J. M. Kyoto Univ.
10-3 (1970) 475-492

On a small drift of Cauchy process

By

Masaaki TSUCHIYA

(Communicated by April 14, 1970)

Introduction

The purpose of this paper is to show the existence and the uniqueness of a Markov process corresponding to the operator $Af(x)=a(x)f'(x)+Bf(x)$, where B is the infinitesinal generator of 1-dimensional symmetric Cauchy process and a is a bounded measurable function. If a is Lipschitz continuous, this problem can be solved as a particular case of the general theory of stochastic integral equations due to K. Ito. The difficulty arises if a is not Lipschitz continuous. In this paper, by making use of the method initiated by D. W. Stroock and S. R. S. Varadhan [13], we shall solve the above problem when a is a measurable function which lies in a sufficiently small neighborhood (with respect to the supremum norm) of a constant function. The problem has important meaning in the so-called boundary problems of diffusion processes since the process corresponding to A becomes the Markov process on the boundary of a Brownian motion with an oblique reflection on the upper half plane. As for the Markov process on the boundary of diffusion processes and its role in the boundary problems of diffusion processes, we refer to M. Motoo [8], K. Sato-T. Ueno [11] and N. Ikeda [2].

Now we summarize the content of this paper. In §1, we prepare the notations and some preliminary facts. In §2, we construct the operator K_λ such that $K_\lambda(\lambda I-A)=I$. Formally, K_λ is expressed as

$$K_\lambda = G_\lambda(I-T_\lambda)^{-1},$$

where $T_\lambda f(x)=a(x)\frac{d}{dx}G_\lambda f(x)$ and G_λ is the resolvent operator of the Cauchy process, i.e., $G_\lambda=(\lambda I-B)^{-1}$. The above expression is justified in § 2. In § 3, following [13], the problem is formulated as a martingale problem, i.e., a problem to find a probability measure on a function space such that a certain functional will be a martingale. In § 4, the uniqueness of solutions of the martingale problem is proved, while the existence of solution is shown in § 5. In § 6, we collect all the previous results to get the main theorem that there exists a unique Markov process corresponding to A if a lies in some neighborhood of a constant function. Some supplements are given in § 7.

ACKNOWLEDGEMENT

The author wishes to express his hearty thanks to Professors M. Motoo, M. Nagasawa, K. Sato, H. Tanaka and S. Watanabe for their valuable advices. Professor Motoo suggested the author the problem treated here.

§ 1. Notations and preliminaries

In this paper we are mainly concerned with the symmetric Cauchy process on the real line R. First of all let us racall the definitions and some properties of the process. It is a spatially homogeneous Markov process on R with the transition function $P(t, x, E)=\int_E p(t, x-y)dy$, where $p(t, x)=\frac{1}{\pi}\frac{t}{t^2+x^2}$ for $t>0$ and $x\in R$. The corresponding semigroup given by $H_t f(x)=\int_{-\infty}^{\infty} f(x+y)$ $p(t, y)dy$, is a strongly continuous on $C_0(R)$[1] and if B is the infinitesimal generator of $\{H_t\}$, then its domain contains $C_K^2(R)$[2] and for $f\in C_K^2(R)$, $Bf(x)=\int_{-\infty}^{\infty}\left[f(x+y)-f(x)-\frac{y}{1+y^2}f'(x)\right]\frac{dy}{\pi y^2}$. Let

1) $C_0(R)$ is the space of continuous functions on R tending to 0 at infinity.

2) $C_K^m(R)$ is the space of C^m functions on R with compact supports.

$G_\lambda f(s, x)=\int_s^\infty\int_{-\infty}^\infty e^{-\lambda t}f(t, y)p(t-s, x-y)dydt$. It is the resolvent of the space-time Cauchy process. When $f(s, x)=f(x)$, $G_\lambda f(s, x) \equiv G_\lambda f(x)=\int_0^\infty e^{-\lambda t}H_t f(x)dt$ and it is the resolvent of $\{H_t\}$.

The following facts are verified easily.

Proposition 1.1. (i) *For each p in $1\leq p\leq\infty$, G_λ is a bounded operator on $L_p(E)(L_p(R))$*[3] *with the norm $\|G_\lambda\|_p\leq\frac{1}{\lambda}$.*

(ii) *For each $t>0$,*

$$(1.1)\qquad \sup_{x\in R}|H_t f(x)|\leq\frac{1}{\pi}\left\|\frac{1}{1+y^2}\right\|_q\frac{1}{t^{1-(1/q)}}\|f\|_p\text{[4]}\quad \textit{for all}\quad f\in L_p(R),$$

provided $1<p$, $q<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$.

(iii) *For each p in $1<p<\infty$, there is a constant $C_{\lambda,p}$ depending only on p and λ such that*

$$(1.2)\qquad \sup_{x\in R}|G_\lambda f(x)|\leq C_{\lambda,p}\|f\|_p\qquad \textit{for all } f\in L_p(R)$$

and G_λ is a bounded operator from $L_p(R)$ to $C_b(R)$.[5]

§ 2. Construction of K_λ

First we shall state some lemmas which lean upon a fundamental theorem of B.F. Jones [5] and D.W. Stroock—S.R.S. Varadhan [13].

Lemma 2.1. *Set $G^*g(t, x)=\int_0^t\int_{-\infty}^\infty p(t-s, x-y)g(s, y)dyds$ for $g\in C_K^\infty(E)$. Then, for each p in $1<p<\infty$, there is a constant B_p (depending only on p) such that*

---

3) $E=[0, \infty)\times R$, $L_p(E)$ (resp. $L_p(R)$) is the L_p-space with respect to the Lebesgue measure on E (resp. R).

4) $\|f\|_p=\left(\int_{-\infty}^\infty|f(x)|^p dx\right)^{1/p}$.

5) $C_b^m(R)$ is the space of bounded functions on R which have bounded continuous derivatives up to m-th order. $C_b^0(R)=C_b(R)$.

(2.1) $\qquad ||DG^*g||_p \leq B_p ||g||_p \qquad$ *for all* $g \in C_K^\infty(E)$,[6]

where $D = \dfrac{\partial}{\partial x}$ (*spatial derivative*).

Proof. Let us set $k(t, x) = \dfrac{\partial}{\partial x} p(t, x) = -\dfrac{2}{\pi} \dfrac{tx}{(t^2+x^2)^2}$, $\phi(t) = t$ and $\Omega(x) = -\dfrac{2}{\pi} \dfrac{x}{(1+x^2)^2}$, then all conditions of B. F. Jones' theorem [5] pp. 443–444 are satisfied. Hence, the statement follows at once. Q. E. D.

Lemma 2.2. *For* $f \in C_K^\infty(E)$, *let us set* $h = G_\lambda f$. *Then for each* p *in* $1 < p < \infty$, *there is a constant* A_p *depending only on* p (*independent of* λ) *such that*

(2.2) $\qquad ||Dh||_p \leq A_p ||f||_p$[7] $\qquad$ *for all* $f \in C_K^\infty(E)$.

and

(2.3) $\qquad ||Dh||_p \leq A_p ||f||_p \qquad$ *for all* $f \in C_K^\infty(R)$.

Proof. We can define $Gf(s, x) = \int_s^\infty \int_{-\infty}^\infty p(t-s, x-y) f(t, y) dy dt$ for $f \in C_K^\infty(E)$ and show that $Gf(s, x) = \lim_{\lambda \downarrow 0} G_\lambda f(s, x) \in C_0^\infty(E)$. Therefore, using the resolvent equation,

$$h(s, x) = \int_s^\infty \int_{-\infty}^\infty p(t-s, x-y)[f(t, y) - \lambda h(t, y)] dy dt$$

for $f \in C_K^\infty(E)$. Let us set $u = f - \lambda h$. Then for $g \in C_K^\infty(E)$,

$$\begin{aligned} I &= \int_0^\infty \int_{-\infty}^\infty g(s, x) Dh(s, x) dx ds = -\int_0^\infty \int_{-\infty}^\infty Dg(s, x) h(s, x) dx ds \\ &= -\int_0^\infty \int_{-\infty}^\infty Dg(s, x) \left[\int_s^\infty \int_{-\infty}^\infty p(t-s, x-y) u(t, y) dy dt \right] dx ds \\ &= -\int_0^\infty \int_{-\infty}^\infty u(t, y) \left[\int_0^t \int_{-\infty}^\infty p(t-s, x-y) Dg(s, x) dx ds \right] dy dt \\ &= -\int_0^\infty \int_{-\infty}^\infty u(t, y) DG^* g(t, y) dy dt. \end{aligned}$$

---

6) $C_K^\infty(E)$ is the space of C^∞ functions on E with compact supports.

7) For a function f on E (resp. R), $||f||_p$ denotes the norm of f in $L_p(E)$ (resp. $L_p(R)$).

Hence from Hölder's inequality, it follows that

$$|I| \leq B_q \|u\|_p \|g\|_q ,$$

provided $1<p,\ q<\infty,\ \frac{1}{p}+\frac{1}{q}=1$. Therefore,

$$\begin{aligned}\|Dh\|_p \leq B_q \|u\|_p \leq B_q(\|f\|_p+\lambda\|h\|_p) \\ \leq 2B_q \|f\|_p .\end{aligned}$$

Setting $A_p=2B_q$, we have (2.2). (2.3) can be proved similarly by setting $f(s, x)=f(x)e^{-\lambda s}$ for $f\in C_K^\infty(R)$. Q. E. D.

We define the operator T_λ as follows:

$$T_\lambda f(x) = (A-B)G_\lambda f(x) = a(x)\frac{d}{dx}G_\lambda f(x) \qquad \text{for } f\in C_K^\infty(R).$$

Then for each p in $1<p<\infty$,

(2.4) $\|T_\lambda f\|_p \leq \|a\| A_p \|f\|_p$.[8)]

Therefore, we can uniquely extend T_λ to be a bounded operator on $L_p(R)$.

Now we will put the following assumption on a.

Assumption (p): $\|a\|A_p<1$.

If a is a measurable function on R satisfying the ASSUMPTION (p) for some p, $1<p<\infty$, we define the operator K_λ as follows:

(2.5) $K_\lambda = G_\lambda(I-T_\lambda)^{-1}$,

which is a bounded operator on $L_p(R)$ and from (1.2) and (2.3) we have at once the following

Proposition 2.2. *K_λ is a bounded operator from $L_p(R)$ to $C_b(R)$ and*

$$\sup_{x\in R} |K_\lambda f(x)| \leq \frac{C_{\lambda,p}}{1-\|a\|A_p}\|f\|_p \qquad \text{for all } f\in L_p(R).$$

---

8) $\|f\|=\sup_{x\in R}|f(x)|$.

§ 3. Martingale problem and stochatic integral equation

Let $\Omega=D([0,\infty)\to R)$ be the space of right continuous functions having the left-hand limit at each $t>0$ with Skorohod topology. The space Ω is a Polish space. We denote generic element of Ω by ω and $\omega(t)=x(t,\omega)=x(t)$. $\mathcal{F}_t$ is the smallest σ-algebra with respect to which $x(s,\omega)$ are measurable for $0\leq s\leq t$ and $\mathcal{F}=\mathcal{F}_\infty=\bigvee_{t\geq 0}\mathcal{F}_t$.

We will say that a probability measure P on $(\Omega,\mathcal{F})$ is a solution of the *martingale problem* for A starting from x, if $P[x(0)=x]=1$ and

$$(3.1)\qquad X_\theta(t)=\exp\Big[i\theta(x(t)-x(0))-i\theta\int_0^t a(x(s))ds -t\int_{-\infty}^{\infty}\Big(e^{i\theta u}-1-\frac{i\theta u}{1+u^2}\Big)\frac{du}{\pi u^2}\Big],\qquad t\geq 0,$$

is a P-martingale[9] for all $\theta\in R$, and also that P is a solution of the *stochastic integral equation* for A starting from x, if there is a Cauchy process ξ with respect to P (i.e., $P[\xi(0)=0]=1$ and

$$E[e^{i\theta(\xi(t)-\xi(s))}/\mathcal{F}_s]=\exp\Big[(t-s)\int_{-\infty}^{\infty}\Big(e^{i\theta u}-1-\frac{i\theta u}{1+u^2}\Big)\frac{du}{\pi u^2}\Big],\ (t>s)\Big)$$

such that

$$(3.2)\qquad x(t)=x+\int_0^t a(x(s))ds+\xi(t) = x+\int_0^t \tilde{a}(x(s))ds+\int_0^t\int_{|u|\leq 1}uq(ds,du)+\int_0^t\int_{|u|>1}up(ds,du),$$

where p, q are random measures appearing in tne Lévy-Ito decomposition of $\xi(t)$, [cf. 3, 4, 12] and $\tilde{a}(x)=a(x)+\frac{1}{\pi}\int_{|u|\leq 1}\frac{u}{1+u^2}du -\frac{1}{\pi}\int_{|u|>1}\frac{u}{1+u^2}\frac{du}{u^2}$.

Proposition 3.1. *The following statements are equivalent:*

(i) *P is a solution of the martingale problem for A starting from x.*

---

9) If $(X_\theta(t),\mathcal{F}_t,R)$ is a martingale, then we call that $X_\theta(t)$ is a P martingale.

(ii) *P is a solution of the stochastic equation for A starting from x.*

Proof. The implication "(i)→(ii)" is clear since if we set $\xi(t)=x(t)-x-\int_0^t a(x(s))ds$, then $\xi(t)$ is a Cauchy process with respect to P. Now we shall prove "(ii)→(i)"; Let us set $y(t)=\frac{1}{i\theta}\log X_\theta(t)=\int_0^t\int_{|u|\le 1} uq(ds, du)+\int_0^t\int_{|u|>1} up(ds, du)-\frac{t}{i\theta}\Big[\int_{|u|\le 1}(e^{i\theta u}-1-i\theta u)\frac{du}{\pi u^2}+\int_{|u|>1}(e^{i\theta u}-1)\frac{du}{\pi u^2}\Big]$ and $F(x)=e^{i\theta x}$. Then using the formula of H. Kunita and S. Watanabe [5] on the stochastic integral,

$$X_\theta(t) = F(y(t)) = \int_0^t\int_{-\infty}^{\infty}[F(y(s)+u)-F(y(s)]q(ds, du)$$

is a P-martingale. Q. E. D.

It should be noticed that Stroock-Varadhan's Theorem ([13] Theorem 3.1, p. 356) is valid in our case, and their theorem is used to prove the Markov property of the solution of the martingaleproblem.

§4. Uniqueness of the solution of the stochastic integral equation

In this section, it is assumed that a is a measurable function satisfying the ASSUMPTION (p) for some p, $1<p<\infty$, and this p is fixed. The following Lemma is essential to show the uniqueness of the solution, which is similar to the corresponding Lemma 5.1 of Stroock-Varadhan [13].

Lemma 4.1. *Let $x_0\in R$ be an arbitrary but fixed point, P be any solution of the martingale problem for A starting from x_0, and set $\mu_\lambda(f)=E[\int_0^\infty e^{-\lambda t}f(x(t))dt]$ for each bounded measurable function f and $\lambda>0$. Then, $\mu_\lambda(f)$ is given as*

$$\mu_\lambda(f) = \int_{-\infty}^{\infty} f(x)g_\lambda(x)dx$$

by some non-negative integrable function $g_\lambda(x)$ such that $g_\lambda(x)$

$\in L_q(R)$[10] *and*

$$|\mu_\lambda(f)| \leq \frac{C_{\lambda,p}}{1-\|a\|A_p}\|f\|_p \qquad \textit{for all } f\in L_p(R).$$

Proof. In the preceding section, we see that $x(t)$ satisfies the equation

$$\begin{aligned} x(t) &= x_0+\int_0^t a(x(s))ds+\xi(t) \\ &= x_0+\int_0^t \tilde{a}(x(s))ds+\int_0^t\int_{|u|\leq 1} uq(ds, du)+\int_0^t\int_{|u|>1} up(ds, du). \end{aligned}$$

Let $\phi_n(s)$ be defined as follows;

$$\phi_n(s) = \begin{cases} \frac{k}{2^n} & \text{for } \frac{k}{2^n}\leq s<\frac{k+1}{2^n} \quad k=0, \pm 1, \cdots, \pm n2^n \\ n & \text{fo } s\geq n \\ -(n+1) & \text{for } s<-n \end{cases}$$

Then, there exist a subsequence $\{n'\}$ and a point $s_0\in[0, 1]$ satisfying the following condition: if we define

$$(4.1) \qquad x_{n'}(t) = x_0+\int_0^t a(x(\phi_{n'}(t-s_0)+s_0))ds+\xi(t),$$

then $x_{n'}(t)\to x(t)$ in probability as $n'\to\infty$ for each $t\geq 0$.(*) Therefore,

$$\lim_{n'\to\infty} E\left[\int_0^\infty e^{-\lambda t}f(x_{n'}(t))dt\right] = E\left[\int_0^\infty e^{-\lambda t}f(x(t))dt\right]$$

for any bounded continuous function $f(x)$. This means $\mu_\lambda^{(n')}\to\mu_\lambda$ for every $f\in C_b(R)$ $(n'\to\infty)$, where we denote

$$\mu_\lambda^{(n')}(f) = E\left[\int_0^\infty e^{-\lambda t}f(x_{n'}(t))dt\right].$$

Applying the formula of H. Kunita and S. Watanabe [6] on the stochastic integrals to (4.1), we have

---

10) $q=\frac{p}{p-1}$.

(*) This will be proved in §7 (1°).

$$f(x_{n'}(t)) = f(x_0)+\int_0^t a(x(\phi_{n'}(s-s_0)+s_0)f'(x_{n'}(s))ds$$
$$+\int_0^t\int_{|u|\leq 1}[f(x_{n'}(s)+u)-f(x_{n'}(s))]q(ds,\ du)$$
$$+\int_0^t\int_{|u|\leq 1}[f(x_{n'}(s)+u)-f(x_{n'}(s))-f'(x_{n'}(s))u]\frac{du}{\pi u^2}$$
$$+\int_0^t\int_{|u|>1}[f(x_{n'}(s)+u)-f(x_{n'}(s))]p(ds,\ du)$$

for any $f\in C_b^2(R)$. Multiplying both sides by $\lambda e^{-\lambda t}$, integrating from 0 to ∞ and taking the expectation with respect to P, we have

$$\mu_\lambda^{(n')}[\lambda f-Bf] = f(x_0)+E[\int_0^\infty e^{-\lambda t}a(x(\phi_{n'}(s-s_0)+s_0))f'(x_{n'}(t))dt].$$

If we choose $f=G_\lambda h$ where h belongs to $C_K^\infty(R)$, then

(4.2) $\quad |\mu_\lambda^{(n')}(h)|\leq C_{\lambda,p}||h||_p+||a||\,\mu_\lambda^{(n')}(|DG_\lambda h|)\,.$

On the other hand, $\mu_\lambda^{(n')}$ is a bounded linear functional on $L_p(R)$ for each n'(**). Therefore from (4.2) and (2.3), it follows that

$$||\mu_\lambda^{(n')}||_p\leq\frac{C_{\lambda,p}}{1-||a||A_p},$$

where $||\mu_\lambda^{(n')}||_p$ is the norm of the bounded linear functional $\mu_\lambda^{(n')}$ on $L_p(R)$. Since $\mu_\lambda^{(n')}\to\mu_\lambda$ for every $f\in C_b(R)$ when $n'\to\infty$,

$$|\mu_\lambda(f)|\leq\frac{C_{\lambda,p}}{1-||a||A_p}||f||_p$$

for all $f\in C_b(R)\cap L_p(R)$. Therefore, we can extend $\mu_\lambda|C_b(R)\cap L_p(R)$ to be a bounded linear functional $\bar{\mu}_\lambda$ on $L_p(R)$, where $\mu_\lambda|C_b(R)\cap L_p(R)$ is the restriction of μ_λ to the subset $C_b(R)\cap L_p(R)$. Hence there exists a function g_λ of $L_q(R)$ such that

$$\bar{\mu}_\lambda(f) = \int_{-\infty}^\infty f(x)g_\lambda(x)dx$$

for all $f\in L_p(R)$. On the other hand, μ_λ is defined by a bounded measure on R; $\mu_\lambda(f)=\int_{-\infty}^\infty f(x)\mu_\lambda(dx)$, $f\in C_b(R)$ and thus,

(**) This will be proved in §7 (2°).

$\int_{-\infty}^{\infty} f(x)\mu_\lambda(dx)=\int_{-\infty}^{\infty} f(x)g_\lambda(x)dx$ for any $f\in C_b(R)\cap L_p(R)$. Therefore, g_λ is non-negative and integrable with respect to the Lebesgue measure dx. Thus, $\mu_\lambda(dx)=g_\lambda(x)dx$ and

$$|\mu_\lambda(f)|\leq\frac{C_{\lambda,p}}{1-\|a\|A_p}\|f\|_p. \quad \text{for all } f\in L_p(R).$$

Q. E. D.

Lemma 4. 2. *Let P be the same as in Lemma 4. 1. Then,*

$$\mu_\lambda(f) = K_\lambda f(x_0) \qquad \textit{for all } f\in L_p(R).$$

Proof. Applying the formula on the stochastic integral to (3. 2) and using the same calculation of Lemma 4. 1, we have $\mu_\lambda[\lambda f-Bf]=f(x_0)+E[\int_0^\infty e^{-\lambda t}a(x(s))f'(x(s))ds]$ and hence if $f=G_\lambda h$, $h\in C_K^\infty(R)$, we have $\mu_\lambda(h)=f(x_0)+\mu_\lambda(T_\lambda h)$. Thus, $\mu_\lambda(g)=f(x_0)$ $=G_\lambda(I-T_\lambda)^{-1}g(x_0)=K_\lambda g(x_0)$ for $g=(I-T_\lambda)h$, $h\in C_K^\infty(R)$. Hence, by Lemma 4. 1, $\mu_\lambda(f)=K_\lambda f(x_0)$ for all $f\in L_p(R)$. Q. E. D.

§ 5. Existence

Assume that a is a measurable function satisfying ASSUMPTION (p) for some p ($1<p<\infty$). Then, there exists a sequence $\{a_n\}$ such that every a_n belongs to $C_0^2(R)$, $\|a_n\|\leq\|a\|$ and $a_n(x)\to a(x)$ in $L_1^{loc}(n\to\infty)$.[11] We consider the sequence of the following equations on some probability space (W, Q);

$$X_n(t) = x_0+\int_0^t a_n(X_n(s))ds+\eta(t),$$

where $\eta(t)$ is a Cauchy process on (W, Q). Since X_n satisfies the following conditions;

(1) for every $T<\infty$,

$$\lim_{k\uparrow\infty}\sup_{1\leq n<\infty} Q[\sup_{0\leq s\leq T}|X_n(s)|>k] = 0$$

(2) for every $T<\infty$ and $\varepsilon>0$,

---

11) If $a_n\to a$ in $L_1(I)$ $(n\to\infty)$ for any finite interval I, then it is denoted as $a_n\to a$ in L_1^{loc} $(n\to\infty)$

$$\lim_{\delta\downarrow 0}\sup_{1\le n<\infty} Q[w'_{X_n,T}(\delta)>\varepsilon]=0^{12)},$$

from a result of Skorohod [12] (cf. [1], [9]) it follows that there exists a sequence of processes $\{(\tilde{X}_n, \tilde{\eta}_n)\}$ on a probability space $(\tilde{W}, \tilde{Q})$ such that finite dimensional distributions of the processes $(\tilde{X}_n, \tilde{\eta}_n)$ and (X_n, η) coincide and $(\tilde{X}_n, \tilde{\eta}_n)$ converges in probability to a certain limit $(\tilde{X}_0, \tilde{\eta}_0)$ whose trajectries belong to $D([0,\infty)\to R)$ and they satisfy the following equations:

$$\tilde{X}_n(t) = x_0+\int_0^t a(\tilde{X}_n(s))ds+\tilde{\eta}_n(t), \quad n=1,2,\cdots.$$

We can show that

$$\int_0^t a_n(\tilde{X}_n(s))ds\to\int_0^t a(\tilde{X}_0(s))ds \text{ in probability } \quad (n\to\infty)$$

for each $t\ge 0$, using the similar arguments as Krylov ([7] pp. 344-345). In fact, we have

$$\begin{aligned}
I(n) &= \tilde{Q}\left[\left|\int_0^t a_n(\tilde{X}_n(s))ds-\int_0^t a(\tilde{X}_0(s))ds\right|>\varepsilon\right]\\
&\le\tilde{Q}\left[\int_0^t \chi_r(\tilde{X}_0(s))|a(\tilde{X}_0(s))-a_N(\tilde{X}_0(s))|\,ds\ge\frac{\varepsilon}{3}\right]\\
&+\tilde{Q}\left[\left|\int_0^t \{a_N(\tilde{X}_0(s))-a_N(\tilde{X}_n(s))\}\,ds\right|\ge\frac{\varepsilon}{3}\right]\\
&+\tilde{Q}\left[\int_0^t \chi_l(\tilde{X}_n(s))|a_N(\tilde{X}_n(s))-a_n(\tilde{X}_n(s))|\,ds\ge\frac{\varepsilon}{3}\right]\\
&+\tilde{Q}[\sup_{0\le s\le t}|\tilde{X}_0(s)|\ge r]+\sup_{1\le n<\infty}\tilde{Q}[\sup_{0\le s\le t}|\tilde{X}_n(s)|>l],
\end{aligned}$$

where $r\le l$ and $\chi_b(x)$ is the indicator function of the set $(-b, b)$. On the other hand, using the same argument as in Lemma 4. 1, we can prove the following iequalities;

$$E[\int_0^t |f(\tilde{X}_n(s))|\,ds]\le\frac{e^{\lambda t}C_{\lambda,p}}{1-\|a\|\|A_p\|}\|f\|_p$$

for $f\in L_p(R)$ and $n=0, 1, 2, \cdots$, where $E[\cdot]$ denotes the expectation with respect to $\tilde{Q}$. Therefore,

12) See [1] for the definition of w'.

$$I(n)\leq\tilde{Q}\left[\left|\int_0^t\{a_N(\tilde{X}_0(s))-a_N(\tilde{X}_n(s))\}ds\right|\geq\frac{\varepsilon}{3}\right]$$
$$+\frac{3}{\varepsilon}\frac{e^{\lambda t}C_{\lambda,p}}{1-\|a\|A_p}(\|a_N-a\|_{p,r}+\|a_N-a_n\|_{p,l})$$
$$+\sup_{1\leq n<\infty}Q[\sup_{0\leq s\leq t}|X_n(s)|\geq l]+\tilde{Q}[\sup_{0\leq s\leq t}|\tilde{X}_0(s)|\geq r],$$

where

$$\|g\|_{p,b}=(\int_{-b}^{b}|g(x)|^p ds)^{1/p}.$$

Therefore, $\tilde{X}_0(t)$ satisfies the following equation;

$$\tilde{X}_0(t) = x_0+\int_0^t a(\tilde{X}_0(s))ds+\tilde{\eta}_0(t).$$

Thus we have the following

Proposition 5.1. *Let a be a measurable function satisfying the Assumption (p) for some $p(1<p<\infty)$. Then there is a solution of the martingale problem.*

§6. Main theorem

Theorem 6.1. *Let a be a measurable function satisfying the Assumption (p) for some p, $1<p<\infty$. Then the martingale problem has unique solution P_x for each starting point x. Moreover, (P_x) is a strong Feller process which satisfies the following equation;*

(6.1) $$T_t f(x)-f(x) = \int_0^t T_s Af(x)ds^{13)}$$

for each $t\geq 0$ and $f\in C_0^2(R)$.

Proof. From Lemma 4.2 it follows that $P(t, x, dy)=P_x(x(t)\in dy)$ is uniquely determined. Therefore, (P_x) becomes a strong Markov process(***) and consequently P_x is uniquely determined. The formula (6.1) is verified from (3.2) using the formula on the stochastic integral. Q. E. D.

Remark 6.1. If the function a satisfies the same condition

13) $T_t f(x)=E_x[f(x(t))]$.

(***) A proof is given in §7 (3°).

as in the above Theorem, then positive strongly continuous contraction semigroup $\{T_t\}$ on $C_0(R)$ satisfying (6. 1) is uniquely determined.

Remark 6. 2. If $a(x)$ is a measurable function and $a(x)-c$ satisfies the Assumption (p) for a constant c, then Theorem 6. 1 remains valid. For we can prove that Proposition 1. 1, Lemma 2. 1 and Lemma 2. 2 are valid for same constants as in the case $c=0$ respectively when we replace $p(t, x)$ by $\tilde{p}(t, x)=p(t, x-ct) =\frac{1}{\pi}\frac{t}{t^2+(x-ct)^2}$.

§ 7. Some proofs

(1°) **Proof of (*)**[14]. Let us set $f(t, \omega)=a(x(t, \omega))$ and for each positive integer m, set

$$\begin{aligned} f_m(t, \omega) &= f(t, \omega) \quad \text{for } 0\leq t\leq m \\ &= 0 \qquad\quad\ \ \text{for } t<0,\ t>m. \end{aligned}$$

Then $f_m(t, \omega)\in L_2(R\times\Omega, dt\times dP)$. Therefore $f_m(\cdot, \omega)\in L_2(R)$ for each m and almost all ω. Hence, for all t, almost all ω and all m,

$$\int_{-\infty}^{\infty}[f_m(\phi_n(t)+s, \omega)-f_m(t+s, \omega)]^2 ds\to 0 \qquad (n\to\infty)$$

and the above integral is smaller than $4\int_{-\infty}^{\infty} f_m(s, \omega)^2 ds$. Therefore, for each m,

$$\int_{-1}^{m}\int_{\Omega}\int_{-\infty}^{\infty}[f_m(\phi_n(t)+s, \omega)-f_m(t+s, \omega)]^2 ds\, dP\, dt\to 0 \qquad (n\to\infty).$$

Thus there exist a subsequence $\{n_k(m)\}$ of $\{n\}$ and a null set $N(m)$ such that $\{n_k(m+1)\}\subset\{n_k(m)\}$ and

$$\int_{-1}^{m}\int_{\Omega}[f_m(\phi_{n_k(m)}(t)+s, \omega)-f_m(t+s, \omega)]^2 dt\, dP\to 0 \qquad (k\to\infty)$$

for any $s\in[0, m]\cap N(m)^c$.[15] Since

14) The proof is a slight variation of K. Ito ([4] Lemma, p. 336).
15) $N(m)^c$ is the complement of $N(m)$.

$$\int_0^m\int_\Omega[f_m(\phi_{n_{k(m)}}(t-s)+s,\omega)-f_m(t,\omega)]^2dt\,dP$$
$$\leq\int_{-1}^m\int_\Omega[f_m(\phi_{n_{k(m)}}(t)+s,\omega)-f_m(t+s,\omega)]^2dt\,dP,$$

it follows that

$$I^m_{n_{k(m)}}(s)=\int_0^m\int_\Omega[f_m(\phi_{n_{k(m)}}(t-s)+s,\omega)-f_m(t,\omega)]^2dt\,dP\to 0$$

($k\to\infty$) for any $s\in[0,m]\cap N(m)^c$. Therefore, using the diagonal method, we can choose a point $s_0\in[0,1]\cap\bigcap_{m=1}^{\infty}N(m)^c$ and a subsequence $\{n_k\}$ such that $I^m_{n_k}(s_0)\to 0$ ($k\to\infty$) for every m. Now, we denote

$$\begin{aligned}f^{(k)}(t,\omega)&=f(\phi_{n_k}(t-s_0)+s_0,\omega)\\&=a(x(\phi_{n_k}(t-s_0)+s_0,\omega)),\end{aligned}$$

then for each m,

$$\int_0^m f^{(k)}(t,\omega)\,dt\to\int_0^m f(t,\omega)dt \quad \text{in probability } (k\to\infty).$$

(2°) **Proof of (**).** First, we note that $\xi(t)=x(t)-x_0-\int_0^t a(x(s))ds$ is measurable with respect to $\mathcal{F}_t$ and $\xi(t)-\xi(s)$ is independent of $\mathcal{F}_s$ ($s<t$). To simplify the calculation, we assume that $s_0=0$ and $\{n'\}=\{n\}$. Since

$$\left|E\left[\int_0^\infty e^{-\lambda t}f(x_n(t))dt\right]\right|\leq\sum_{k=1}^{n2^n}\left|E\left[\int_{(k-1)/2^n}^{k/2^n}e^{-\lambda t}f(x_n(t))dt\right]\right|+$$
$$+\left|E\left[\int_n^\infty e^{-\lambda t}f(x_n(t))dt\right]\right|,$$

it is sufficient to estimate $E[\int_0^{1/2^n}e^{-\lambda t}f(x_n(t))dt]$, etc. From Proposition 1.1 (ii), it follows that

$$\left|E\left[\int_0^{1/2^n}e^{-\lambda t}f(x_n(t))dt\right]\right|=\left|\int_0^{1/2^n}e^{-\lambda t}dt\,E[f(x_0+a(x_0)t+\xi(t))]\right|$$
$$\leq\left(\frac{1}{\pi}\left\|\frac{1}{1+y^2}\right\|_q\int_0^{1/2^n}e^{-\lambda t}\frac{1}{t^{1-1/q}}dt\right)\|f\|_p.$$

Let us set $Y=\xi(t)-\xi\left(\frac{1}{2^n}\right)\left(t>\frac{1}{2^n}\right)$, $Z=x_0+a(x_0)\frac{1}{2^n}+a\left(x\left(\frac{1}{2^n}\right)\right)\left(t-\frac{1}{2^n}\right)+\xi\left(\frac{1}{2^n}\right)$, $P_Y(dy)=P(Y\in dy)=p\left(t-\frac{1}{2^n}, y\right)dy$ and $P_Z(dz)=P(Z\in dz)$. Then,

$$\begin{aligned}
&E\left[\int_{1/2^n}^{2/2^n} e^{-\lambda t} f(x_n(t))dt\right] \\
&= E\left[\int_{1/2^n}^{2/2^n} e^{-\lambda t} f\left(x_0+a(x_0)\frac{1}{2^n}+a\left(x\left(\frac{1}{2^n}\right)\right)\left(t-\frac{1}{2^n}\right)+\xi(t)\right)dt\right] \\
&= \int_{1/2^n}^{2/2^n} e^{-\lambda t}dt\, E[f(Y+Z)] \\
&= \int_{1/2^n}^{2/2^n} e^{-\lambda t}dt \int_{-\infty}^{\infty} P_Z(dz)\int_{-\infty}^{\infty} f(y+z)P_Y(dy) \\
&= \int_{1/2^n}^{2/2^n} e^{-\lambda t}dt \int_{-\infty}^{\infty} P_Z(dz)\int_{-\infty}^{\infty} f(y+z)p\left(t-\frac{1}{2^n}, y\right)dy.
\end{aligned}$$

Hence from Proposition 1. 1 (ii), it follows that

$$\left|E\left[\int_{1/2^n}^{2/2^n} e^{-\lambda t} f(x_n(t))dt\right]\right| \leq \left(\frac{1}{\pi}\left\|\frac{1}{1+y^2}\right\|_q \int_0^{1/2^n} \frac{dt}{\left(t-\frac{1}{2^n}\right)^{1-1/q}}\right)\|f\|_p.$$

Also, the ramainders are estimated by the same calculations as the above and we have the proof of (**).

(3°) **Proof of the Markov property of (P_x).** The Markov property of (P_x) is proved by the same method as Stroock-Varadhan [13], but we give another proof as follows. In order to show the Markov and the strong Markov properties of (P_x), it is sufficient to prove the following fact.

Proposition 7. 1.[16] *Let us set $u=\mu_\lambda(f)$ for $f\in L_p(R)$. Then*

$$e^{-\lambda\tau}u(x(\tau))=E_x\left[\int_\tau^\infty e^{-\lambda t}f(x(t))dt/\mathcal{F}_\tau\right] \ a.s.\ P_x({}^\forall x),$$

where τ is $\{\mathcal{F}_t\}$-Markov time[17] *and $\mathcal{F}_\tau=\{A\in\mathcal{F}: A\cap\{\tau\leq t\}\in\mathcal{F}_t,\ t\geq 0\}$. Also, the above equation is valid when we replace τ by*

16) This Proposition is suggested by Krylov [7].

17) If $\tau\geq 0$ and $\{\tau\leq t\}\in\mathcal{F}_t$ for any $t\geq 0$, then τ is called $\{\mathcal{F}_t\}$-Markov time.

$\{\mathcal{F}_{t+}\}$[18]*-Markov time and* $\mathcal{F}_\tau$ *by* $\mathcal{F}_{\tau+}$.[19]

Proof. The following formula is verified from (3.2) using the formula on the stochastic integral;

(7.1) $\quad e^{-\lambda\tau}g(x(\tau)) = E_x[\int_\tau^\infty e^{-\lambda t}(\lambda-A)g(x(t))dt/\mathcal{F}_\tau]$ a.s. $P_x(^\forall x)$

for any $g\in C_b^2(R)$. Let us set $f=(I-T_\lambda)h$ where $h\in C_K^\infty(R)$. Then

$$u = \mu_\lambda(f) = K_\lambda f = G_\lambda h\in C_b^2(R).$$

Hence from (7.1)

$$e^{-\lambda\tau}u(x(\tau)) = E_x[\int_\tau^\infty e^{-\lambda t}f(x(t))dt/\mathcal{F}_\tau] \text{ a.s. } P_x(^\forall x).$$

Therefore if H is a bounded $\mathcal{F}_\tau$-measurable function, then for $f=(I-T_\lambda)h$, $h\in C_K^\infty(R)$,

$$E_x[He^{-\lambda\tau}K_\lambda f(x(\tau))] = E_x[H\int_\tau^\infty e^{-\lambda t}f(x(t))dt].$$

On the other hand, $E_x[He^{-\lambda\tau}K_\lambda f(x(\tau))]$ and $E_x[H\int_\tau^\infty e^{-\lambda t}f(x(t))dt]$ are bounded linear functionals on $L_p(R)$. Therefore for all $f\in L_p(R)$

$$E_x[He^{-\lambda\tau}K_\lambda f(x(\tau))] = E_x[H\int_\tau^\infty e^{-\lambda t}f(x(t))dt].$$

Thus we proved the proposition. Q. E. D.

§8. Remarks

In this section we will give some supplementary remarks.

(1°) Space-time case.

Let $a(t, x)$ be a measurable function on E such that

$$\|a-c\| \equiv \sup_{(t,x)\in E} |a(t, x)-c| < \frac{1}{A_p}$$

for some $p>2$ and constant c and set

$$Af(t,x) = \frac{\partial}{\partial t}f(t, x)+a(t, x)\frac{\partial}{\partial x}f(t, x)+Bf(t, x)$$

---

18) $\mathcal{F}_{t+}=\bigcap_{u>t}\mathcal{F}_u$.

19) $\mathcal{F}_{\tau+}=\{A\in\mathcal{F}: A\cap\{\tau\le t\}\in\mathcal{F}_{t+}, t\ge 0\}$.

for $f \in C_K^2(E)$, where B is the infinitesimal generator of 1-dimensional symmetric Cauchy process. Then, we can prove that Therem 6.1 is valid in this case.

(2°) **Multi-dimensional case.**

Let $a(x)=(a_1(x), a_2(x), \cdots, a_N(x))$ be measurable mapping on R^N such that

$$||a-c|| \equiv \max_{1 \le i \le N} ||a_i - c_i|| < \frac{1}{A_p}$$

for some $p>N$ and constant vector $c=(c_1, c_2, \cdots, c_N)$. The operator A is defined as follows;

$$Af(x) = \sum_{i=1}^{N} a_i(x) \frac{\partial}{\partial x_i} f(x) + Bf(x)$$

for $f \in C_K^2(R^N)$ and $x=(x_1, x_2, \cdots, x_N) \in R^N$, where B is the infinitesimal generator of N-dimensional symmetric Cauchy process. Then Theorem 6.1 is valid in this case.

TOKYO INSTITUTE OF TECHNOLOGY

REFERENCES

[1] P. Billingsley: Convergence of probability measures. Wiley, New York, 1968.
[2] N. Ikeda: On the construction of two-dimensional diffusion processes satisfying Wentzell's boundary conditions and its application to boundary value problems. Mem. Coll. Sci. Univ. Kyoto, Ser. A, 33 (1961), 367-427.
[3] K. Ito: Stochastic processes. Iwanami-Koza, Gendai-Oyo-Sugaku, Tokyo, 1957 (Japanese).
[4] K. Ito: Theory of probability. Iwanami, Tokyo 1953 (Japanese).
[5] B. F. Jones: A class of singular integrals. Amer. J. Math. 86 (1964), 441-462.
[6] H. Kunita-S. Watanabe: On square integrable martingales. Nagoya Math. Jour. 30 (1967), 209-245.
[7] N. V. Krylov: On Ito's stochastic equations. Teor. Veroyat. Primen. vol. **XIV**, No. 2 (1969), 340-348 (in Russian).
[8] M. Motoo: Application of additive functionals to the boundary problem of Markov processes (Lévy's system of U-processes), Proc. 5-th. Berkeley Symp. vol. II, part 2, Univ. Calif. Press (1967), 75-110.
[9] K. R. Parthasarathy: Probability measures on metric space. Academic press, New York, 1967.
[10] Irving E. Segal-Ray A. Kunze: Integrals and operators, McGraw-Hill, 1968.
[11] K. Sato-T. Ueno: Multi-dimensional diffusion and the Markov process on the boundary, Jour. Math. Kyoto Univ. 4 (1965), 529-605.

[12] A. V. Skorohod: Studies in the theory of random processes, Kiev, 1961, (English translation, Addison Wesley, 1965).
[13] Daniel W. Stroock-S. R. S. Varadhan: Diffusion processes with continuous coefficients I, II, Comm. Pure Appl. Math. **XXII** (1969), 345-400 and 479-530.
[14] H. Tanaka-M. Hasegawa: Stochastic differential equations, Semi. on Prob. **19** (1964) (Japanese).