J. Math. Kyoto Univ. (JMKYAZ)
11-1 (1971) 113-147

The decomposition of L^2 $(\Gamma\backslash SL(2, R))$ and Teichmüller spaces

By

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(Communicated by Professor Yoshizawa, July 18, 1970)

§0. Introduction

Let H be the complex upper half plane and let Γ be a discrete subgroup of the group G of conformal automorphisms of H. We assume that $\Gamma\backslash H$ is compact.

For each unitary matrix representation χ of Γ, we consider an eigenvalue problem (called the (Γ, χ)-eigenvalue problem in §1,) following [S]. The spectra of this eigenvalue problem and its generalizations have been investigated since the famous paper of A. Selberg [S] appeared in 1956. But, at present, not much is known even in the above special case.

In this paper, we want to study "How do the spectra of (Γ, χ)-problem behave when Γ varies?"

There is another (more group-theoretical) interpretation of our problem. We give it in the following.

Let G, Γ, χ be as above and let $U=\operatorname{Ind}_{\Gamma\uparrow G}\chi$ be the unitary representation of G induced from χ. As is well known, U can be decomposed into the discrete sum $\sum_i \oplus U_i$ of irreducible unitary representations U_i of G. We call the set $S_U=\{U_i; i=1, 2, \cdots\}$ the spectra of $U=\operatorname{Ind}_{\Gamma\uparrow G}\chi$ and decompose it into the disjoint union of two subsets, the C-part S_U^C and the D-part S_U^D, where S_U^C consists of those elements of S_U contained in the continuous series (that is, the continuous principal series and the supplementary series) of irreducible unitary representations of G and S_U^D consists of those elements of S_U contained in the discrete series.

Now, our problem can be rephrased as follows[1]: "How do the C-part S_U^C of the spectra S_U of the induced representation $U=\operatorname{Ind}_{\Gamma\uparrow G}\chi$ of G behave when Γ varies?" (We need not study the behaviour of the D-part S_U^D, for $S_U^D(U=\operatorname{Ind}_{\Gamma\uparrow G}\chi)$ is completely known.)

Some problems of the similar nature were also discussed by J.M.G. Fell [F] in a more general situation.

Obviously, the first thing we must do is to given a precise meaning to the phrase "Γ varies". In our case, however, there exists a very suitable theory for that purpose, at least when Γ's have no elliptic elements. It is the so-called moduli theory of closed Riemann surfaces. The purpose of this theory is, roughly speaking, to give a "natural" topology and a "natural" complex analytic structure to the set M_g of conformal equivalence classes of closed Riemann surfaces of a given genus g. The transcendental approach to the moduli problem of closed Riemann surfaces was originated some thirty years ago by O. Teichmüller and has been developed by L. V. Ahlfors, L. Bers, and H. E. Rauch. Since M_g is not a manifold in its natural topology, Teichmüller introduced a covering space T_g of it, called Teichmüller space. One of the main theorem in the theory is that T_g can be endowed with a structure of a complex analytic manifold[2]. Another important structure of T_g is that of Riemannian manifold due to A. Weil and L. V. Ahlfors. For the sake of completeness, the outline of the theory of Teichmüller spaces is presented in §2. Almost all proofs are omitted

1) About the relation between the spectra of (Γ, χ)-eigenvalue problem and the spectra of the induced representation, see [G, Chapter 1, §5].

2) This theorem implies that the multiplicity in which each irreducible unitary representation in the discrete series appears in the decomposition $\sum_i \oplus U_i$ of $U=\operatorname{Ind}_{\Gamma\uparrow G}\chi$ must be a constant on every T_g. (Note that representations in the discrete series are parametrized by integers.) In fact, we can assure the above observation using the explicit values of the multiplicities.

referring to [A1-3]. The only exception is Lemma 2.6., which plays an essential role in our later task (§§4-5). Its proof, being rather long, is given in the Appendix at the end of this paper.

§1 is preliminaries.

The contents of §2 and their important roles in this paper are explained above.

§3 is devoted to give the precise formulation of our problems in terms of the moduli theory. We here explain it in the simplest case when $\chi=1$: the identity representation.

Since each point of T_g represents a conformal equivalence class of closed Riemann surface, it determines a Fuchsian group Γ (up to inner automorphisms of G). Hence, we can associate it with the spectra of $(\Gamma, 1)$-problem. Then, we can ask, "Is there a series $\Lambda^{(j)}$ $(j=1, 2, \cdots)$ of continuous functions on T_g whose values at each point represent all the eigenvalues of the eigenvalue problem associated with the point?"

Moreover, if such $\Lambda^{(j)}$ exist, we may ask whether each $\Lambda^{(j)}$ can be considered as a real analytic function or not.

§§4-5 is the main part of this paper.

In §4, the followings are proved:

'For every real analytic curve $\mathscr{C}=\mathscr{C}(t)$ (t; real parameter) in T_g, there exists a series $\Lambda^{(j)}=\Lambda^{(j)}(t)$ of real analytic functions in t whose values at each point represent all the eigenvalues of the associated problem.' (Theorem 4.1.)

'Given a point P of T_g and a simple eigenvalue λ_0 of the eigenvalue problem associated with P, there exists a neighbourhood V of P in T_g and a real analytic function Λ on V such that $\Lambda(P)=\lambda_0$ and at each point the value of Λ represents one of the eigenvalues of the associated problem.' (Theorem 4.2.)

In §5, we prove an interesting formula for the differential coefficients of the above functions $\Lambda^{(j)}(t)$.

Namely, Theorem 5.1., the main theorem in this section, implies that

'In the above notations, let, $\lambda_0 = \Lambda^{(j)}(t_0)$ be a simple eigenvalue of the eigenvalue problem associated with the point $P=\mathscr{C}(t_0)$ on the curve $\mathscr{C}(t)$ and let F be an eigenvector belonging to λ_0. Then we can find an element ν_{λ_0} (calculable from F) of the tangent space $\mathscr{T}_P$ of T_g at P such that

$$\left[\frac{d}{dt}\Lambda^{(j)}(t)\right]_{t=t_0} = g_p(\nu, \nu_{\lambda_0}),$$

where $g_p(\cdot, \cdot)$ is the innerproduct in $\mathscr{T}_P$ and ν is the element of $\mathscr{T}_P$ tangential to the curve $\mathscr{C}=\mathscr{C}(t)$.'

Now, a word for our method. Since, our problem is, roughly speaking, to ask, "How do the eigenvalues behave when Γ varies?", we are tempted to consider this as a sort of perturbation problem at least locally. In fact, our problems resemble the ones which have long been known as boundary perturbation problems in mathematical physics. In particular, L. A. Segel showed[3] that the use of conformal mappings is very effective to turn some of such problems into ordinary perturbation problems. It will become clear in §§4–5 that his method, suitably modified, is also applicable to our problems. For perturbation theory, we use the terminologies and theorems of the excellent book of T. Kato [K] rather frequently.

Finally, the author wants to express his hearty thanks to Prof. K. Shibata for his kind advices on the theory of Teichmüller spaces.

§1. Preliminaries

1.1. Notations

Let $H=\{z=x+iy;\ y>0\}$ be the complex upper half plane. Then, $G=SL(2, R)/\{\pm e\}$ operates transitively on H from the left by

$$g(z)=\frac{az+b}{cz+d} \tag{1.1}$$

3) Archive for rational mechanics and analysis, 8 (1961), pp. 228-237.

for z in H and $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G. Hence, G can be identified with the group of all conformal automorphisms of H.

The G-invariant metric on H is

$$ds^2=\frac{dx^2+dy^2}{y^2}, \tag{1.2}$$

hence the G-invariant measure on H is

$$dz=\frac{dx\,dy}{y^2}. \tag{1.3}$$

And the ring of G-invariant differential operators on H is generated by

$$\Delta=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right). \tag{1.4}$$

1.2. (Γ, χ)-eigenvalue problems

Let Γ be a discrete subgroup of G such that $\Gamma\backslash H$ is compact and let χ be a finite-dimensional representation of Γ by $n\times n$ unitary matrices. Following [S], we call an n-dimensional column vector, whose components are scalar functions on H, a function vector on H.

Consider the complex vector space $\mathscr{H}(\Gamma, \chi)$ of all function vectors F on H which satisfy the following conditions:

(i) F is (componentwisely) measurable;

(ii) $F(Az)=\chi(A)F(z)$ for all z in H and A in Γ;

(iii) $\int_{\mathscr{F}} {}^tF(z)\overline{F(z)}dz<\infty$,

where $\mathscr{F}$ is a measurable fundamental domain of Γ in H and dz is the G-invariant measure on H given by (1.3).

Introducing an innerproduct

$$(F_1, F_2)_{\mathscr{H}(\Gamma,\chi)}=\int_{\mathscr{F}} {}^tF_1(z)\overline{F_2(z)}dz, \tag{1.5}$$

we can consider $\mathscr{H}(\Gamma, \chi)$ as a Hilbert space. This innerproduct does not depend on the choice of a fundamental domain $\mathscr{F}$ because of the above condition (ii).

For a differential operator L on H and a function vector F, we define LF by

$$LF(z)=\begin{pmatrix} Lf_1(z) \\ Lf_2(z) \\ \vdots \\ Lf_n(z) \end{pmatrix} \qquad \text{for } F=\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

We also use the following notation:

$$(1.6) \qquad |F|=({}^tF\cdot\bar{F})^{\frac{1}{2}} \qquad \text{for } F\in\mathscr{H}(\Gamma, \chi).$$

Now, we consider the fundamental eigenvalue problem:

$$(1.7) \qquad \Delta F=\lambda F \qquad (F\in\mathscr{H}(\Gamma, \chi)),$$

where Δ is the differential operator given by (1.4).

It is well known and can be easily verified that the operator Δ in $\mathscr{H}(\Gamma, \chi)$ is symmetric and positive. Hence, it has the selfadjoint extension $\tilde{\Delta}$ (the Friedrichs extension). The quadratic form $\mathfrak{p}$ associated with $\tilde{\Delta}$ (in the sense of [K, pp. 322-323]) is given by

$$(1.8) \qquad \mathfrak{p}[F]=\int_{\mathscr{F}} y^2\left\{\left|\frac{\partial F}{\partial x}\right|^2+\left|\frac{\partial F}{\partial y}\right|^2\right\}dz \qquad (F\in\mathscr{D}(\mathfrak{p})),$$

where $\mathscr{D}(\mathfrak{p})$ is the domain of $\mathfrak{p}$.

For brebity, we shall use the notation Δ for $\tilde{\Delta}$ in the following.

Definition 1.1. The eigenvalue problem (1.7) interpreted as above is called the (Γ, χ)-eigenvalue problem or the (Γ, χ)-problem.

The following theorem is well known.

Theorem 1.1.

(i) $\mathscr{H}(\Gamma, \chi)$ *has a complete orthonormal system* $\{F_j; j=1, 2, 3, \cdots\}$

which consists of eigenvectors of the (Γ, χ)*-eigenvalue problem* (1.7).

(ii) *The eigenvalues* λ_j $(j=1, 2, \cdots)$ *of the* (Γ, χ)*-problem are all real and nonnegative and of finite multiplicity.*

Moreover, they have no finite point of accumulation on the real line.

§2. The moduli theory of closed Riemann surfaces

As stated in §0, we give here a brief survey of the moduli theory of closed Riemann surfaces following [A1-3].

2.1. Teichmüller spaces

Let H, G, Γ be as in §1. In this section and below, we shall assume, furthermore, that Γ has no elliptic elements. Then, as is well known, any closed Riemann surface of genus $g>1$ can be represented as a quotient space $\Gamma\backslash H$ for some such Γ. Moreover, two Riemann surfaces $W_1=\Gamma_1\backslash H$ and $W_2=\Gamma_2\backslash H$ are conformally equivalent if and only if Γ_1 and Γ_2 are conjugate to each other as subgroups of G.

Definition 2.1. We say that Γ is a Fuchsian group of genus $g(>1)$, if

(i) Γ is a discrete subgroup of $G=SL(2, R)/\{\pm e\}$ such that $\Gamma\backslash H$ is compact,

(ii) Γ has no elliptic element,

(iii) as a closed Riemann surface, the genus of $W=\Gamma\backslash H$ is g.

The following is also well known.

Lemma 2.1. *Any Fuchsian group of a given genus g is isomorphic to an abstract group generated by elements* $a_1, a_2, \cdots, a_{2g}$ *which satisfy the only one relation*

$$a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1} = 1.$$

Let Γ be a Fuchsian group of genus g. By the above lemma, it is generated by elements $A_1, A_2, \cdots, A_{2g}$ of G which satisfy the only one relation

$$A_1 A_2 A_1^{-1} A_2^{-1} \cdots A_{2g-1} A_{2g} A_{2g-1}^{-1} A_{2g}^{-1} = E, \tag{2.1}$$

where E is the unit matrix of the second order. Using this fact, we can state the formal definition of Teichmüller spaces:

Definition 2.2. Consider the set $\tilde{T}_g$ of pairs $(\Gamma, \{A_i\})$ where Γ is a Fuchsian group of genus g and $\{A_i; i=1, 2, \cdots, 2g\}$ is a set of generators of Γ which satisfies (2.1).

We define an equivalence relation $\sim$ in $\tilde{T}_g$ by setting $(\Gamma, \{A_i\}) \sim (\Gamma', \{A_i'\})$ if and only if there exists an element B of G such that

$$A_i = B A_i' B^{-1} \qquad (i=1, 2, \cdots, 2g).$$

The set of all equivalence classes under this relation is called the Teichmüller space T_g.

Lemma 2.2. *In each equivalence class of T_g (in the sense of Definition 2.2), there exists a unique element $(\Gamma, \{A_i\})$ such that the fixed points of A_1 are 0, ∞ and the attractive fixed point of A_2 is 1.*

We call such $(\Gamma, \{A_i\})$ a normalized pair in $\tilde{T}_g$. It is clear that T_g can be identified with the set of all normalized pairs in $\tilde{T}_g$. Hence, we arrived at the second definition of Teichmüller spaces, that is,

Definition 2.3. Let $(\Gamma_0, \{A_i\})$ be a fixed normalized pair in $\tilde{T}_g$ (in the sense above). Consider the set of all pairs $[\Gamma, \theta]$ where Γ is a Fuchsian group of genue g and θ is an isomorphism from Γ_0 onto Γ such that $(\Gamma, \{\theta(A_i)\})$ is a normalized pair in $\tilde{T}_g$.

The set of all pairs $[\Gamma, \theta]$ is called the Teichmüller space $T(\Gamma_0)$.

The natural topology (moreover, the structure of real analytic manifold) of $T_g = T(\Gamma_0)$ is given in [A2, pp. 177–179].

2.2. Quasiconformal mappings and Beltrami equations.

Definition 2.4. Let Ω be a domain in the complex plane. A homeomorphic mapping f of Ω onto itself is said to be quasiconformal, if f has locally integrable distributional derivatives and satisfies

$$f_{\bar{z}} = \mu f_z \qquad \text{a.e.,} \tag{2.2}$$

where μ is an element of $L^\infty(\Omega)$ such that $\|\mu\|_\infty < 1$ and

$$f_z \equiv \frac{1}{2}(f_x - if_y), \quad f_{\bar{z}} \equiv \frac{1}{2}(f_x + if_y).$$

Remark 2.1. Any quasiconformal mapping f is sense-preserving. In fact, the Jacobian of $f (= |f_z|^2 - |f_{\bar{z}}|^2)$ is nonnegative by the above definition.

In the following, we consider only such cases when Ω is the whole complex plane $\boldsymbol{C}$ or the complex upper half plane H.

Theorem 2.1. (*Morrey*, [A1], [AB])

(i) *Let* μ *be an element of* $L^\infty(\boldsymbol{C})$ *which satisfies* $\|\mu\|_\infty < 1$. *There exists a unique homeomorphic mapping of* $\boldsymbol{C}$ *onto itself, to be denoted by* W^μ, *which satisfies*

$$W^\mu_{\bar{z}} = \mu \cdot W^\mu_z \qquad a.e.$$

and is normalized by

$$W^\mu(0) = 0, \quad W^\mu(1) = 1, \quad W^\mu(\infty) = \infty.$$

(ii) *Let* $\mu \in L^\infty(H)$ *satisfies* $\|\mu\|_\infty < 1$. *There exists a unique homeomorphic mapping of* H *onto itself, to be denoted by* f^μ, *which satisfies*

$$f_{\bar{z}}^{\mu}=\mu f_{z}^{\mu} \qquad a.e.$$

and is normalized by

$$f^{\mu}(0)=0, \quad f^{\mu}(1)=1, \quad f^{\mu}(\infty)=\infty.$$

Remark 2.2. The normalization in (ii) makes sense, for it is known that f^{μ} has a unique extension to a homeomorphism of closed half plane.

Remark 2.3. Let μ be as in (ii). Extend the definition of μ by

$$\hat{\mu}(z)=\overline{\mu(\bar{z})}.$$

Then we can define $W^{\hat{\mu}}$, and f^{μ} is the restriction of $W^{\hat{\mu}}$ to the upper half plane H.

Remark 2.4. In the particular case when $\mu=0$, we have $W^{\mu}(z) \equiv z$ and $f^{\mu}(z)\equiv z$.

We also need the following concept.

Definition 2.5. Let Γ be a Fuchsian group of genus g and let μ be an element of $L^{\infty}(H)$.

We say that μ is a Beltrami coefficient with respect to Γ or a Γ-Beltrami coefficient, if

$$\mu(Az)=\mu(z)\frac{\overline{(cz+d)^2}}{(cz+d)^2} \qquad (z\in H) \tag{2.3}$$

for all $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ.

The linear space of Γ-Beltrami coefficients will be denoted by $B(\Gamma)$, its open unit ball $\{\mu\in B(\Gamma);\ \|\mu\|_{\infty}<1\}$ by $B_1(\Gamma)$.

The importance of the above introduced concepts: quasiconformal mappings and Beltrami coefficients, becomes clear by the following

Lemma 2.3. *Let Γ be a Fuchsian group of genus g and let μ*

be an element of $B_1(\Gamma)$. *We can define the mapping* f^μ *by Theorem* 2.1 (ii). *Then, for any* $A\in\Gamma$, $A^\mu\equiv f^\mu\circ A\circ(f^\mu)^{-1}$ *is a conformal automorphism of* H, *that is, contained in* G. *Thus,*

$$\theta^\mu:\ A\mapsto A^\mu=f^\mu\circ A\circ(f^\mu)^{-1} \tag{2.4}$$

is an isomorphism from Γ *onto another Fuchsian group* Γ^μ *of genue* g.

Theorem 2.2. (*Ahlfors*)

Let $(\Gamma, \{A_i\})$ *be a fixed normalized pair in* $\tilde{T}_g$. *For each* $\mu\in B_1(\Gamma)$, $(\Gamma^\mu, \{A_i^\mu\})$ *is also a normalized pair in* T_g, *that is,* $[\Gamma^\mu, \theta^\mu]\in T(\Gamma)$ (*in the sense of Definition* 2.3.).

2.3. The complex analytic structure on T_g

Let Γ be a Fuchsian group of genus g and let ν be a Γ-Beltrami coefficient. Then, the limits

$$\lim_{t\to 0}\frac{f^{t\nu}(z)-z}{t}\qquad (z\in H)$$

and

$$\lim_{t\to 0}\frac{A^{t\nu}(z)-A(z)}{t}\qquad (z\in H,\ A\in\Gamma)$$

exist ([A1, p. 103]). We denote them by $\dot{f}[\nu]$ and $\dot{A}[\nu]$ respectively.

Definition 2.6. We say that $\nu\in B(\Gamma)$ is locally trivial (or stationary), if $\dot{A}[\nu]=0$ for all $A\in\Gamma$.

The set of all locally trivial Γ-Beltrami coefficients is obviously a linear subspace of $B(\Gamma)$. We denote it by $N(\Gamma)$.

Let $Q(\Gamma)$ be the space of all automorphic forms of weight 2 with respect to Γ. It is well known that $Q(\Gamma)$ is a complex vector space of dimension $3g-3$, where g is the genus of Γ.

Lemma 2.4. ([A1], [A3])

(i) *There is an antilinear isomorphism* Φ *from the complex vector space* $B(\Gamma)/N(\Gamma)$ *onto* $Q(\Gamma)$:

$$\Phi: \nu \mapsto \Phi[\nu](\xi) = \frac{12}{\pi}\iint_H \frac{\overline{\nu(z)}}{(\bar{z}-\xi)^4}\, dx\, dy. \tag{2.5}$$

Hence, the dimension of $B(\Gamma)/N(\Gamma)$ *is* $3g-3$.

(ii) *On the other hand, we may define an antilinear mapping* Φ^* *from* $Q(\Gamma)$ *into* $B(\Gamma)$:

$$\Phi^*: \varphi \mapsto y^2\bar{\varphi}. \tag{2.6}$$

(iii) $\Phi\Phi^* =$ *identity, and*

$$\nu - \Phi^*\Phi[\nu] \in N(\Gamma) \qquad \text{for all } \nu \text{ in } B(\Gamma).$$

Definition 2.7. Any basis of the $(3g-3)$-dimensional vector space $B(\Gamma)/N(\Gamma)$ is called a Γ-Beltrami basis. By the above lemma, we know that $\{y^2\bar{\varphi}_i;\ \varphi_i (i=1, 2, \cdots, 3g-3)$ form a basis of $Q(\Gamma)\}$ is a Γ-Beltrami basis. Such one will be called a Weil's Γ-Beltrami basis.

Lemma 2.5. (Bers)

Let Γ *be a Fuchsian group of genus* g *and let* $\{\nu_i;\ i=1, 2, \cdots, 3g-3\}$ *be a* Γ*-Beltrami basis.*

Then, for every element μ *of* $B(\Gamma)$ *which has a sufficient small norm* $\|\mu\|_\infty$, *there corresponds unique complex numbers* $\zeta_1(\mu), \zeta_2(\mu), \cdots, \zeta_{3g-3}(\mu)$ *such that*

$$A^{\zeta_1(\mu)\nu_1+\zeta_2(\mu)\nu_2+\cdots+\zeta_{3g-3}(\mu)\nu_{3g-3}} = A^{\mu}$$

for all A *in* Γ.

Now, we can state the main theorem in the theory of Teichmüller spaces.

Theorem 2.3. (Bers)

(i) *Let $[\Gamma, \theta]$ be any element of $T(\Gamma_0)$, where Γ_0 is a Fuchsian group of genus g. Then,*

$$[\Gamma^\mu, \theta^\mu \theta] \mapsto \zeta = (\zeta_1(\mu), \zeta_2(\mu), \cdots, \zeta_{3g-3}(\mu)) \tag{2.7}$$

$$(\|\mu\|_\infty: \textit{sufficiently small})$$

defines a topological mapping from a neighbourhood of $[\Gamma, \theta]$ in $T(\Gamma_0)$ onto a neighbourhood of the origin of $\boldsymbol{C}^{3g-3}$.

(ii) *$T(\Gamma_0)$ has a structure of a $(3g-3)$-dimensional complex analytic manifold defined by the coordinates mappings* (2.7).

Furthermore, $T(\Gamma_0)$ with the above structure is complex analytically equivalent to a bounded domain in $\boldsymbol{C}^{3g-3}$.

Definition 2.8. The above local coordinates $\zeta = (\zeta_1, \cdots, \zeta_{3g-3})$ defined in a neighbourhood of $[\Gamma, \theta] \in T(\Gamma_0)$ is called Bers' coordinates with respect to a Γ-Beltrami basis $\{\nu_j\}$.

The point $[\Gamma, \theta]$ is called the origin of the Bers' coordinates.

2.4. The Riemannian structure on T_g, two lemmas of special importance

Clearly, the (real) tangent space of $T(\Gamma_0)$ at $P = [\Gamma, \theta]$ can be naturally identified with $B(\Gamma)/N(\Gamma)$ considered as a real vector space. [A2, 3] defined a natural hermitian structure of T_g. It is given by the following:

$$g_P(\nu, \nu') = 2\mathrm{Re.}\int_{\Gamma\backslash H} \overline{\Phi[\nu]}\, \Phi[\nu']\, y^2\, dx\, dy \tag{2.8}$$

$$= 2\mathrm{Re.}\int_{\Gamma\backslash H} \nu\, \Phi[\nu']\, dx\, dy,$$

for all $\nu, \nu' \in B(\Gamma)$. (See Lemma 2.4.)

The following lemmas are essential ones for our later work.

Lemma 2.6.[4)]

(i) *Let* $\{\nu_1, \cdots, \nu_n\}$ *be a set of elements of* $L^\infty(\boldsymbol{C})$. *We assume that every* ν_i *is indefinitely differentiable in a common domain* $\mathscr{D}(\subset \boldsymbol{C})$. *We set*

$$\mu=\zeta_1\nu_1+\zeta_2\nu_2+\cdots+\zeta_n\nu_n,$$

where $\zeta=(\zeta_1, \cdots, \zeta_n)\in \boldsymbol{C}^n$. *Then, for any compact subset* K *of* $\mathscr{D}$ *and nonnegative integers* n_1, n_2, *there is a positive number* ε *such that*

$$\left(\frac{\partial}{\partial z}\right)^{n_1}\left(\frac{\partial}{\partial \bar{z}}\right)^{n_2} W^{\mu(\zeta)}(z)$$

is holomorphic in ζ *and continuous in* (z, ζ) *both for* $z\in K$ *and* $\max_{1\leq i\leq n}|\zeta_i|<\varepsilon$.

(ii) *Let* $\{\nu_1, \cdots, \nu_n\}$ *be a set of elements of* $L^\infty(H)$. *We assume that every* ν_i *is indefinitely differentiable in* H. *We set*

$$\mu=s_1\nu_1+s_2\nu_2+\cdots+s_n\nu_n,$$

where $S=(s_1, s_2, \cdots, s_n)\in R^n$. *Then, for any compact subset* K *of* H *and nonnegative integers* n_1, n_2, *there is a positive number* ε *such that*

$$\left(\frac{\partial}{\partial z}\right)^{n_1}\left(\frac{\partial}{\partial \bar{z}}\right)^{n_2} W^{\hat{\mu}(\zeta)}(z). \tag{2.9}$$

is holomorphic in ζ *and continuous in* (z, ζ) *both for* $z\in K$ *and* $\max_{1\leq i\leq n}|\zeta_i|<\varepsilon$, *where*

$$\hat{\mu}(\zeta)=\zeta_1\hat{\nu}_1+\zeta_2\hat{\nu}_2+\cdots+\zeta_n\hat{\nu}_n,\ \zeta=(\zeta_1, \cdots, \zeta_n)\in \boldsymbol{C}^n$$

4) In [A2, 3], the following is asserted without proof:

Let $\mu\in L^\infty(H)$ be real analytic on H. If, furthermore, μ depends real analytically on a number of parameters, $f^\mu(z)$ is simultaneously real analytic in z and parameters.

But, the author does not know any proof of this assertion.

$$\text{and} \quad \hat{\nu}_i(z) \equiv \begin{cases} \nu_i(z) & z \in H \\ \overline{\nu_i(\bar{z})} & \bar{z} \in H. \end{cases}$$

Moreover, if we restrict (2.9) *to* $z \in H$ *and* $\zeta \in \boldsymbol{R}^n$, *we get*

$$\left(\frac{\partial}{\partial z}\right)^{n_1}\left(\frac{\partial}{\partial \bar{z}}\right)^{n_2} f^{\mu(S)}(z).$$

Proof. (ii) is a consequence of (i) and Remark 2.3.. A proof of (i) is given in the Appendix at the end of this paper.

Lemma 2.7. ([A2])

(i) *For any* $\mu \in L^\infty(H) \cap C^\infty(H)$,

$$\frac{\partial}{\partial \bar{z}} f^{t\mu} \text{ and } \frac{d}{dt}\left(\frac{\partial}{\partial \bar{z}} f^{t\nu}\right) \text{ exist (for sufficiently small } t\text{), and}$$

$$\left[\frac{d}{dt}\left(\frac{\partial}{\partial \bar{z}} f^{t\mu}\right)\right]_{t=0} = \mu.$$

(ii) *Put* $\rho^\mu = (|f_z^\mu|^2 - |f_{\bar{z}}^\mu|^2)\,(\mathrm{Im}.f^\mu(z))^{-2}$, *where* $\mathrm{Im}.f^\mu(z)$ *means the imaginary part of* $f^\mu(z)$. *If* $\nu = y^2\bar{\varphi}$ $(\varphi \in Q(\Gamma))$, *we get that* $\left[\frac{d}{dt}\rho^{t\nu}\right]_{t=0} = 0.$

§ 3. The statement of our problems

Let Γ_0 be a fixed Fuchsian group of genus g (Definition 2.1.) and let χ be its fixed finite dimensional representation by unitary matrices.

With each element $[\Gamma, \theta]$ in $T(\Gamma_0)$ (Definition 2.3.), we associate the (Γ, χ_θ)-eigenvalue problem (Definition 1.1.), where χ_θ is a unitary matrix-representation of Γ defined by

(3.1) $$\chi_\theta(A) = \chi(\theta^{-1}A) \qquad (A \in \Gamma).$$

By the naturality of the above correspondence, we can expect that there is a series $\Lambda_j = \Lambda_j([\Gamma, \theta])$ of continuous function with some regularity such that at each point $[\Gamma, \theta] \in T(\Gamma_0)$, they represents all the repeated eigenvalues of the eigenvalue problem associated with the point.

Remark 3.1. Even if we use other normalization in Lemma 2.2., the obtained series $\Lambda_j([\Gamma, \theta])$ will be the same.

§4. The reduction of our problems to the perturbation theory for differential operators

Let $[\Gamma, \theta]$ be an arbitrary point in the Teichmüller space $T(\Gamma_0) = T_g$ (Definition 2.3.). Fix a Weil's Γ-Beltrami basis $\{\nu_j; j=1, 2, \cdots, 3g-3\}$ (Definition 2.8.). Then, following Theorem 2.3., we can introduce Bers' coordinates $\zeta=(\zeta_1, \zeta_2, \cdots, \zeta_{3g-g})$ with respect to $\{\nu_j\}$ in a neighbourhood V of a point $[\Gamma, \theta] \in T(\Gamma_0)$. (The origin of the coordinates is $[\Gamma, \theta]$.) We shall often use the notation $[\Gamma_\zeta, \theta_\zeta]$ instead of $[\Gamma^\mu, \theta^\mu\theta]$, if ζ is the Bers' coordinates of $[\Gamma^\mu, \theta^\mu\theta]$ in V.

Now, fix an n-dimensional unitary matrices-representation χ of Γ_0 and consider (Γ, χ_θ)-eigenvalue problem for each point $[\Gamma, \theta]$ in $T(\Gamma_0)$.

As explained in §1, $[\Gamma_\zeta, \chi_{\theta_\zeta}]$-eigenvalue problem is associated with the quadratic form $\mathfrak{p}_\zeta$ in $\mathscr{H}(\Gamma_\zeta, \chi_{\theta_\zeta})$:

$$\mathfrak{p}_\zeta[F]=\int_{\mathscr{F}_\zeta}\left\{\left|\frac{\partial F}{\partial X}\right|^2+\left|\frac{\partial F}{\partial Y}\right|^2\right\}dX\,dY \qquad (F\in\mathscr{D}(\mathfrak{p}_\zeta)), \tag{4.1}$$

where $\mathscr{F}_\zeta$ is a measurable fundamental domain of Γ in H.

On the other hand, the norm in $\mathscr{H}(\Gamma_\zeta, \chi_{\theta_\zeta})$ is defined by

$$\mathfrak{a}_\zeta[F]=\|F\|^2_{\mathscr{H}(\Gamma_\zeta,\chi_{\theta_\zeta})}=\int_{\mathscr{F}_\zeta}|F(Z)|^2\frac{dX\,dY}{Y^2}. \tag{4.2}$$

To avoid the difficulty that the underlying Hilbert space $\mathscr{H}(\Gamma_\zeta, \chi_{\theta_\zeta})$ depends on ζ, we use here a device which is analogous to the one used in the so-called boundary perturbation problems. (See [K, pp. 423–426].)

Now suppose that $[\Gamma_\zeta, \chi_{\theta_\zeta}]=[\Gamma^\mu, \theta^\mu\theta]$ be a point in V and let f^μ be the quasiconformal mapping of H defined in Theorem 2.1. (ii).

Then, for any element $F(Z)$ of $\mathscr{H}(\Gamma_\zeta, \chi_{\theta_\zeta})$, the function vector $\hat{F}(z)$ defined by

$$(4.3) \qquad \hat{F}(z)=F(Z) \qquad (Z=f^\mu(z))$$

is an element of $\mathscr{H}=\mathscr{H}(\Gamma, \chi_\theta)$. (Note that the condition "$\hat{F}(Az) =\chi(A)\hat{F}(z)$ $(A\in\Gamma)$" follows from Lemma 2.3.)

By means of this linear mapping; $F\mapsto\hat{F}$, we can define new quadratic forms $\hat{\mathfrak{p}}_\zeta$ and $\hat{\mathfrak{a}}_\zeta$ in $\mathscr{H}$ depending on ζ:

$$(4.4) \qquad \hat{\mathfrak{p}}_\zeta[\hat{F}]=\mathfrak{p}_\zeta[F] \qquad \text{(defined by (4.1))}$$

$$=\int_{\mathscr{F}}\left\{\left|\frac{\partial x}{\partial X}\frac{\partial \hat{F}}{\partial x}+\frac{\partial y}{\partial X}\frac{\partial \hat{F}}{\partial y}\right|^2 +\left|\frac{\partial x}{\partial Y}\frac{\partial \hat{F}}{\partial x}+\frac{\partial y}{\partial Y}\frac{\partial \hat{F}}{\partial y}\right|^2\right\}J(z, \zeta)\,dx\,dy,$$

$$(4.5) \qquad \hat{\mathfrak{a}}_\zeta[\hat{F}]=\mathfrak{a}_\zeta[F] \qquad \text{(defined by (4.2))}$$

$$=\int_{\mathscr{F}}|\hat{F}(z)|^2\frac{J(z, \zeta)}{Y^2}\,dx\,dy,$$

where $\mathscr{F}$ is a measurable fundamental domain of Γ in H, $Z=f^\mu(z)$, $X=X(z, \zeta)=\text{Re.}\{Z(z, \zeta)\}=\text{Re.}\{f^\mu(z)\}$, $Y=Y(z, \zeta)=\text{Im.}\{Z(z, \zeta)\} =\text{Im.}\{f^\mu(z)\}$, and

$$J(z, \zeta)=\frac{\partial(X, Y)}{\partial(x, y)}(\zeta)\geqq 0 \qquad \text{(Remark 2.1.)}.$$

Lemma 4.1.

(i) *For each ζ, $\hat{\mathfrak{a}}_\zeta$ is a bounded symmetric quadratic form in $\mathscr{H}$. So, there is a bounded selfadjoint operator $\mathfrak{A}_\zeta$ in $\mathscr{H}$ associated with $\hat{\mathfrak{a}}_\zeta$:*

$$(\mathfrak{A}_\zeta F_1, F_2)=\hat{\mathfrak{a}}_\zeta[F_1, F_2] \qquad (F_1, F_2\in\mathscr{H}).$$

Moreover, $\mathfrak{A}_\zeta$ is a multiplication by

$$(4.6) \qquad A_\zeta=\frac{y^2J(z, \zeta)}{Y^2}.$$

(ii) *For each* ζ, $\hat{\mathfrak{p}}_\zeta$ *is a densely defined closed positive symmetric quadratic form in* $\mathscr{H}$. *So, there is a unique positive selfadjoint operator* $\mathfrak{P}_\zeta$ *in* $\mathscr{H}$ *associated with* $\mathfrak{p}_\zeta$:

$$\mathscr{D}(\mathfrak{P}_\zeta)\subseteq\mathscr{D}(\hat{\mathfrak{p}}_\zeta) \quad and \quad (\mathfrak{P}_\zeta F_1, F_2)=\hat{\mathfrak{p}}_\zeta[F_1, F_2]$$

$$(F_1\in\mathscr{D}(\mathfrak{P}_\zeta),\ F_2\in\mathscr{D}(\hat{\mathfrak{p}}_\zeta)).$$

Proof. By §1 and [K, pp. 322–323], we have only to show the closedness of $\hat{\mathfrak{p}}_\zeta$. But this can be easily verified by using (4.4) and the closedness of $\mathfrak{p}_\zeta$.

Thus, if $|\zeta|$ is sufficiently small, the $(\Gamma_\zeta, \chi_{\theta_\zeta})$-problem is equivalent to the following eigenvalue problem:

$$\mathfrak{Q}_\zeta G=\lambda G \qquad (G\in\mathscr{H}(\Gamma, \chi_\theta)), \tag{4.7}$$

where $\mathfrak{Q}_\zeta=\mathfrak{A}_\zeta^{-\frac{1}{2}}\mathfrak{P}_\zeta\mathfrak{A}_\zeta^{\frac{1}{2}}$.

Given a real analytic curve in V which passes through the origin $[\Gamma, \theta]$ of the above Bers' coordinates ζ. This can be represented by a real analytic function of a real small parameter t:

$$\zeta=\zeta(t), \tag{4.8}$$

where
$$\zeta(0)=(0, 0, \cdots, 0).$$

In such a situation, we shall use notations $\mathfrak{Q}_t$, $\mathfrak{q}_t$, $\cdots$, instead of $\mathfrak{Q}_{\zeta(t)}$, $\mathfrak{q}_{\zeta(t)}$, $\cdots$.

Now, we give an important lemma.

Lemma 4.2. *The family of operators* $\mathfrak{Q}_t$ *(defined above) is a selfadjoint holomorphic family of type* (*B*) (*in the sense of* [K, pp. 385–386 *and* p. 395]) *for sufficiently small* t.

Proof. For each t, $\mathfrak{Q}_t$ is clearly a positive selfadjoint operator. So, the only thing we have to show is that the family of quadratic forms $\mathfrak{q}_t$ associated with $\mathfrak{Q}_t$ satisfies the assumption of [K, p. 398, Theorem 4.8.].

The quadratic form $\mathfrak{q}_t$ is defined by

$$(4.9) \qquad \mathfrak{q}_t[G]=\hat{\mathfrak{p}}_t[\mathfrak{A}_t^{-\frac{1}{2}}G] \qquad \text{(defined by (4.4))}$$

$$=\int_{\mathscr{F}}\left\{\left|\frac{\partial x}{\partial X}\frac{\partial(A(t)^{-\frac{1}{2}}G)}{\partial x}+\frac{\partial y}{\partial X}\frac{\partial(A(t)^{-\frac{1}{2}}G)}{\partial y}\right|^2\right.$$

$$\left.+\left|\frac{\partial x}{\partial Y}\frac{\partial(A(t)^{-\frac{1}{2}}G)}{\partial x}+\frac{\partial y}{\partial Y}\frac{\partial(A(t)^{-\frac{1}{2}}G}{\partial y}\right|^2\right\}$$

$$\times J(z, t)\,dx\,dy.$$

If we denote the above integrand by $I(z, t)$, it can be shown that

$$(4.10) \qquad I(z, t)=P_1(z, t)\left|\frac{\partial G}{\partial x}\right|^2+P_2(z, t)^t\left(\frac{\partial G}{\partial x}\right)\left(\overline{\frac{\partial G}{\partial y}}\right)$$

$$+P_3(z, t)\left|\frac{\partial G}{\partial y}\right|^2+P_4(z, t)\,{}^tG\left(\overline{\frac{\partial G}{\partial x}}\right)$$

$$+P_5(z, t)\,{}^tG\left(\overline{\frac{\partial G}{\partial y}}\right)+P_6(z, t)\,|G|^2$$

where

$$P_i(z, t)=\sum_{n=0}^{\infty}P_{i,n}(z)\,t^n \qquad (\text{: convergent for } z\in\mathscr{F},\ |t|<\varepsilon)$$

and

$$(4.11) \qquad |P_{i,n}(z)|\leq bc^{n-1} \qquad (z\in\mathscr{F})$$

with positive constants b, c.

Here, we used the following facts,

a) Lemma 2.6. (ii).

b) $f^{\mu(0)}(z)\equiv z$ (Remark 2.4.) and hence $J(z, 0)\equiv 1$.

c) Lemma 4.1. (ii).

d) Cauchy inequality for Taylor coefficients of holomorphic functions.

Further more,

$$\left|{}^t\left(\frac{\partial G}{\partial x}\right)\left(\overline{\frac{\partial G}{\partial y}}\right)\right| \leq \frac{1}{2}\left\{\left|\frac{\partial G}{\partial x}\right|^2+\left|\frac{\partial G}{\partial y}\right|^2\right\}, \text{ etc.} \tag{4.12}$$

Using (4.10), (4.11) and (4.12), we get that

$$\mathfrak{q}_t[G]=\mathfrak{q}_0[G]+t\,\mathfrak{q}^{(1)}[G]+t^2\mathfrak{q}^{(2)}[G]+ \qquad ,$$

where

$$\mathfrak{q}_0[G]=\int_{\mathscr{F}}\left\{\left|\frac{\partial G}{\partial x}\right|^2+\left|\frac{\partial G}{\partial y}\right|^2\right\}dx\,dy$$

and

$$\mathfrak{q}^{(n)}[G]\leqq d^{n-1}(e\|G\|^2_{\mathscr{H}(\Gamma,\chi_\theta)}+f\mathfrak{q}_0[G]),$$

$(G\in C^2(H)\cap\mathscr{H}(\Gamma,\chi_\theta))$ with positive constants d, e, f.

Thus we verified the conditions.

Therefore Lemma 4.2. is proved.

Now, we state our principal results in this section. As above, we fix a Fuchsian group Γ_0 of genus g and its finite dimensional representation χ by unitary matrices.

Theorem 4.1. *Given any real analytic curve $\mathscr{C}=\mathscr{C}(t)=[\Gamma(t),\theta(t)]$ in $T_g=T(\Gamma_0)$, where t varies in an open interval I of the real line.*

Then, there is a sequence of real analytic functions $\Lambda^{(j)}(t)$ $(j=1, 2, \cdots)$ such that for each t, all the repeated eigenvalues of the $(\Gamma_{(t)}, \chi_{\theta(t)})$-problem are represented by $\{\Lambda^{(j)}(t)\}$.

Proof. Let $[\Gamma,\theta]\in T(\Gamma_0)$ be any point on the curve $\mathscr{C}$ and let $\{\nu_1, \nu_2, \cdots, \nu_{3g-3}\}$ be a Weil's Γ-Beltrami basis. Then we can introduce Bers' coordinates ζ with respect to $\{\nu_i\}$ in a neighbourhood V

of $[\Gamma, \theta]$. (The origin is $[\Gamma, \theta]$.) In V, the curve $\mathscr{C}$ can be represented in the form (4.9). Hence, as far as the curve $\mathscr{C}$ is contained in V, this theorem follows from Lemma 4.2. and Theorem 1.1 (ii). (See [K, p. 408, Remark 4.22])

Applying the above arguments at each point on $\mathscr{C}$, we can easily show that the theorem also holds globally.

Theorem 4.2. *Given a point $[\Gamma, \theta]$ in $T(\Gamma_0)$ and a simple eigenvalue λ of the (Γ, χ_θ)-problem, there is a real analytic function $\Lambda=\Lambda([\Gamma', \theta'])$ defined in a neighbourhood V of $[\Gamma, \theta]$ in $T(\Gamma_0)$ such that at each point $[\Gamma', \theta']$ its value represents one of the eigenvalues of the $(\Gamma', \chi_{\theta'})$-problem.*

Proof. Although, analytic perturbation problems with several parameters are discussed in [K] only for the finite dimensional cases ([K, p. 119, Theorem 5.16]), it can be generalized to the infinite dimensional ones by almost the same method as in the one-parameter-theory ([K, pp. 365–371]). Then, Theorem 4.2. can be proved by the same manner as Theorem 4.1..

Remark 4.1. Using Theorem 4.1. the above function Λ $=\Lambda([\Gamma', \theta'])$ can be extended to the whole space $T(\Gamma_0)$ so that at each point its values represent some eigenvalues of the associated problem. (Note that the extended function may be many-valued.)

§5. Calculation of the differential coefficients of the functions $\Lambda^{(j)}(t)$

In this section, we are going to calculate the differential coefficients of the functions $\Lambda^{(j)}(t)$ given in Theorem 4.1.. Since our method depends on a lemma in the perturbation theory, we shall first explain it and then turn to our problem.

Let $T(x)$ be a selfadjoint holomorphic family of type (B) ([K, p. 403]) depending on a real parameter x.

If λ_0 is one of the eigenvalues of the unperturbed operator $T(x_0)$ with multiplicity $m(<\infty)$, there are real analytic functions $\lambda^{(1)}(x)$, $\lambda^{(2)}(x), \cdots, \lambda^{(m)}(x)$ (defined for sufficiently small $|x-x_0|$) such that for each x, $\lambda^{(i)}(x)$ are eigenvalues of $T(x)$ and $\lambda^{(1)}(x_0)=\lambda^{(2)}(x_0)=\cdots=\lambda^{(m)}(x_0)=\lambda_0$.

The following is proved in [K, p. 405]

Lemma 5.1. *Let the situation be as above and let $t(x)$ be the sesquilinear form associated with $T(x)$. Then,*

$$\left[\frac{d}{dx}\hat{\lambda}(x)\right]_{x=x_0}=\frac{1}{m}\sum_{i=1}^{m}\left[\frac{d}{dx}(t(x)[\varphi_i])\right]_{x=x_0} \tag{5.1}$$

where

$$\hat{\lambda}(x)=\frac{1}{m}\sum_{i=1}^{m}\lambda_i(x)$$

and φ_i $(i=1, \cdots, m)$ form an orthonormal basis of the eigenspace of $T(x_0)$ corresponding to the eigenvalue λ_0.

Now, we shall apply this lemma to our case explained in §4.

Let the situation be as in Theorem 4.1. Given any point $[\Gamma, \theta]\in T(\Gamma_0)$ on the curve $\mathscr{C}=\mathscr{C}(t)$, we introduce Bers' coordinates $\zeta=(\zeta_1, \cdots, \zeta_{3g-3})$ with respect to a Weil's Γ-Beltrami basis $\{\nu_i;\ i=1, 2, \cdots, 3g-3\}$ in a neighbourhood V of $[\Gamma, \theta]$. In V, the curve $\mathscr{C}$ can be represented in the form (4.8). Hence by Lemma 4.2. and Lemma 5.1., our problem is reduced to calculate

$$\frac{d}{dt}[\mathfrak{q}_t[G]]_{t=0}, \tag{5.2}$$

where $\mathfrak{q}_t$ is given by (4.9) with $Z=f^{\mu(t)}(z)$, $\mu(t)=\mu(\zeta(t))=\sum_{i=1}^{3g-3}\zeta_i(t)\nu_i$, (We assumed, without loss of generality, that $[\Gamma, \theta]=\mathscr{C}(0)$.)

As can be easily seen, we can replace $\mu(\zeta(t))$ by $\left[\frac{d}{dt}\mu(\zeta(t))\right]_{t=0}$.

By the above reason, we shall assume that $\mu(t)=t\nu$, $\nu=y^2\bar{\varphi}$ $(\varphi\in Q(\Gamma))$ in calculating (5.2).

The following lemma, which follows from Lemma 2.7. (ii), makes our task much easier.

Lemma 5.2. *Let $A(z, t)$ be as (4.6), where $Z=f^{\mu(t)}(z)$, $\zeta=\zeta(t)$. Then, we have that*

$$\left[\frac{d}{dt}(A(z, t)^{-\frac{1}{2}})\right]_{t=0}\equiv 0.$$

By this lemma and $A(z, 0)\equiv 1$, it can be shown that (5.2) is equal to

$$\left[\frac{d}{dt}\hat{\mathfrak{p}}_t[G]\right]_{t=0}, \tag{5.3}$$

where $\hat{\mathfrak{p}}_t$ is given by (4.4) with $Z=f^{t\nu}(z)$, $\zeta=\zeta(t)$.

$$\hat{\mathfrak{p}}_t[G]=\int_{\mathcal{F}}\Big[\left|\frac{\partial f^{t\nu}}{\partial y}\right|^2\left|\frac{\partial G}{\partial x}\right|^2+\left|\frac{\partial f^{t\nu}}{\partial x}\right|^2\left|\frac{\partial G}{\partial y}\right|^2$$

$$-2\,\mathrm{Re}\Big\{\Big(\frac{\partial G}{\partial x}\Big)\Big(\overline{\frac{\partial G}{\partial y}}\Big)\Big(\frac{\partial Y}{\partial y}\frac{\partial Y}{\partial x}+\frac{\partial X}{\partial y}\frac{\partial X}{\partial x}\Big)\Big\}\Big]$$

$$\times\frac{1}{J(z, t)}\,dx\,dy.$$

Now, let us begin to calculate (5.3), using Lemma 2.7. (i), that is, $\left[\frac{d}{dt}\{f^{t\nu}_{\bar{z}}\}\right]_{t=0}=\nu.$

$$\left[\frac{d}{dt}\Big\{\left|\frac{\partial f^{t\nu}}{\partial y}\right|^2\Big\}\right]_{t=0}=\left[\frac{d}{dt}\Big\{\Big(\frac{\partial f^{t\nu}}{\partial z}-\frac{\partial f^{t\nu}}{\partial \bar{z}}\Big)\Big(\overline{\frac{\partial f^{t\nu}}{\partial z}-\frac{\partial f^{t\nu}}{\partial \bar{z}}}\Big)\Big\}\right]_{t=0}$$

$$=(\omega-\nu)+\overline{(\omega-\nu)},$$

where $\omega\equiv\left[\frac{d}{dt}\Big(\frac{\partial f^{t\nu}}{\partial z}\Big)\right]_{t=0}.$

Here, we used the fact that $f^0(z)\equiv z$ (Remark 2.4.).

In the same manner, we get that

$$\left[\frac{d}{dt}\left\{\left|\frac{\partial f^{t\nu}}{\partial x}\right|^2\right\}\right]_{t=0}=(\omega+\nu)+\overline{(\omega+\nu)},$$

$$\left[\frac{d}{dt}\left(\frac{\partial Y}{\partial y}\frac{\partial Y}{\partial x}+\frac{\partial X}{\partial y}\frac{\partial X}{\partial x}\right)\right]_{t=0}=\frac{1}{i}(\nu-\bar{\nu}),$$

$$\left[\frac{d}{dt}\left(\frac{1}{J(z,t)}\right)\right]_{t=0}=-\omega-\bar{\omega}.$$

Using these formulas, we get that

$$\left[\frac{d}{dt}\hat{\mathfrak{p}}_t[G]\right]_{t=0}$$

$$=2\,\mathrm{Re.}\int_{\mathscr{F}}\bar{\nu}\left[\left|\frac{\partial G}{\partial y}\right|^2-\left|\frac{\partial G}{\partial x}\right|^2-i\left\{{}^t\!\left(\frac{\partial G}{\partial x}\right)\left(\overline{\frac{\partial G}{\partial y}}\right)+{}^t\!\left(\overline{\frac{\partial G}{\partial x}}\right)\left(\frac{\partial G}{\partial y}\right)\right\}\right]dx\,dy$$

$$=2\,\mathrm{Re.}\int_{\mathscr{F}}\varphi\left[{}^t\!\left(\frac{\partial G}{\partial \bar{z}}\right)\left(\frac{\partial \bar{G}}{\partial \bar{z}}\right)(z-\bar{z})^2\right]dx\,dy.$$

(Recall that $\nu=y^2\bar{\varphi}$, where φ is an element of $Q(\Gamma)$.) We here remark that, if we put

$$\beta_G={}^t\!\left(\frac{\partial G}{\partial \bar{z}}\right)\left(\frac{\partial \bar{G}}{\partial \bar{z}}\right)(z-\bar{z})^2 \tag{5.4}$$

for an element G of $\mathscr{H}(\Gamma,\chi)$, β_G is a Γ-Beltrami coefficient.

Now, we can state our principal result in this section.

Let the situation be as in Theorem 4.1..

Given a point $P=\mathscr{C}(t_0)=[\Gamma_{t_0},\theta_{t_0}]$ on the curve $\mathscr{C}$ and one of the eigenvalues $\lambda_0=\Lambda^{(j_1)}(t_0)=\cdots=\Lambda^{(j_m)}(t_0)$ of the $(\Gamma_{t_0},\chi_{\theta_{t_0}})$-problem of multiplicity m, we put

$$\nu_{\lambda_0}=\frac{1}{m}\sum_{i=1}^{m}\Phi^*\Phi[\beta_{F^{(j_i)}}] \qquad \text{(see Lemma 2.4.),} \tag{5.5}$$

where $\{F^{(j_1)}, F^{(j_2)}\ldots, F^{(j_m)}\}$ is an orthonormal basis of the eigenspace belonging to the eigenvalue λ_0.

Theorem 5.1. *The weighted mean of the differential coefficients of $\Lambda^{(j_i)}(t)$ $(i=1, 2, \cdots m)$ at $t=t_0$ is given by*

$$(5.6)\qquad \frac{1}{m}\sum_{i=1}^{m}\left[\frac{d}{dt}\Lambda^{(j_i)}(t)\right]_{t=t_0}$$

$$=\frac{1}{m}\sum_{i=1}^{m}\left\{2\mathrm{Re.}\int_{\mathscr{F}}\varphi\left[{}^t\left(\frac{\partial F^{(j_i)}}{\partial\bar z}\right)\left(\frac{\partial \bar F^{(j_i)}}{\partial\bar z}\right)(z-\bar z)^2\right]dx\,dy\right\}$$

$$=g_p(\nu, \nu_{\lambda_0}),$$

where ν_{λ_0} is the element of $\mathscr{T}_p$ given by (5.5) *and $\nu=y^2\bar\varphi$ $(\varphi\in Q(\Gamma_{t_0}))$ is the element of $\mathscr{T}_p$ tangential to the curve $\mathscr{C}$ and $g_p(\cdot,\cdot)$ is the innerproduct in $\mathscr{T}_p$ given in* §2.4.

Remark 5.1. When $\Lambda^{(j_1)}(t)=\Lambda^{(j_2)}(t)=\cdots=\Lambda^{(j_m)}(t)$ at least for small $|t-t_0|$ (hence for all t, by the real analyticity of $\Lambda^{(j_i)}(t)$), the above value (5.6) coincide with the differential coefficients of $\Lambda^{(j_i)}(t)$ $(i=1, 2, \cdots, m)$ at t_0. In particular, such case occurs when $m=1$, that is, λ_0 is a simple eigenvalue of the $(\Gamma_{t_0}, \chi_{\theta_{t_0}})$-problem.

Appendix. A proof of Lemma 2.6. (i)

1. From now on, $\boldsymbol{C}$ is the whole complex plane. Let $p>2$ be a real number and $n\geqq 0$ be an integer. We introduce the Banach space $H_{p,n}$ of functions defined on the whole plane $\boldsymbol{C}$, which satisfy the following conditions:

a) $f\in C^n(\boldsymbol{C})$

b) $\partial_z^{n_1}\partial_{\bar z}^{n_2}f$ $(0\leq n_1+n_2\leq n)$ are contained in $L^p(\boldsymbol{C})$,

c) $\partial_z^{n_1}\partial_{\bar z}^{n_2}f$ $(0\leq n_1+n_2\leq n)$ satisfy the global Hölder conditions of order $1-2/p$,

where $\partial_z f\equiv f_z\equiv\frac{1}{2}(f_x-if_y)$, $\partial_{\bar z}f\equiv f_{\bar z}\equiv\frac{1}{2}(f_x+if_y)$. The norm is defined by

$$<f>_{p,n}=\max_{0\leq n_1+n_2\leq n}\{|f|_{p,n_1,n_2}\}$$

where

$$|f|_{p,n_1,n_2}=\max\left\{\|\partial_z^{n_1}\partial_{\bar z}^{n_2}f\|_{L^p},\ \sup_{z_1,z_2\in C}\frac{|\partial_z^{n_1}\partial_{\bar z}^{n_2}f(z_1)-\partial_z^{n_1}\partial_{\bar z}^{n_2}f(z_2)|}{|z_1-z_2|^{1-2/p}}\right\}.$$

The fact that $H_{p,n}$ is a Banach space follows easily from

Lemma A. $\|\partial_z^{n_1}\partial_{\bar z}^{n_2}f\|_\infty\leqq 2<f>_{p,n}$ *for* $0\leqq n_1+n_2\leqq n$.

Proof. An easy generalization of [A4, (2.8)].

2. Following [A4], we introduce two operators P and T defined by

$$(Ph)(\xi)=\frac{-1}{\pi}\iint_C h(z)\left(\frac{1}{z-\xi}-\frac{1}{z}\right)dx\,dy$$

for $h\in L^p(C)$ $(p>2)$, and

$$(Th)(\xi)=\frac{-1}{\pi}\lim_{\varepsilon\to 0}\iint_{|z-\xi|>\varepsilon}\frac{h(z)}{(z-\xi)^2}\,dx\,dy$$

for $h\in H_{p,0}$.

Lemma B. ([A4, pp. 7–8]) *For* $h\in H_{p,0}$, *Ph is continuously differentiable and*

$$(Ph)_{\bar z}=h,\qquad (Ph)_z=Th.$$

Lemma C. *For* $h\in C_0^1(C)\cap H_{p,1}$, *Th is continuously differentiable and*

$$(Th)_{\bar z}=h_z,\qquad (Th)_z=T(h_z).$$

This follows from [Al, p. 88, (9)] and Lemma B.

The following is proved in [Al, Chapter V].

Lemma D. *For every $p>1$, there is a constant C_p such that*

$$||Th||_p \leqq C_p||h||_p \qquad \text{for all } h \in C_0^2(\boldsymbol{C})$$

Moreover, $C_p \to 1$ as $p \to 2$.

Clearly, this enables us to extend T to $L^p(\boldsymbol{C})$.

Lemma E.

(i) *If f, $g \in H_{p,n}$, $f \cdot g$ is also contained in $H_{p,n}$*

$$<f \cdot g>_{p,n} \leqq \alpha_n <f>_{p,n} <g>_{p,n},$$

where α_n is a positive constant which depends only on n.

(ii) *For $f \in C_0^n(\boldsymbol{C}) \cap H_{p,n}$, Tf is contained in $H_{p,n}$ and*

$$<Tf>_{p,n} \leqq \beta_{p,n} <f>_{p,n},$$

where $\beta_{p,n}$ is a positive constant which depends only on p and n.

Proof. (i) is easily shown from Lemma A. Let us prove (ii) by induction on n. For $n=0$, this is (essentially) proved in [A4, p. 9]. Assume that this is true for $n=m-1$. If $f \in C_0^m \cap H_{p,m}$, f_z and $f_{\bar{z}}$ is contained in $C_0^{m-1} \cap H_{p,m-1}$ and

$$<f_z>_{p,m-1} \leqq <f>_{p,m}, \quad <f_{\bar{z}}>_{p,m-1} \leqq <f>_{p,m}.$$

Hence

$$\begin{aligned} <Tf>_{p,m} &= \max\{<Tf>_{p,0},\ <(Tf)_z>_{p,m-1},\ <(Tf)_{\bar{z}}>_{p,m-1}\} \\ &\leqq \max\{\beta_{p,0} <f>_{p,0},\ \beta_{p,m-1} <f_z>_{p,m-1},\ <f_z>_{p,m-1}\} \\ &\leqq \max\{\beta_{p,0},\ \beta_{p,m-1},\ 1\} <f>_{p,m}, \end{aligned}$$

where we used **Lemma C.** So, if we put $\beta_{p,m} = \max\{\beta_{p,1}, \beta_{p,m-1}, 1\}$, (ii) is also true for $n=m$. Thus, (ii) is proved.

3. Let μ be an element of $L^\infty(\boldsymbol{C})$ which satisfies $||\mu||_\infty \leqq k$ with a fixed constant $k<1$. By Lemma D., there is a number $p>2$ such that $kC_p<1$. We fix such p from now on.

Theorem A. ([AB, Theorem 1])

If $\sigma\in L^p(\boldsymbol{C})$ the equation $\omega_{\bar z}=\mu\omega_z+\sigma$ has a unique solution $\omega^{\mu,\sigma}$ which satisfies

$$\omega(0)=0, \qquad \omega_z\in L^p(\boldsymbol{C}).$$

Moreover, this solution can be represented as $\omega^{\mu,\sigma}=P(\mu q+\sigma)$, where q is the solution of the equation: $q=T\mu q+T\sigma$ in L^p, that is, $q=T\sigma+T\mu T\sigma+T\mu T\mu T\sigma+\cdots$ (: convergent in $L^p(\boldsymbol{C})$).

Theorem B. *Let μ, σ in Theorem A. be contained in $C_0^n\cap H_{p,n}$. If $<\mu>_{p,n}\leqq\dfrac{1}{\alpha_n\beta_{p,n}}$, q, $\omega_z^{\mu,\sigma}$ and $\omega_{\bar z}^{\mu,\sigma}$ are in $H_{p,n}$ and*

$$<q>_{p,n}\leqq\frac{\beta_{p,n}}{1-\alpha_n\beta_{p,n}<\mu>_{p,n}}<\sigma>_{p,n},$$

$$<\omega_z^{\mu,\sigma}>_{p,n}\leqq\frac{\beta_{p,n}}{1-\alpha_n\beta_{p,n}<\mu>_{p,n}}<\sigma>_{p,n},$$

$$<\omega_{\bar z}^{\mu,\sigma}>_{p,n}\leqq\frac{1}{1-\alpha_n\beta_{p,n}<\mu>_{p,n}}<\sigma>_{p,n}.$$

Proof. The norm of the transformation $T\mu$ in $H_{p,n}$ is $\leqq\alpha_n\beta_{p,n}<\mu>_{p,n}$ by Lemma E. (i), (ii). So, if $<\mu>_{p,n}\leqq\dfrac{1}{\alpha_n\beta_{p,n}}$,

$$q=T\sigma+T\mu T\sigma+T\mu T\mu T\sigma+\cdots$$

is convergent in $H_{p,n}$. Hence $q\in H_{p,n}$ and

$$\omega_z^{\mu,\sigma}=\{P(\mu q+\sigma)\}_z=T(\mu q+\sigma)=q$$

$$\omega_{\bar z}^{\mu,\sigma}=\{P(\mu q+\sigma)\}_{\bar z}=\mu q+\sigma \qquad \text{(by Lemma B.)}$$

are also in $H_{p,n}$.

From $q=T\mu q+T\sigma$, we get that

$$<q>_{p,n}\leq\alpha_n\beta_{p,n}<\mu>_{p,n}<q>_{p,n}+\beta_{p,n}<\sigma>_{p,n}$$

and thus

$$<q>_{p,n}\leq\frac{\beta_{p,n}}{1-\alpha_n\beta_{p,n}}<\sigma>_{p,n}.$$

Other estimates follows from this easily.

We also need

Lemma F. *Let* μ, σ *be as in Theorem* A. *and let* μ_0, σ_0 *also satisfy the same conditions as* μ, σ, *namely*, $\mu_0\in L^\infty$, $\sigma_0\in L^P$ *and* $||\mu_0||\leq k<1$.

If we put $\Omega=\omega^{\mu,\sigma}-\omega^{\mu_0\sigma_0}$, *we get that*

$$\Omega=\omega^{\mu,\xi}$$

with $$\xi=(\mu-\mu_0)q_0+\sigma-\sigma_0,$$

where q_0 *is the solution of* $q_0=T(\mu_0 q_0)+T\sigma_0$ *in* L^P.

This is proved in [AB, p. 388].

4. Here we recall Theorem 2.1, (i), namely,

Theorem C. ([AB, Lemma 12])

For any $\mu\in L^\infty(\boldsymbol{C})$ *such that* $||\mu||_\infty\leq k<1$, *there is a unique homeomorphic mapping* W^μ *from* $\boldsymbol{C}$ *onto itself which satisfies the Beltrami equation*:

$$W^\mu_{\bar z}=\mu W^\mu_z$$

and is normalized by

$$W^\mu(0)=0,\quad W^\mu(1)=1,\quad W^\mu(\infty)=\infty.$$

Lemma G. ([AB, Theorem 7 and Theorem 9])

(i) *For the solution* W^{μ} *in Theorem* C.,

$$\|W^{\mu}\|_{R,P} \leqq c(R)$$

where $c(R)$ *is a constant which depends only on* R, k *and* p, *and*

$$\|W\|_{R,P} = \sup_{|z_1|,|z_2| \leqq R} \frac{|W(z_1) - W(z_2)|}{|z_1 - z_2|^{1-2/p}}.$$

(ii) $\|W^{\mu_n} - W^{\mu}\|_{R,p} \to 0$, *if* $\mu_n \to \mu$ *almost everywhere.*

Lemma H. ([AB], p. 397)

Let $\lambda(z)$ *be a fixed* C_0^{∞}*-function with* $0 \leq \lambda \leq 1$, $\lambda(z) = 1$ *for* $|z| \leq R - \delta$, $\lambda(z) = 0$ *for* $|z| \geq R - \delta/2$, *and let* $\chi(z)$ *be also a fixed* C_0^{∞}*-function with* $0 \leq \chi \leq 1$, $\chi(z) = 1$ *for* $|z| \leq R - \delta/2$, $\lambda(z) = 0$ *for* $|z| \geqq R - \delta/4$, *where* δ *is an arbitrary positive number.*

Then, we have that

$$\lambda W^{\mu} = \omega^{\chi\mu, \eta} \qquad (\textit{Theorem A.}),$$

where

$$\eta = (\lambda_{\bar{z}} - \mu\lambda_z) W^{\mu}.$$

5. Let ν_i $(i = 1, 2, \cdots)$ be $(n+1)$-times continuously differentiable in $\{z : |z| \leqq R\}$.

Theorem D. *Put* $\mu(t) = \sum_{i=1}^{l} t_i \nu_i$, *where* $t = (t_1, t_2, \cdots t_l)$ *are real parameters. Then, for any positive number* δ, *there is a positive number* ε *such that*

$$\partial_z^{n_1} \partial_{\bar{z}}^{n_2} W^{\mu(t)}(z) \qquad (0 \leq n_1 + n_2 \leq n + 1)$$

are continuous in (z, t) *for* $|z| \leq R - \delta$, $|t| < \varepsilon$.

Proof. Let λ and χ be as in Lemma H.

We first prove that

(*) $\lambda W^{\mu(t)} \in H_{p,n+1}$ for sufficiently small t,

by induction on n. For $n=0$, using

$$(**) \quad \lambda W^{\mu(t)} = \omega^{\chi\mu(t), \eta(t)},$$

$$\text{where } \eta(t) = (\lambda_{\bar z} - \mu(t)\lambda_z) W^{\mu(t)} \text{ (Lemma H),}$$

we can see that $\lambda W^{\mu(t)} \in H_{p,1}$ by Theorem B. (Note that $\eta(t) \in H_{p,0}$ by Lemma G.)

Now assume that (*) is proved for $n=m$. Then, $\eta(t) \in H_{p,m+1}$, hence by Theorem B $\lambda W^{\mu(t)} \in H_{p,m+2}$, namely, (*) is also true for $n=m+1$. Thus (*) is proved for all n.

Next, we shall prove that

$$(***) \quad <\lambda W^{\mu(t')} - \lambda W^{\mu(t)}>_{p,n+1} \to 0 \text{ as } t' \to t,$$

also by induction on n.

For that purpose we need some estimates:

$$\text{(a)} \quad <(\lambda W^{\mu(t')} - \lambda W^{\mu(t)})_z>_{p,n}$$

$$= <(\omega^{\chi\mu(t'), \eta(t')} - \omega^{\chi\mu(t), \eta(t)})_z>_{p,n}$$

$$= <\omega_z^{\chi\mu(t'), \xi(t', t)}>_{p,n} \qquad \text{(Lemma F.)}$$

$$\leqq \frac{\beta_{p,n}}{1 - \alpha_n \beta_{p,n} <\chi\mu(t)>_{p,n}} <\xi(t', t)>_{p,n} \qquad \text{(Theorem B.),}$$

where $\xi(t', t) = \chi\{\mu(t') - \mu(t)\} q(t) + \eta(t') - \eta(t)$

with $q(t) = T(\chi\mu(t) q(t)) + T\eta(t)$.

$$\text{(b)} \quad <\xi(t', t)>_{p,n}$$

$$\leqq <\chi\{\mu(t') - \mu(t)\} q(t)>_{p,n} + <\eta(t') - \eta(t)>_{p,n}$$

$$\leqq \alpha_n <\chi\{\mu(t') - \mu(t)\}>_{p,n} <q(t)>_{p,n}$$

$$+ <\lambda_{\bar z}\{W^{\mu(t')} - W^{\mu(t)}\}>_{p,n}$$

$$+ <\lambda_z\{\mu(t') W^{\mu(t')} - \mu(t) W^{\mu(t)}\}>_{p,n}.$$

(c) $<\lambda W^{\mu(t')}-\lambda W^{\mu(t)}>_{p,n+1}$

$$=\max\{<(\lambda W^{\mu(t')}-\lambda W^{\mu(t)})_z>_{p,n},\ <(\lambda W^{\mu(t')}-\lambda W^{\mu(t)})_{\bar{z}}>_{p,n},$$
$$<\lambda W^{\mu(t')}-\lambda W^{\mu(t)}>_{p,0}\}.$$

For $n=0$, (***) follows easily from (a), (b), (c) and Lemma G.

Assume that (***) is proved for $n=m$. Then, we get from (b), (c) that

$$<\xi(t', t)>_{p,m}\to 0 \quad \text{as} \quad t'\to t.$$

Hence follows that

$$<(\lambda W^{\mu(t')}-\lambda W^{\mu(t)})_z>_{p,m}\to 0 \qquad (t'\to t)$$
$$<(\lambda W^{\mu(t')}-\lambda W^{\mu(t)})_{\bar{z}}>_{p,m}\to 0 \qquad (t'\to t).$$

By this and (d), we see that (***) is true for $n=m+1$. Thus (***) is true for all m.

By Lemma A, (*) implies that

$\partial_z^{n_1}\partial_{\bar{z}}^{n_2} W^{\mu(t)}$ $(0\leq n_1+n_2\leq n)$ are continuous in z for $|z|\leq R-2\delta$, and (***) implies that they are uniformly continuous in t for $|t|\leq\varepsilon$ and $|z|\leq R-\delta$. Combining these, we have proved the theorem.

Theorem E. *Let ν_i $(i=1, 2, \cdots, l)$ and $\mu(t)$ be as in Theorem* D.. *Then, for any positive number δ, there is a positive number ε such that*

$\partial_z^{n_1}\partial_{\bar{z}}^{n_2}\dfrac{d}{dt_i} W^{\mu(t)}$ $(0\leq n_1+n_2\leq n,\ 1\leq i\leq l)$ are continuous in (z, t) for $|z|\leqq R-\delta$ and $|t|\leqq\varepsilon$.

Proof. We borrow the following result from [A1, p. 104]:

$$\left(\frac{d}{dt_i} W^{\mu(t)}\right)(\xi)=\frac{-1}{\pi}\iint_C (L^{\mu(t)}\nu_i)(z)R(z, W^{\mu(t)}(\xi))\,dx\,dy,$$

where $\quad L^{\mu(t)}\nu_i=\left\{\nu_i\dfrac{(W_z^{\mu(t)})^2}{|W_z^{\mu(t)}|^2-|W_{\bar{z}}^{\mu(t)}|^2}\right\}\circ(W^{\mu(t)})^{-1}$

and

$$R(z, \xi)=\frac{1}{z-\xi}-\frac{\xi}{z-1}+\frac{\xi-1}{z}.$$

Let χ be as in Lemma H. Then,

$$(\sharp)\qquad \left(\frac{d}{dt_i}W^{\mu(t)}\right)(\xi)$$

$$=-\frac{1}{\pi}\iint_C(\chi L^{\mu(t)}\nu_i)(z)R(z, W^{\mu(t)}(\xi))\,dx\,dy$$

$$-\frac{1}{\pi}\iint_C\{(1-\chi)L^{\mu(t)}\nu_i\}(z)R(z, W^{\mu(t)}(\xi))\,dx\,dy.$$

From the fact that $W^{\mu(0)}(\xi)\equiv\xi$ and the uniform continuity of $W^{\mu(t)}$ in t for $|\xi|\leqq R$(Lemma G.), we can show that $|W^{\mu(t)}(\xi)|\leq R-\frac{\delta}{2}$ for $|\xi|\leq R-\delta$ and sufficiently small t. Hence, the second term of the right member of $(\sharp)$ can be n-times continuously differentiable in ξ for $|\xi|\leq R-\delta$ and sufficiently small $|t|$. The obtained derivatives are continuous in (ξ, t).

On the other hand, the first term is equal to

$$-\frac{1}{\pi}\iint_C(\chi L^{\mu(t)}\nu_i)(z)\frac{1}{z-W^{\mu(t)}(\xi)}\,dx\,dy$$

$$-\frac{1}{\pi}\iint_C(\chi L^{\mu(t)}\nu_i)(z)\left\{\frac{-W^{\mu(t)}(\xi)}{z-1}+\frac{W^{\mu(t)}(\xi)-1}{z}\right\}dx\,dy,$$

and again the second term is n-times continuously differentiable in ξ for $|\xi|\leq R-\delta$ and sufficiently small $|t|$, the obtained derivatives being continuous in (ξ, t).

The remained term is

$$G=-\frac{1}{\pi}\iint_C(\chi L^{\mu(t)}\nu_i)(z)\frac{1}{z-W^{\mu(t)}(\xi)}\,dx\,dy.$$

By Lemma B,

$$G_{\bar{\xi}}=\{T(\chi L^{\mu(t)}\nu_i)\circ W^{\mu(t)}\}W_{\bar{\xi}}^{\mu(t)}+\{(\chi L^{\mu(t)}\nu_i)\circ W^{\mu(t)}\}\overline{W_{\xi}^{\mu(t)}},$$

$$G_{\xi}=\{T(\chi L^{\mu(t)}\nu_i)\circ W^{\mu(t)}\}W_{\xi}^{\mu(t)}+\{(\chi L^{\mu(t)}\nu_i)\circ W^{\mu(t)}\}\overline{W_{\bar{\xi}}^{\mu(t)}}.$$

We can easily show that

$$T(\chi L^{\mu(t)}\nu_i)\in H_{p,n} \qquad (\text{for small } |t|)$$

by Theorem D and Lemma E (ii). Hence the only thing we must prove is that

$$< T(\chi L^{\mu(t')}\nu_i)-T(\chi L^{\mu(t)}\nu_i)>_{p,n}\to 0 \quad \text{as} \quad t'\to t.$$

For that purpose it is sufficient to show that

$$<\chi L^{\mu(t')}\nu_i-\chi L^{\mu(t)}\nu_i>_{p,n}\to 0 \quad \text{as} \quad t'\to t. \qquad (\text{Lemma E (ii)})$$

But this is obvious from (***) in the proof of Theorem D and the fact that

$$(W^{\mu})^{-1}=W^{\nu} \text{ with } \nu=-\left(\frac{W_z^{\mu}}{\overline{W_z^{\mu}}}\mu\right)\circ(W^{\mu})^{-1}.$$

6.

Theorem F. ([AB, Theorem 11])
Let ν_i $(i=1, 2, \cdots l)$ be elements of $L^{\infty}(\boldsymbol{C})$. We set

$$\mu(\zeta)=\zeta_1\nu_1+\zeta_2\nu_2+\cdots+\zeta_l\nu_l,$$

where $\mu(\zeta)=(\zeta_1, \cdots, \zeta_l)\in\boldsymbol{C}^n$ and $|\zeta|$ is so small that $\|\mu(\zeta)\|_{\infty}<1$. Then, $W^{\mu(\zeta)}(z)$ is holomorphic in ζ for each z.

Theorem G.

Let ν_i $(i=1, 2, \cdots l)$ in Theorem F *be indefinitely differentiable in a domain $\mathscr{D}$ in* **C**. *Then for each compact subset K of $\mathscr{D}$ and all integers*

n_1, $n_2 \geqq 0$, *there is a positive number* ε *such that* $\partial_z^{n_1}\partial_{\bar z}^{n_2} W^{\mu(\zeta)}(z\in K)$ *are holomorphic in* ζ *for* $|\zeta|<\varepsilon$.

Proof. When $\mathscr{D}=\{z;|z|<R\}$ for some $R>0$, this follows from Theorem E. and Theorem F.. In fact,

$$\partial_{\bar\zeta_i}\partial_z^{n_1}\partial_z^{n_2} W^{\mu(\zeta)}=\partial_z^{n_1}\partial_{\bar z}^{n_2}\partial_{\bar\zeta_i} W^{\mu(\zeta)}=0.$$

There is no essential difference in proving the theorem for general domain $\mathscr{D}$.

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