J. Math. Kyoto Univ.
10-3 (1970) 419-431

Rings with nonzero singular ideals

By

Edward T. WONG

(Communicated by Prof. Nagata, April 7, 1970)

In recent years, there have been many results about rings and their various types of ring of quotients including the case of classical quotient ring. However all significant results are so far limited to the case where the singular ideal is identically zero. The difficulty in the general case lies in the nonclosed property of the singular ideal. In this paper we study some of the properties of the closure of the singular ideal of a ring R and the relations between the rings of quotients of R and the rings of quotients of factor rings of R.

Let R be a ring with identity element 1. If S is a subset of R and $x\in R$, we denote $x^{-1}(S)=\{r\in R \mid xr\in S\}$. We also denote the right and left annihilators of S in R by $\gamma(S)$ and $l(S)$ respectively. The singular ideal (right singular ideal) $J(R)$ of R is defined as $J(R)=\{r\in R \mid \gamma(r)$ is an essential right ideal of $R\}$. The closure $K(R)$ of $J(R)$ is defined as $K(R)=\{k\in R \mid k^{-1}(J(R))$ is an essential right ideal of $R\}$. $K(R)$ is a two sided ideal in R and is the unique maximal essential extension of $J(R)$ in R as right R-module. Let $\tilde{R}=R/J(R)$. Since the inverse image of an essential right ideal in $\tilde{R}$ is essential in R. $\widetilde{K(R)}$, the image of $K(R)$ in $\tilde{R}$, contains the singular ideal $J(\tilde{R})$ of $\tilde{R}$. It is not true that they are equal always. In the case where $J(R)$ is essential in R, $K(R)=R$ whereas $J(\tilde{R})\neq\tilde{R}$.

Lemma 1.1. *The following statements are equivalent*

1. $\widetilde{K(R)}=J(\tilde{R})$
2. $k_1, k_2\in K(R)$ *there exists* $r\in R$ *such that* $(k_1k_2)r\in J(R)$, $k_2r\notin J(R)$ *if* $k_2\notin J(R)$.

3. $k_1, k_2 \in K(R)$, $k_2 \notin J(R)$, $k_2^{-1}(J(R)) \subset (k_1k_2)^{-1}(J(R))$ *properly.*

Proof. Verifications follow the definitions directly.

Theorem 1.2. *If R satisfies any one of the following condition then $\widetilde{K(R)} = J(\tilde{R})$.*

1. *$J(R)$ is closed, i.e. $K(R) = J(R)$.*
2. *The cannonical mapping of R onto $\tilde{R}$ preserves essential right ideals.*
3. *$K^2(R) \subset J(R)$.*
4. *$J(R)$ is prime and is not essential in R.*
5. *If A is an essential right ideal containing $J(R)$ in R then its image in $\tilde{R}$ is essential in $\tilde{R}$.*

Proof. 1., 2., and 5. are obvious. 3. follows 2. by lemma 1.1. For 4., let A be a nonzero right ideal of R such that $A \cap J(R) = 0$. Then $A \cap K(R) = 0$ and $AK(R) = 0$. Hence $K(R) = J(R)$. In other words in a prime ring either $K(R) = 0$ or $K(R) = R$.

If $\widetilde{K(R)} = J(\tilde{R})$ then consequently $J(R)$ can not be essential in R. But it is still too difficult to study under the assumption where $J(R)$ is not essential only. In this paper we always assume $\widetilde{K(R)} = J(\tilde{R})$ unless stated otherwise.

Lemma 1.3. $k \in K(R)$. *If $k^n \notin J(R)$ for all natural number n, then the following two sequences of right ideals of R are strictly increasing.*

1. $k^{-1}(J(R)) \subset (k^2)^{-1}(J(R)) \subset \cdots$
2. $\gamma(k) \subset \gamma(k^2) \subset \cdots$

Proof. If $k^i \notin J(R)$ then there exists $r \in R$ such that $(kk^i)r \in J(R)$ and $k^i r \notin J(R)$. At the same time there exists $t \in R$ such that $k^{i+1}rt = 0$ and $k^i rt \neq 0$.

Theorem 1.4. *If R satisfies the a.c.c. on annihilating right ideals then $K(R)$ is nil.*

Proof. Since $J(R)$ is nil if R satisfies the a.c.c. on annihilating right ideals.

Consequently, if R is a right noetherian ring then $K(R)$ is nilpotent.

Lemma 1.5. *$a \in R$, if there exists a natural number n such that $a^n \in J(R)$ then*

1. $\gamma(1-a)=0$
2. $\gamma(a^i) \subset (1-a)R$ *for all i, the inclusion is proper unless* $a^i=0$.

Proof. Let $x \in \gamma(1-a)$, $x=a^n x$. If $x \neq 0$ then $xR \cap \gamma(a^n) \neq 0$. There exist $r, r' \in R$, $0 \neq r = xr'$ and $a^n r=0$. But $r=xr'=a^n xr' = a^n r = 0$. Contradiction. From $1-a^i=(1-a)(1+a+\cdots+a^{i-1}) \subset (1-a)R$, $\gamma(a^i) \subset (1-a^i)R \subset (1-a)R$ and $\gamma(a^n) \subset (1-a^n)R \subset (1-a)R$. If $a^n=0$ then $(1-a^n)R=(1-a)R=R$. If $a^n \neq 0$ then $(1-a^n)R \cap a^n R \neq 0$. There exist r, r' in R such that $(1-a^n)r=a^n r'$. $r-a^n(r+r')=0$. $r'=r'+r-a^n(r+r')=(1-a^n)(r+r') \in (1-a^n)R$. Thus $\gamma(a^n)$ is contained in $(1-a^n)R$ and hence in $(1-a)R$ properly. Now suppose $a^i \neq 0$. If $i<n$, then $\gamma(a^i) \subset \gamma(a^n) < (1-a^n)R \subset (1-a)R$. If $a^n=0$ then $(1-a)R=R$ and $\gamma(a^i) \neq R$. If $a^n \neq 0$ then $\gamma(a^i) \subset \gamma(a^n) < (1-a^n)R \subset (1-a)R$. If $i \geqslant n$ then $a^i \in J(R)$ and $\gamma(a^i) < (1-a^i)R \subset (1-a)R$.

Corollary 1.5.1. *If $k \in K(R)$ then*

1. $(1-k)^{-1}(J(R))=J(R)$,
2. *$K(R)$ contains no nonzero idempotent,*
3. *for any i, $(k^i)^{-1}(J(R)) \subset (1-k)R+J(R)$. If $k^n \in J(R)$ for some n then $(1-k)R+J(R)=R$.*

Proof. By our assumption $\widetilde{K(R)}=J(\tilde{R})$, $k \in K(R)$, $(1-k)r \in J(R)$, if and only if $(\tilde{1}-\tilde{k})\tilde{r}=0$. By the lemma $r \in J(R)$. It is a well known fact that $J(R)$ contains no nonzero idempotent. If $k \in K(R)$ and $k^2=k$. Then $(1-k)k=0$, $k \in J(R)$ and $k=0$. 3. follows from 2. of the lemma.

Theorem 1.6. *If R satisfies the a.c.c. on annihilating right ideals then $k \in K(R)$:*

1. $\gamma(1-k)=0$,
2. $(1-k)R=R$.

Proof. Since $k \in K(R)$, there exists n such that $k^n \in J(R)$ and k is a nilpotent. Therefore $\gamma(1-k)=0$ and $\gamma(k^i)=R=(1-k)R$.

In some sense an element $a \in R$, $a^n \in J(R)$ for some natural number n can be called a generalized nilpotent element of R.

Lemma 1.7. *If eR is a minimal right ideal of R where $e=e^2$, then $K(R)\subset(1\text{-}e)R$.*

Proof. $eRJ(R)=0$. Otherwise $eR\subset J(R)$ and $(eR)^2=0$. Thus $J(R)\subset(1-e)R$. $k\in K(R)$, $k=ek+(1-e)k$. $kk^{-1}(J(R))=ekk^{-1}(J(R))+(1-e)kk^{-1}(J(R))$. $ekk^{-1}(J(R))\subset eR\cap(1-e)R=0$. $ek\in J(R)$ and hence $ek=0$. $k=(1-e)k\in(1-e)R$.

Lemma 1.8. *N is any maximal right ideal of R. If $kN=0$, $k\in K(R)$, then k is nilpotent.*

Proof. If $k^2\notin J(R)$ then $k^{-1}(J(R))<(k^2)^{-1}(J(R))$. But $N\subset k^{-1}(J(R))$. This implies $(k^2)^{-1}(J(R))=R$. Now $y=k^2\in J(R)$. If y is not a nilpotent then $\gamma(y)<\gamma(y^2)$. But then contradicts the maximality of N. Hence k is nilpotent.

If M is a right unitary R-module and N is a submodule of M. A. W. Goldie defined the closure $cl(N)$ of N in M as $cl(N)=\{x\in M\,|\,x^{-1}(N)$ is an essential right ideal of $R\}$ [1]. The singular submodule $J(M)$ of M is defined to be the closure of the zero submodule. Goldie proves that for any submodule N, $clclN=clclcl(N)$ [1]. Hence $cl(J(M))=clcl(0)=clclcl(0)=clcl(J(M))$. If we let $K(M)=cl(J(M))$ then $cl(K(M))=K(M)$. In the case where $M=R$ then $K(R)$ is closed. Consequently, $J(\bar{R})=0$ where $\bar{R}=R/K(R)$.

Lemma 2.1. *T is a right ideal of R, $T\not\subset K(R)$, then there exists a right ideal N, $0\neq N\subset T$ such that $N\cap K(R)=0$.*

Proof. Let $t\in T$, $t\notin K(R)$ then $t^{-1}(K(R))$ is not essential. There exists a nonzero right ideal W of R such that $W\cap t^{-1}(K(R))=0$. $N=tW$ will do.

Lemma 2.2. *If $J(R)$ is semi-prime then $K(R)$ is semi-prime.*

Proof. Suppose $A^2\subset K(R)$ and $A\not\subset K(R)$ where A is a right ideal of R. By the above lemma there exists a nonzero right ideal N in A such that $N\cap K(R)=0$. $N^2\subset A^2\subset K(R)\cap N=0$. $N\subset J(R)\subset K(R)$. Contradiction.

Corollary 2.2.1. *If R is semi-prime then $K(R)$ is semi-prime.*

Proof. As above $N^2=0$ implies $N=0$.

Theorem 2. 3. *If $K(R)$ is semi-prime then $K(R)=\gamma(W)$ where W is a right ideal of R.*

Proof. Let $W=\sum N_i$ where $N_i \cap K(R)=0$. Since $N_iK(R)=0$ for each N_i, $WK(R)=0$. Suppose there exists $y\in R$ such that $Wy=0$ and $y\notin K(R)$. There exists a right ideal N, $0\neq N\subset yR$ and $N\cap K(R)=0$. $N\subset W$, $N^2\subset WN\subset WyR=0$. $N\subset K(R)$. Contradiction. Thus $\gamma(W)=K(R)$.

Corollary 2. 3. 1. *If $K(R)$ is semi-prime and R satisfies the maximal condition on annihilating right ideals then $\bar{R}=R/K(R)$ is semi-prime with maximal condition on annihilating right ideals.*

Proof. Since $K(R)$ is itself an annihilating right ideal and the inverse image of an annihilating right ideal of $\bar{R}$ in R is an annihilating right ideal in R.

A right ideal A in R is said to be uniform if every nonzero right ideal in A is essential in A.

Lemma 2. 4. *If N is a uniform right ideal in R then $\bar{N}$ is uniform in $\bar{R}$ where $\bar{N}$ is the cannonical image of N in $\bar{R}$.*

Proof. Let $\bar{T}_1$, $\bar{T}_2$ be nonzero right ideals in $\bar{N}$ and $T_1=\{x\in N \mid \bar{x}\in \bar{T}_1\}$, $T_2=\{x\in N \mid \bar{x}\in \bar{T}_2\}$. There exist nonzero right ideals T_1', T_2' such that $T_1'\subset T_1$, $T_2'\subset T_2$ and $T_i'\cap K(R)=0$, $i=1,2$. Since N is uniform, $T_1'\cap T_2'\neq 0$. This implies $\bar{T}_1\cap \bar{T}_2\neq 0$ and $\bar{N}$ is uniform.

Corollary 2. 4. 1. *If R has a uniform right ideal N and $N\not\subset K(R)$, then $\bar{R}$ has a nonzero uniform right ideal.*

Lemma 2. 5. *A is a right ideal R. $\bar{A}$ is essential in $\bar{R}$, if and only if, there exists a right ideal $N\subset A$, $N\cap K(R)=0$, and $N+K(R)$ is essential in R.*

Proof. If there exists $N\subset A$ such that $N\cap K(R)=0$ and $N+K(R)$ is essential in R. $x\notin K(R)$, there exists a right ideal T, $xT\neq 0$ and $xT\cap K(R)=0$. $xT\cap(N+K(R))\neq 0$. $xt=n+k$, $0\neq n\in N$, $k\in K(R)$. $\bar{x}\bar{t}=\bar{n}\neq 0$. $\bar{N}$ is essential in $\bar{R}$. $\bar{A}$ is essential in $\bar{R}$ follows from $\bar{N}\subset \bar{A}$.

Conversely, if $\bar{A}$ is essential in $\bar{R}$. Let N be a maximal right ideal contained in A such that $N\cap K(R)=0$. $A+K(R)$ is essential in R since it is the inverse image of $\bar{A}$. If $N+K(R)$ is not essential in R, then there exists a nonzero right ideal $T\subset A+K(R)$ such that $T\cap(N+K(R))=0$. First we claim $T\cap A\neq 0$. Since $0\neq t\in T$, $t=a+k$, $a\in A$, $k\in K(R)$. There exists $d\in k^{-1}(J(R))$ such that $td\notin K(R)$. $td=ad+kd$. There exists $d'\in\gamma(kd)$ such that $tdd'\notin K(R)$. $tdd'=add'\neq 0$, $T\cap A\neq 0$. Let $W=T\cap A$. Then $W\cap(N+K(R))=0$. But $N\subset W+N\subset A$ and $(W+N)\cap K(R)=0$. This contradicts the maximality of N. Hence $N+K(R)$ is essential in R.

Corollary 2.5.1. *If A is an essential right ideal in R then $\bar{A}$ is essential in $\bar{R}$.*

Proof. Since there exists $N\subset A$ such that $N+K(R)$ is essential and $N\cap K(R)=0$.

Lemma 2.6. *$a\in R$, if $\gamma(a)=0$ then $\gamma(\bar{a})=0$.*

Proof. If $ar\in K(R)$ and $r\notin K(R)$. Let $d\in(ar)^{-1}(J(R))$ such that $rd\notin K(R)$. Now $ard\in J(R)$, $ardD=0$ for some essential right ideal D of R. Since $\gamma(a)=0$, $rd\in J(R)\subset K(R)$. Contradiction.

Corollary 2.6.1. *$k\in K(R)$, $l(k)\neq 0$ and $\gamma(k)\neq 0$.*

Proof. $\gamma(k)\neq 0$ follows the lemma and $K(R)\neq R$. $l(k)\neq 0$ follows $K(R)\neq R$.

Lemma 2.7. *A right ideal C containing $K(R)$ in R is closed in R, if and only if, $\bar{C}$ is closed in $\bar{R}$.*

Proof. If C is a closed right ideal in R, then $\bar{C}$ is clearly closed in $\bar{R}$. If $\bar{C}$ is closed in $\bar{R}$ and $xL\subset C$, $x\in R$ and L is an essential right ideal of R. Since $\bar{L}$ is essential in $\bar{R}$ and C contains $K(R)$. $x\in C$ and C is closed.

Since $J(\bar{R})=0$, either chain condition imposed on the set of closed right ideals of $\bar{R}$ implies the other [2]. By the above lemma this property also holds in R. In [2] we proved that if the a.c.c. holds for the set of closed right ideals in $\bar{R}$, then $\bar{a}\bar{R}$ is an essential right ideal of $\bar{R}$ if $\gamma(\bar{a})=0$. In this case $\bar{a}$ has an inverse in the maximal ring of right quotients of $\bar{R}$. Thus we have the following theorem.

Theorem 2. 8. *If the set of closed right ideals of R satisfies the maximal condition then*:

1. $a\in R$, *if* $\gamma(a)=0$ *then* $\bar{a}\bar{R}$ *is essential in* $\bar{R}$. *Or equivalently, for any* $b\notin K(R)$ *there exist* $r, r'\in R$ *such that* $ar-br'\in K(R)$ *and* $br'\notin K(R)$.

2. $a\in R$, *if* $\gamma(a)=0$ *then* $ax\in K(R)$, $ya\in K(R)$, *if and only if*, $x\in K(R)$ *and* $y\in K(R)$.

If $\bar{A}$ is a right ideal of $\bar{R}$ and $\bar{f}\in \mathrm{Hom}_{\bar{R}}(\bar{A}, \bar{R})$. Let $\bar{W}=\bar{f}(\bar{A})$ and W be the inverse image of $\bar{W}$ in R. Choose N to be a maximal right ideal in W such that $N\cap K(R)=0$ and $\bar{A}'=\{\bar{a}\in\bar{A}\,|\,\bar{f}(\bar{a})\in\bar{N}\}$. For each $a'\in A'$, the inverse image of $\bar{A}'$ in R, $a'\to\bar{a}'\in\bar{A}'\to\bar{f}(\bar{a}')=\bar{n}\in\bar{N}$. Since $N\cap K(R)=0$ there exists a unique $n\in N$ whose canonical image in $\bar{R}$ is $\bar{n}$. Let f be the composite mapping from A' to N defined by:

$$f(a')=n \quad \text{where} \quad \bar{n}=\bar{f}(\bar{a}') \quad \text{or}$$
$$\overline{f(a')}=\bar{f}(\bar{a}')\,, \quad a'\in A'\,.$$

It is routine to verify $f\in \mathrm{Hom}_R(A', R)$.

Lemma 2. 9. *$\bar{A}'$ is an essential right ideal of $\bar{R}$, if and only if, $\bar{A}$ is an essential right ideal of $\bar{R}$.*

Proof. Suppose $\bar{A}$ is essential. $\bar{a}\in\bar{A}$, and $\bar{a}\neq 0$. If $\bar{f}(\bar{a})=0$, then $\bar{a}\in\bar{A}'$ and $\bar{a}\bar{R}\cap\bar{A}'\neq 0$. If $\bar{f}(\bar{a})=\bar{b}\neq 0$. Suppose $\bar{b}=b+K(R)$, then $b\in W$ and $b\notin K(R)$. Let T be a nonzero right ideal in R such that $T\cap b^{-1}(K(R))=0$. If $bT\cap N\neq 0$ then there exists $t\in T$ and $n\in N$ such that $bt=n$. $\bar{f}(\bar{a}\bar{t})=\bar{f}(\bar{a})\bar{t}=\bar{b}\bar{t}=\bar{n}\neq 0$. $0\neq\bar{a}\bar{t}\in\bar{A}'$ and hence $\bar{a}\bar{R}\cap\bar{A}'\neq 0$. If $bT\cap N=0$. First we claim that $(bT+N)\cap K(R)=0$. Since if there exist $x\in bT$, $n\in N$, and $k\in K(R)$ such that $x+n=k\neq 0$ ($x\neq 0$ and $n\neq 0$). Let $d\in k^{-1}(J(R))$ such that $xd\notin K(R)$. $xd+nd=kd$. Let $g\in\gamma(kd)$ such that $xdg\notin K(R)$. $xdg+ndg=0$. This shows $bT\cap N\neq 0$. Contradiction. Therefore, $(bT+N)\cap K(R)=0$. But $bT+N\subset W$ and $bT+N$ contains N properly. This contradicts the maximality of N. Hence $bT\cap N\neq 0$. Thus we have proved that for each $0\neq\bar{a}\in\bar{A}$, $\bar{a}\bar{R}\cap\bar{A}'\neq 0$. Since $\bar{A}$ is essential it is sufficient to conclude $\bar{A}'$ is also essential.

Theorem 2.10. *If $\bar{f}\in Hom_{\bar{R}}(\bar{A}, \bar{R})$ where $\bar{A}$ is an essential right ideal if $\bar{R}$, then there exists an essential right ideal A' in R containing $K(R)$ and $f\in Hom_R(A', R)$ such that $\overline{f(a')}=\bar{f}(\bar{a}')$ for all $a'\in A'$.*

The concept of dense right ideal in a ring R was introduced first by Y. Utumi [4] under a different name and was later modified somewhat by J. Lambek [3]. In the case $J(R)=0$, dense right ideals and essential right ideals coincide. Utumi calls a ring S a ring of right quotients of R if R is a subring of S and for each $q\in S$, $q^{-1}(R)=\{r\in R\mid qr\in R\}$ is a dense right ideal of R and $q(q^{-1}(R))\neq 0$ if $q\neq 0$. He also proved that for any ring R with identity element has a unique maximal ring of right quotients Q [4]. Each $q\in Q$, q can be realized as a R-homomorphism of a dense right ideal of R into R [3], [4]. If $J(R)=0$ then Q is a self right injective regular ring [2]. It is easy to show that $a\in R$, a has an inverse in Q, if and only if, $\gamma(a)=0$ and aR is a dense right ideal of R. If M is a multiplicatively closed subset of regular elements of R and satisfies the right Ore's condition ($a\in M$, $b\in R$, $b^{-1}(aR)\cap M\neq 0$), then aR is dense, $a\in M$, and $Q_M=\{xa^{-1}\mid x\in R, a\in M\}\subset Q$ is the classical right quotient ring of R relative to M. For the sake of completeness, we mention some of the properties of dense right ideals here again.

Definition. A right ideal D of R is *dense* in R if for all r_1, r_2 in R, $r_1\neq 0$, $r_1 r_2^{-1}(D)\neq 0$.

Theorem 3.1.

1. *If D is a dense right ideal of R then D is essential and $l(D)=0$.*
2. *If D is a dense right ideal of R and S is a ring of right quotients of R, then for any $q\in S$, $q^{-1}(D)$ is a dense right ideal of R.*
3. *Intersection of any finite collection of dense right ideals of R is dense.*
4. *Considering R as a right R-module, let I be its injective hull and $H=Hom_R(I, I)$ then a right ideal D of R is dense,*

if and only if, $h \in H$, $hD=0$ implies $hR=0$.

Proof. Proofs can be found in [3].

Lemma 3. 2. *S is a ring of right quotients of R. A right ideal C of S is essential in S, if and only if, $C \cap R$ is an essential right ideal of R. If A is an essential right ideal of R then AS is an essential right ideal in S.*

Proof. If A is an essential right ideal of R, then it is obvious AS is an essential right ideal in S. Suppose C is an essential right ideal in S. $0 \neq r \in R$, $rS \cap C \neq 0$. There exists $q \in S$ such that $0 \neq rq \in C$. $0 \neq rqq^{-1}(R) \subset rR \cap (C \cap R)$. $C \cap R$ is essential in R.

Corollary 3. 2. 1. *If S is a ring of right quotients of R and C is a right ideal of S, then the closure $cl(C)$ of C in S is $\{x \in S \mid x^{-1}(C)$ is an essential right ideal of $R\}$.*

Proof. Follows the lemma directly.

Corollary 3. 2. 2. *If S is a ring of right quotients of R, then $J(R)=J(S) \cap R$ and $K(R)=K(S) \cap R$.*

Proof. Follows corollary 3. 2. 1 directly.

Corollary 3. 2. 3. *S is a ring of right quotients of R. If C is a closed right ideal in S then $C \cap R$ is a closed right ideal in R. If A is a closed right ideal of R then $A^* = \{x \in S \mid x^{-1}(A)$ is an essential right ideal of $R\}$ is a closed right ideal in S and $A^* \cap R = A$.*

Proof. $cl(C \cap R) \subset cl(C) = C$, if C is a closed right ideal in S. $r \in R \cap cl(C \cap R)$, $r \in R \cap C$. $C \cap R$ is closed in R. For any right ideal A of R, $A^* \cap R =$ closure of A in R. Thus if A is closed in R then $A^* \cap R = A$. If $q \in S \cap cl(A^*)$ then $q^{-1}(R) \cap q^{-1}(A^*) = B$ is an essential right ideal of R. Since $qB \subset A^* \cap R = A$, $q \in A^*$ and A^* is closed in S. A^* is a right ideal of S follows from the fact that S is an essential extension of R as a right R-module.

Notice that if C is a closed right ideal in S then $C=(C \cap R)^*$. Thus we have a natural correspondence between the closed right ideals of R and S. If A is a closed right ideal of R then:

$$A \to A^* \to A^* \cap R = A\,.$$

If C is a closed right ideal in S then:

$$C \to C \cap R \to (C \cap R)^* = C\,.$$

Since $r \in R \cap K(S)$, if and only if, $r \in K(R)$. We can consider $\bar{R} = R/K(R)$ as a subring of $\bar{S} = S/K(S)$. $0 \neq \bar{q} \in \bar{S}$, $\bar{q} = q + K(S)$ with $q \notin K(S)$. Let D be a dense right ideal of R such that $qD \subset R$. Since $q \notin K(S)$ there exists $d \in D$ such that $qd \notin K(S)$. $0 \neq \bar{q}\bar{d} = qd + K(S) \in \bar{R}$. $\bar{S}$ is a ring of right quotients of $\bar{R}$.

Let Q be the maximal ring of right quotients of R and $\bar{W}$ be the maximal ring of right quotients of $\bar{R}$. We have shown that $\bar{W}$ is a ring of right quotients of $\bar{Q} = Q/K(Q)$. It is interesting to ask under what condition $\bar{Q} = \bar{W}$? Or equivalently, under what condition every ring of right quotients of $\bar{R}$ is an image of a subring of Q?

D-condition: A right ideal D in R is dense in R if it is an inverse image of an essential right ideal in $\bar{R}$ where $\bar{R} = R/K(R)$.

Obviously D-condition is equivalent to that every essential right ideal containing $K(R)$ in R is dense.

Lemma 3.4. *The followings are equivalent.*

1. *D-condition.*
2. *If A is an essential right ideal containing $K(R)$ in R, then $l(A) = 0$.*

Proof. Trivial.

Theorem 3.5. *If R satisfies D-condition then $\bar{Q} = \bar{W}$.*

Proof. Let $\bar{w} \in \bar{W}$ then $\bar{w} \in \mathrm{Hom}_{\bar{R}}(\bar{A}, \bar{R})$ where $\bar{A}$ is an essential right ideal in $\bar{R}(\bar{A} = \bar{w}^{-1}(\bar{R}))$. By theorem 2.10 there exist an essential right ideal A' containing $K(R)$ in R and $f \in \mathrm{Hom}_R(A', R)$ such that

$$\overline{f(a')} = \bar{w}(\bar{a}') = \bar{w}\bar{a}' \quad \text{for all } a' \text{ in } A'\,.$$

D-condition implies A' is dense. There exists $q \in Q$ such that $qa' = f(a')$ for all $a' \in A'$. $\overline{qa'} = \bar{q}\bar{a}' = \bar{w}\bar{a}'$ for all $a' \in A'$. Since $J(\bar{R}) = 0$, $\bar{q} = \bar{w}$ and $\bar{Q} = \bar{W}$.

Theorem 3. 6. *Assume the D-condition and the maximal condition for the set of closed right ideals are satisfied in R. If $a \in R$, $\gamma(a)=0$, and $aR \supset K(R)$, then a has an inverse in Q.*

Proof. If $a \in R$, $\gamma(a)=0$, then by theorem 2. 8 $\bar{a}\bar{R}$ is essential in $\bar{R}$. The inverse image of $\bar{a}\bar{R}$ is aR since $aR \supset K(R)$. By D-condition aR is dense and hence a has an inverse in Q.

If R is a semi-prime ring then Q is obviously semi-prime. Suppose A is a right ideal in Q, $A \cap K(Q)=0$, and $f \in \mathrm{Hom}_Q(A, Q)$. Define $\bar{f}: \bar{A} \to \bar{Q}$ by $\bar{f}(\bar{a})=\overline{f(a)}$ where $\bar{a}=a+K(Q)$, $a \in A$. $\bar{f} \in \mathrm{Hom}_{\bar{Q}}(\bar{A}, \bar{Q})$. Since $\bar{W}$ is self injective. So if $\bar{Q}=\bar{W}$ there exists $\bar{q} \in \bar{Q}$ such that $\bar{f}(\bar{a})=\bar{q}\bar{a}$ for all $\bar{a} \in \bar{A}$. If $\bar{q}=q+K(Q)$ then $f(a)=qa+k_a$ where $k_a \in K(Q)$. $f(a)K(Q)=f(aK(Q))=qaK(Q)+k_aK(Q)$. Since $aK(Q) \subset A \cap K(Q)=0$. $k_aK(Q)=0$. Let $T=\{k \in K(Q) \mid$ there exists $a \in A$, $f(a)=qa+k\}$. T is a right ideal of Q and is contained in $K(Q)$. Hence $T^2 \subset TK(Q)=0$. If R is semi-prime then $T=0$ and $f(a)=qa$ for all $a \in A$.

Theorem 3. 7. *If R is semi-prime and satisfies the D-condition, then for any $f \in Hom_Q(A, Q)$, where A is a right ideal of Q and $A \cap K(Q)=0$, there exists $q \in Q$ such that $f(a)=qa$ for all $a \in A$.*

In order to obtain more properties about R and Q from the informations of $\bar{R}$, $\bar{Q}$, and $\bar{W}$, we must know more about $K(Q)$ in related to $K(R)$ and $J(R)$. For instance, under what situation $K(Q)$ will be nil?

J-condition: D is a dense right ideal of R, $f \in \mathrm{Hom}_R(D, J(R))$ then the kernel of f is an essential right ideal of R.

It is clear that J-condition is equivalent to that $q \in Q$, $qR \cap R \subset J(R)$, if and only if $q \in J(Q)$.

Theorem 4. 1. *If every dense right ideal D of R contains an element a such that $\gamma(a)=0$ and aR is dense, then R satisfies the J-condition.*

Proof. $f \in \mathrm{Hom}_R(D, J(R))$ where D is a dense right ideal of R. Let $q \in Q$ such the $qd=f(d)$ for all $d \in D$. Let $a \in D$ such that $\gamma(a)=0$ and aR is dense. $qa=j \in J(R)$. a has an inverse a^{-1} in

Q. $q=ja^{-1}\in J(Q)$. $\ker(f)=\gamma(q)\cap D$ is an essential right ideal of R.

Theorem 4.2. *If R satisfies the J-condition then:*

1. *If D is a dense right ideal of R then the image $\tilde{D}$ of D in $\tilde{R}=R/J(R)$ is dense in $\tilde{R}$.*
2. *$\tilde{S}=S/J(S)$ is a ring of right quotients of $\tilde{R}$ where S is a ring of right quotients of R.*
3. *If $J(R)$ is closed in R, i.e. $J(R)=K(R)$, then $J(S)=K(S)$ where S is any ring of right quotients of R.*

Proof. For 1., let $\tilde{x}\neq 0$, $\tilde{y}\in\tilde{R}$. $\tilde{x}=x+J(R)$ and $\tilde{y}=y+J(R)$. $y^{-1}(D)=B$ is a dense right ideal of R. $xB\not\subset J(R)$. Otherwise $x\in J(R)$ and $\tilde{x}=0$. There exists $b\in B$ such that $yb\in D$ and $xb\notin J(R)$. $\tilde{D}$ is dense in $\tilde{R}$. For 2., Since $J(R)=J(S)\cap R$, $\tilde{R}=R/J(R)$ can be considered as a subring of $\tilde{S}=S/J(S)$. $q\in S$, $\tilde{q}=q+J(Q)$. $\tilde{D}=\widetilde{q^{-1}(R)}$ is a dense ring ideal of $\tilde{R}$ by 1. $\tilde{q}\tilde{D}=0$, if and only if, $q\in J(S)$. Since $\tilde{q}\tilde{D}\subset\tilde{R}$, $\tilde{S}$ is a ring of right quotients of $\tilde{R}$. For 3., $q\in K(S)$, $q(q^{-1}(R))\subset K(S)\cap R=K(R)=J(R)$. $q\in J(S)$.

Recall that at the begining, we assume $J(\tilde{R})=\widetilde{K(R)}$. From this we can prove $J(\tilde{R})$ is closed in $\tilde{R}$, i.e., $K(\tilde{R})=J(\tilde{R})$. If R satisfies the J-condition and S is a ring of right quotients of R, then $\tilde{S}=S/J(S)$ is a ring of right quotients of $\tilde{R}$. If $\tilde{R}$ also satisfies the J-condition, then of course $J(\tilde{S})$ would be closed in $\tilde{S}$. Consequently, $\widetilde{K(S)}=J(\tilde{S})$. $q\in K(S)$, $A=q^{-1}(J(S))$ is an essential right ideal of R and contains $J(R)$. If we assume the property 5. in theorem 1.2 holds in R then $\tilde{A}$ would be essential in $\tilde{R}$ and $\tilde{q}\in J(\tilde{S})$. From now on we assume R has such property. That is, if A is an essential right ideal containing $J(R)$ in R then $\tilde{A}$ is essential in $\tilde{R}$.

Now suppose $q\in K(S)$, then $\tilde{A}=\widetilde{q^{-1}(J(S))}=\gamma(\tilde{q})$ is an essential right ideal of $\tilde{R}$. If $q'\notin J(S)$ then $q'D\not\subset J(R)$ where $D=q'^{-1}(R)$. Let $d\in D$ such that $0\neq\tilde{q}'\tilde{d}$ in $\tilde{R}$. Since $\tilde{A}$ is essential in $\tilde{R}$, there exists $t\in R$ such that $\tilde{q}'\tilde{d}\tilde{t}\neq 0$ and $\tilde{q}\tilde{q}'\tilde{d}\tilde{t}=0$ in $\tilde{R}$. Since $q'dt\in R$, $q'dt\notin J(S)$. If we let $r=dt$ then $q'r\notin J(S)$ and $qq'r\in J(S)$. Consequently, if $q\in K(S)$ and $q^{n}\notin J(S)$ for all natural number n, then the following sequence of right ideals of R is strictly increasing

$$q^{-1}(J(S))<(q^{2})^{-1}(J(S))<\cdots$$

$x \in J(S)$, $\gamma(x) < \gamma(x^2)$ if $x \neq 0$. Thus if x is not a nilpotent, $x \in J(S)$, then $\gamma(x) < \gamma(x^2) < \cdots$ is a strictly increasing sequence of right ideals of R.

Theorem 4. 3. *If R satisfies the J-condition and the maximal condition on right ideals, then $J(S)$ and $K(S)$ are nil ideals where S is any ring of right quotients of R.*

Proof. $q \in K(S)$, there exists n such that $q^n \in J(S)$. q is a nilpotent follows q^n is a nilpotent.

Theorem 4. 4. *If R satisfies the D-condition, the J-condition, and the maximal condition on right ideals, then $a \in R$, a has an inverse in Q if $\gamma(a) = 0$ where Q is the maximal ring of right quotients of R.*

Proof. If $a \in R$, $\gamma(a) = 0$, then $\gamma(\bar{a}) = 0$ and $\bar{a}\bar{R}$ is essential in $\bar{R} = R/K(R)$ by theorem 2. 8. Therefore $\bar{a}$ has an inverse in $\bar{W}$, the maximal ring of right quotients of $\bar{R}$. By D-condition there exists $q \in Q$ such that $aq = 1 + k$ for some $k \in K(Q)$. Since k is nilpotent by the previous theorem, a has an inverse in Q.

OBERLIN COLLEGE, OBERLIN, OHIO

REFERENCES

[1] Goldie, A. W., Torsion-Free Modules and Rings. J. Algebra **1** (1964), 268–287.
[2] Lambek, J., Lecture on Rings and Modules. Blaisdell Publishing Co., Waltham, Massachusetts, 1966.
[3] Johnson, R. E. and Wong, E. T., Quasi-Injective Modules and Irreducible Rings. J. London Math. Soc. **36** (1961), 260–268.
[4] Utumi, Y., On Quotient Rings. Osaka Math. J. **8** (1956), 1–18.