\infty$ has the boundary value $\bar{u}$ belonging to $L^\infty(\partial E, \mu_0)$ and*

$$\int \sum a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, h\, dx = \tilde{D}^h(\bar{u}, \bar{u}).$$

Theorem 2 and 3 will be proved in Section 4.

1.4. Let us now return to our main subject. Let R_α be an L-diffusion resolvent. We will assume that this R_α satisfies the following two conditions.

(R.1) *The adjoint R_α^* in $L^2(E, dx)$ satisfies $(\alpha - L^*)R_\alpha^* f=f$.*

(R.2) *There exists a strictly positive function h such that $\frac{\partial h}{\partial x_i} \in L^2(E, dx)$ and $\int h(x)\,R_\alpha f(x)\,dx = \int h(x) f(x)\,dx$ holds for all f.*

The measure $m(dx)=h dx$ is called an *invariant measure* of R_α.

It can be proved that the above h is represented as (1.6) with $\bar{h} \in L^1(\partial E, \mu_0^*)$. Define $d\nu = \frac{1}{2}(\bar{h}\,d\mu_0^* + d\mu_0)$ and denote by $\mathfrak{B}'$ the least σ-field in which the family of functions $\{\bar{u} \mid u=R_\alpha f, f\in L^\infty(E, dx)\}$[3)]

3) u is written as $u=G_\alpha f + H_\alpha \bar{u}$ with bounded measurable function $\bar{u}$. It can be shown that the fine limit of u to the boundary exists and equals $\bar{u}$ a.e. μ_0. See footnote 16).

are measurable and by $L^2(\partial E, \mathfrak{B}', \nu)$ the subspace of $L^2(\partial E, \nu)$ consisting of all $\mathfrak{B}'$-measurable functions. We denote the orthogonal projection from $L^2(\partial E, \nu)$ to $L^2(\partial E, \mathfrak{B}', \nu)$ as P. Then we have

Theorem 4. *Let R_α be a conservative*[4] *L-diffusion resolvent with* (R.1) *and* (R.2). *Then there exists a unique linear operator Q of $L^2(\partial E, \mathfrak{B}', \nu)$ admitting the following properties*:

(Q.1) *Q is a generator of a strongly continuous and conservative Markovian semigroup in $L^2(\partial E, \mathfrak{B}', \nu)$ with ν as its invariant measure.*

(Q.2) $D(Q) \subset D(\tilde{D}^h)$ and

$$\int \varphi^+ Q\varphi \, d\nu + \tilde{D}^h(\varphi^+, \varphi^-)$$

$$-\int \Theta(\xi, \eta)\, \varphi^+(\xi)\varphi^-(\eta)\, \bar{h}(\xi)\, \mu_0(d\xi)\, \mu_0(d\eta) \leqq 0$$

holds for all $\varphi \in D(Q)$, where $\varphi^+ = \max\{\varphi, 0\}$, $\varphi^- = \varphi - \varphi^+$ and $D(\tilde{D}^h) = \{\varphi \in L^2(\partial E, \nu) \mid \tilde{D}^h(\varphi, \varphi) < \infty\}$.

(Q.3) *The generator's domain of the resolvent R_α is characterized as*

$$D(A) = \{u \in C(E) \mid Lu \in L^\infty(E, dx),\ \bar{u} \in D(Q)$$

$$\text{and } Q\bar{u} + P\frac{\partial}{\partial g}(u - H\bar{u}) = 0\},$$

where $H\bar{u}$ is the harmonic function taking value $\bar{u}$ at the boundary.

Conversely we have the following assertion.

Theorem 5. *Let h be a strictly positive function represented as* (1.6) *and $\bar{h}$, its boundary value. Let $\mathfrak{B}'$ be a sub σ-field of the Borel field of ∂E. Suppose we are given a linear operator Q of $L^2(\partial E, \mathfrak{B}', \nu)$ which fulfills* (Q.1) *and* (Q.2). *Then there is a unique conservative L-diffusion resolvent R_α whose boundary condition is* (Q.3). *Furthermore, this R_α satisfies* (R.1) *and* (R.2).

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4) R_α is called conservative if $\alpha R_\alpha 1 = 1$ holds for all α.

Theorems 4 and 5 will be proved at Section 6.

1.5. Theorems 4 and 5 show that there is a one to one correspondence between conservative L-diffusion resolvent satisfying (R.1) and (R.2), and linear operator Q on the boundary ∂E satisfying (Q.1) and (Q.2) through the relation (Q.3).

Condition (R.1) means, roughly, that the sample paths of the Markov process associated with the resolvent R_α has no jumps from the boundary ∂E to the inside E. However, we do not investigate the property of the sample paths in this paper.

The operator Q is the generator of the Ueno's Markov process on the boundary. But in case that $\mathfrak{B}'$ is not equal to the topological Borel field of ∂E, the Markov process (semigroup) on the boundary is defined not on the boundary ∂E, but on a suitable identification (or partition) of ∂E subject to the σ-field $\mathfrak{B}'$.

Let us investigate the meaning of (Q.2) and (Q.3) in some special cases. In what follows we assume that the coefficients b_i of the operator L are identically 0. Let us denote by $D(\tilde{D})$ the set of all $\varphi \in L^2(\partial E, \mu_0)$ such that $\tilde{D}(\varphi, \varphi) < \infty$[5]. Then $(D(\tilde{D}), \tilde{D})$ is a Dirichlet space in the sense of Beurling-Deny [1], namely $D(\tilde{D})$ is a vector lattice and $\tilde{D}(U\varphi, U\varphi) \leqq \tilde{D}(\varphi, \varphi)$ holds for $\varphi \in D(\tilde{D})$, where $U\varphi = \varphi^+ \wedge 1$. For the Dirichlet space $(D(\tilde{D}), \tilde{D})$, there corresponds a unique generator denoted by $\frac{\partial H}{\partial g}$ of a strongly continuous Markovian semigroup in $L^2(\partial E, \mu_0)$ such that $-\tilde{D}(\varphi, \psi) = \int \frac{\partial H}{\partial g} \varphi \cdot \psi \, d\mu_0$ holds for $\varphi \in D\left(\frac{\partial H}{\partial g}\right)$ and $\varphi \in D(\tilde{D})$. $\frac{\partial H}{\partial g} \varphi$ is an analogue of the normal derivative of the harmonic function $H\varphi$ taking the value φ on the boundary. We define the normal derivative analogue for the function u such that $Lu \in L^\infty(E, dx)$ and $\bar{u} \in D\left(\frac{\partial H}{\partial g}\right)$ as follows.

$$\frac{\partial u}{\partial g} = \frac{\partial H}{\partial g} \bar{u} + \frac{\partial}{\partial g} (u - H\bar{u}), \qquad \bar{u} \in D\left(\frac{\partial H}{\partial g}\right).$$

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5) When $h=1$, we omit superfix h in D^h etc.

Now let R_α be an L-diffusion resolvent satisfying (R.2) with $h=1$. Suppose in addition that the σ-field $\mathfrak{B}'$ on the boundary associated with R_α coincides with the topological Borel field. Then the boundary condition stated in (Q.3) is rewritten as

$$Q'\bar{u}+\frac{\partial u}{\partial g}=0 \qquad \text{for } \bar{u}\in D\Big(\frac{\partial H}{\partial g}\Big)\cap D(Q),$$

where $Q'=Q-\dfrac{\partial H}{\partial g}$.

The above operator Q' has an interesting property. It can be shown that (Q.2) is equivalent to

$$\text{(Q}'.2) \qquad \int(\varphi-c)^{-}Q\varphi\, d\mu_0+\tilde{D}((\varphi-c)^{-},\varphi)\leqq 0$$

$$^{\forall}c\leqq 0,\ ^{\forall}\varphi\in D(D),$$

where c are constant functions. Obviously the above inequality implies

$$\int(\varphi-c)^{+}Q'\varphi\, d\mu_0\leqq 0 \qquad \text{for } \varphi\in D(Q)\cap D\Big(\frac{\partial H}{\partial g}\Big).$$

Generally, the operator satisfying the above inequality is called completely dispersive. It is known that the generator of a contraction and sub-Markov semigroup is completely dispersive and conversely if a completely dispersive operator becomes a generator of a semigroup, the semigroup has to be sub-Markov (See [12]).

Keeping this in mind, we have

Corollary to Theorem 5. *Let Q' be a generator of a conservative Markovian semigroup in $L^2(\partial E,\mu_0)$ with μ_0 as its invariant measure. Suppose that $D(Q')\cap D\Big(\dfrac{\partial H}{\partial g}\Big)$ and range $R\Big(\lambda-Q'-\dfrac{\partial H}{\partial g}\Big)$ are both dense in $L^2(\partial E,\mu_0)$. Then the smallest closed extension of $Q'+\dfrac{\partial H}{\partial g}$ denoted by Q exists and this Q posesses all properties* (Q.1) *and* (Q.2).

It is possible to have the similar explanation of the generator Q even in the case $b_i\neq 0$. These problems are discussed in Section 7.

§2. The minimal L-diffusion process.

Let us first introduce several notations. $C^1(E)$ stands for the set of all continuously differentiable functions in E. We introduce norms $\| \|_{H^{1,p}(E)}$, $p \geqq 1$ to $C^1(E)$ as

$$\|u\|_{H^{1,p}(E)}=\left(\int\left(\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^p+|u|^p\right)dx\right)^{\frac{1}{p}}$$

and denote the completion of $C^1(E)$ by these norms as $H^{1,p}(E)$. u is said to belong to $H^{1,p}_{\mathrm{loc}}(E)$ if $u\in H^{1,p}(U)$ for any open set U such that $\bar{U}\subset E$. $H_0^{1,p}(E)$ stands for the closure of $C_0^\infty(E)$ in $H^{1,p}(E)$. The dual space of $H_0^{1,p}(E)$ is denoted by $H^{-1,q}(E)$, where q is the number such that $p^{-1}+q^{-1}=1$. The norm of $H^{-1,q}(E)$ is denoted as $\| \|_{H^{-1,q}(E)}$. When $p=2$, the super-index p is often dropped and written as $H^1(E)$ etc. The space of continuous functions vanishing at ∞ is denoted by $C_\infty(E)$ and the space of bounded continuous functions is denoted as $C_b(E)$. The norm of $C_\infty(E)$ or $C_b(E)$ is defined as the spremum of the absolute value of the function.

2.1. Let us define a bilinear form in $H_0^1(E)\times H_0^1(E)$ as

$$B(u, v)=\int_E\left(\sum_{i,j=1}^{n}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}-u\sum_{i=1}^{n}b_i\frac{\partial v}{\partial x_i}\right)dx$$

and $B_\alpha(u, v)$ as $B(u, v)+\alpha(u, v)$, where $(,)$ is the inner product of $L^2(E, dx)$. Then B is continuous, i.e. there exists a positive constant K such that $|B(u, v)|\leqq K\|u\|_{H^1(E)}\|v\|_{H^1(E)}$. Further, $B_\alpha(u, v)$ is coercive for sufficiently large α, i.e. there exists positive constants β_0 and k such that $B_{\beta_0}(u, u)\geqq k\|u\|^2_{H^1(E)}$. (See [21]). Suppose now $\alpha>\beta_0$. Then for each $T\in H^{-1}(E)$ there exists a unique u of $H_0^1(E)$ which satisfies

$$B_\alpha(v, u)=\langle T, v\rangle \qquad \text{for all } v\in H_0^1(E), \tag{2.1}$$

by the Lax-Milgram theorem ([23], p. 92). Conversely for a given $u\in H_0^1(E)$ there exists a unique $T\in H^{-1}(E)$ which satisfies the above equality. Write the mapping $T\in H^{-1}(E)\to u\in H_0^1(E)$ as $u=G_\alpha T$.

Then G_α is a continuous, one-one and onto linear mapping from $H^{-1}(E)$ to $H_0^1(E)$. We prove this fact for arbitrary $\alpha>0$, namely,

Lemma 2.1. *For each $\alpha>0$, there exists a unique one-one, onto and continuous linear operator G_α from $H^{-1}(E)$ to $H_0^1(E)$ which satisfies* (2.1). *Furthermore, G_α restricted to $L^\infty(E, dx)$ is sub-Markov.*

Proof. It is well known that $H_0^1(E)$ is a vector lattice and is easily seen to satisfies

$$B((u-c)^+, u\wedge c)=0 \qquad \text{for all } u\in H_0^1(E) \text{ and } c\geqq 0.$$

Hence there exists a strongly continuous sub-Markov semigroup T_t with $\|T_t\|\leqq e^{\beta_0 t}$ such that $\int e^{-\alpha t}T_t f\,dt=G_\alpha f$ for $\alpha>\beta_0$ (See [12]). Then the Laplace transform $\int e^{-\alpha t}T_t f\,dt$ is well defined and is in $L^\infty(E, dx)$ for all $\beta_0\geqq\alpha>0$ if $f\in L^\infty(E, dx)$. We shall again write this as $G_\alpha f$. Clearly it satisfies the resolvent equation $G_\alpha f-G_\beta f+(\alpha-\beta)G_\beta G_\alpha f=0$ for $\beta>\beta_0\geqq\alpha>0$. Hence the range of $I+(\alpha-\beta)G_\beta$ is dense in $L^2(E, dx)$. On the other hand, since G_β, $\beta>\beta_0$ is a compact operator in $L^2(E, dx)$[6], the range of $I+(\alpha-\beta)G_\beta$ is closed. This and the above argument show that the range of $I+(\alpha-\beta)G_\beta$ coincides with the whole space $L^2(E, dx)$. Consequently, the equation $u+(\alpha-\beta)G_\beta u=G_\beta T$ has a unique solution u for any $T\in H^{-1}(E)$ by the Riesz-Schauder theorem. It is easily seen that this u is the unique solution of (2.1). We denote this u as $G_\alpha T$. The "one-one and onto" property is immediate from the resolvent equation $G_\alpha T-G_\beta T+(\alpha-\beta)G_\alpha G_\beta T=0$ for $\beta>\beta_0\geqq\alpha>0$.

The following Stampacchia's result is fundamental in our later discussion.

Theorem (*Stampacchia* [21]). (1) *Let u be a solution of*

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6) By the coersiveness of $B_\alpha(u, v)$, we have
$$K\|G_\alpha f\|^2_{H^1(E)}\leqq B_\alpha(G_\alpha f, G_\alpha f)=(G_\alpha f, f)\leqq\|f\|\,\|G_\alpha f\|_{H^1(E)}$$
which process $\|G_\alpha f\|_{H^1(E)}\leqq K^{-1}\|f\|$. The compactness of G_α follows from Rellich's theorem.

$(\alpha-L)u=T$ belonging to $H^1_{\mathrm{loc}}(E)$. If $T\in H^{-1,p}(E)$ with some $p>n$, then u is (Hölder) continuous in E. (2) *G_α is a continuous linear operator from $H^{-1,p}(E)$ into $C_b(E)$. In particular, if the boundary ∂E is regular*[7], *$G_\alpha T$ $(T\in H^{-1,p})\in C_\infty(E)$.*

Appealing the above theorem, we have

Proposition 2.1. *Suppose that ∂E is regular. Then G_α is a sub-Markov resolvent in $C_\infty(E)$ whose range is dense in $C_\infty(E)$.*

Proof. The assertions except the denseness of the range are immediate from Lemma 2.1 and Stampacchia's theorem. Let u be of $C_0^\infty(E)$. Then Lu belongs to $H^{-1,p}(E)$ because a_{ij} are bounded and $b_i\in L^p(E, dx)$. Therefore $u=G_\alpha(\alpha-L)u$ belongs to the range $G_\alpha(H^{-1,p}(E))$. This proves that the range $G_\alpha(H^{-1,p}(E))$ is dense in $C_\infty(E)$. On the other hand since $C_\infty(E)$ is dense in $H^{-1,p}(E)$ and since G_α is a continuous mapping from $H^{-1,p}(E)$ into $C_\infty(E)$, the range $G_\alpha(C_\infty(E))$ has to be dense in $C_\infty(E)$.

The above G_α is called the *minimal L-diffusion resolvent.*

2.2. We shall prove Theorem 1 in the case where ∂E is regular. By virtue of Proposition 2.1 and the Hille-Yosida theorem, there exists a positive, contraction and strongly continuous semigroup of linear operators T_t, $t>0$ in $C_\infty(E)$ whose Laplace transform is the minimal L-diffusion resolvent. Thus there exists a Hunt process (x_t, ζ, P_x), $x\in E$ associated with the above semigroup. In order to prove the continuity of the sample paths x_t, we require

Lemma 2.2. *Let $T\in E^{-1,p}(E)$, $p>n$, and $u=G_\alpha T$. Let U be an open set such that $<T, \varphi>=0$ for all $\varphi\in C_0^\infty(U)$. Then $u(x)=E_x(e^{-\alpha\tau_U}u(x_{\tau_U}))$ holds for all x, where τ_U is the hitting time for the set U^c.*

Proof. Let us choose $T_n\in L^p(E, dx)$ converging to T in $H^{-1,p}(E)$ such that $<T_n, \varphi>=0$ for all $\varphi\in C_0^\infty(U)$. Then for $u_n(x)=G_\alpha T_n$ we

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7) ∂E is of C^1-class, for example.

have $u_n(x)=E_x(e^{-\alpha\tau_U}u_n(x_{\tau_U}))$ by Dynkin's formula. Making n tend to ∞, $u_n(x)$ converges to $u(x)$ uniformly by Stampacchia's theorem and hence the assertion holds.

We now prove the continuity of the sample paths x_t following Kanda [9]. Let U be an open set such that $\bar{U}\subset E$ and V, an open set such that $V\subset E$ and $V\cap\bar{U}=\phi$. Then there exists $u=G_\alpha T\in C_0^\infty(E)$ $\subset H^{-1,p}(E)$ such that $u(x)=0$ in U and $u(x)>0$ in V. Then since $T=(\alpha-L)u=0$ in U, we have $u(x)=E_x(e^{-\alpha\tau_U}u(x_{\tau_U}))$. This proves $P_x(x_{\tau_U}\in V,\ \tau_U<\infty)=0$. Since V is arbitrary, we see

$$P_x(x_{\tau_U}\in E-U,\ \tau_U<\infty)=0 \text{ or } P_x(x_{\tau_U}\in E-\partial U,\ \tau_U<\infty)=0.$$

This concludes that the sample paths are continuous.

The following proposition asserts that the minimal L-diffusion process is transient.

Proposition 2.2. *$Gf(x)=E_x\left(\int_0^\zeta f(x_t)\,dt\right)$ is well defined and belongs to $C_\infty(E)$ if $f\in L^p(E, dx)$, $p>n$.*

Proof. Suppose for a moment that for each $\alpha>0$,

$$u-\alpha G_\alpha u=G_\alpha f \tag{2.2}$$

has a unique solution $u\in L^2(E, dx)$ for a given $f\in L^p(E, dx)$, $f\geqq 0$. Then u belongs to $H_0^1(E)$ and satisfies $-Lu=f$. (Hence u does not depend on α). Since u is bounded and continuous by Stampacchia's theorem, $\alpha G_\alpha u$ is bounded in α. Making α tend to 0 in (2.2), we see that $Gf(x)$ is bounded and hence it coincides with u.

The existence of the solution of (2.2) is equivalent to the uniqueness of the solution of the homogeneous equation $u-\alpha G_\alpha u=0$ (i.e., $u=0$), by the Riesz-Schauder theorem. Let u be a solution of the homogeneous equation. Then $Lu=0$ is satisfied, so that $u(x)=E_x(u(x_{\tau_K}))$ holds for all compact set K[8]. Letting $K\uparrow E$, we see $u=0$ because

8) The condition $u=\alpha G_\alpha u$ is equivalent to $u(x)=E_x(u(x_t))$ holds. On the other hand, it is known that $u(x)\geqq E_x(u(x_{t\wedge\tau_K}))\geqq E_x(u(x_t))$. Making t tend to infinity, we have $u(x)=E_x(u(x_{\tau_K}))$.

$u \in C_\infty(E)$.

2.3. Before we proceed to the proof of the existence of the minimal L-diffusion process in arbitrary open set, we develope some potential theory in case where ∂E is regular.

Let us define the adjoint operator G_α^* of G_α as $<G_\alpha T, S> = <T, G_\alpha^* S>$, where $T, S \in H^{-1}(E)$. Then G_α^* is one-one and onto mapping from $H^{-1}(E)$ to $H_0^1(E)$ and $u = G_\alpha^* T$ is a unique solution of $(\alpha - L^*) u = T$. Furthermore, we have

Proposition 2.3. (1) *G_α is a continuous mapping from $H^{-1,p}(E)$ into $C_\infty(E)$ if $p > n$.* (2) *The range $G_\alpha^*(C_\infty(E))$ is dense in $C_\infty(E)$.*

The first half of the above proposition is again due to [21]. The latter half can be proved similarly as that of Proposition 2.1.

We have thus shown that the minimal L-diffusion process satisfies Hypothesis (B) of [14] and hence all discussions of [14] is applicable to the minimal L-diffusion. We state here some of them. A nonnegative function u defined on E is called *α-excessive* if $\beta G_{\alpha+\beta} u(x) \leqq u(x)$ holds for all $\beta > 0$ and $\beta G_{\alpha+\beta}$ converges to u as $\beta \to \infty$. The function u is called *α-co-excessive* if G_α is replaced by G_α^* in the above definition. In case $\alpha = 0$, α-(co)-excessive function is called *(co)-excessive.*

a) *Existence of the potential kernels. For each $\alpha \geqq 0$, there exists a unique kernel $g_\alpha(x, y)$ which satisfies*

(i) $G_\alpha f(x) = \int g_\alpha(x, y) f(y)\, dy. \qquad G_\alpha^* f(x) = \int g_\alpha(y, x) f(y)\, dy.$

(ii) *$g_\alpha(x, y)$ is α-excessive function of x for each y and, α-co-excessive function of y for each x.*

In case that $\alpha = 0$, $g_\alpha(x, y)$ is written as $g(x, y)$.

We next define harmonic and superharmonic functions. Let A be a Borel set and let σ_A be the hitting time for the set A. The α-distribution of x_{σ_A}: $H_A^\alpha(x, dy) = E_x(e^{-\alpha\sigma_A}; x_{\sigma_A} \in dy)$ is called α-harmonic measure for the set A. A function u defined in E is called *α-superharmonic* in V if it is lower semi-continuous, finite from below, finite a.e. dx and satisfies

(2.3) $$u(x) \geqq H^{\alpha}_{U^c} u(x) \quad \left(=\int H^{\alpha}_{U^c}(x, dy) u(y)\right)$$

for all open sets U such that $\bar{U} \subset V$. If the inequality (2.3) holds for $\alpha=0$, u is called *superharmonic* in V. When the equality holds in (2.3) for all such U, u is called *α-harmonic* (or *harmonic* if $\alpha=0$). It is well known that a nonnegative function u is α-superharmonic if and only if it is α-excessive.

Let μ be a positive measure with finite mass on each compact set. The function $G_\alpha(\mu)=\int g_\alpha(x, y)\mu(dy)$ is called an α-potential if it is finite almost everywhere. α-potential is α-superharmonic.

b) *Riesz decomposition* ([14]). *Let u be an α-superharmonic function. Then u is decomposed to the sum of an α-harmonic function and an α-potential if and only if there exists an α-harmonic function dominated by u. Furthermore such decomposition is unique.*

c) *Direct decomposition of $H^1(E)$. Let H_α or H^*_α be the set of all u of $H^1(E)$ such that $(L-\alpha)u=0$ or $(L^*-\alpha)u=0$, respectively. Then*

$$H^1(E)=H_\alpha \oplus H^1_0(E)=H^*_\alpha \oplus H^1_0(E).$$

Furthermore, $B_\alpha(u, v)=0$ is satisfied for $u \in H^*_\alpha$ and $v \in H^1_0(E)$, or $v \in H_\alpha$ and $u \in H^1_0(E)$. In fact, let $u \in H^1(E)$ and $v=G_\alpha(\alpha-L)u$ ($(\alpha-L)u \in H^{-1}(E)$), then $v \in H^1_0(E)$ and $u-v \in H_\alpha$ as is easily seen. It is also obvious that $H_\alpha \cap H^1_0(E)=0$. The latter direct decomposition can be proven similarly. The orthogonality relation $B_\alpha(u, v)=0$ is immediate from the definition of H_α and H^*_α.

2.4. A function $u \in H^{1,1}_{loc}(E)$ is called an *α-subsolution* if $B_\alpha(v, u) \geqq 0$ holds for all $v \in C^\infty_0(E)$ such that $v \geqq 0$.

Theorem 2.1. (cf. [8]). 1) *Every α-potential is an α-subsolution belonging to $\bigcap_{1<q<\frac{n}{n-1}} H^{1,q}_{loc}(E)$*[9]. *Conversely every α-subsolution belonging to $\bigcap_{1<q<\frac{n}{n-1}} H^{1,q}_{loc}(E)$ is decomposed to the sum of an α-potential and a*

9) n is the dimension of the space E.

solution of $(L-\alpha)v=0$. 2) *A function* u *is* α*-harmonic if and only if it is a solution of* $(L-\alpha)u=0$ *belonging to* $H^1_{\mathrm{loc}}(E)$.

Proof. We divide the proof into several steps.

1^0. Suppose u is an α-potential represented as $\int g_\alpha(x, y)\,\mu(dy)$ with finite measure μ. We prove that $u \in \bigcap_{1<q<\frac{n}{n-1}} H^{1,q}(E)$ and $B_\alpha(v, u) = \int v\,d\mu$ holds for $v \in C_0^\infty(E)$. Choose $f_n \geqq 0$ in $L^\infty(E, dx)$ such that $\mu_n = f_n\,dx$ converges weakely to μ. Then

$$\left|\int G_\alpha(\mu_n)(x)\,g(x)\,dx\right| = \left|\int G_\alpha^* g(x)\,\mu_n(dx)\right| \leqq \sup|G_\alpha^* g|\,\mu_n(E).$$

Since G_α^* is a continuous linear operator from $H^{-1,p}(E)$ into $C_b(E)$ by Proposition 2.3, there exists $K>0$ such that $\sup\,|G_\alpha^* g| \leqq K\|g\|_{H^{-1,p}}$, where $p=(1-q^{-1})^{-1}>n$. This proves that $\|G_\alpha(\mu_n)\|_{H^{1,q}(E)}$ is bounded in n. Consequently a subsequence of $G_\alpha(\mu_n)$ converges weakly to u' in $H^{1,q}(E)$. It is easy to see that u' coincides with u. Making n tend to $+\infty$ in $B_\alpha(v, G_\alpha(\mu_n)) = \int v\,d\mu_n$, we obtain $B_\alpha(v, u) = \int v\,d\mu$.

2^0. Let us prove that the assertion of 1^0 is valid for any α-potential $u=G_\alpha(\mu)$, if we replace $u \in \bigcap_{1<q<\frac{n}{n-1}} H^{1,q}(E)$ by $u \in \bigcap_{1<q<\frac{n}{n-1}} H^{1,q}_{\mathrm{loc}}(E)$. It is known in [14] that the potential measure μ has finite mass on each compact set. Decompose the potential as $u=u_1+u_2$, where $u_1(x) = \int_V g_\alpha(x, y)\,\mu(dy)$ and V is an open set such that $\bar{V} \subset E$. It suffices to verify that u_2 is in $H^{1,q}_{\mathrm{loc}}(E)$ for $q<\dfrac{n}{n-1}$. Notice that $H^\alpha_{\partial V} u_2$ coincides with u_2 in V and that it is an α-potential with finite mass in ∂V([14]). Then $H^\alpha_{\partial V} u_1(x)$ is of $H^{1,q}(E)$ by 1^0. This shows that $u \in H^{1,q}(V)$. Since V is arbitrary, we obtain the assertion.

3^0. Suppose now u is α-harmonic. We may assume without loss of generality that u is nonnegative. Notice that $u(x) = H^\alpha_{\partial V} u(x)$ holds in V and $H^\alpha_{\partial V} u$ is an α-potential. Then we have

$$B_\alpha(v, u) = B_\alpha(v, H^\alpha_{\partial V} u) = \int_{\partial V} v\,d\mu = 0 \qquad \text{if } v \in C_0^\infty(V),$$

where μ is the potential measure of $H^\alpha_{\partial V} u$. Since this holds for arbitrary V with $\bar{V}\subset E$, we see that $u\in H^{1,p}_{loc}(E)$ and satisfies $(L-\alpha)u=0$. The fact that $u\in H^1_{loc}(E)$ follows from $H^\alpha_{\partial V}u\in H^1_{loc}(V)$ (See [21, §9]).

4^0. Let u be a solution of $(L-\alpha)u=0$ belonging to $H^1_{loc}(E)$. We first assume that u is nonnegative and belongs to $H^1(E)$. Since $B_\alpha(v, \beta G_{\alpha+\beta}u)=(v, u-\beta G_{\alpha+\beta}u)$ and $B_\alpha(v, u)=0$ holds for $v\in H^1_0(E)$, we have

$$B_\alpha(v, \beta G_{\alpha+\beta}u-u)=(v, u-\beta G_{\alpha+\beta}u).$$

Set $w=\beta G_{\alpha+\beta}u-u$. Notice that w^+ belongs to $H^1_0(E)$. Then the above equality implies $B_\alpha(w^+, w)=-\beta\|w^+\|^2$. On the other hand, since $B_\alpha(w^+, w)=B_\alpha(w^+, w^+)\geqq(\alpha-\beta_0)\|w^+\|^2$, we have $w^+=0$ or equivalently, $u\leq\beta G_{\alpha+\beta}u$ if $\alpha+\beta>\beta_0$. This proves that u is α-excessive or α-superharmonic.

Assume now that the above u is not α-harmonic. Then it has a nontrivial potential in the Riesz decomposition. Then $B_\alpha(v, u)=\int v\,d\mu$ hold by 1^0 and 2^0, where μ is the potential measure. This contradicts $(L-\alpha)u=0$.

Let us consider the general case. Let U be an open set with regular boundary in E and let G^U_α be the minimal diffusion resolvent in U. Then u belongs to $H^1(E)$ and is bounded from below. Set $c=\min_{x\in\partial U}u(x)$ and define $v=u-cH^\alpha_{\partial V}1$. Then v satisfies $(\alpha-L)v=0$ in U and is nonnegative in U. Then the argument of the above can be applied to v and we see that v is α-harmonic in U, proving that u is α-harmonic relative to the minimal L-diffusion in U. Then Proposition 2.4 of the next small section shows that u is α-superharmonic relative to the diffusion considered.

5^0. Let u be an α-subsolution belonging to $\bigcap_{1<q<\frac{n}{n-1}}H^{1,q}_{loc}(E)$. We may assume without loss of generality that $u\in\bigcap_{1<q<\frac{n}{n-1}}H^{1,q}(E)$. We prove that u is decomposed into the sum of an α-potential and a solution of $(L-\alpha)v=0$. Let $p_0>n$ be a number such that $b_i\in L^{p_0}(E, dx)$ and q_0, the conjugate of p_0. Choose q so that $q_0<q$ and $q/q_0<n(n-1)^{-1}$, and let p be the conjugate of q. Then

$$|B_\alpha(v,u)| \leqq \|v\|_{H^{1,p}(E)}\|u\|_{H^{1,q}(E)}$$
$$+\sum \|b_i\|_{L^{p_0}(E,dx)}\|u\|_{H^{1,p}(E)}\|v\|_{H^{1,p}(E)} \leqq k\|v\|_{H^{1,p}(E)}\|u\|_{H^{1,q}(E)}.$$

Hence $F(v)\equiv B_\alpha(v, u)$ is a positive linear functional on $H^{1,p}(E)$. On the other hand, $H^{1,p}(E)$ is included in $C_\infty(E)$ densely and the norm $\| \ \|_{H^{1,p}}$ is dominated by the supremum norm up to constant multiple, by Sobolev's lemma. Hence $F(v)$ is extended to a continuous linear functional on $C_\infty(E)$. Then there exists a unique positive measure μ such that $F(v)=\int v\,d\mu$ holds for all $v\in C_\infty(E)$.

The function $G_\alpha(\mu)=\int g_\alpha(x, y)\mu(dy)$ is finite almost everywhere, because $\int G_\alpha(\mu)\,g\,dx=\int G_\alpha^* g(y)\mu(dy)$ holds for $g\in L^\infty(E, dx)$ and the right hand is finite. Hence $G_\alpha(\mu)$ is a potential. Since $B_\alpha(v, G_\alpha(\mu)) =\int v\,d\mu$ holds by 1^0, $u-G_\alpha(\mu)$ satisfies $(L-\alpha)(u-G_\alpha(\mu))=0$. This completes the proof.

Corollary. *α-harmonic function is (Hölder) continuous.*

Proof is obvious since the solution of $(L-\alpha)u=0$ belonging to $H^1_{\mathrm{loc}}(E)$ is Hölder continuous by Stampacchia's theorem.

2.5. Let V be an open set of E such that $\bar{V}\subset E$. We denote by τ_V the hitting time for the set $\bar{V}^c$. Then the process terminated at τ_V, denoted by (x_t, τ_V, P_x), $x\in V$ is again a standard diffusion process. On the other hand, to the operator L restricted to $V\subset E$, we can associate the minimal L-diffusion resolvent G_α^V. That is, $u=G_\alpha^V f$ is defined as a unique solution of $(\alpha-L)u=f$ in V belonging to $H^1_0(V)$.

Proposition 2.4. G_α^V coinsides with the resolvent of (x_t, τ_V, P_x), $x\in V$.

Proof. Set $G'_\alpha f(x)=E_x\Big(\int_0^{\tau_V} e^{-\alpha t}f(x_t)\,dt\Big)$. We have to prove $G_\alpha^V =G'_\alpha$. Since $G'_\alpha f(x)=G_\alpha f(x)-E_x(e^{-\alpha\tau_V}G_\alpha f(x_{\tau_V}))$ holds and the latter term of the right hand is α-harmonic in V, $G'_\alpha f$ and $G_\alpha f$ are both solutions of $(L-\alpha)u=-f$ in V. Consequently, $w=G_\alpha f-G'_\alpha f$ satisfies $(L-\alpha)w=0$ in V, i.e, w is α-harmonic in V. Furthermore, if ∂V is regular, $w=0$ on ∂V because both of $G_\alpha f$ and $G'_\alpha f$ are 0 on ∂V. Therefore w is identically 0 in V.

In case that ∂V is not regular, some modification is necessary. Let U be an open set with regular boundary such that $\bar{U}\subset V$ and let G^U_α be the corresponding minimal L-diffusion resolvent. Then it is easily seen that $G^U_\alpha f\leqq G^V_\alpha f$ if $f\geqq 0$. Choose U_n so that $\bigcup_n U_n=V$. Then $G^{U_n}_\alpha f$ increase to $G'_\alpha f$ because τ_{U_n} increase to τ_V. On the other hand, it is easy to see that $G^{U_n}_\alpha f$, $n\geqq 1$ are bounded set in $H^1_0(V)$. Therefore $G'_\alpha f$ coincides with a weak limit of a subsequence of $G^{U_n}_\alpha f$ in $H^1_0(V)$. This proves that $G'_\alpha f=G^V_\alpha f\in H^1_0(V)$. The proof is completed.

We shall now give the

Proof of Theorem 1. Let $\tilde{E}$ be a bounded domain with regular boundary such that $\bar{E}\subset\tilde{E}$. We shall extend the given operator L to the operator $\tilde{L}$ in $\tilde{E}$ by setting $a_{ij}=\delta_{ij}$, $b_i=0$ on $\tilde{E}-E$, for example. Let $(x_t, \zeta, \tilde{P}_x)$ be the minimal $\tilde{L}$-diffusion process on $\tilde{E}$. Then $(x_t, \tau_E, \tilde{P}_x)$ is the desired diffusion process by Proposition 2.4. Other assertions of the theorem is obvious.

Remark. Theorem 2.1. is valid for arbitrary minimal diffusion process.

2.6. At the end of this section, let us consider the adjoint diffusion process. Let h be a strictly positive co-excessive function such that $L^*h=0$. We define the h-transform of G^*_α as $G^{*,h}_\alpha f(x)=h(x)^{-1}G^*_\alpha hf(x)$. Then $G^{*,h}_\alpha$ is a sub-Markov resolvent and it is the adjoint of G_α relative to the measure $h(x)\,dx$. It has been shown in [15, Theorem 6.1] that there exists a diffusion process $(x_t, \zeta, P^{*,h}_x)$, $x\in E$ having $G^{*,h}_\alpha$ as its resolvent. Set

$$L^{*,h}=\sum\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial x_j}\Big)-\sum_i\Big(b_i-2\sum_j a_{ij}\frac{\partial\log h}{\partial x_j}\Big)\frac{\partial}{\partial x_i}\,.$$

Then,

Proposition 2.5. *$u=G^{*,h}_\alpha f$ is a solution of $(\alpha-L^{*,h})u=f$.*

Further, if h is bounded and uniformly positive, $(x_t, \zeta, P_x^{*,h})$ is the minimal $L^{*,h}$-diffusion process.

Proof. Set $u = G_\alpha^{*,h} f$, $f \in L^\infty(E, dx)$. Then

$$B(uh, v) + \alpha \int uhv\,dx = \int hfv\,dx \qquad v \in C_0^\infty(E), \tag{2.4}$$

where B is the bilinear form defined by (2.1). We have on the other hand,

(2.5) $B(uh, v)$

$$= \int \Big[\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} h - uh \sum_i \Big\{ b_i - \sum_j a_{ij} \frac{\partial \log h}{\partial x_j} \Big\} \frac{\partial v}{\partial x_i} \Big] dx,$$

(2.6) $B(h, uv)$

$$= \int \Big[uh \sum_{i,j} a_{ij} \frac{\partial \log h}{\partial x_j} \frac{\partial v}{\partial x_i} + uh \sum_{i,j} a_{ij} \frac{\partial \log h}{\partial x_j} \frac{\partial u}{\partial x_i} \Big] dx$$

$$- \int \Big[vh \sum_i b_i \frac{\partial u}{\partial x_i} + uh \sum_i b_i \frac{\partial v}{\partial x_i} \Big] dx.$$

Since h satisfies $L^* h = 0$ and since $uv \in H_0^1(E)$, $B(h, uv) = 0$ holds. Subtract the right hand of (2.6) from the right hand of (2.5). Then,

(2.7) $B(uh, v)$

$$= \int \Big[\sum a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} - v \sum_i \Big\{ b_i - \sum_{i} a_{ij} \frac{\partial \log h}{\partial x_j} \Big\} \frac{\partial u}{\partial x_i} \Big] h\,dx$$

$$= \int \Big[\sum a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial vh}{\partial x_j} - vh \sum \Big\{ b_i - \sum_j 2a_{ij} \frac{\partial \log h}{\partial x_j} \Big\} \frac{\partial u}{\partial x_j} \Big] dx.$$

Combining this with (2.4), we see that $u = G_\alpha^{*,h} f$ is a solution of $(\alpha - L^{*,h}) u = f$. In case that h is bounded and uniformly positive, it is obvious that $u \in H_0^1(E)$. This completes the proof.

The following is the counterpart of Theorem 2.1 and can be proved similarly, making use of Proposition 2.5.

Theorem 2.1′. (i) *If u is an α-co-potential, it is $(L^*-\alpha)$-subsolution belonging to* $\bigcap_{1<q<\frac{n}{n-1}} H^{1;q}_{\mathrm{loc}}(E)$. *Conversely if u is a $(L^*-\alpha)$-subsolution of* $\bigcap_{1<q<\frac{n}{n-1}} H^{1;q}_{\mathrm{loc}}(E)$, *it is the sum of α-co-potential and a solution of $(L^*-\alpha)v=0$.*

(ii) *u is a solution of $(L^*-\alpha)u=0$ belonging to $H^1_{\mathrm{loc}}(E)$ if and only if u/h is an α-harmonic function of the $L^{*,h}$-diffusion.*

§3. Additive and multiplicative functionals of the minimal L-diffusion processes.

In the paper [11], we have discussed the relation of the generators of two Markov processes whose Markovian measures are mutually absolutely continuous. We shall apply these results to the present case in a slightly modified form. The results of this section will be applied to the proof of Theorem 2.

3.1. Let (x_t, ζ, P_x) be the minimal L-diffusion process. Let us recall the class of additive functionals $\mathfrak{M}$ and $\mathfrak{A}$ defined in [18] or [13]. $\mathfrak{M}$ stands for the set of all continuous additive functional (AF) X_t such that $E_x(X_t^2)<\infty$ and $E_x(X_t)=0$ for $0\leqq t<\infty$ and $x\in E$. A stands for the set of all continuous AF φ_t which is written as the difference of two nonnegative ($=$increasing) AF $\varphi_t^i (i=1, 2)$ such that $E_x(\varphi_t^i)<\infty$ for $0\leqq t<\infty$ and $x\in E$.

It is convenient to extend the classes $\mathfrak{M}$ and $\mathfrak{A}$ in the following manner. We shall write "quasi everywhere" for "except for a polar set". $\overline{\mathfrak{M}}$ is defined as the set of all AF X_t such that for all $0\leqq t<\infty$, $E_x(X_t^2)<\infty$ and $E_x(X_t)=0$ holds quasi-everywhere. (The excepted set may depend on each AF X_t). The class of AF $\overline{\mathfrak{A}}$ is defined similarly.

Similarly as is the case of $\mathfrak{M}$, it can be shown that for each X, Y of $\overline{\mathfrak{M}}$, there exists a unique $<X, Y>$ of $\overline{\mathfrak{A}}$ such that

$$E_x(X_t Y_t)=E_x(<X, Y>_t) \qquad 0\leqq t<\infty$$

holds quasi-everywhere. Now let $\overline{L^2}(<X, X>)$ be the set of all nearly

Borel measurable functions f such that $E_x\left(\int_0^t f(x_s)^2\, d<X, X>_s\right)$ $<\infty$ holds for all $t<\infty$ quasi-everywhere. Then for $X\in\mathfrak{M}$ and $f\in \bar{L}_2(<X, X>)$, the stochastic integral $Y_t=\int_0^t f(x_s)\,dX_s$ is defined as an element of $\mathfrak{M}$ satisfying

$$<Y, Z>_t=\int_0^t f d<X, Z> \qquad \text{for all } Z\in\mathfrak{M}.$$

The existence and the uniqueness of the stochastic integral is proved similarly as Motoo-Watanabe [18].

3.2. Let us define X_t^u for $u\in D(A)$ as

$$X_t^u=u(x_t)-u(x_0)-\int_0^t Lu(x_s)\,ds. \tag{3.1}$$

Then $X_t^u\in\mathfrak{M}$ ([18]).

Lemma 3.1. *For each* u, $v\in D(A)$, *we have*

$$<X^u, X^v>_t=2\int_0^t\left(\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\right)ds \tag{3.2}$$

Proof. Notice

$$X_\infty^u=-u(x_0)-\int_0^\zeta Lu(x_s)\,ds, \qquad u(x)=-E_x\left(\int_0^\zeta Lu(x_s)\,ds\right)$$

etc., we have

$$E_x(X_\infty^u X_\infty^v)=E_x\left(\left(\int_0^\zeta Lu(x_s)\,ds\right)\left(\int_0^\zeta Lv(x_s)\,ds\right)\right)-u(x)\,v(x)$$

$$=E_x\left(\int_0^\zeta Lu(x_s)\left(\int_s^\zeta Lv(x_t)\,dt\right)ds\right)+E_x\left(\int_0^\zeta Lv(x_s)\left(\int_s^\zeta Lu(x_t)\,dt\right)ds\right)$$

$$-u(x)\,v(x)$$

$$=-E_x\left(\int_0^\zeta Lu(x_s)\,v(x_s)\,ds\right)-E\left(\int_0^\zeta Lv(x_s)\,u(x_s)\,ds\right)-u(x)\,v(x)$$

$$=G(Luv)(x)-G(vLu)(x)-G(uLv)(x).$$

On the other hand, it is easy to see

$$Luv=uLv+vLu+2\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}.$$

Therefore we get

$$E_x(X^u_\infty X^v_\infty)=2E_x\left(\int_0^\zeta \sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}ds\right), \tag{3.3}$$

which implies

$$E_x(X^u_t X^v_t)=2E_x\left(\int_0^t \sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}ds\right) \tag{3.4}$$

if we notice the additivety of X^u_t and that of the integrand of the right hand of (3.4) and use the Markov property. The equality (3.4) proves the lemma.

Remark. Lemma 3.1 shows, in particular, that all $\langle Z, Z\rangle$ $(Z\in\mathfrak{M})$ are absolutely continuous with respect of $t\wedge\zeta$, because X^u_t, $u\in D(A)$ generates $\mathfrak{M}$ (See [18]).

We shall extend the above lemma to arbitrary $u\in H^1(E)$. But before doing this let us introduce the following function family. $\overline{L^p(E)}$ stands for the set of all measurable functions f in E such that $\int G(x, y)|f(y)|^p dy$ is finite quasi-everywhere. $\overline{H^1(E)}$ stands for the set of all $u\in\overline{L^2(E)}$ such that $\int g(x, y)\left|\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right|dy$ is finite quasi-everywhere. Then $\overline{L^p(E)}\supset L^p(E, dx)$ and $\overline{H^1(E)}\supset H^1(E)$ holds. In fact, let $g\in L^\infty(E, dx)$; then

$$\left|\int gG|f|^p dx\right|=\left|\int |f|^p G^* g dx\right|\leqq \sup|G^* g|\int|f|^p dx$$

if $f\in L^p(E, dx)$. This shows that $G|f|^p$ is finite almost everywhere, which implies that $G|f|^p$ is finite quasi-everywhere. The latter assertion $\overline{H^1(E)}\supset H^1(E)$ is proved similarly.

Proposition 3.1. *Let* $\{f_i\}$, $i=1, 2, \cdots, n$ *be a set of functions in* $\overline{L^2(E)}$. *Then there exists a unique AF* X_t *of* $\overline{\mathfrak{M}}$ *which satisfies*

$$(3.5)\qquad <X, X^v>_t=\int_0^t 2\Big(\sum a_{ij}f_i\frac{\partial v}{\partial x_j}\Big)ds \qquad \text{for all } v\in D(A).$$

Proof. Suppose there is a linear mapping from $\overline{\mathfrak{M}}$ into $\overline{L^2(E)}$ which satisfies the following (F.1)$\sim$(F.3).

(F.1) $F\Big(\int f dZ\Big)=fF(Z)$ for all $f\in \overline{L^2}(<Z, Z>)$,

(F.2) $F(Z)\leqq(2\sum_{i,j}a_{ij}f_if_j)(Z, Z)$, where (Z, Z) is a function such that $<Z, Z>_t=\int_0^t(Z, Z)(x_s)\,ds$.

(F.3) $F(X^v)=2\Big(\sum_{i,j}a_{ij}f_i\frac{\partial v}{\partial x_j}\Big)$ for all $v\in D(A)$.

Then, since $E_x\Big(\int_0^\zeta\sum a_{ij}f_if_j\,ds\Big)<\infty$ holds quasi-everywhere, there exists a unique X_t of $\overline{\mathfrak{M}}$ such that $<X, Z>_t=\int_0^t F(Z)(x_s)\,ds$ holds for all Z by virtue of Proposition 2.4 of [11]. This X is the desired one by (F.3).

The existence of F satisfying (F.1)$\sim$(F.3) is proved as follows. Let $\mathfrak{N}$ be the set of $X\in\overline{\mathfrak{M}}$ which is written as $X=\sum_{k=1}^m\int g_k dX^{v_k}$, $g_i\in\overline{L^2}$ $(<X^{v_k}, X^{v_k}>)$. We define F': $\mathfrak{N}\to\overline{L^1(E)}$ as $F'(X)=\sum_k g_k\sum_{i,j}a_{ij}f_i\frac{\partial v_k}{\partial x_j}$ for such X. Then this F' satisfies (F.1)$\sim$(F.3). It is known, on the other hand, that the space $\mathfrak{N}$ is dense in $\overline{\mathfrak{M}}$ ([18]). Thus F' is extended uniquely to a linear mapping from $\overline{\mathfrak{M}}$ into $\overline{L^1(E)}$. The proof is complete.

Corollary. *For each* $u\in\overline{H^1(E)}$ *there exists a unique* X_t^u *of* $\overline{\mathfrak{M}}$ *which satisfies* (3.2) *for all* $v\in D(A)$.

Remark. If $\{f_i\}$ belongs to $L^p(E, dx)$ with a suitable $p>n$, the *AF* X_t of Proposition 3.1 belongs to $\mathfrak{M}$, because $E_x\Big(\int_0^\zeta\sum a_{ij}f_if_j\,ds\Big)$ is finite for all x by Proposition 2.2.

Theorem 3.1. *There exists $X^1, \cdots, X^n$ of $\mathfrak{M}$ such that*

$$<X^i, X^v>_t = 2\int_0^t \sum_j a_{ij} \frac{\partial v}{\partial x_j} ds \qquad \text{for all } v \in D(A).$$

Furthermore, every X_t of $\overline{\mathfrak{M}}$ is represented as $X_t = \sum_{i=1}^n \int f_i(x_s) dX_s^i$ where $f_i \in \overline{L^2(E)}$.

Proof. The existence of such $X_t^1, \cdots, X_t^n$ is obvious from Proposition 3.1 and Remark after that. The latter assertion follows from that X_t^u is represented as $X_t^u = \sum_{i=1}^n \int \frac{\partial u}{\partial x_i} dX_s^i$ and $\{X_t^u, u \in D(A)\}$ generates $\mathfrak{M}$.

3.3. It is desirable to get the expression of X^u for arbitrary $u \in H^1(E)$, similarly as (3.1) of the case $u \in D(A)$.

Proposition 3.2. *Let u be a function of $\overline{H^1(E)}$ such that $Lu \in L^1(E)$. Then*

$$X_t^u = u(x_t) - u(x_0) - \int_0^t Lu(x_s) ds$$

holds for $t < \zeta$. In particular, if $Lu = 0$, X_t^u is of $\mathfrak{M}$.

Proof. We first consider the case $u \in H^1(E)$. Set $u_2 = -GLu$ and $u_1 = u - u_2$. The both of u_1 and u_2 belong to $H^1(E)$ by c) of §2.3. It is sufficient to prove the assertion for u_1 and u_2. Set

$$X_t = u_2(x_t) - u_2(x_0) - \int_0^t Lu(x_s) ds.$$

Then the proof of Lemma 3.1 is applicable with no essential change and we see that $X_t = X_t^{u_2}$.

For the proof of the case u_1, let us assume for the moment that u_1 is bounded. Then since $u_1(x_t)$ is a martingale relative to $(\mathfrak{F}_t, P_x)$, $^\forall x \in E$, $\tilde{u}_1(\omega) = \lim_{t \uparrow \infty} u_1(x_t)$ exists and the AF defined by

$$X_t = u_1(x_t) - u_1(x_0) \quad \text{if } t < \zeta, \tag{3.6}$$
$$= \tilde{u}_1(\omega) - u_1(x_0) \quad \text{if } t \geqq \zeta,$$

has mean 0 relative to P_x, $x \in E$. Hence X_t belongs to $\mathfrak{M}$ as is easily seen. Furthermore, for $v \in D(A)$ we have

$$E_x(X_\infty X_\infty^v) = 2E_x\left(\int_0^\zeta \sum a_{ij} \frac{\partial u_2}{\partial x_i} \frac{\partial v}{\partial x_j} ds\right) \tag{3.7}$$

In fact,

$$E_x(X_\infty X_\infty^v) = E_x((\tilde{u}_1(\omega) - u_1(x_0))(-v(x_0) - \int_0^\zeta Lv(x_s)ds))$$
$$= u_1(x)GLv(x) - E(\tilde{u}_1 \int_0^\zeta Lv(x_s)ds).$$

Notice that $E_x(\tilde{u}_1 | \mathfrak{F}_{t\wedge\zeta}) = u_1(x_t)$, then the last expression of the above is equal to

$$u_1(x)GLv(x) - G(u_1 Lv)(x) = 2E_x\left(\int_0^\zeta \sum a_{ij} \frac{\partial u_1}{\partial x_i} \frac{\partial v}{\partial x_j} ds\right)$$

for all $v \in D(A)$.

Therefore we get $X_t = X_t^u$ by the uniqueness of X_t^u.

Let us consider the case that u_1 is unbounded. Choose a sequence of open sets E_n with regular boundary such that $\bar{E}_n \subset E_{n+1} \subset E$ and $\cup E_n = E$. Let T_n be the hitting time for the set E_n^c. Then $X_{t\wedge T_n} \in \mathfrak{M}$, where X_t is the AF defined by (3.6). Furthermore, we have

$$E_x(X_{t\wedge T_n} X_{t\wedge T_n}^v) \tag{3.8}$$
$$= E_x((u_1(x_{t\wedge T_n}) - u_1(x_0)(v(x_{t\wedge T_n}) - v(x_0) - \int_0^{t\wedge T_n} Lv(x_s)ds))$$
$$= 2E_x\left(\int_0^{t\wedge T_n} \sum a_{ij} \frac{\partial u_1}{\partial x_i} \frac{\partial v}{\partial x_j} ds\right)$$

by the argument of the preceding paragraph. On the other hand, since u_1 is harmonic we have

$$(3.9)\quad E_x(X^2_{t\wedge T_n})=E_x(u_1(x_{t\wedge T_n})^2)-u_1(x)^2=2E_x\left(\int_0^{t\wedge T_n}\sum a_{ij}\frac{\partial u_1}{\partial x_i}\frac{\partial u_1}{\partial x_j}ds\right)$$
$$\leqq 2E_x\left(\int_0^{\zeta}\sum a_{ij}\frac{\partial u_1}{\partial x_i}\frac{\partial u_1}{\partial x_j}ds\right).$$

Since the last term of the above is finite quasi-everywhere, $E_x(X^2_{t\wedge T_n})$ is bounded in n quasi-everywhere. This proves $X_t\in\overline{\mathfrak{M}}$. Making n tend to infinity in (3.8), we see $X_t=X^u_t$.

We shall next consider the case $u\in\overline{H^1(E)}$. Let us notice that $\overline{H^1(E)}\subset H^1_{loc}(E)$. In fact if $u\in\overline{H^1(E)}$ the function $\int g(x, y)\sum a_{ij}\frac{\partial u}{\partial x_i}\cdot\frac{\partial u}{\partial x_j}dy$ is locally integrable, i.e. for any open set U such that $\bar U\subset E$, we have

$$\int_U dx\int g(x, y)\left[\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right]dy=\int G^*I_U(y)\left[\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right]dy<\infty.$$

Since we can choose U so that G^*I_U is strictly positive in E, u must belong to $H^1_{loc}(E)$. Consequently, the above argument applied to the stopped process (x_t, τ_U, P_x) can conclude that

$$X^u_{t\wedge\tau_U}=u(x_{t\wedge\tau_U})-u(x_0)-\int_0^{t\wedge\tau_U}Lu(x_s)ds$$

holds. Since U is arbitrary, we obtain the first assertion.

It remains to prove the latter assertion. Let u be a harmonic function of $H^1(E)$. Then

$$E_x((X^u_{T_n})^2)=E_x(u(x_{T_n})^2)-u(x)^2=2E_x\left(\int_0^{T_n}\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}ds\right)$$

by (3.9). Hence $w(x)=\lim_{n\to+\infty}E_x(u(x_{T_n})^2)$ is finite quasi-everywhere. On the other hand, it is easy to see that w is a harmonic function, which implies $w(x)$ is continuous and, in particular, is finite everywhere. This proves $E_x(X^2_t)<\infty$ for all $0<t<\infty$ and $x\in E$. The proof is completed.

Corollary. *Under the same assumption as Proposition* 3.2, $\{u(x_{T_n})\}$ *are uniformly integrable relative to* P_x *for* x *except a polar set. In particular, if* u *is harmonic,* $E_x(u(x_{T_n})^2)$ *is bounded in* n *for all* $x \in E$.

3.4. It is possible to obtain similar results as the aboves for the adjoint diffusion process. $\overline{L^p(E)}^*$ is defined as the set of all f such that $\int g(y, x)|f(y)|^p dy$ is finite quasi-everywhere. $\overline{H^1(E)}^*$ is defined similarly. Now let h be a bounded and uniformly positive co-excessive function such that $L^*h=0$. The existence of such h will be proved at §7. Let $(x_t, \zeta, P_x^{*,h})$ be the minimal $L^{*,h}$-diffusion. Then Lemma 3.1, Proposition 3.1 and Theorem 3.1 are valid for the minimal $L^{*,h}$-diffusion with the obvious modification. However as to Proposition 3.2, we have only a slightly weaker result, namely,

Proposition 3.2′. *Let* u *be a locally bounded function of* $\overline{H^1(E)}^*$ *such that* $L^*u \in \overline{L^1(E)}^*$. *Then*

$$X_t^u = u(x_t) - u(x_0) - \int_0^t h^{-1} L^*(hu)ds$$

belongs to $\mathfrak{M}$ *and*

$$<X^u, X^v>_t = \int_0^t \sum a_{ij} \cdot \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} ds$$

with respect to the minimal $L^{*,h}$*-diffusion.*

We omit the proof.

Corollary. *Under the same condition as Proposition* 3.2′, $\{u(x_{T_n})\}$ *is uniformly integrable with respect to* $P_x^{*,h}$ *except* x *of a polar set.*

3.5. Let $\{c_i\}$, $i=1, \cdots, n$ be functions of $L^2(E, dx)$. Set

$$L' = L + \sum_i \{\sum_j a_{ij} c_j\} \frac{\partial}{\partial x_i} .$$

A method of constructing a diffusion process corresponding to the operator L' is the transformation of the minimal L-diffusion by multiplicative functional. Set

$$M_t=\exp\Big[\sum_i\int_0^t c_i(x_s)dX_s^i-\frac{1}{2}\int_0^t\sum_{i,j}a_{ij}c_ic_j\,ds\Big]\qquad \text{if } t<\zeta,\tag{3.10}$$

$$=0 \qquad\qquad \text{if } t\geqq\zeta.$$

Then it is a multiplicative functional (MF) such that $E_x(M_t)\leqq 1$ holds quasi-everywhere (See [13]). Set $E'=\{x\,;\,P_x(M_0=1)=1\}$. It is well known that there is a standard diffusion process (x_t, ζ, P_x^M), $x\in E'$ such that

$$P_x^M(B\cap\{t<\zeta\})=E_x(M_t;\ B\cap\{t<\zeta\}),\qquad {}^\forall B\in\mathfrak{F}_t$$

holds if we define (x_t, ζ, P_x^M) on the same basic space $(\Omega, \mathfrak{F})$ as that of (x_t, ζ, P_x). We shall denote the semigroup of (x_t, ζ, P_x) as T_t^M. Then Theorem 4.1 of [11] asserts

Proposition 3.3. *Let u be a function of $H_0^1(E)\cap L^\infty(E, dx)$ such that $Lu\in L^1(E, dx)$ and $L'u\in L^\infty(E, dx)$. Then we have*

$$T_t^Mu(x)-u(x)=\int_0^t T_s^M(L'u)(x)ds.\tag{3.11}$$

However, we do not know in general whether such $(T_t^M, t\geqq 0)$ is the unique semigroup satisfying (3.10). A sufficient condition for this is that each c_i belongs to $L^p(E, dx)$ with some $p>n$. In this case, the transformed process coincides with the minimal L'-diffusion process. In fact, under this condition there exists the minimal L'-diffusion semigroup T_t' by Theorem 1. Obviously, T_t' satisfies

$$T_t'u-u=\int_0^t T_s'L'u\,ds\qquad u\in D(A'),\tag{3.12}$$

where $D(A')$ is the domain of the generator of the semigroup T_t'. Taking the Laplace transform in the equalities (3.11) and (3.12), we obtain

$$u=G_\alpha'(\alpha-L')u=G_\alpha^M(\alpha-L')u\qquad u\in D(A').$$

Since $\{(\alpha-L')u \mid u\in D(A')\}=L^{\infty}(E, dx)$ holds, $G'=G_{\alpha}^{M}$. This proves that (x_t, ζ, P_x^M) is the minimal L'-diffusion process. Furthermore we have

Theorem 3.2. *Let (x_t, ζ, P_x') be the minimal L'-diffusion with $c_i\in L^p(E, dx)$, $p>n$. Then there exists a strictly positive $\mathfrak{F}$-measurable function $\xi(\omega)$ such that $\xi=M_t\xi(\theta_t)$, $E_x(\xi)=1$ and*

$$P_x'(B)=E_x(\xi\,;\,B), \qquad B\in\mathfrak{F}$$

holds quasi-everywhere.

Proof. By virtue of Dynkin [3, Chapt. X] and Kunita [10], it is enough to verify that M_t defined by (3.10) is of the class (D), that is, for arbitrary family of stopping times $\{T\}$, the family $\{M_T\}$ is P_x-uniformly integrable quasi-everywhere.

Apply the formula on stochastic integral of [15] to $F(x)=xe^x$ and $A_t=\sum_{i=1}^{n}\int_0^t c_i(x_s)dX_s^i-\frac{1}{2}\int_0^t\sum_{i,j}a_{ij}c_ic_j\,ds$. Then we obtain

$$F(A_t)-F(A_0)=M_t\log M_t=Z_t+\frac{1}{2}\int_0^t M_s(\sum_{i,j}a_{ij}c_ic_j)ds,$$

where Z_t is a local martingale with mean 0.[10] Therefore if T is a stopping time such that $Z_{t\wedge T}$ is a martingale, we have

$$E_x(M_T\log M_T)=\frac{1}{2}E_x'\Big(\int_0^T\sum_{i,j}a_{ij}c_ic_j\,ds\Big)\leqq\frac{1}{2}G'(\sum a_{ij}c_ic_j)(x).$$

Since the last expression of the above is finite quasi-everywhere, we see that $E_x(M_T\log M_T)$ is uniformly bounded in $\{T\}$ quasi-everywhere, which proves that M_t is of the class (D).

The condition $c_i\in L^p(E, dx)$, $p>n$ is not always necessary for Theorem 3.2. Actually we have only used that $G'(\sum_{i,j}a_{ij}c_ic_j)$ is finite quasi-everywhere. Consequently, we can prove, similarly as the above,

10) Precisely, there is a sequence of stopping times such that $T_n\to+\infty$ and Z_{T_n} is a martingale with mean 0.

the following

Theorem 3.2′. *Let h be a bounded and uniformly positive co-excessive function such that $L^*h=0$. Let $(x_t, \zeta, P_x^{*,h})$ be the minimal $L^{*,h}$-diffusion process. Then there exists a MF M_t and a strictly positive $\mathfrak{F}$-measurable function ξ such that $\xi=M_t\xi(\theta_t)$, $E_x(\xi)=1$ and*

$$P_x^{*,h}(B)=E_x(\xi; B) \qquad {}^{\forall}B\in\mathfrak{F}$$

holds quasi-everywhere.

Proof is obvious because $\dfrac{\partial \log h}{\partial x_i}\in L^2(E, dx)$ and hence

$$G^{*,h}\left[\sum a_{ij}\left(b_i-2\sum_k a_{ik}\frac{\partial \log h}{\partial x^k}\right)\left(b_j-2\sum_k a_{jk}\frac{\partial \log h}{\partial x^k}\right)\right]$$

is finite quasi-everywhere.

§4. Martin boundary

4.1. The minimal L-diffusion process satisfies Hypothesis (B) of [14], by virtue of Theorem 1. Hence the discussions concerning Martin boundary in [14, 15] are applicable in our situation. We have also shown in Theorem 2.1 that a solution of $Lu=0$ belonging to $H^1_{\mathrm{loc}}(E)$ is harmonic. Hence if the function u is nonnegative and γ-integrable, it has the unique Martin representation

$$u(x)=\int_{\partial M_1} K(x, \eta)\,\mu(d\eta), \tag{4.1}$$

where ∂M_1 is the set of all η such that $K(\cdot, \eta)$ is minimal harmonic and $\int K(x, \eta)\,\gamma(dx)=1$ holds. In particular, the constant function 1 satisfies $L1=0$, so that it has the representation, making use of the canonical measure μ_0.

It is interesting to know under which condition the harmonic function u is represented as

$$u(x)=\int_{\partial M_1} K(x, \eta)\,\bar{u}(\eta)\,\mu_0(d\eta) \tag{4.2}$$

with $\bar{u}$ of $L^p(\partial M_1. \mu_0)$. The following is due to T. Watanabe[11].

Proposition 4.1. *Let $u(x)$ be a harmonic function.*
1) *u is represented as (4.2) with $\bar{u} \in L^1(\partial M_1, \mu_0)$ if and only if $\{u(x_{T_n})\}$ is uniformly integrable relative to the measure $P_\gamma = \int P_x \gamma(dx)$.*
2) *u is represented as (4.2) with $\bar{u} \in L^p(\partial M_1, \mu_0)$, $p>1$ if and only if $\sup E_\gamma(|u(x_{T_n})|^p) < \infty$.*
Here, T_n is the hitting time for E_n^c, where $\{E_n\}$ is a sequence of open sets such that $\bar{E}_n \subset E_{n+1} \subset E$ and $\cup E_n = E$.

Proof. We follows the proof of T. Watanabe for the completeness. It is known ([14]) that $x_{\zeta-} = \lim_{t \uparrow \zeta} x_t$ exists and $P_x(x_{\zeta-} \in B) = \int_B K(x, \eta) \mu_0(d\eta)$ holds. Hence if u is represented as (4.2) with $\bar{u} \in L^1(\partial M_1, \mu_0)$, we have $u(x) = E_x(\bar{u}(x_{\zeta-}))$ and $P_\gamma(|\bar{u}(x_{\zeta-})|) < \infty$. Since $u(x_{T_n}) = E_\gamma(u(x_{\zeta-}) | \mathfrak{F}_{T_n})$ holds, $\{u(x_{T_n})\}$ are uniformly integrable with respect to P_γ.

Conversely assume that $\{u(x_{T_n})\}$ are uniformly integrable relative to P_γ. Then $v = \lim_{n-\infty} E_x(|u(x_{T_n})|)$ exists, integrable relative to the measure γ and hence it is harmonic[12]. As a consequence of this, both of $u_1(x) = \lim_{n \to \infty} E_x(u^+(x_{T_n}))$ and $u_2(x) = \lim_{n \to \infty} E_x(u^-(x_{T_n}))$ are harmonic functions integrable relative to γ and $u = u_1 - u_2$. This proves that both of u and v have Martin representation, and the canonical measure of v is the absolute value of the canonical measure of u.

It is now sufficient to prove that the canonical measure of v, denoted by μ', is absolutely continuous with respect to μ_0. Let $f(\omega)$ be the limit of $u(x_{T_n})$. Set $v' = E_x(|f|)$. Then $v = v'$ holds. In fact, it is easy to see that $v - v'$ is nonnegative harmonic function such that $\int (v - v') \gamma(dx) = 0$. Hence $v - v'$ is identically 0. Now consider the reduced function $H_B v$ for the Borel set B of ∂M. Then

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11) T. Watanabe, "Some topics related to Martin boundaries induced by contable Markov processes", Proc. of 32nd Section of *ISU* (1960).

12) If an excessive function u is integrable relative to the measure, it is finite quasi-everywhere. See [16].

$$H_B v(x) = E_x(|f|\,;\, x_{\zeta-} \in B) = \int_B K(x,\eta)\,\mu'(d\eta)$$

holds ([14]). Consequently, $P_\gamma(x_{\zeta-} \in B)=0$ implies $\int H_B u(x)\gamma(dx)=0$ or $\mu'(B)=0$. This proves that μ' is absolutely continuous with respect to μ_0.

The second assertion can be proved similarly.

Corollary 1. *u is represented as* (4.2) *with $\bar{u} \in L^2(\partial M_1, \mu_0)$ if and only if* $\int u(x)^2 \gamma(dx) < \infty$ *and* $\int \gamma g(y) \Big[\sum a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \Big](y)\, dy < \infty$

Proof is immediate from the above proposition, Proposition 3.2 and its corollary.

Corollary 2. (Doob [2]) *Suppose that the reference measure γ is concentrated in a single point. Then every harmonic function of $H^1(E)$ is represented as* (4.2) *with $\bar{u} \in L^2(\partial M_1, \mu_0)$.*

Proof is immediate because $E_x(u(x_{T_n})^2)$ is bounded in n for all x by Proposition 3.2 and its corollary.

4.2. Let h_0 be a bounded and uniformly positive co-excessive function with $L^* h_0 = 0$. We fix the function h_0 hereafter. Let $(x_t, \zeta, P_x^{*,h_0})$ be the minimal L^{*,h_0}-diffusion process. It has the potential kernel $g_\alpha^{h_0}(x,y) = g_\alpha(y,x) h_0(y)^{-1} h_0(x)$. If we take $h_0(x)\gamma(dx)$ as its reference measure, the Martin kernel of the L^{*,h_0}-diffusion is defined as

$$K^{h_0}(x,y) = \frac{g(y,x)}{h_0(x)\, g\gamma(x)} = \frac{K^*(y,x)}{h_0(x)}.$$

This proves that the Martin (exit) boundary of L^{*,h_0}-diffusion coincides with (homeomorphic to) the Martin (entrance) boundary of L-diffusion. Furthermore, the associated Martin kernel $K^{h_0}(x,\xi)$, $\xi \in \partial M^*$ coincides with $h_0(x)^{-1} K^*(\xi, x)$. Then any nonnegative harmonic function u of L^{*,h_0}-diffusion, integrable with respect to $h_0(x)\gamma(dx)$, is represented as

$$u(x) = \int_{\partial M_1^*} h_0(x)^{-1} K^*(\xi, x)\, \mu(d\xi),$$

where ∂M_1^* is the set of all $\xi \in \partial M^*$ such that $K^*(\xi, \cdot)$ is minimal harmonic. Consequently, if u is a nonnegative solution of $L^*u=0$ such that $\int u(x)\gamma(dx)<\infty$ and $u \in H^1_{\mathrm{loc}}(E)$, it is represented as

$$u(x)=\int_{\partial M_1^*} K^*(\xi, x)\mu(d\xi) \tag{4.3}$$

by virtue of Theorem 2.1$'$. The measure μ is called the canonical measure of u. We shall denote the canonical measure of h_0 as μ_0^* and fix it.

Let us investigate the condition that u is represented as

$$u(x)=\int K^*(\xi, x)\bar{u}(\xi)\mu_0^*(d\xi). \tag{4.4}$$

Proposition 4.1$'$. *Let u be a solution of $L^*u=0$ belonging to $H^1_{\mathrm{loc}}(E)$. The assertion of Proposition 4.1 is valid if we replace P_γ, $L^2(\partial M, \mu_0)$ and formula (4.2) by $P^{*,h_0}_{h_0\gamma}$, $L^2(\partial M_1^*, \mu_0^*)$ and formula (4.4), respectively.*

Proof. Since h_0 is bounded and uniformly positive, the uniform integrability of $u(x_{T_n})/h(x_{T_n})$ etc. are equivalent to that of $u(x_{T_n})$. Hence if we notice u/h_0 is harmonic with respect to L^{*,h_0}-diffusion, the assertion is the consequence of Proposition 4.1.

Corollary 1. *Let u be a solution of $L^*u=0$. Suppose*

$$\int \gamma(dx)\,u(x)^2<\infty \quad \textit{and} \quad \int g\gamma(x)\Big[\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big](x)\,dx<\infty.$$

Then u is represented as (4.4) with $\bar{u} \in L^1(\partial M_1^*, \mu_0^*)$.

Corollary 2. *Suppose that γ is concentrated in a single point. Then every solution of $L^*u=0$ belonging to $H^1(E)$ is represented as (4.4) with $\bar{u} \in L^1(\partial M_1^*, \mu_0^*)$.*

Corollary 3. *Let u be a nonnegative solution of $L^*u=0$ such that $\int u(x)\gamma(dx)<\infty$ and $u \in \overline{H^1(E)}^*$. Then it is represented as (4.4)* *with* $\bar{u}\in L^1(\partial E, \mu_0^*)$.

4.3. Let us recall the definition of the kernel $K_\alpha(x, \eta)$ (See (1.3)). Since $K(\cdot, \eta)$ is excessive, $K_\alpha(\cdot, \eta)$ increases to a kernel $\hat{K}(\cdot, \eta)$ which is smaller than $K(\cdot, \eta)$, as α decreases to 0. Noting that $K(\cdot, \eta)$, $\eta\in\partial M_1$ is minimal, we can show easily that either $\hat{K}(\cdot, \eta)=K(\cdot, \eta)$ or $\hat{K}(\cdot, \eta)=0$ holds. We denote by M_{ex} the set of all $\eta\in\partial M_1$ such that $\hat{K}(\cdot, \eta)=K(\cdot, \eta)$. We shall show $\mu_0(\partial M_1-\partial M_{ex})=0$. Since

$$E_x(e^{-\alpha\zeta})=1-\alpha G_\alpha 1(x)=\int_{\partial M_1}K_\alpha(x, \eta)\,\mu_0(d\eta)$$

and since $E_x(\zeta)=G1(x)<\infty$, $P_x(\zeta<\infty)=1$ holds. Therefore, making α tend to 0 in the above equality, we have $1=\int_{\partial M_1}\hat{K}(x, \eta)\,\mu_0(d\eta)$. This proves $\mu_0(\partial M_1-\partial M_{en})=0$[13].

Observe that the kernel $K_\alpha(x, \eta)$ coincides with $g_\alpha(x, \eta)/\gamma g(\eta)$ if $\eta\in E$. Then by a simple calculation, we obtain

$$K_\alpha(x, \eta)-K_\beta(x, \eta)+(\alpha-\beta)\int K_\alpha(x, z)K_\beta(z, \eta)\,\gamma g(z)\,dz=0$$

for all α, $\beta>0$. Therefore, $S_\alpha^* f(\eta)=\int g(z)f(z)K_\alpha(z, \eta)dz$ satisfies the resolvent equation. In what follows we prove that S_α^* is sub-Markov. Since $K_\alpha(x, \eta)$ is positive, it is obvious that $f\geqq 0$ implies $S_\alpha^* f\geqq 0$. We have on the other hand,

$$\alpha S_\alpha^* 1(\eta)=\int\{\gamma g(x)-\alpha\int\gamma g(z)\,g_\alpha(z, x)\,dz\}\;K(x, \eta)\,dx.$$

Note that $\gamma g(x)$ is co-excessive. Then the quantity in the blacket $\{\ \}$ of the above equality is positive. This implies that $\alpha S_\alpha^* 1(\eta)$ is lower semi-continuous, since the kernel $K(x, \eta)$ is lower semi-continuous in η. Furthermore, $\alpha S_\alpha^* 1(\eta)$ coincides with $G_\alpha^*\gamma g(\eta)/\gamma g(\eta)$ for $\eta\in E$, which

13) The point $\eta\in M_{ex}$ is oftenly called an active boundary point and $\eta\in\partial M_1-\partial M_{en}$, a passive boundary point. Let us denote the $K(\cdot, \eta)$-path process as (x_t, ζ, P_x^η). Then $P_x^\eta(\zeta<\infty)=0$ or $=1$, according to η is passive or active, respectively.

is dominated by 1. Hence $\alpha S_\alpha^* 1(\eta) \leqq 1$ holds for all $\eta \in \partial M_{ex}$.

A similar argument can be applied to the entrance kernel $K^*(\xi, x)$. We can prove $\mu_0^*(\partial M_1^* - \partial M_{en}^*) = 0$ and the operator S_α defined by (1.3) is again a sub-Markov resolvent.

It is easily seen that S_α restricted to E is a $g\gamma$-transform of G_α, namely, $S_\alpha f(x) = G_\alpha(g\gamma f)(x)/g\gamma(x)$. Therefore, if u is G_α-excessive, $u/g\gamma$ is S_α-excessive in E. This fact permits us to define the fine continuous extension of $u/g\gamma$ to the space M_{en}, by the following lemma.

Lemma 4.1. *Let $G_\alpha(x, dy)$, $\alpha > 0$ be a sub-Markov resolvent kernel defined on a measurable space (S, B). Suppose that each $G_\alpha(x, dy)$, $\alpha > 0$, $x \in S$ is absolutely continuous relative to a suitable measure m. If v is a nonnegative function defined* a.e. *m and satisfies $\alpha G_\alpha v \leqq v$ for each $\alpha > 0$,* a.e. *m, then $\alpha G_\alpha v(x)$ increases with α for all $x \in E$. Furthermore, $\hat{v}(x) = \lim_{\alpha-\infty} \alpha G_\alpha v(x)$ is excessive relative to G_α and $G_\alpha \hat{v}(x) = G_\alpha v(x)$ holds for all $x \in S$.*

Definition. The function v is called a *supermedian* and v, its regularization.

Proof. Assume that the above stated function u is bounded. If $\alpha \leqq \beta$, we have

$$\begin{aligned}\alpha G_\alpha v(x) - \beta G_\beta v(x) &= \alpha G_\alpha v(x) - \beta[G_\alpha v(x) - (\beta-\alpha) G_\alpha G_\beta v(x)] \\ &= (\alpha-\beta) G_\alpha v(x) + \beta(\beta-\alpha) G_\alpha G_\beta v(x) \\ &\leqq (\alpha-\beta) G_\alpha v(x) + (\beta-\alpha) G_\alpha v(x) \\ &\leqq 0\end{aligned}$$

for all $x \in S$. Hence $\alpha G_\alpha v$ increases with α. By the definition of $\hat{v}$ we have

$$G_\beta \hat{v} = \lim_{\alpha \to \infty} \alpha G_\beta G_\alpha v = \lim_{\alpha \to \infty} \alpha(\alpha-\beta)^{-1}[G_\beta v - G_\alpha v] = G_\beta v.$$

The above inequalities imply $\beta G_\beta \hat{v} = \beta G_\beta v \leqq \hat{v}$ and $\lim_{\beta \to \infty} \beta G_\beta \hat{v} = \hat{v}$.

In case that v is unbounded it is easy to see that $v_n = v \wedge n$ is a supermedian. Let $\hat{v}_n$ be the regularization of v_n. Then clearly $\hat{v}_n$ increases with n. Hence the limit of $\{\hat{v}_n\}$ denoted by $\hat{v}$ is excessive. Since $G_\alpha \hat{v}_n = G_\alpha v_n$, we see $G_\alpha \hat{v} = G_\alpha v$ by letting n tend to infinity.

Remark. Let u be a potential represented as $u(x) = \int g(x, y)\mu(dy)$. Then the normal derivative $\partial u/\partial g$ (defined in §1) is represented as

$$\frac{\partial u}{\partial g}(\xi) = \int K^*(\xi, y)\mu(dy).$$

The following is an analogue of Green's formula involving a potential and a harmonic function.

Proposition 4.2. *Let u be a potential of the minimal L-diffusion represented as $u(x) = \int g(x, y)\mu(dy)$ and let v be a harmonic function of the minimal L^{*,h_0}-diffusion represented as $v(x) = \int \bar{v}(\xi)\frac{K^*(\xi, x)}{h_0(x)}\mu_0^*(d\xi)$ Then*

$$\int v(y) h_0(y)\mu(dy) = \int_{\partial M_{en}} \bar{v}(\xi)\frac{\partial u}{\partial g}(\xi)\mu_0^*(d\xi)$$

holds. In particular, if $v = 1$,

$$\mu(E) = \int_{\partial M_{en}} \frac{\partial u}{\partial g}(\xi)\mu_0^*(d\xi).$$

Proof is immediate from

$$\int_{\partial M_{en}} \bar{v}(\xi)\frac{\partial u}{\partial g}(\xi)\mu_0^*(d\xi) = \int \mu_0^*(d\xi)\bar{v}(\xi)K^*(\xi, y)\mu(dy)$$

$$= \int v(y) h_0(y)\mu(dy).$$

4.4. We have defined the kernel $\Theta(x, y)$ by (1.5) in the case x and y are in E. We shall prove

Proposition 4.3.[14] *The kernel* $\Theta(x, y)$ *defined by* (1.5) *has a unique finely continuous extension to* $M_{en} \times M_{ex}$. *Furthermore, the kernel*

$$\Theta_\alpha(\xi, \eta) = \alpha \int K_\alpha^*(\xi, x) K(x, \eta)\, dx \tag{4.5}$$

increases to $\Theta(\xi, \eta)$ *as* $\alpha \to +\infty$ *for all* $\xi \in M_{en}$ *and* $\eta \in M_{ex}$.

Proof. Let $\{\xi_n\}$ be a sequence in E converging to $\xi \in M_{en} - E$ in the fine topology of M_{en}. Since the kernel $K^*(\xi, x)$ is excessive function of ξ for each $x \in E$ relative to the resolvent S_α, it is finely continuous in M_{en}. Thus $\Theta(\xi_n, x)$ converges to $K^*(\xi, x)/\gamma g(x)$ as $n \to \infty$, that is, $\Theta(\xi, x) = K^*(\xi, x)/\gamma g(x)$ holds. Observe that $\Theta(\xi, x)$ is a supermedian relative to S_α^* for each fixed ξ, that is,

$$\int S_\alpha^*(\eta, dx) \Theta(\xi, x) \leqq \Theta(\xi, \eta) \tag{4.6}$$

holds for $\eta \in E$. Then the left hand of the above increases with α for all $\xi \in M_{en}$ and $\eta \in M_{ex}$ by Lemma 4.1. Denote its limit as $\hat{\Theta}(\xi, \eta)$. Then it is S_α-excessive in ξ and S_α^*-excessive in η. Furthermore, the above regularization $\hat{\Theta}(\xi, \eta)$ coincides with $\Theta(\xi, \eta)$ if $\xi, \eta \in E$. This proves that $\hat{\Theta}(\xi, \eta)$ coincides with the finely continuous extension of $\Theta(\xi, \eta)$. The latter assertion is obvious since the left hand of (4.6) coincides with the right hand of (4.5).

4.5. We now proceed to the proof of Theorem 2. Let (x_t, ζ, P_x) be the minimal L-diffusion and (x_t, ζ, P^{*,h_0}) be the minimal L^{*,h_0}-diffusion. We have shown in Theorem 3.2′ that the measure P_x and P_x^{*,h_0} are mutually absolutely continuous except x of a polar set. Thus in particular, the sample paths $x_t(\omega)$ converge to both of the Martin exit boundary and the Martin entrance boundary as $t \to \zeta$, for almost all ω relative to P_x and P_x^{*,h_0} except x of a polar set[15]. We can

14) The latter half of this theorem has been pointed out by Fukushima "On Feller's kernel and the Dirichlet norm", Nagoya Math. J. **24** (1964), 167-175.

15) The polar set of the minimal L-diffusion and that of the minimal L^{*,h_0}-diffusion coincides. See [16].

prove the above fact for all $x \in E$. In fact, since $w(x) = P_x(x_{\zeta-} = \lim_{t \uparrow \zeta} x_t$ exists in M and M^*) is harmonic and coincides with 1 quasi-everywhere, it has to be 1 for all x, by the continuity of harmonic function. Similarly as the above, $x_{\zeta-}$ exists in M and M^* a.e. P_x^{*,h_0} for all $x \in E$.

The absolute continuity of the measure P_x and $P_x^{*,h}$ (quasi-everywhere) implies the mutual absolute continuity of the terminal distributions $P_x(x_{\zeta-} \in E)$ and $P_x^{*,h_0}(x_{\zeta-} \in B)$ (B is a Borel set of ∂M or ∂M^*), for all x. (We can remove "quasi-everywhere" by a similar reason as the above).

Now let us notice that the convergence to a.e. points (μ_0) of the Martin boundary in the fine topology coincides with the convergence to the boundary along with the sample paths a.e. P_x[16]. Then the assertion of Theorem 2 follows.

4.6. Let u be a harmonic function of the minimal L-diffusion. Then $-u^2$ is superharmonic. In particular, if u is represented as (4.2) with $\bar{u} \in L^2(\partial E, \mu_0)$, $-u^2$ dominates the harmonic function $-\int K(x, \eta)\, \bar{u}(\eta)^2 \mu_0(d\eta)$. Hence it has the Riesz decomposition. Denote the potential part of $-u^2$ as u_p and, by μ the corresponding potential measure. Then we have

Lemma 4.2. (Doob [2]). *Let u be a harmonic function represented as* (4.2) *with $\bar{u} \in L^2(\partial E, \mu_0)$. Then*

$$\frac{\partial u_p}{\partial g}(\xi) = \int [\bar{u}(\xi) - \bar{u}(\eta)]^2 \Theta(\xi, \eta)\, \mu_0(d\eta) \qquad \text{a.e. } \mu_0^*.$$

Proof. Since

$$u_p - u^2 = -\int K(x, \eta)\, \bar{u}(\eta)^2 \mu_0(d\eta)$$

holds,

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16) This fact has been proved independently by Föllmer, Meyer and the author. See [16], H. Föllmer "Feine Topologie am Martinrand eines Standard processes," Z. Wahrscheinlichkeits theorie Verw. Geb. **12** (1969), 127-144. H. Kunita, "Markov process and Martin boundary," Sem. on Prob. **17** (1963) (Japanese).

$$\frac{\partial u_p}{\partial g}(y)=-\int_{\partial E}\frac{\bar{u}(\eta)^2-u(y)^2}{g\gamma(y)}K(y,\eta)\mu_0(d\eta)$$

$$=\int_{\partial E}[\bar{u}(\eta)-\bar{u}(\xi)]^2\Theta(y,\eta)\mu_0(d\eta)-\frac{1}{g\gamma(y)}[\bar{u}(\xi)-u(y)]^2$$

$$\leqq\int_{\partial E}[\bar{u}(\eta)-\bar{u}(\xi)]^2\Theta(y,\eta)\mu_0(d\eta).$$

Since the last term of the above has the finely continuous extension to $E\cup\partial E$, we have

$$\frac{\partial u}{\partial g}(\xi)\leqq\int[\bar{u}(\eta)-\bar{u}(\xi)]^2\Theta(\xi,\eta)\mu_0(d\eta). \tag{4.7}$$

On the other hand, notice

$$-\int\frac{\bar{u}(\eta)^2-u(y)^2}{g\gamma(y)}\mu_0(d\eta)=\int[\bar{u}(\eta)-u(y)]^2\Theta(y,\eta)\mu_0(d\eta)$$

and the fine limit of u is $\bar{u}$. Then we obtain

$$\frac{\partial u_p}{\partial g}(\xi)\geqq\int[\bar{u}(\xi)-\bar{u}(\eta)]^2\Theta(\xi,\eta)\mu_0(d\eta) \tag{4.8}$$

by Fatou's lemma. The above two inequalities (4.7) and (4.8) imply the assertion.

Proof of Theorem 3. Since

$$-Lu_p=Lu^2=2\sum_{i,j}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$$

the Riesz measure of the potential u_p is given by $2\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$. Consequently Proposition 4.2 and Lemma 4.2 conclude the theorem.

§5. Invariant measures of L-diffusion resolvents

In this section, we discuss several properties of L-diffusion resolvents involving invariant measures.

Let R_α, $\alpha>0$ be an L-diffusion resolvent satisfying (R.1) and

(R.2), and R_α^*, the adjoint of R_α in $L^2(E, dx)$. We may assume without loss of generality that R_α^* maps $L^\infty(E, dx)$ into $C(E)$, by the assumption (R.1) and Stampacchia [21]. The function h of (R.2) satisfies $\alpha R_\alpha^* h = h$ for all $\alpha > 0$, so that h satisfies $L^* h = 0$ and is continuous. We define a new resolvent $R_\alpha^{*,h} f$ by $h^{-1} R_\alpha^* hf$. Then $R_\alpha^{*,h}$ is a conservative Markovian resolvent. Similarly as Proposition 2.5, we have

Proposition 5.1. *$R_\alpha^{*,h}$ is an $L^{*,h}$-diffusion resolvent.*

Proposition 5.2. *Both of R_α and $R_\alpha^{*,h}$ map $L^\infty(E, dx)$ into $C_b(E)$, and for all $f \in C_b(E)$ both of $\alpha R_\alpha f$ and $\alpha R_\alpha^{*,h} f$ converge to $f(x)$ as $\alpha \to \infty$ at every point x.*

Proof. The first assertion is obvious. Let us notice

$$\alpha R_\alpha f(x) = \alpha G_\alpha f(x) + E_x(e^{-\alpha\zeta} \bar{u}_\alpha(x_{\zeta-})),$$

where $\bar{u}_\alpha$ is the boundary value of $\alpha R_\alpha f$. Since ess sup $|\bar{u}_\alpha(x_{\zeta-})| \leqq \sup|f(x)| < \infty$, the second term of the right hand converges to 0 as $\alpha \to \infty$, which proves $\lim_{\alpha\to\infty} \alpha R_\alpha f(x) = \lim_{\alpha\to\infty} \alpha G_\alpha f(x) = f(x)$. The convergence of $\alpha R_\alpha^{*,h} f(x)$ is proved similarly.

Let us denote the inner product and the norm of $L^2(E, m)$ as $(\ ,\ )_m$ and $\|\ \|_m$, where m is the measure defined by $m(dx) = h dx$. Since m is an invariant measure of R_α, R_α and $R_\alpha^{*,h}$ can be regarded as resolvents in $L^2(E, m)$ with norm conditions $\alpha\|R_\alpha\|_m \leqq 1$ and $\alpha\|R_\alpha^{*,h}\|_m \leqq 1$. (e.g. Yosida [23]). Further the ranges $R(R_\alpha)$ and $R(R_\alpha^{*,h})$ are both dense in $L^2(E, m)$ by Proposition 5.2. Thus there exists strongly continuous and contraction semigroups T_t and $T_t^{*,h}$ associated with R_α and $R_\alpha^{*,h}$, respectively. We denote the generators of T_t^* and $T_t^{*,h}$ as A and $A^{*,h}$. It is well known that the contraction property implies that A and $A^{*,h}$ are dissipative, i.e. $(u, Au)_m \leqq 0$ for all $u \in D(A)$ etc. However, we can mention a stronger assertion in this situation.

Proposition 5.3. *The following two inequalities are satisfied.*

$$(5.1) \qquad -(u, Au)_m \geqq \int \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dm \qquad \textit{for all } u \in D(A)$$

$$(5.2) \qquad -(u, A^{*,h} u)_m \geqq \int \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dm \qquad \textit{for all } u \in D(A^{*,h}).$$

Proof. We shall only verify (5.1), because the proof of (5.2) is carried over similarly. Set $A^\beta = A\beta R_\beta = \beta(\beta R_\beta - I)$. Since m is an invariant measure of R_α, we have $\int A^\beta u^2 \, dm = 0$ or equivalently,

$$(5.3) \qquad -2(A^\beta u, u)_m = \int (A^\beta u^2 - 2u A^\beta u) \, dm.$$

The integrand of the right hand side of the above is nonnegative because

$$(A^\beta u^2 - 2u A^\beta u)(x) = \beta(\beta R_\beta u^2 - 2u \beta R_\beta u + u^2)(x)$$

$$\geqq \beta \int R_\beta(x, dy) (u(x) - u(y))^2 \geqq 0.$$

Therefore for any v with compact support in E such that $0 \leqq v \leqq 1$,

$$(5.4) \qquad -2(A_\beta u, u)_m \geqq \int \{A(\beta R_\beta u^2) - 2u A \beta R_\beta u\} vh dx.$$

Now we can choose the above v so that $L^* v$ is a bounded function. In fact let $\tilde{G}_\alpha$ be the minimal L-diffusion resolvent in U, where U is an open set with regular boundary and $\bar{U} \subset E$. Then $v = \tilde{G}_1^* f$, $0 \leqq f \leqq 1$ has all these properties. Now it is easy to see that $L^*(vh)$ and $L^*(uvh)$ belong to $L^2(U, dx)$. Consequently, (5.4) can be rewritten as

$$-2(A_\beta u, u)_m \geqq \int \{L^*(vh) \beta R_\beta u^2 - 2L^*(uvh) \beta R_\beta u\} dx.$$

Making β tend to ∞, we get

$$-2(Au, u)_m \geqq \int \{L^*(vh) u^2 - 2L^*(uvh) u\} dx$$

$$\geqq \int vh(Lu^2 - 2uLu)\,dx$$

$$\geqq 2\int\Big(\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)v\,dm.$$

Since v is arbitrary, we get the inequality (5.1).

Let $D(D^h)$ be the set of all functions u whose first derivatives $\frac{\partial u}{\partial x_i}$ (in the sense of the distribution) satisfy $\int \sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}h\,dx < \infty$. Proposition 5.3 shows that both of $D(A)$ and $D(A^{*,h})$ are included in $D(D^h)$. We define the symmetric bilinear form D^h in $D(D^h)\times D(D^h)$ as follows;

$$D^h(u, v) = \int_E \sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}h\,dx \tag{5.5}$$

Then $D(D^h)$ is a vector lattice and $D^h((u-c)^+, u\wedge c)=0$ is satisfied for all $u\in D(D^h)$ and nonnegative constant c, where $(u-c)^+ = u - u\wedge c$. The following proposition is sharper than the preceding one.

Proposition 5.4. Let c be an arbitrary nonnegative constant. Then

$$-((u-c)^+, Au)_m \geqq D^h((u-c)^+, u) \qquad {}^\forall u\in D(A). \tag{5.6}$$

$$-((u-c)^+, A^{*,h}u)_m \geqq D^h((u-c)^+, u) \qquad {}^\forall u\in D(A^{*,h}). \tag{5.7}$$

Proof. We only prove (5.6). Since $m = h\,dx$ is the invariant measure, we have

$$-2((u-c)^+, A^\beta u)$$

$$= \beta\int[\beta R_\beta\{(u-c)^+\}^2 - 2(u-c)^+\beta R_\beta u + (u-c)^+(u+u\wedge c)]\,dm.$$

But the integrand of the right hand is greater than or equal to

$$\int \beta R_\beta(\,\cdot\,, dx)\{(u-c)^+(\cdot) - (u-c)^+(x)\}^2$$

$$-2(u-c)^+\{\beta R_\beta(u\wedge c) - u\wedge c\} \geqq 0.$$

Let v be the function used in the proof of Proposition 5.3. Then

$$-2((u-c)^+, A^\beta u)_m$$

$$\geqq \int v[A^\beta\{(u-c)^+\}^2 - 2(u-c)^+ A^\beta u] h\,dx$$

$$= \int L^*(vh)[\beta R_\beta\{(u-c)^+\}^2 - 2L^*(vh(u-c)^+)\beta R_\beta u]\,dx$$

$$\xrightarrow[\beta\to\infty]{} \int [L^*(vh)\{(u-c)^+\}^2 - 2uL^*(vh(u-c)^+)]\,dx$$

$$= 2D^h((u-c)^+, (u-c)^+) + 2B(vh(u-c)^+, u\wedge c)$$

$$= 2D^h((u-c)^+, u)$$

This proves (5.6).

Remark. Let us introduce the norm $|||\ |||_m$ as

$$|||u|||_m = \sqrt{D^h(u, u) + (u, u)_m}$$

and denote the completion of $C_0^\infty(E)$ by the above norm as $D_{0,h}$. Let $A^{*,h}$ be the adjoint of A in $L^2(E, m)$. Then both of $D(A)$ and $D(A^{*,h})$ contain dense subsets of $D_{0,h}$. In fact, since $(Lu, v)_m = (u, L^{*,h} v)_m$ is satisfied for all u with $Lu \in L^2(E, m)$ and for $v \in H_0^1(U)\,(\bar{U} \subset E)$ with $L^{*,h} v \in L^2(E, m)$, $D(A^{*,h})$ contains such v, which implies that $D(A^{*,h})$ contains a dense subset of $D_{0,h}$. This shows that $A^{*,h}$ is exactly the generator of the adjoint semi-group $T_t^{*,h}$ of T_t in $L^2(E, m)$. As a consequence of this fact, it turns out that $D(A)$ contains a dense subset of $D_{0,h}$, too. ([23, Chapt. IX]).

At the end of this section, we give a sufficient condition that an L-diffusion resolvent has an invariant measure $m = h\,dx$.

Proposition 5.5. *Suppose that R_α is a conservative L-diffusion resolvent in $L^2(E, dx)$ such that $(\alpha-\beta_0)\|R_\alpha^m\| \leqq 1$ holds for $m = 1, 2, \cdots$. Then there exists a strictly positive function h of $L^2(E, dx)$ such that $T_t^* h = h$, where T_t^* is the semi-group of the adjoint resolvent.*

Proof. Let $R(I-\alpha R_\alpha)$ be the range of $L^2(E, dx)$ by the mapping $I-\alpha R_\alpha$ and $N(I-\alpha R_\alpha)$ be the kernel. Then $L^2(E, dx) = R(I-\alpha R_\alpha)\oplus N(I-\alpha R_\alpha)$ holds (See Yosida [23. Chapt. VIII]). Since $1\in N(I-\alpha R_\alpha)$ by the conservativity of R_α, $R(I-\alpha R_\alpha)\neq L^2(E, dx)$. Then there exists g of $L^2(E, dx)$ such that $(g, (I-\alpha R_\alpha)f)=0$ for all $f\in L^2(E, dx)$. This means that g satisfies $g=\alpha R_\alpha^* g$ and hence $\alpha R_\alpha^* |g| \geqq |g|$ or equivalently, $T_t^*|g|\geqq |g|$. On the other hand, we have

$$\int T_t^* |g|\, dx = \int T_t 1 |g|\, dx = \int |g|\, dx.$$

Consequently $h\equiv |g|$ satisfies $T_t^* h=h$. Since h is not identically 0, it is strictly positive by virtue of the relation $h=\alpha R_\alpha^* h\geqq \alpha G_\alpha^* h$.

§6. Boundary conditions for L-diffusion resolvents.

We now come to the place of proving Theorems 4 and 5. A crucial point for the proof of Theorem 4 is to find the operator Q, which is done in series of propositions. We will fix a conservative L-diffusion resolvent R_α satisfying (R.1) and (R.2), unless otherwise mentioned.

6.1. Let h be a function of (R.2). It is convenient to choose the reference measure γ for the Martin kernel such as $\gamma(dx)=(h(x)+1)dx$ or slightly generally, $\gamma(dx)=f(x)(h(x)+1)\,dx$, where f is a bounded measurable function. Since $\alpha R_\alpha^* h=h$ is satisfied by (R.2), it satisfies $L^*h=0$ by (R.1). Hence h is represented as (4.4) by Corollary 3 to Proposition 4.1′.

Let us first introduce several notations. Let $\bar{h}$ be the boundary value of h. Set $\nu=\frac{1}{2}(\mu_0+\bar{h}\mu_0^*)$, $c(\eta)=\frac{d\mu_0}{d\nu}(\eta)$ and $c^*(\xi)=\bar{h}\frac{d\mu_0}{d\nu}(\xi)$. Since μ_0 and μ_0^* are mutually absolutely continuous by Theorem 2, the functions c and c^* are bounded and c is strictly positive a.e. ν. Define new kernels as

$$H_\alpha(x, \eta)=K_\alpha(x, \eta)c(\eta), \qquad H_\alpha^*(\xi, x)=K_\alpha^*(\xi, x)c^*(\xi),$$

$$U_\alpha(\xi, \eta)=\int \hat{H}_\alpha^*(\xi, x)H(x, \eta)dx=\Theta_\alpha(\xi, \eta)c^*(\xi)c(\eta).$$

Then $U_\alpha(\xi, \eta)$ increases to $\Theta(\xi, \eta)c^*(\xi)c(\eta)$ as $\alpha\to\infty$, by Proposition 4.2. The following notations are often used later.

$$H_\alpha\varphi(x)=\int H_\alpha(x, \eta)\varphi(\eta)\nu(d\eta),$$

$$H_\alpha^*\varphi(x)=\int H_\alpha^*(\xi, x)\varphi(\xi)\nu(d\xi),$$

$$\hat{H}_\alpha^* f(\xi)=\int H_\alpha^*(\xi, x)f(x)dx,$$

$$U_\alpha\varphi(\xi)=\int U_\alpha(\xi, \eta)\varphi(\eta)\nu(d\eta).$$

In case $\alpha=0$, we drop the suffix α.

Lemma 6.1. *U_α is a bounded operator in $L^2(\partial E, \nu)$.*

Proof. The kernels U_α admits the following properties. (i) $U_\alpha(\xi, \eta)$ increases with α, (ii) $\alpha^{-1}U_\alpha(\xi, \eta)$ decreases as α increasing (Proposition 4.3), and (iii) $U_\alpha 1$ and $U_\alpha^* 1$ are bounded functions, because of the following two inequalities;

$$U_\alpha 1(\xi)=c^*(\xi)\int K_\alpha^*(\xi, x)dx\leqq c^*(\xi)\int K^*(\xi, x)dx\leqq c^*(\xi),$$

$$U_\alpha^* 1(\eta)=c(\eta)\int H_\alpha^* 1(x)K(x, \eta)dx\leqq c(\eta)\int K(x, \eta)h(x)dx\leqq c(\eta).$$

Here U_α^* is the adjoint of U_α in $L^2(\partial E, \nu)$. Consequently, we have

$$\begin{aligned}|(\psi, U_\alpha\varphi)_\nu|&\leqq\left[\iint U_\alpha(\xi, \eta)\psi(\xi)^2\nu(d\xi)\nu(d\eta)\right]^{\frac{1}{2}}\\&\quad\times\left[\iint U_\alpha(\xi, \eta)\varphi(\eta)^2\nu(d\xi)\nu(d\eta)\right]^{\frac{1}{2}}\\&\leqq\left[\int U_\alpha 1(\xi)\psi(\xi)^2\nu(d\xi)\right]^{\frac{1}{2}}\left[\int U_\alpha^* 1(\eta)\varphi(\eta)^2\nu(d\eta)\right]^{\frac{1}{2}}\end{aligned}$$

$$\leqq \text{const } \|\psi\|_{\nu} \|\varphi\|_{\nu}.$$

This proves that U_α is a bounded operator.

6.2. By virtue of Theorem 1 of [14], there exists a unique kernel $r_\alpha(x, y)$ which satisfies the following three conditions;

(i) $R_\alpha f(x) = \int r_\alpha(x, y) f(y) dy, \quad R_\alpha^* f(x) = \int r_\alpha(y, x) f(y) dy.$

(ii) $r_\alpha(x, y)$ is α-excessive function of x and y relative to R_α and R_α^*, respectively.

(iii) For each $\alpha, \beta > 0$ and $x, y \in E$,

$$r_\alpha(x, y) - r_\alpha(x, y) + (\alpha - \beta) \int r_\alpha(x, z) r_\beta(z, y) dz = 0.$$

Furthermore,

Proposition 6.1. *For each fixed x (or y), $h_\alpha(x, y) - g_\alpha(x, y)$ is a nonnegative solution of $(L - \alpha)u = 0$ (or $(L^* - \alpha)u = 0$).*

Proof. Observe that $\int h_\alpha(x, y) f(y) dy$ $(f \in L^\infty(E, dx))$ is α-harmonic function of x. Then for each open ball B of E and $x \in B$,

$$\int H_{\partial B}^\alpha(x, dz) h_\alpha(z, y) = h_\alpha(x, y) \tag{6.1}$$

holds for almost all y since $R_\alpha f - G_\alpha f$ is an α-harmonic function. Take the regularization for the both sides of (6.1), as the function of y. Then we see that equality (6.1) holds for every y of E. Thus $h_\alpha(x, y)$ is α-harmonic or a solution of $(L - \alpha) u = 0$ by Theorem 2.1. A similar discussion proves that $h_\alpha(x, y)/h_0(y)$ is α-harmonic relative to L^{*, h_0}-diffusion. Hence $h_\alpha(x, y)$ is a solution of $(L^* - \alpha)u = 0$ by Theorem 2.1′.

Our next task is to get the Feller representation of $r_\alpha(x, y)$.

Proposition 6.2. *There exists a nonnegative $\partial E \times \partial E$-measurable function $M_\alpha(\eta, \xi)$ such that*

$$(6.2)\qquad h_\alpha(x, y)=\iint K_\alpha(x, \eta)M_\alpha(\eta, \xi)K_\alpha^*(\xi, y)\mu_0(d\eta)\bar{h}(\xi)\mu_0^*(d\xi)$$

for every $x, y\in E$. *Furthermore, such* M_α *is unique up to* $\mu_0\times\mu_0^*$-*measure* 0.

Proof. Let $f\in L^\infty(E, dx)$. Since $u=R_\alpha f-G_\alpha f$ is a bounded α-harmonic function, there exists a bounded measurable function $\tilde{M}_\alpha(f)(\eta)$ on ∂E such that $R_\alpha f(x)-G_\alpha f(x)=\int K_\alpha(x, \eta)\tilde{M}_\alpha(f)(\eta)\mu_0(d\eta)$ holds. The function $\tilde{M}_\alpha(f)$ enjoys the following properties; $\tilde{M}_\alpha(f+g)=\tilde{M}_\alpha(f)+\tilde{M}_\alpha(g)$ a.e. μ_0 and $\tilde{M}_\alpha(f_n)$ decrease to 0 a.e. μ_0 if f_n decrease to 0 a.e. dx. Consequently, there exists a function $\tilde{M}_\alpha(\eta, y)$ such that $\tilde{M}_\alpha(f)(\eta)=\int\tilde{M}_\alpha(\eta, y)f(y)dy$. Therefore for almost all y,

$$h_\alpha(x, y)=\int K_\alpha(x, \eta)\tilde{M}_\alpha(\eta, y)\mu_0(d\eta)$$

holds for all x. We shall show the above holds for all x and y, by taking a suitable version of $\tilde{M}_\alpha(\eta, y)$. Since $h_\alpha(x, y)$ is an α-excessive function of y relative to G_α^*, $\int\beta G_{\alpha+\beta}^*(y, dz)\tilde{M}_\alpha(\eta, z)$ increases to a function $M_\alpha(\eta, y)$. This $M_\alpha(\eta, y)$ is obviously a desired version of $\tilde{M}_\alpha$.

Now the function $v=R_\alpha^*hf-G_\alpha^*hf$ is a solution of $(L^*-\alpha)v=0$ belonging to $H_{\mathrm{loc}}^1(E)$ by (R.1), (R.2) and Proposition 5.1. Hence it has the Martin representation. Set $\varphi(\eta)=\int h(x)f(x)K(x, \eta)\,dx$. Since v/h is a bounded function, there exists a bounded measurable function $\tilde{M}_\alpha(\varphi)$ such that $v=\int K_\alpha^*(\xi, x)\tilde{M}_\alpha(\varphi)\bar{h}\mu_0^*(d\xi)$. Hence we have the following equality

$$v(y)=\int K_\alpha^*(\xi, y)\tilde{M}_\alpha(\varphi)\bar{h}(\xi)\mu_0^*(d\xi)=\int\varphi(\eta)M_\alpha(\eta, y)\mu_0(d\eta)$$

This proves that for almost all η, there is $M_\alpha(\eta, \xi)$ such that $\int M_\alpha(\eta, \xi)K_\alpha^*(\xi, x)\bar{h}(\xi)\mu_0^*(d\xi)=M_\alpha(\eta, y)$.

It remains to verify that there is a jointly measurable version of $M_\alpha(\eta, \xi)$. Let B'_+ be the set of all $\varphi\geqq 0$ of $L^\infty(\partial E, \mu_0^*)$ for which

$\int M_\alpha(\eta, \xi)\varphi(\xi)\bar{h}(\xi)\mu_0^*(d\xi)$ is measurable function of η. Then B'_+ contains $\{K_\alpha^*(\xi, y),\ y \in E\}$. Set $u^\eta(y) = h_0(y)^{-1}\int M_\alpha(\eta, \xi)K_\alpha^*(\xi, y)\bar{h}(\xi)\mu_0^*(d\xi)$. The reduced function $H_A^{*,h_0}u^\eta(y)$ relative to the minimal L^{*,h_0}-diffusion is measurable function of η and coincides with $h_0(y)^{-1}\int_A M_\alpha(\eta, \xi)K^*(\xi, y)\bar{h}(\xi)\mu_0^*(d\xi)$, where A is a closed subset of ∂M_{en} (See [14]). Hence B'_+ contains all nonnegative functions of $L^\infty(\partial E, \mu_0^*)$. Thus by an usual argument we can get a jointly measurable version of M_α. The uniqueness of M_α follows from that of the Martin representation. This completes the proof.

6.3. Let $D(M)$ be the set of all $\varphi \in L^\infty(\partial E, \nu)$ such that $\varphi U_\alpha 1^{-1}$ are bounded functions. Then $D(M)$ does not depend on α, because of the properties (i) and (ii) of $U_\alpha 1$ stated in the proof of Lemma 6.1. Then

$$M_\alpha\varphi(\xi) = \int M_\alpha(\xi, \eta)\varphi(\eta)\nu(d\eta)$$

is a linear mapping from $D(M)$ into $L^\infty(\partial E, \nu)$, since $M_\alpha U_\alpha 1 = 1$ following from the conservativeness of R_α. Using this notation, the Feller representation is written as

$$R_\alpha = G_\alpha + H_\alpha M_\alpha \hat{H}_\alpha^*. \tag{6.3}$$

The operator H_α and H_α^* are determined by the minimal L-diffusion. Therefore, all informations of R_α such as (Q.1)$\sim$(Q.2) are included in M_α. We shall investigate its properties in this small section.

We denote by $\mathfrak{B}'$ the smallest σ-field for which the function family $\{\bar{u} \mid u \in D(A)\}$ are measurable, where $\bar{u}$ is the boundary value of u. The L^2-subspace of $L^2(\partial E, \nu)$ consisting of all $\mathfrak{B}'$-measurable functions are denoted by $L^2(\partial E, \mathfrak{B}', \nu)$. Then we have

Proposition 6.3. *For each $\alpha > 0$, there exists a sub-Markov and contraction semigroup T_t^α in $L^2(\partial E, \nu)$ such that $M_\alpha = \int_0^\infty T_t^\alpha dt$. Furthermore, T_t^α restricted to $L^2(\partial E, \mathfrak{B}', \nu)$ forms a strongly continuous* *semigroup.*

Before the proof, we prepare the following lemma.

Lemma 6.2. *For each* $\alpha \geqq 0$, *we have*

$$\int (L-\alpha) u H_\alpha^* \{(\bar{u}-c)^+\} dx \leqq 0, \qquad {}^\forall u \in D(A) \text{ and } c \geqq 0. \tag{6.4}$$

Proof. Let us denote the inner product relative to the measure $m(dx)=hdx$ as $(\ ,\ )_m$. Set $u_p=G_\alpha(\alpha-L)u$, $u \in D(A)$. Then there exists $\{u_{p_n}\}$ of $D(A) \cap H_0^1(E)$ converging to u_p with respect to the norm $\|\!|\ \|\!|_m=\{D^h(\ ,\ )+(\ ,\ )_m\}^{\frac{1}{2}}$, by the remark after Proposition 5.4. Moreover,

$$((u-u_{p_n}-c)^+,\ (L-\alpha)(u-u_{p_n}))_m \leqq 0$$

holds for every n, because $L-\alpha$ restricted to the domain $D(A)$ is completely dispersive. The left hand is rewritten as

$$((u-u_{p_n}-c)^+,\ (L-\alpha)u)_m + B_\alpha^h((u-u_{p_n}-c)^+,\ u_{p_n})$$

where $B^h(v,\ u)=B(vh,\ u)$. Making n tend to $+\infty$ above, we obtain

$$0 \geqq ((H_\alpha \bar{u}-c)^+,\ (L-\alpha)u)_m + B_\alpha^h((H_\alpha \bar{u}-c)^+,\ u_p), \tag{6.5}$$

$$0 \geqq (h(H_\alpha \bar{u}-c)^+,\ (L-\alpha)u) + B_\alpha(h(H_\alpha \bar{u}-c)^+,\ u_p).$$

Now $hH_\alpha(\bar{u}-c)^+$ is decomposed to the sum of $H_\alpha^*\{(\bar{u}-c)^+\} \in H_\alpha^*$ and $v \in H_0^1(E)$, and further

$$B_\alpha(h(H_\alpha \bar{u}-c)^+,\ u_p)=B_\alpha(v,\ u_p)=-(v,\ (L-\alpha)u)$$

holds (§2.2). Hence the last expression of (6.5) coincides with $(H_\alpha^*\{(\bar{u}-c)^+\},\ (L-\alpha)u)$. This completes the proof.

Remark. Similarly as the above, we can prove

$$\int (L-\alpha) u H_\alpha^* \bar{u}\, dx + D^h(H_\alpha \bar{u},\ H_\alpha \bar{u}) + \alpha(H_\alpha \bar{u},\ H_\alpha \bar{u})_m \leqq 0, \qquad {}^\forall u \in D(A).$$

In fact choose $\{u_{p_n}\}$ as before, substitute $u-u_{p_n}$ in the place of u in the equality (5.1), and next make n tend to infinity. Then we get

$$((L-\alpha)u, H_\alpha\bar{u})_m+B_\alpha(hH_\alpha\bar{u}, u_p)+D^h(H_\alpha\bar{u}, H_\alpha\bar{u})+\alpha(H_\alpha\bar{u}, H_\alpha\bar{u})_m\leqq 0.$$

We can prove as before that $B_\alpha(hH_\alpha\bar{u}, u_p)=((L-\alpha)u, H_\alpha^*\bar{u}-hH_\alpha\bar{u})$ and hence we have the inequality.

Proof of Proposition 6.3. The boundary value of $u\in D(A)$ satisfies $\bar{u}=M_\alpha\hat{H}_\alpha^*(\alpha-L)u$ by the Feller representation. Cosequently, (6.4) is rewritten as

$$((M_\alpha\varphi-c)^+, \varphi)_\nu\geqq 0, \qquad \varphi=\hat{H}_\alpha^*(\alpha-L)u.$$

It can be easily seen that the above holds for all $\varphi\in D(M)$[17]. We can now apply Theorem 4 of [12] and we obtain the proposition.

We shall denote the generator of $\hat{T}_t^\alpha$ in $L^2(\partial E, \mathfrak{B}', \nu)$ as Q_α.

Proposition 6.4. *The domains of Q_α, $\alpha>0$ are independent of α. Furthermore, the operator $Q_\alpha+PU_\alpha$ does not depend on $\alpha>0$, where P is the orthogonal projection from $L^2(\partial E, \nu)$ to $L^2(\partial E, \mathfrak{B}', \nu)$.*

For the proof of this proposition we prepare

Lemma 6.3. (cf. Fukushima-Ikeda [7]). *M_α and M_β are related by*

$$M_\alpha-M_\beta+M_\alpha(U_\alpha-U_\beta)M_\beta=0.$$

Proof. Let us write as $R_\alpha f=G_\alpha f+H_\alpha M_\alpha\hat{H}_\alpha^*f$ and subtract the resolvent equation of $G_\alpha f$ from that of $R_\alpha f$. Then

$$H_\alpha M_\alpha\hat{H}_\alpha^*f-H_\beta M_\beta\hat{H}_\beta^*f+(\alpha-\beta)G_\alpha H_\beta M_\beta\hat{H}_\beta^*f$$
$$+(\alpha-\beta)H_\alpha M_\alpha\hat{H}_\alpha^*G_\beta f+(\alpha-\beta)H_\alpha M_\alpha\hat{H}_\alpha^*H_\beta M_\beta\hat{H}_\beta^*f=0.$$

Hence at the boundary,

$$M_\alpha\hat{H}_\alpha^*f-M_\beta\hat{H}_\beta^*f+(\alpha-\beta)M_\alpha\hat{H}_\alpha^*G_\beta f \tag{6.6}$$
$$+(\alpha-\beta)M_\alpha\hat{H}_\alpha^*H_\beta M_\beta\hat{H}_\beta^*f=0.$$

17) $\{\hat{H}_\alpha^*f|f\in L^\infty(E, dx)\}$ is dense in $L^2(\partial E, \nu)$.

On the other hand,

$$(\alpha-\beta)\hat{H}_\alpha^* G_\beta f=\hat{H}_\beta^* f-\hat{H}_\alpha^* f,$$

$$(6.7)\qquad (\alpha-\beta)\hat{H}_\alpha^* H_\beta=(\alpha-\beta)\hat{H}_\alpha^* H+\beta(\alpha-\beta)\hat{H}_\alpha^* G_\beta H$$

$$=(\alpha-\beta)\hat{H}_\alpha^* H+\beta(\hat{H}_\alpha^*-\hat{H}_\beta^*)H$$

$$=\alpha\hat{H}_\alpha^* H-\beta\hat{H}_\beta^* H=U_\alpha-U_\beta.$$

Substitute (6.7) into (6.6) and we get the desired relation for $\varphi=\hat{H}_\beta^* f$. It is easy to extend this for all $\varphi\in D(M)$.

Proof of Proposition 6.4. We know that U_α maps $L^\infty(\partial E, \nu)$ into $D(M)$. Hence by Lemma 6.3, the ranges $\{M_\alpha\varphi\mid\varphi\in D(M)\}$ are independent of α. Since Q_α coincides with M_α^{-1} on the range of M_α and since the equality of the above lemma is written as $M_\alpha-M_\beta+M_\alpha(PU_\alpha-PU_\alpha)M_\beta=0$ for $\varphi\in D(M)$, $Q_\alpha+PU_\alpha$ does not depend on α on the range $\{M_\alpha\varphi\mid\varphi\in D(M)\}$. This proves that

$$(6.8)\qquad M_\alpha^\lambda\varphi-M_\beta^\lambda\varphi+M_\alpha^\lambda(PU_\alpha-PU_\beta)M_\beta^\lambda\varphi=0$$

holds for every $\varphi\in L^2(\partial E, \mathfrak{B}', \nu)$, where $M_\alpha^\lambda=\int_0^t e^{-\lambda t}T_t^\alpha dt$. Repeating the same argument to M_α^λ, we see that $\{M_\alpha^\lambda\varphi\mid\varphi\in L^2(\partial E, \mathfrak{B}', \nu)\}$ is independent of α and λ, and $Q_\alpha+PU_\alpha$ is independent of α on the common ranges $\{M_\alpha^\lambda\varphi\mid\varphi\in L^2(\partial E, \mathfrak{B}', \nu)\}$.

6.4. We denote the common operator $Q_\alpha+PU_\alpha$ as Q and its domain as $D(Q)$. We will show in this small section that this operator Q possesses all properties required in Theorem 4.

Proposition 6.5. *The operator Q is a generator of a conservative Markovian semigroup with ν as its invariant measure.*

Proof. Since U_α is a bounded operator by Lemma 6.1, Q becomes a generator of a semigroup by Phillips' perturbation theorem of semigroup. Furthermore, the operator Q is completely dispersive because it is the limit of the completely dispersive operators $\{Q_\alpha\}$ (as α

tend to 0). Consequently, the semigroup associated to the generator Q is sub-Markov. Now, the property $M_\alpha U_\alpha 1=1$ implies $Q_\alpha 1=-PU_\alpha 1$ or equivalently, $Q1=0$, which proves the conservativeness of the semigroup.

Since $m=hdx$ is the invariant measure of R_α, the adjoint resolvent $R_\alpha^{*,h}$ with respect to m is again conservative $L^{*,h}$-diffusion resolvent. Therefore semigroup $\hat{T}_t^{\alpha,*}$, which is the adjoint of $\hat{T}_t^\alpha$ in $L^2(\partial E, \mathfrak{B}', \nu)$ is again strongly continuous and sub-Markov by [12, Corollary to Theorem 3]. This implies that the adjoint Q_α^* of Q_α is exactly the generator of $\hat{T}_t^{\alpha,*}$. Repeating the same discussion to Q^*, we see $Q^*1=0$ or $(1, Q\varphi)_\nu=0$ for all $\varphi\in D(Q)$. This proves that ν is the invariant measure of the semigroup associated with the generator Q. This completes the proof.

The property (Q.2) is proved at the next proposition.

Proposition 6.6. (cf. Lemma 5.5 of [7]). *$D(Q)\subset D(\tilde{D}^h)$ and*

$$(\varphi^+, Q\varphi)_\nu+\frac{1}{2}\tilde{D}^h(\varphi^+, \varphi^+) \tag{6.9}$$

$$-\int\Theta(\xi, \eta)\varphi^-(\xi)\varphi^-(\eta)\bar{h}(\xi)\mu_0^*(d\xi)\mu_0(d\eta)\leqq 0$$

holds for all $\varphi\in D(Q)$.

Proof. Set $\tilde{Q}_\alpha=Q_\alpha+\frac{1}{2}(PU_\alpha 1+PU_\alpha^* 1)I$, where I is the identity operator. Since $\frac{1}{2}(PU_\alpha 1+PU_\alpha^* 1)I$ is a bounded operator, $\tilde{Q}_\alpha$ constitutes a generator of a strongly continuous semigroup. We shall prove that the associated semigroup is contraction. Observe $(\varphi, \tilde{Q}_\alpha\varphi)_\nu=(\varphi, Q\varphi)_\nu+\tilde{D}_\alpha^h(\varphi, \varphi)$, where

$$\tilde{D}_\alpha^h(\varphi, \psi)=\frac{1}{2}\iint U_\alpha(\xi, \eta)(\varphi(\xi)-\varphi(\eta))(\psi(\xi)-\psi(\eta))\nu(d\xi)\nu(d\eta).$$

Since $\tilde{D}_\alpha^h(\varphi, \varphi)\leqq\tilde{D}^h(\varphi, \varphi)\leqq+\infty$, we obtain

$$(\varphi, \tilde{Q}_\alpha\varphi)_\nu\leqq(\varphi, Q\varphi)_\nu+\tilde{D}^h(\varphi, \varphi).$$

Now, substitute $\bar{u}=M_\beta \hat{H}^*_\beta(\beta-A)u$ in the place of φ above and next make β tend to 0. Then we obtain

$$(\bar{u},\ \tilde{Q}_\alpha \bar{u})_\nu \leqq (H^*\bar{u},\ Au)+\tilde{D}^h(\bar{u},\ \bar{u}).$$

But the right hand of the above inequality coincides with $(H^*\bar{u},\ Au)+D^h(H\bar{u},\ H\bar{u})$ by Theorem 3 and hence it is nonpositive by Remark after Lemma 6.2. Since $\tilde{Q}_\alpha$ is the smallest closed extension of $\tilde{Q}_\alpha$ restricted to $\{\bar{u}\mid u\in D(A)\}$, we have $(\varphi,\ \tilde{Q}_\alpha\varphi)_\nu\leqq 0$ for all $\varphi\in D(Q)$. Therefore the semigroup associated with $\tilde{Q}_\alpha$ has contraction property.

Incidentally, we have $D(Q)\subset D(\tilde{D}^h)$ and $(\varphi,\ Q\varphi)_\nu+\tilde{D}^h(\varphi,\ \varphi)\leqq 0$. In fact, notice $(\varphi,\ Q\varphi)_\nu+\tilde{D}^h_\alpha(\varphi,\ \varphi)=(\varphi,\ \tilde{Q}_\alpha\varphi)_\nu\leqq 0$ and make α tend to $+\infty$, then we have $0\leqq \tilde{D}^h(\varphi,\ \varphi)\leqq -(\varphi,\ Q\varphi)_\nu<\infty$.

On the other hand, since

$$(\varphi^+,\ \tilde{Q}_\alpha\varphi)_\nu=(\varphi^+,\ Q\varphi)_\nu+((\varphi^+)^2,\ \frac{1}{2}(U_\alpha 1+U^*_\alpha 1))_\nu\leqq K\|\varphi^+\|^2_\nu,$$

$\tilde{Q}_\alpha$ is K-dispersive in the terminology of [12], where $K=\text{ess sup}\,\frac{1}{2}(U_\alpha 1+U^*_\alpha 1)$. Therefore, the associated semigroup is positive. Consequently, the operator $\tilde{Q}_\alpha$ has to satisfy the inequality $(\varphi^+,\ \tilde{Q}_\alpha\varphi)_\nu\leqq 0$ for all $\varphi\in D(Q)$ (See [12]). However $(\varphi^+,\ \tilde{Q}_\alpha\varphi)_\nu$ is rewritten as

$$(\varphi^+, Q\varphi)_\nu-\iint U_\alpha(\xi,\ \eta)\varphi^+(\xi)\varphi^-(\eta)\nu(d\xi)\nu(d\eta)+\frac{1}{2}\tilde{D}^h_\alpha(\varphi^+,\ \varphi^+).$$

Making n tend to infinity, we obtain (6.9). The proof is completed.

We have so far proved that the operator Q satisfies (Q.1) and (Q.2) of Theorem 4. We shall prove now the boundary condition

(6.10) $D(A)=\{u\in C_b\mid Lu\in L^\infty(E,\ dx),\ u\in D(Q)$ and $Q\bar{u}-P\hat{H}^*Lu=0\}$

Denote the right hand of the above as $\tilde{D}$. Let u be of $D(A)$. Then $-(Q-PU_\alpha)\bar{u}=P\hat{H}^*_\alpha(\alpha-L)u$ holds. Letting α tend to 0, we get the equality $Q\bar{u}-P\hat{H}^*Lu=0$. This proves $D(A)\subset\tilde{D}$. Conversely, take u from $\tilde{D}$, set $f=(\alpha-L)u$ and define $v=R_\alpha f$ by (6.4). Then $w=u-v$ is an α-harmonic function belonging to $\tilde{D}$. Hence $Q\bar{w}+P\hat{H}^*(-L)w=0$

or equivalently, $Q\bar{w}-PU_\alpha\bar{w}=0$. Hence we have

$$\frac{1}{2}(U_\alpha 1+U_\alpha^* 1, \bar{w}^2)_\nu=(\bar{w}, Q\bar{w})_\nu+\tilde{D}_\alpha^h(\bar{w}, \bar{w})=(\bar{w}, \tilde{Q}_\alpha\bar{w})_\nu\leqq 0.$$

Since $\frac{1}{2}(U_\alpha 1+U_\alpha^* 1)$ is strictly positive a.e. ν, $\bar{w}$ is identically 0. Thus we have proved $\tilde{D}\subset D(A)$. The proof of Theorem 4 is now completed.

6.5. Proof of Theorem 5. Let Q be an operator satisfying (Q.1) and (Q.2). Note that

$$\Theta(\xi, \eta)\bar{h}(\xi)\mu_0^*(d\xi)\mu_0(d\eta)\geqq U_\alpha(\xi, \eta)\nu(d\xi)\nu(d\eta),$$

then the following inequality is immediate.

$$\tilde{D}^h(\varphi^+, \varphi^+)-\iint\Theta(\xi, \eta)\varphi^-(\xi)\varphi^-(\eta)\bar{h}(\xi)\mu_0^*(d\xi)\mu_0(d\eta)$$

$$\geqq\tilde{D}_\alpha^h(\varphi^+, \varphi^+)-\iint U_\alpha(\xi, \eta)\varphi^+(\xi)\varphi^-(\eta)\nu(d\xi)\nu(d\eta).$$

Hence the inequality of (Q.2) implies

$$(\varphi^+, Q\varphi)_\nu+\tilde{D}_\alpha^h(\varphi^+, \varphi^+)-\iint U_\alpha(\xi, \eta)\varphi^-(\xi)\varphi^-(\eta)\nu(d\xi)\nu(d\eta)\leqq 0.$$

Setting $\tilde{Q}_\alpha=Q+\left(\frac{1}{2}PU_\alpha 1+\frac{1}{2}PU_\alpha^* 1\right)I-PU_\alpha$, the above inequality is equivalent to $(\varphi^+, \tilde{Q}_\alpha\varphi)_\nu\leqq 0$. This proves

$$(\varphi^+, (Q-PU_\alpha)\varphi)_\nu=-\left(\varphi^+, \frac{1}{2}\{PU_\alpha 1+PU_\alpha^* 1\}\varphi\right)_\nu\leqq 0.$$

Since Q is completely dispersive by (Q.1), we have

$$((\varphi-c)^+, (Q-PU_\alpha)\varphi)_\nu\leqq c((\varphi-c)^+, -PU_\alpha 1)_\nu\leqq 0.$$

This shows that the semigroup associated with the generator $Q-PU_\alpha$ is sub-Markov.

Now the inequality (Q.2) implies $(Q\varphi, \varphi)_\nu + \tilde{D}^h(\varphi, \varphi) \leqq 0$. In fact, substituting $-\varphi$ in the place of (Q.2), we obtain

$$(\varphi^-, Q\varphi)_\nu + \frac{1}{2} \tilde{D}^h(\varphi^-, \varphi^-)$$

$$+ \iint \Theta(\xi, \eta) \varphi^-(\xi) \varphi^-(\xi) \bar{h}(\xi) \mu_0^*(d\xi) \mu_0(d\eta) \leqq 0.$$

Summing up this and (Q.2), we obtain

$$0 \geqq (\varphi, Q\varphi)_\nu + \frac{1}{2} \tilde{D}^h(\varphi^-, \varphi^-) + \frac{1}{2} \tilde{D}^h(\varphi^-, \varphi^-) + \tilde{D}^h(\varphi^+, \varphi^-)$$

$$\geqq (\varphi, Q\varphi)_\nu + \frac{1}{2} \tilde{D}^h(\varphi, \varphi).$$

Therefore we have

$$0 \geqq (\varphi, Q\varphi)_\nu + \frac{1}{2} \tilde{D}^h(\varphi, \varphi)$$

$$\geqq (\varphi, (Q - PU_\alpha)\varphi)_\nu + \frac{1}{2} \int (U_\alpha 1 + U_\alpha^* 1) \varphi^2 d\nu.$$

Consequently, if $(\varphi, (Q - PU_\alpha)\varphi)_\nu = 0$, φ has to be 0, which proves the existence of $(Q - PU_\alpha)^{-1}$. We shall write this as M_α. Then M_α satisfies equality of Lemma 6.3 by the definition, and $M_\alpha U_\alpha 1 = 1$ by $Q1 = 0$. From these two properties it is easy to see that $R_\alpha f$ defined by (6.3) satisfies the resolvent equation and is actually conservative Markovian resolvent.

The above defined resolvent R_α satisfies (R.1) obviously. We shall prove that (R.2) is also satisfied for this resolvent. Let us notice that Q is a generator of a strongly continuous, conservative Markovian semigroup in $L^2(\partial E, \mathfrak{B}', \nu)$. Then the adjoint operator Q^* is also a generator of a strongly continuous semigroup by [12, Corollary to Theorem 3]. Since ν is the invariant measure, the associated adjoint semigroup is also conservative Markovian so that $M_\alpha^* U_\alpha^* 1 = 1$, where M_α^* and U_α^* are adjoints in $L^2(\partial E, \mathfrak{B}', \nu)$. This equality proves that

$R_\alpha^{*,h}$ is conservative, so that (R.2) is satisfied.

It remains to verify the boundary condition (6.10). Denote the right hand of (6.10) by $\tilde{D}$, and take u from $D(A)$. Then by (6.3), $-(Q-PU_\alpha)\bar{u}=P\hat{H}_\alpha^*(\alpha-L)u$ or equivalently, $Q\bar{u}=PU_\alpha\bar{u}-P\hat{H}_\alpha^*(\alpha-L)u$. Making α tend to 0, we see $Q\bar{u}=+P\hat{H}^*Lu$. This proves $u\in\tilde{D}$. Conversely, take u from $\tilde{D}$, set $f=(\alpha-L)u$ and define $v=R_\alpha f$ by (6.3). Then $w=u-v$ is an $(L-\alpha)$-solution belonging to $\tilde{D}$. Hence $Q\bar{w}-P\hat{H}^*Lw=0$ or equivalently, $Q\bar{w}-PU_\alpha\bar{w}=0$, which proves $\bar{w}=0$. Thus u belongs to $D(A)$. The proof is completed.

6.6. It is possible to weaken condition (R.2) to get a similar result as Theorem 4. We shall introduce the following condition (R.2′) instead of (R.2).

(R.2′) There exists a strictly positive function belonging to $L^2(E, dx) \cap \overline{H^1(E)}$ for which (5.1) of Proposition 5.3 holds.

Then Theorem 4 is modified as the following way.

Theorem 4′. *Let R_α be a conservative L-diffusion resolvent with* (R.1) *and* (R.2′). *Then there exists a unique operator Q in $L^2(\partial E, \mathfrak{B}', \nu)$ satisfying* (Q.2), (Q.3) *and the following* (Q.1′).

(Q.1′) *Q is a generator of a strongly continuous and conservative Markovian semigroup.*

The proof is similar and is omitted. Theorem 5 can be modified in a obvious way in this direction.

§7. Some special cases

7.1. Condition (Q.2) of Theorem 4 is of special importance. We shall discuss the meaning of (Q.2) in some special cases.

We will assume in this small section that the operator L is self adjoint i.e., $L=\sum \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$. Then, since the minimal L-diffusion resolvent is self adjoint with respect to the Lebesgue measure, the Naim's kernel $\Theta(\xi, \eta)$ is symmetric in ξ and η so that we have

$$-\int\Theta(\xi, \eta)\varphi(\xi)^{-}\varphi(\eta)^{-}\mu_0(d\xi)\mu_0(d\eta)=\frac{1}{2}\tilde{D}(\varphi^{+}, \varphi^{-})$$

Now let R_α be an L-diffusion resolvent satisfying (R.1) and (R.2′) with $h(x)=1$, and let Q be the associated operator on the boundary. Then making use of the above expression, (Q.3) is rewritten as

$$(\varphi^{+}, Q\varphi)_{\mu_0}+\tilde{D}(\varphi^{+}, \varphi)\leqq 0 \qquad \text{for all } \varphi\in D(Q),$$

or

$$\text{(Q.2′)} \qquad ((\varphi-c)^{+}, Q\varphi)_{\mu_0}+\tilde{D}((\varphi-c)^{+}, \varphi)\leqq 0 \qquad \text{for all } \varphi\in D(Q).$$

In order to study the meaning of (Q.2′), we will introduce the reflecting L-diffusion. For each f of $L^2(E, dx)$, we can associate a unique element $u\in H^1(E)$ in such a way that

$$D(u, v)+\alpha(u, v)=(v, f) \qquad \text{for all } v\in H^1(E).$$

Write this u as $R_\alpha^r f$. Then R_α^r is an L-diffusion resolvent. In fact $u=R_\alpha^r f$ is a solution of $(\alpha-L)u=f$ and hence it is continuous if $f\in L^\infty(E, dx)$; The sub-Markov property follows from the fact that $H^1(E)$ is a Dirichlet space in the sense of Beurling-Deny. We call this R_α^r as the *reflecting L-diffusion.* We denote by Q^r the operator of Theorem 4 associated with the reflection L-diffusion. Then Q^r satisfies

$$\text{(7.1)} \qquad (\varphi, Q^r\psi)_{\mu_0}+\tilde{D}(\varphi, \psi)=0$$

for all $\psi\in D(Q^r)$ and $\varphi\in D(\tilde{D})$. In fact, the relation

$$\text{(7.2)} \qquad (H\bar{u}, Av)+D(H\bar{u}, H\bar{v})=0 \qquad v\in D(A) \text{ and } u\in H^1(E),$$

is immediate from the equality $D(u, v)+(u, Av)=0$, $u\in H^1(E)$ and $v\in D(A)$. Let us notice the relation $\hat{H}Lu=Q^r u$ and $D(H\bar{u}, H\bar{v})=\tilde{D}(\bar{u}, \bar{v})$. Then we obtain (7.1) for $\varphi=\bar{u}$ and $\psi=\bar{v}$. The extension to the general $\psi\in D(Q)$ and $\varphi\in D(\tilde{D})$ is obvious.

The operator Q^r coincides with $\frac{\partial H}{\partial g}$ defined in §1. Therefore, the boundary condition of the reflecting L-diffusion is characterized as

$\frac{\partial u}{\partial g}=0$, where $\frac{\partial u}{\partial g}$ is the normal derivative introduced in §1.

Now, the condition (Q′.2) shows

$$(7.3)\quad ((\varphi-c)^+, Q\varphi)_{\mu_0}-\left((\varphi-c)^+, \frac{\partial H}{\partial g}\varphi\right)_{\mu_0}\leqq 0,\ \varphi\in D(Q)\cap D\left(\frac{\partial H}{\partial g}\right).$$

This shows that both of $\frac{\partial H}{\partial g}$ and $Q-\frac{\partial H}{\partial g}$ are completely dispersive. Corollary to Theorem 4 will now be obvious.

7.2. It is possible to get the decomposition of the operator Q for general L-diffusion resolvent. In this small section, we assume that the coefficients b_i of the operator L is bounded or that the boundary ∂E of the open set E in which the operator L is defined, satisfies the cone condition. Then the bilinear form $B(u, v)$ of (2.1) with the domain $H^1(E)\times H^1(E)$ is continuous and bounded from below ([21]). Furthermore, $B((u-c)^+, u\wedge c)=0$ holds for all $u\in H^1(E)$ and positive constant c. Therefore there exists an L-diffusion resolvent R_α^r such that $u=R_\alpha^r f$ satisfies $B(v, u)+\alpha(v, u)=(v, f)$ for all $v\in H^1(E)$ by Theorem 2 of [12]. We call this R_α^r the reflecting L-diffusion resolvent. The reflecting L-diffusion resolvent is conservative because $u=1$ is the solution of $B(v, u)+\alpha(v, u)=(v, \alpha)$ for all $v\in H^1(E)$. Therefore there is a strictly positive function h of $L^2(E, dx)$ such that $m=hdx$ is an invariant measure of R_α^r by Proposition 5.5. The function h satisfies $\alpha R_\alpha^* h=h$ and hence it belong to $H^1(E)$ (The range of R_α^* is included in $H^1(E)$). Consequently, the reflecting L-diffusion satisfies (R.1) and (R.2). We denote the corresponding operator of Theorem 4 by Q^r. Then

Lemma 7.1. *For $\varphi\in L^2(\partial E, \nu)$ and $\psi\in D(Q^r)$, it holds*

$$(7.4)\qquad (\varphi, Q^r\psi)_\nu+B(H^*\bar{h}, H\psi)=0.$$

Proof. Since $B(v, u)+(v, Au)=0$ holds for $u\in D(A)$ and $v\in H^1(E)$, we get $(\varphi, \hat{H}^*Au)_\nu+B(H^*\bar{h}, H\bar{u})=0$ if we set $v=H^*\varphi$. Notice that $Q^r\bar{u}=\hat{H}^*Au$, we obtain (7.4) for $\bar{u}=\psi$. The extension to general ψ is obvious.

Lemma 7.2. *Let R_α be an L-diffusion resolvent satisfying* (R.1) *and* (R.2′) *with the h. Set $B^h(u, v)=B(uh, v)$. Then*

$$(7.5)\qquad ((u-c)^+, Lu)_m+B^h((u-c)^+, u)\leqq 0 \qquad \text{for all } u\in D(A).$$

Proof. We know by Proposition 5.3 that $((u-c)^+, Lu)+D^h((u-c)^+, u)\leqq 0$ holds. On the other hand we have $B^h(u, u)=D^h(u, u)$. In fact, since $B^h(1, u)=0$ for $u\in H^1(E)$, we have $B^h(1, u^2)=0$ if u is bounded, which is written as

$$\int u\left\{\sum\left(b_i-\sum a_{ij}\frac{\partial \log h}{\partial x_j}\right)\frac{\partial u}{\partial x_i}\right\}h\,dx=0$$

by (2.7). This and the formula (2.7) show that $D^h(u, u)=B^h(u, u)$. Therefore

$$\begin{aligned}B^h((u-c)^+, u)&=B^h((u-c)^+, (u-c)^+)\\ &=D^h((u-c)^+, (u-c)^+)=D^h((u-c)^+, u).\end{aligned}$$

This proves (7.5).

Similarly as Lemma 6.2, we obtain the following lemma.

Lemma 7.3. *Under the same condition as Lemma* 7.2, *we have*

$$(\mathrm{Q}.2'')\qquad ((\varphi-c)^+, Q\varphi)_\nu+B(H^*\{(\varphi-c)\}^+, H\varphi)\leqq 0.$$

Proof is similar as that of Lemma 6.2. Let $u\in D(A)$ and $u=-G(Lu)$. Choose $\{u_{p_n}\}$ of $H_0^1(E)\cap D(A)$ converging to u in $|||\ |||_m$-norm. Substitute $u-u_{p_n}$ in the place of u in the equality (7.5) and then make n tend to infinity. Then we obtain

$$(Lu, H^*\{(\bar{u}-c)^+\})+B(H^*\{(\bar{u}-c)^+\}, H\bar{u})\leqq 0$$

by the same argument as Lemma 6.2. This proves (Q.2′′) in case $\psi=\bar{u}$, because $Q\bar{u}=P\hat{H}^*Lu$.

This lemma combined with Lemma 7.1 shows that both of Q^r and $Q-Q^r$ are completely dispersive.

7.3. Finally we show that there exists a bounded and uniformly positive excessive function such that $\alpha G_{\alpha}^{*} h \leqq h$ and $L^{*} h=0$. Let $\tilde{E}$ be a bounded open set with regular boundary including E. Extend the operator L to $\tilde{E}$ as is done in § 2, which we write as $\tilde{L}$. Then the reflecting $\tilde{L}$-diffusion exists and has an invariant measure $m(dx)=h dx$ by Proposition 5.5. This h satisfies $L^{*} h=0$ and strictly positive in $\tilde{E}$. Since h is continuous, the restriction of h to the space E is bounded and uniformly positive. It is easily verified that $\alpha G_{\alpha}^{*} h \leqq h$.

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