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On some degenerate diffusion system related with a certain reaction system

By

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§1. Introduction.

In this paper we consider a mathematical model which represents the competition order of two antibodies to one antigen in asthmatics. Our problems in the mathematical form are derived by H. Mikawa and M. Mimura and others through their piled discussions and through their medical and numerical experiments [4].

They are formulated as follows: Suppose two antibodies C_1 and C_2 react with one antigen C_4 to form the products C_5 and C_3 respectively

$$C_1+C_4\rightarrow C_5 \tag{1.1}$$

$$C_2+C_4\rightarrow C_3 \tag{1.2}$$

and C_3 reacts with C_1 to form C_2 and C_5,

$$C_3+C_1\rightarrow C_2+C_5. \tag{1.3}$$

Here it is assumed that C_1 and C_2 are diffusible and C_3, C_4 and C_5 are non-diffusible and the all reactions (1.1), (1.2) and (1.3) are all of second order.

We denote by $u_j(x, t)$ the concentrations of C_j at the place $x=$

$(x_1, x_2, \cdots, x_n)$ and at time t for $j=1, 2, \cdots, 5$. Then these processes can be expressed in the following degenerate diffusion system;

$$U_t=\tilde{D}_0 \varDelta U+\tilde{D}_1 F(U) \tag{1.4}$$

where

$$U={}^t(u_1, u_2, u_3, u_4, u_5)$$

$$\tilde{D}_0=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \tilde{D}_1=\begin{pmatrix} -d_1 & -d_2 & 0 \\ 0 & d_2 & -d_3 \\ 0 & -d_2 & d_3 \\ -d_1 & 0 & -d_3 \\ d_1 & d_2 & 0 \end{pmatrix}$$

and

$$F(U)={}^t(u_1u_4, u_1u_3, u_2u_4)$$

and all the coefficients d_1, d_2 and d_3 are positive constants and $\varDelta$ means the Laplace operator.

It is known that our system represents an idealized model of the fibre-regent system when $d_2=d_3=0$ [1].

Here we deal with our system as an initial value problem. Since the behavior of $u_5(x, t)$ is completely determined by those of $u_j(x, t)$ for $j=1, 2, \cdots, 4$, it is sufficient to consider the following system;

$$U_t=D_0\varDelta U+D_1F(U) \quad \text{in } \Omega_n=R^n\times(0, \infty) \tag{1.5}$$

$$U(x, 0)=\Phi(x) \tag{1.6}$$

where

$$U={}^t(u_1, u_2, u_3, u_4)$$

$$D_0=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad D_1=\begin{pmatrix} -d_1 & -d_2 & 0 \\ 0 & d_2 & -d_3 \\ 0 & -d_2 & d_3 \\ -d_1 & 0 & -d_3 \end{pmatrix}$$

$$F(U)={}^t(u_1u_4,\ u_1u_3,\ u_2u_4)$$

and

$$\Phi(x)={}^t(\phi_1(x),\ \phi_2(x),\ 0,\ \phi_4(x)).$$

From the point of view of chemistry, we shall treat the case of non-negative initial data throughout this paper.

Our paper consists of two sections. In the first section we discuss the relations between the initial data and the asymptotic behavior of the solution of the problem (1.5) and (1.6). (See THEOREM 2.1.) Another section is devoted to study the semilinear elliptic equation

$$\Delta u=a(x)(1-e^{-u})-f(x) \tag{1.7}$$

derived from our problem (1.5) and (1.6). There we discuss the existence, uniqueness and the non-existence of the solution of (1.7). (See THEOREM 3.2, 3.3 and 3.4.)

As for the Cauchy problem (1.5) and (1.6), for any non-negative initial data $(\phi_1, \phi_2, \phi_3, \phi_4)\in \mathscr{B}^2\times\mathscr{B}^2\times\mathscr{B}^1\times\mathscr{B}^1$, we can find a unique non-negative, global solution $(u_1(x, t),\ u_2(x, t),\ u_3(x, t),\ u_4(x, t))$ such that

$$(u_1(x, t),\ u_2(x, t),\ u_3(x, t),\ u_4(x, t))\in \mathscr{E}_t^0(\mathscr{B}^2\times\mathscr{B}^2\times\mathscr{B}^1\times\mathscr{B}^1)$$
$$\cap\mathscr{E}_t^1(\mathscr{B}^0\times\mathscr{B}^0\times\mathscr{B}^0\times\mathscr{B}^0).$$

(See Mimura [3].) Here $\mathscr{B}^m$ is the topological vector space of uniformly continuous and bounded functions in R^n together with their derivatives of order up to m.

§2 Asymptotic behavior.

We will derive some sufficient conditions to be imposed on the initial data under which whether or not $u_3(x, t)$ and $u_4(x, t)$ will tend to zero as $t\to\infty$.

In order to state our results, we prepare two lemmas which are so-called "comparison theorem".

Lemma 2.1 *Consider the following three Cauchy problems* (P_1), (P_2) *and* (P_3) *in* Ω_n:

$$(P_1) \quad U_t = D_0 \Delta U + D_1 F(U) \qquad U(x, 0) = \Phi(x)$$

$$(P_2) \quad V_t = D_0 \Delta V + D_2 F(V) \qquad V(x, 0) = \Phi(x)$$

$$(P_3) \quad W_t = D_0 \Delta W + D_3 F(W) \qquad W(x, 0) = \Phi(x),$$

where

$$D_2 = \begin{pmatrix} -d & -d & 0 \\ 0 & d & -d \\ 0 & -D & D \\ -D & 0 & -D \end{pmatrix} \qquad D_3 = \begin{pmatrix} -D & -D & 0 \\ 0 & D & -D \\ 0 & -d & d \\ -d & 0 & -d \end{pmatrix}$$

and $d = \min(d_1, d_2, d_3)$ *and* $D = \max(d_1, d_2, d_3)$. *Then it follows that for non-negative* $\Phi(x)$,

i) $v_1(x, t) \geqq u_1(x, t) \geqq w_1(x, t) \geqq 0$

ii) $v_1(x, t) + v_2(x, t) \geqq u_1(x, t) + u_2(x, t) \geqq w_1(x, t) + w_2(x, t) \geqq 0$

iii) $w_3(x, t) + w_4(x, t) \geqq u_3(x, t) + u_4(x, t) \geqq v_3(x, t) + v_4(x, t) \geqq 0$

iv) $w_4(x, t) \geqq u_4(x, t) \geqq v_4(x, t) \geqq 0.$

Proof. We can prove Lemma 2.1 by using the following simple difference scheme Sch(D_1),

$$\frac{u_1^{m+1,J} - u_1^{m,J}}{k} = \frac{1}{h^2} \sum_{i=1}^{n} T_-^i T_+^i u_1^{m,J} - (d_1 u_1^{m+1,J} u_4^{m,J} + d_2 u_1^{m+1,J} u_3^{m,J})$$

$$\frac{u_2^{m+1,J} - u_2^{m,J}}{k} = \frac{1}{h^2} \sum_{i=1}^{n} T_-^i T_+^i u_2^{m,J} - (d_3 u_2^{m+1,J} u_4^{m,J} - d_2 u_1^{m+1,J} u_3^{m,J})$$

$$\frac{u_3^{m+1,J} - u_3^{m,J}}{k} = \qquad (d_3 u_4^{m+1,J} u_2^{m,J} - d_2 u_3^{m+1,J} u_1^{m,J})$$

$$\frac{u_4^{m+1,J} - u_4^{m,J}}{k} = \qquad -(d_1 u_4^{m+1,J} u_1^{m,J} + d_3 u_4^{m+1,J} u_2^{m,J})$$

and the initial data

$$U^{0,J}=\Phi(Jh)=(\phi_1(j_1h,\ j_2h,\ \cdots,\ j_nh),\ \phi_4(j_1h,\ j_2h,\ \cdots,\ j_nh)),$$

with k and h satisfying $\frac{k}{h^2}\leqq\frac{1}{2n}$. Here $u_1^{m,J}=u_i(j_1h,\ j_2h,\ \cdots,\ mk)$ $(i=1,\ 2,\ 3,\ 4)$ for n-tuple of integers $(j_1,\ j_2,\ \cdots,\ j_n)$ and for a non-negative integer m, h and k are the mesh sizes in x and t directions respectively and $T_\pm^i$ is an operator replacing j_i by $j_i\pm 1$, that is,

$$T_\pm^i u^{m,J}=u(j_1h,\ \cdots,\ j_{i-1}h,\ (j_i\pm 1)h,\ j_{i+1}h,\ \cdots,\ j_nh,\ mk)-u^{m,J}.$$

Considering the problems (P_2) and (P_3) by the difference schemes $\mathrm{Sch}(D_2)$ and $\mathrm{Sch}(D_3)$, we find for any J and m

i) $v_1^{m,J}\geqq u_1^{m,J}\geqq w_1^{m,J}\geqq 0$

ii) $v_1^{m,J}+v_2^{m,J}\geqq u_1^{m,J}+u_2^{m,J}\geqq w_1^{m,J}+w_2^{m,J}\geqq 0$

iii) $w_3^{m,J}+w_4^{m,J}\geqq u_3^{m,J}+u_4^{m,J}\geqq v_3^{m,J}+v_4^{m,J}\geqq 0$

iv) $w_4^{m,J}\geqq u_4^{m,J}\geqq v_4^{m,J}\geqq 0.$

From these inequalities, Lemma 2.1 can be proved. (See Mimura [3].)

Lemma 2.2 *Consider the following Cauchy problem*

$$u_t=\Delta u-duw$$

$$w_t=\quad -d'uw$$

in Ω_n *with the initial data*

$$u(x,\ 0)=u_0(x)$$

$$w(x,\ 0)=w_0(x).$$

If $\tilde{u}_0(x)\geqq u_0(x)\geqq 0$ *and* $w_0(x)\geqq\tilde{w}_0(x)\geqq 0$, *then* $\tilde{u}(x,\ t)\geqq u(x,\ t)\geqq 0$ *and* $w(x,\ t)\geqq\tilde{w}(x,\ t)\geqq 0$, *where* d *and* d' *are positive constants and*

$\tilde{u}(x, t)$ and $\tilde{w}(x, t)$ are the solutions with the initial data and $\tilde{u}_0(x)$ $\tilde{w}_0(x)$.

Proof. The proof of this lemma is easy and hence is omitted.

Now consider the following equations obtained from (P_3),

$$\left(w_1-\frac{D}{d}(w_3+w_4)\right)_t=\Delta w_1 \tag{2.1}$$

$$(w_3+w_4)_t=-d(w_3+w_4)w_1. \tag{2.2}$$

Integrating (2.1) and (2.2) from 0 to t with respect to t, we have

$$w_1(x, t)-\frac{D}{d}(w_3(x, t)+w_4(x, t))-\phi_1(x)+\frac{D}{d}\phi_4(x) = \int_0^t w_3(x, \tau)d\tau \tag{2.3}$$

$$w_3(x, t)+w_4(x, t)=\phi_4(x)\exp\left(-d\int_0^t w_1(x, \tau)\,d\tau\right). \tag{2.4}$$

Eliminating $w_3(x, t)$ and $w_4(x, t)$ from (2.3) and (2.4), we obtain

$$w_1(x, t)-\frac{D}{d}\phi_4(x)\exp\left(-d\int_0^t w_1(x, \tau)d\tau\right)-\phi_1(x)+\frac{D}{d}\phi_4(x) = \int_0^t \Delta w_1(x, \tau)d\tau. \tag{2.5}$$

Lemma 2.3 *Supposing that the initial data ϕ_1, ϕ_2 and ϕ_4 of* (P_3) *are all constant. If $\phi_1 \geqq \frac{D}{d}\phi_4 \geqq 0\,(\phi_1 \neq 0)$, then for the corresponding solution w_1, it holds that*

$$\int_0^\infty w_1(x, t)dt=+\infty.$$

Proof. The solution W of (P_3) is unique and hence it is independent of x. Thus (2.5) implies

$$w_1(t)-\frac{D}{d}\phi_4 \exp\left(-d\int_0^t w_1(\tau)d\tau\right)-\phi_1+\frac{D}{d}\phi_4=0. \tag{2.6}$$

Now assume

$$\int_0^\infty w_1(t)dt<+\infty.$$

Then by $w_1(t)\geqq 0$ and $\left|\frac{dw_1}{dt}\right|\leqq M(=\text{const.})$, we can see that $w_1(t)\to 0$ at as $t\to\infty$. Letting $t\to\infty$ in (2.6), we obtain

$$\frac{D}{d}\phi_4\left\{1-\exp\left(-d\int_0^\infty w_1(t)dt\right)\right\}=\phi_1.$$

This contradicts to $\phi_1\geqq\frac{D}{d}\phi_4\geqq 0$ unless $w_1=0$.

Lemma 2.4 *Supposing that the initial data* ϕ_1, ϕ_2 *and* ϕ_4 *of* (P_2) *are all constant. If* $\frac{d}{D}\phi_4>\phi_1\geqq 0$, *then for the corresponding solution* v_1, *it holds that*

$$\int_0^\infty v_1(x,\, t)dt<+\infty.$$

Proof. Since the proof of this lemma is analogous to that of Lemma 2.3, it may be omitted.

Here we can refine Lemma 2.3 and Lemma 2.4 as follows.

Lemma 2.3′ *Let* $\phi_1(x)$, $\phi_2(x)$ *and* $\phi_4(x)$ *be the initial data of* (P_3) *with*

$$\phi_1(x)\geqq\frac{D}{d}\sup_x\phi_4(x)\geqq 0 \qquad (\phi_1(x)\not\equiv 0),$$

then the corresponding solution $w_1(x,\, t)$ *satisfies*

$$\int_0^\infty w_1(x,\, t)dt=+\infty \qquad \textit{for all } x.$$

Proof. Consider the following system obtained from (P_3),

$$\frac{\partial w_1}{\partial t}=\Delta w_1-D(w_3+w_4)w_1 \tag{2.7}$$

$$\frac{\partial}{\partial t}(w_3+w_4)=-d(w_3+w_4)w_1.$$

Consider pairs of the initial data $(\phi_1(x), \phi_4(x))$ and $\left(\frac{D}{d}\sup_x\phi_4(x), \sup_x\phi_4(x)\right)$ and denote the corresponding solutions by $(w_1(x, t), w_3(x, t)+w_4(x, t))$ and by $(\tilde{w}_1(x, t), \tilde{w}_3(x, t)+\tilde{w}_4(x, t))$ respectively. If

$$\phi_1(x)\geqq\frac{D}{d}\sup_x\phi_4(x),$$

then by Lemma 2.2, we have

$$w_1(x, t)\geqq\tilde{w}_1(x, t)\geqq 0$$

and

$$\tilde{w}_3(x, t)+\tilde{w}_4(x, t)\geqq w_3(x, t)+w_4(x, t)\geqq 0.$$

On the other hand, since (2.7) is independent of $\phi_2(x)$, we can apply Lemma 2.3 and obtain that

$$\int_0^\infty\tilde{w}_1(x, t)dt=+\infty.$$

Hence we see that

$$\int_0^\infty w_1(x, t)dt=+\infty.$$

Lemma 2.4′ *Let* $\phi_1(x), \phi_2(x)$ *and* $\phi_4(x)$ *be the initial data of* (P_2) *with*

$$\frac{d}{D}\inf_x\phi_4(x)>\phi_1(x)\geqq 0,$$

then the corresponding solution $v_1(x, t)$ *satisfies*

$$\int_0^\infty v_1(x, t)\,dt < +\infty.$$

Proof. It is sufficient to consider two pairs of the initial data $(\phi_1(x), \phi_4(x))$ and $\left(\frac{d}{D}\inf_x \phi_4(x)-\varepsilon,\ \inf_x \phi_4(x)\right)$, where $\varepsilon>0$ is sufficiently small so that $\frac{d}{D}\inf_x \phi_4(x)-\varepsilon \geqq \phi_1(x)$.

Together with these facts, we have the following proposition on the asymptotic behavior of the solution of our problem (1.5) and (1.6).

Proposition 2.1 *Let* $U(x, t)={}^t(u_1(x, t), u_2(x, t), u_3(x, t), u_4(x, t))$ *be the solution in the Cauchy problem* (1.5) *and* (1.6).

i) *if* $\phi_1(x) \geqq \frac{D}{d}\sup_x \phi_4(x) \geqq 0$, *then*

$$\lim_{t\to\infty} u_3(x, t)=0 \text{ and } \lim_{t\to\infty} u_4(x, t)=0 \text{ for any } x,$$

ii) *if* $\frac{d}{D}\inf_x \phi_4 > \phi_1(x) \geqq 0$, *then*

$$\lim_{t\to\infty} (u_3(x, t)+u_4(x, t)) \neq 0 \text{ for any } x.$$

Proof. According to Lemma 2.3′, we can see $\lim_{t\to\infty}(w_3(x, t)+w_4(x, t))=0$ from (2.4). Thus it follows $\lim_{t\to\infty}(u_3(x, t)+u_4(x, t))=0$ for any x from iii) of Lemma 2.1 and hence

$$\lim_{t\to\infty} u_3(x, t)=0 \quad \text{and} \quad \lim_{t\to\infty} u_4(x, t)=0$$

by the non-negativity of U. ii) can be proved easily by Lemma 2.4′ and by iii) of Lemma 2.1.

Next we investigate more precisely ii) of Proposition 2.1.

Lemma 2.5 *Supposing that the initial data* ϕ_1, ϕ_2 *and* ϕ_4 *of* (P_3)

are all constant. If

$$\phi_1+\phi_2 \geqq \frac{D}{d}\phi_4 > \frac{d}{D}\phi_4 > \phi_1 \geqq 0,$$

then, for the corresponding solution w_2,

$$\int_0^\infty w_2(t)dt = +\infty$$

holds.

Proof. Consider the following equations obtained from (P_3):

$$\frac{\partial}{\partial t}\left(w_2+\frac{D}{d}w_3\right)=\Delta w_2 \tag{2.8}$$

$$\frac{\partial}{\partial t}w_4=-d(w_1+w_2)w_4. \tag{2.9}$$

Integrating (2.8) and (2.9) from 0 to t with respect to t, we have

$$w_2(x, t)-\phi_2+\frac{D}{d}w_3(x, t)=\int_0^t \Delta w_2(x, \tau)d\tau \tag{2.10}$$

$$w_4(x, t)=\phi_4 \exp\left(-d\int_0^t (w_1(x, \tau)+w_2(x, \tau))\right)d\tau \tag{2.11}$$

from (2.8) and (2.9). Eliminating from (2.10) by (2.4) and (2.11), we obtain

$$\begin{aligned} w_2(x, t)-\phi_2+\frac{D}{d}\phi_4 \exp\left(-d\int_0^t w_1(x, \tau)\right)d\tau \times \\ \times\left\{1-\exp\left(-d\int_0^t w_2(x, \tau)\right)\right\}d\tau=0. \end{aligned} \tag{2.12}$$

Now suppose that

$$\int_0^\infty w_2(x, t)dt < +\infty,$$

then, as is the case of $w_1(x, t)$, we can see $w_2(x, t)\to 0$ as $t\to\infty$, and letting $t\to\infty$ in (2.12), we obtain

$$-\phi_2+\frac{D}{d}\phi_4\exp\left(-d\int_0^\infty w_1(x, t)dt\right)\times \times\left(1-\exp\left(-d\int_0^\infty w_2(x, t)dt\right)=0. \tag{2.13}$$

We remark here that $\int_0^\infty w_1 dt$ exists by Lemma 2.4. Thus from (2.6)′ and (2.13), we have

$$-\phi_2+\left(\frac{D}{d}\phi_4-\phi_1\right)\left\{1-\exp\left(-d\int_0^\infty w_2(x, t)\,dt\right)\right\}=0$$

and this contradicts to $\phi_2+\phi_1\geqq\frac{D}{d}\phi_4\geqq 0$.

Lemma 2.6 *Supposing that the initial data* ϕ_1, ϕ_2 *and* ϕ_4 *of* (P_2) *are all constant. If*

$$\frac{d}{D}\phi_4>\phi_1+\phi_2\geqq 0,$$

then it holds for the corresponding solution v_2

$$\int_0^\infty v_2(t)dt<+\infty.$$

The proof of this lemma is similar to that of Lemma 2.5.

Refine Lemma 2.5 and Lemma 2.6 as follows:

Lemma 2.5′ *Let* $\phi_1(x)$, $\phi_2(x)$ *and* $\phi_4(x)$ *be the initial data of* (P_3) *with*

$$\phi_1(x)+\phi_2(x)\geqq\frac{D}{d}\sup_x\phi_4(x)\geqq\frac{d}{D}\inf_x\phi_4(x)>\phi_1(x)\geqq 0,$$

then the corresponding solution $w_2(x)$ *satisfies*

$$\int_0^\infty w_2(x,\ t)dt=+\infty \qquad \textit{for all } x.$$

Proof. Consider the following system obtained from (P_3),

$$\frac{\partial}{\partial t}(w_1+w_2)=\Delta(w_1+w_2)-D(w_1+w_2)w_4 \tag{2.14}$$

$$\frac{\partial}{\partial t}w_4= \qquad -d(w_1+w_2)w_4 \tag{2.15}$$

and pairs of the initial data $(\phi_1(x)+\phi_2(x),\ \phi_4(x))$ and $\left(\frac{D}{d}\sup_x \phi_4(x),\ \sup_x \phi_4(x)\right)$ and denote the corresponding solutions by $(w_1(x,\ t)+w_2(x,\ t),\ w_4(x,\ t))$ and by $(\tilde{w}_1(x,\ t)+\tilde{w}_2(x,\ t),\ \tilde{w}_4(x,\ t))$ respectively. If

$$\phi_1(x)+\phi_2(x)\geqq\frac{D}{d}\sup_x \phi_4(x),$$

then by Lemma 2.2, we have

$$w_1(x,\ t)+w_2(x,\ t)\geqq\tilde{w}_1(x,\ t)+\tilde{w}_2(x,\ t)$$

and

$$\tilde{w}_4(x,\ t)\geqq w_4(x,\ t).$$

By Lemma 2.3, we obtain

$$\int_0^\infty \tilde{w}_2(x,\ t)dt=+\infty,$$

and hence

$$\int_0^\infty \{\tilde{w}_1(x,\ t)+\tilde{w}_2(x,\ t)\}\,dt=+\infty.$$

On the other hand, we know already

$$\int_0^\infty w_1(x, t)dt < +\infty$$

by Lemma 2.4′ and therefore

$$\int_0^\infty w_2(x, t)dt = +\infty.$$

Lemma 2.6′ *Let* $\phi_1(x)$, $\phi_2(x)$ *and* $\phi_4(x)$ *be the initial data of* (P_2) *with*

$$\frac{d}{D}\inf_x \phi_4(x) > \phi_1(x)+\phi_2(x) \geqq 0,$$

then the corresponding solution $v_2(x, t)$ *satisfies*

$$\int_0^\infty v_2(x, t)dt < +\infty$$

for all x.

Proof. It suffices to consider pairs of the initial data $(\phi_1(x)+\phi_2(x), \phi_4(x))$ and $\left(\frac{d}{D}\inf_x \phi_4(x)-\varepsilon, \inf_x \phi_4(x)\right)$ where $\varepsilon>0$ is sufficiently small so that $\frac{d}{D}\inf_x \phi_4(x)-\varepsilon \geqq \phi_1(x)+\phi_2(x)$.

Summing up all the results obtained above, we attain the following theorem:

Theorem 2.1 *Let* $U(x, t)={}^t(u_1(x, t), u_2(x, t), u_3(x, t), u_4(x, t))$ *be the solution in the Cauchy problem* (1.5) *and* (1.6).

i) *If* $\phi_1(x) \geqq \frac{D}{d}\sup_x \phi_4(x) \geqq 0$, *then for any* x,

$$\lim_{t\to\infty} u_3(x, t)=0 \quad \text{and} \quad \lim_{t\to\infty} u_4(x, t)=0.$$

ii) *If* $\phi_1(x)+\phi_2(x) \geqq \frac{D}{d}\sup_x \phi_4(x) \geqq \frac{d}{D}\inf_x \phi_4(x) > \phi_1(x) \geqq 0$, *then* $\lim_{t\to\infty} u_3(x, t)\neq 0$ *and* $\lim_{t\to\infty} u_4(x, t)=0.$

iii) *If* $\frac{d}{D}\inf_x \phi_4(x)>\phi_1(x)+\phi_2(x)>0$ *and* $\phi_2(x)\not\equiv 0,$

then $\lim_{t\to\infty} u_3(x, t)\not\equiv 0$ *and* $\lim_{t\to\infty} u_4(x, t)\not\equiv 0.$

Proof. i) is nothing but i) of Proposition 2.1. ii) is proved as follows: first note that (P_2) indicates

$$v_3(x, t)+v_4(x)=\phi_4(x)\exp\left(-D\int_0^t v_1(x, \tau)d\tau\right).$$

Since $\int_0^\infty v_1(x, \tau)d\tau<+\infty$ by Lemma 2.4′, we have

$$\lim_{t\to\infty}(v_3(x, t)+v_4(x, t))=\phi_4(x)\exp\left(-D\int_0^\infty v_1(x, \tau)d\tau\right)\not\equiv 0.$$

On the other hand, since

$$w_4(x, t)=\phi_4(x)\exp\left(-d\int_0^t \{w_1(x, \tau)+w_2(x, \tau)\}d\tau\right),$$

we see $\lim_{t\to\infty} w_4(x, t)=0$ by virtue of Lemma 2.5. With the aid of iii) and iv) of Lemma 2.1, we get

$$\lim_{t\to\infty} u_3(x, t)\not\equiv 0 \quad \text{and} \quad \lim_{t\to\infty} u_4(x, t)=0.$$

Next by Lemma 2.4′ and Lemma 2.6′, we have

$$\int_0^\infty v_1(x, t)dt<+\infty \quad \text{and} \quad \int_0^\infty v_2(x, t)dt<+\infty.$$

Hence, with the aid of i) and ii) of Lemma 2.1, we can see

$$\int_0^\infty u_1(x, t)dt<+\infty \quad \text{and} \quad \int_0^\infty u_2(x, t)dt<+\infty.$$

On the other hand, it follows from (P_1) that

$$u_4(x, t)=\phi_4(x)\exp\left(-d_1\int_0^t u_1(x, t)dt-d_3\int_0^t u_2(x, t)dt\right),$$

and this shows that $\lim_{t\to\infty} u_4(x, t)$ exists for all x and it does not vanish. Next note the following relation which can be derived from (P_1),

$$(u_3)_t=-d_2u_1u_3+d_3u_3u_4,$$

then we have

$$u_3(x, t)=\exp\left(-d_2\int_0^t u_1(x, \tau)d\tau\right)\times$$

$$\times\left\{d_3\int_0^t u_2(x, \tau)u_4(x, \tau)\exp\left(d_2\int_0^\tau u_1(x, \sigma)\,d\sigma\right)\right\}d\tau,$$

and we see that $\lim_{t\to\infty} u_3(x, t)$ exists and does not vanish.

Remark. There will arise naturally the question whether or not the following occurs:

$$\lim_{t\to\infty} u_3(x, t)=0 \quad \text{and} \quad \lim_{t\to\infty} u_4(x, t)\not\equiv 0.$$

As for this question one can say that if

$$\frac{d}{D}\inf_x \phi_4(x)>\phi_1(x)\geqq 0 \quad \text{and} \quad \phi_2(x)\equiv 0,$$

then the situation above is true and that if $\phi_2(x)\not\equiv 0$, then it never occurs.

§3. On some semilinear elliptic equation.

In this section we assume the diffusible matters $\phi_1(x)$ and $\phi_2(x)$ are of class L^1 and the non-diffusible matter $\phi_4(x)$ is of class $\mathscr{B}^1$. It will be natural from the chemical meaning.

Remember (2.5) and a similar relation for $v_1(x, t)$:

$$v_1(x,t)-\frac{d}{D}\phi_4(x)\exp\left(-D\int_0^t v_1(x,\tau)d\tau\right)-\phi_1(x)+$$

$$+\frac{d}{D}\phi_4(x)=\int_0^t \Delta v_1(x,\tau)d\tau.$$

If we assume

$$\int_{R^n}\int_0^\infty v_1(x,t)dtdx<+\infty,$$

we obtain

$$\Delta w=\frac{D}{d}\phi_4(x)\{1-\exp(-dw)\}-\phi_1(x) \tag{3.1}$$

$$\Delta v=\frac{d}{D}\phi_4(x)\{1-\exp(-Dv)\}-\phi_1(x) \tag{3.2}$$

in the sense of distribution,
where

$$w(x)=\int_0^\infty w_1(x,t)dt$$

and

$$v(x)=\int_0^\infty v_1(x,t)dt.$$

Thus, observing Lemma 2.1, it will be interesting to investigate whether or not (3.1) or (3.2) has a solution for given $\phi_1(x)$ and $\phi_4(x)$.

We study the following semilinear elliptic equation,

$$\Delta u=a(x)(1-e^{-u})-f(x), \tag{3.3}$$

which is of same type as (3.1) and (3.2), with $a(x)\in\mathscr{B}^1$, $\alpha^2\geqq a(x)\geqq 0$ and $f(x)(\geqq 0)\in L^1(R^n)$.

We call that is $u(x)$ a solution of (3.3) if and only if $u(x)$ is of class $L^1(R^n)$ with $u(x) \geqq 0$ and satisfies (3.3) in the sense of distribution.

Remark. If $f(x) \equiv 0$, then it can be proved that (3.3) has only trivial solution. Therefore we assumed that $f(x) \not\equiv 0$ in what follows.

We consider the sequence of functions $\{u_\mu(x)\}$ defined by the following equations:

$$(3.4) \qquad \Delta u_\mu - \alpha^2 u_\mu = a(x)(1-e^{-u_{\mu-1}}) - \alpha^2 u_{\mu-1} - f(x) \qquad (\mu=1, 2, \cdots)$$

$$u_0(x)=0.$$

As for the above sequence $\{u_\mu(x)\}$, we have

Proposition 3.1 *Each $u_\mu(x)$ is non-negative and of class $L^1(R^n)$, and moreover $u_\mu(x)$ is monotone increasing in μ.*

In order to prove Proposition 3.1, we prepare some lemmas:

Lemma 3.1 *Put $k(x)=\mathscr{F}^{-1}\left[\frac{1}{\alpha^2+4\pi^2|\xi|^2}\right]$, where $|\xi|^2=\xi_1^2+\xi_2^2+\cdots+\xi_n^2$ and $\mathscr{F}^{-1}$ denotes the inverse Fourier transformation, then it follows*

i) *$k(x)$ depends on only $|x|=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$ and $k(x)>0$*

ii) $k(x) \in L^1(R^n)$

iii) $\|k\|_{L^1(R^n)} = \frac{1}{\alpha^2}$

iv) $\frac{dk}{d|x|}<0.$

We denote by K the convolution operator with its kernel $k(x)$,

$$(K\varphi)(x)=\int_{R^n} k(x-y)\varphi(y)dy.$$

Lemma 3.2 *Let $\varphi(x)$ be of class $L^1(R^n)$, then we have*

i) $$\|K\varphi\|_{L^1(R^n)} \leqq \frac{1}{\alpha^2}\|\varphi\|_{L^1(R^n)}$$

especially for non-negative φ,

$$\|K\varphi\|_{L^1(R^n)} = \frac{1}{\alpha^2}\|\varphi\|_{L^1(R^n)}$$

ii) $$\|K(\beta\varphi)\|_{L^1(R^n)} \leqq \frac{1}{\alpha^2}\|\beta\|_{L^\infty(R^n)}\|\varphi\|_{L^1(R^n)}$$

for any $\beta(x)$ in $L^\infty(R^n)$.

Now we are in a position to prove Proposition 3.1.

Proof of Proposition 3.1. If $u_{\mu-1}(x)$ is of class $L^1(R^n)$ and non-negative, then we see that $a(x)(1-e^{-u_{\mu-1}})-\alpha^2 u_{\mu-1}$ is also of class $L^1(R^n)$ and non-negative. We have $u_\mu(x)=(Kf)(x)-K(a(1-e^{-u_{\mu-1}})-\alpha^2 u_{\mu-1})(x)$ and it is of class $L^1(R^n)$ and non-negative. On the other hand, we can easily see that $u_1(x)=(Kf)(x)$ is of class $L^1(R^n)$ and non-negative. This shows that each $u_\mu(x)$ is of class $L^1(R^n)$ and non-negative.

Next, from (3.4) we have for $\mu=1, 2, \cdots$,

$$\Delta(u_{\mu+1}-u_\mu)-\alpha^2(u_{\mu+1}-u_\mu)=a(x)(e^{-u_{\mu-1}}-e^{-u_\mu})-\alpha^2(u_\mu-u_{\mu-1}),$$

hence

$$\Delta(u_{\mu+1}-u_\mu)-\alpha^2(u_{\mu+1}-u_\mu)=(u_\mu-u_{\mu-1})(a(x)e^{-u_\mu+\theta(u_\mu-u_{\mu-1})}-\alpha^2)$$

for some θ satisfying $0<\theta<1$. Thus if we see $u_\mu-u_{\mu-1}\geqq 0$, then we can obtain $u_{\mu+1}-u_\mu\geqq 0$. On the other hand, $u_1-u_0=u_1\geqq 0$. This completes the proof.

In treating our equation, it is sufficient to consider the scheme (3.4). In fact we have

Theorem 3.1 *A necessary and sufficient condition in order that*

(3.3) *has a solution is that*

$$||u_\mu||_{L^1(R^n)} \leqq M$$

where $\{u_\mu\}$ *is of constructed in* (3.4) *and* M *is a constant independent of* μ.

Proof. Necessity: Let $u(x)$ be an arbitraly solution of (3.3), then we have

$$\Delta v_\mu - \alpha^2 v_\mu = -v_{\mu-1}(\alpha^2 - a(x)e^{-u+\theta(u-u_\mu)})$$

for $\mu = 1, 2, \cdots$, where $v_\mu = u - u_\mu$ and θ satisfies $0 < \theta < 1$. This shows $v_\mu \geqq 0$, since $v_0 = u - u_0 \geqq 0$. Thus

$$||u_\mu||_{L^1(R^n)} \leqq ||u||_{L^1(R^n)}$$

for $\mu = 1, 2, \cdots$.

Sufficiency: Assume that

$$||u_\mu||_{L^1(R^n)} \leqq M,$$

then since $u_\mu(x)$ is monotone increasing in μ, we see that $\lim_{\mu \to \infty} u_\mu(x) = u(x)$ is of class $L^1(R^n)$ in virtue of Beppo Levi's theorem. It is easy to see that $u(x)$ satisfies (3.3) as a distribution.

We give a sufficient condition for the existence of solution of (3.3), which can be stated as

Theorem 3.2 *For some positive* $Q(<\alpha^2)$, $mE_Q^a = m\{x; a(x) \leqq Q\} < +\infty$, *then there exists a solution* $u(x)$ *of* (3.3).

For the proof if this theorem, we shall have to prepare some lemmas.

Lemma 3.3 *For any fixed positive number* γ,

$$\delta(\gamma)=\sup_{mB\leqq\gamma}\int_B k(x)dx<\frac{1}{\alpha^2}$$

where the supremum is taken for all measurable sets in R^n *with its measure* $mB\leqq\gamma$.

For the proof of this lemma, it will be sufficient to note Lemma 3.1.

Lemma 3.4 *Suppose* $\psi(x)$ *be a measurable function in* R^n *such that*

i) $0\leqq\psi(x)\leqq\alpha^2$

ii) $mE^{\psi}_{\alpha^2-S}=m\{x;\ \psi(x)\geqq\alpha^2-S\}<+\infty$

for some S *with* $0<S<\alpha^2$. *Then we have*

$$\sup_x\int_{R^n}k(x-y)\psi(y)dy<1.$$

Proof. We note first

$$(K\psi)(x)=\int_{E^{\psi}_{\alpha^2-S}}k(x-y)\psi(y)dy+\int_{R^n-E^{\psi}_{\alpha^2-S}}k(x-y)\psi(y)dy.$$

Hence

$$(K\psi)(x)\leqq\alpha^2\int_{E^{\psi}_{\alpha^2-S}}k(x-y)dy+(\alpha^2-S)\int_{R^n-E^{\psi}_{\alpha^2-S}}k(x-y)dy$$

$$=(\alpha^2-S)/\alpha^2+S\int_{E^{\psi}_{\alpha^2-S}}k(x-y)dy$$

$$=(\alpha^2-S)/\alpha^2+S\int_{x-E^{\psi}_{\alpha^2-S}}k(y)dy.$$

Thus by virtue of Lemma 3.3, we have

$$\sup_x(K\psi)(x)=(\alpha^2-S)/\alpha^2+S\sup_x\int_{x-E^{\psi}_{\alpha^2-S}}k(y)dy<1.$$

Lemma 3.5 *Put* $w_\mu = u_\mu - u_{\mu-1}$ *for* $\mu = 1, 2, \cdots$, *then* w_μ *are bounded, if* $\mu \geqq \left[\frac{n}{2}\right] + 1$

Proof. Remember $u_\mu = Kf + K(\alpha^2 u_{\mu-1}) - K(a(x)(1 - e^{-u_{\mu-1}}))$, then we have that,

$$w_\mu = \alpha^2 K w_{\mu-1} + K(a(x)(e^{-u_{\mu-1}} - e^{-u_{\mu-2}})),$$

hence we have

$$0 \leqq w_\mu \leqq \alpha^2 K w_{\mu-1}$$

and that $0 \leqq w_\mu \leqq \alpha^{2(\mu-1)} K^\mu f$. Since Fourier image of $\alpha^{2(\mu-1)} K^\mu f$ is integrable when $\mu \geqq \left[\frac{n}{2}\right] + 1$, $\alpha^{2(\mu-1)} K^\mu f$ is bounded and so are w_μ.

Lemma 3.6 *Under the assumption on* $a(x)$ *in Theorem* 3.2, *put* $\sup_x w_\mu(x) = A_\mu$ *for* $\mu \geqq \left[\frac{n}{2}\right] + 1$, *then* $A_{\mu+1} \leqq c A_\mu$, *where* c *is a constant with* $0 < c < 1$.

Proof. Since $w_\mu = K(\alpha^2 w_{\mu-1} + a(x)(e^{-u_{\mu-1}} - e^{-u_{\mu-2}}))$, we obtain

$$w_\mu \leqq K((\alpha^2 - a(x) e^{-u_{\mu-1}}) w_{\mu-1}).$$

Thus, when $\mu \geqq \left[\frac{n}{2}\right] + 1$ we have

$$w_{\mu+1} \leqq A_\mu K(\alpha^2 - a(x) e^{-u_\mu}) \leqq A_\mu K(\alpha^2 - a(x) e^{-u}),$$

where $u = \lim_{\mu \to \infty} u_\mu$. On the other hand, from

$$\int_{R^n} a(x)(1 - e^{-u_\mu})\, dx \leqq \int_{R^n} f(x) dx, \tag{3.5}$$

it follows that $a(x)(1 - e^{-u})$ is of class $L^1(R^n)$ by virtue of Beppo Levi's lemma.

Now consider the following two sets $E_N^u=\{x;\ u\geqq N\geqq 0\}$ and $E_T^{ae^{-u}}=\{x;\ a(x)e^{-u}\leqq T,\ 0<T<\alpha^2\}$. We have

$$(1-e^{-N})\int_{E_N^u} a(x)dx<+\infty$$

by (3.5). Since $a(x)$ is not integrable in any measurable set of infinite measure from the assumption of Theorem 3.2, we see $mE_N^u<+\infty$ for ${}^\forall N>0$. If x belongs to $CE_N^u\cap E_T^{ae^{-u}}$, then $T\geqq a(x)e^{-u}\geqq a(x)e^{-N}$. Thus choosing T and N such as $e^N T<Q$, we may assume

$$m(CE_N^u\cap E_T^{ae^{-u}})<+\infty$$

by the assumption on $a(x)$ and then we see

$$mE_T^{ae^{-u}}\leqq mE_N^u+m(E_T^{ae^{-u}}\cap CE_T^u)<+\infty$$

from $E_T^{ae^{-u}}=(E_T^{ae^{-u}}\cap E_N^u)\cup(E_T^{ae^{-u}}\cap CE_N^u)$. Therefore, if we replace $\psi(x)$ and S in Lemma 3.4 by $\alpha^2-a(x)e^{-u}$ and T respectively, we have, for $\mu\geqq\left[\frac{n}{2}\right]+1$,

$$w_{\mu+1}\leqq cA_\mu \qquad (0<c<1)$$

and hence $A_{\mu+1}\leqq cA_\mu$.

As an immediate consequence of Lemma 3.6, we obtain,

Proposition 3.2 *The sequence* $\{u_\mu\}$ *defined by* (3.3) *satisfies*

$$u_\mu(x)\leqq A+\sum_{s=1}^{\left[\frac{n}{2}\right]} K^s f$$

for $\mu=1, 2, \cdots$, *where* A *is a positive constant.*

Now we can prove Theorem 3.2 as follows;

Proof of Theorem 3.2. Consider the set $E_B^g=\{x;\ g(x)\equiv\sum_{s=1}^{[\frac{n}{2}]}K^s f\leqq B\}$ with an arbitrary positive constant B. Since $g(x)$ belongs to $L^1(R^n)$, it follows

$$mCE_B^g<+\infty.$$

If y belongs to $E_B^g\cap CE_Q^a$, then $u_\mu(y)\leqq A+B$ and $a(x)\geqq Q$. Hence, if y belongs to $E_B^g\cap CE_Q^a$, it follows

$$\alpha^2 u_\mu(y)+a(x)(e^{-u_{\mu-1}})\leqq\alpha^2 u_\mu(y)+Q(e^{-u_{\mu-1}}) \tag{3.6}$$

$$\leqq\frac{1}{A+B}\{\alpha^2(A+B)+Q(e^{-A-B}-1)\}\,u_\mu(y).$$

Remember again,

$$u_\mu(x)=Kf+K(\alpha^2 u_{\mu-1}+a(x)(e^{-u_{\mu-1}}-1)) \tag{3.7}$$

$$\leqq Kf+K(\alpha^2 u_\mu+a(x)(e^{-u_\mu}-1)).$$

Consider the second term in (3.6),

$$K(\alpha^2 u_\mu+a(x)(e^{-u_\mu}-1)) \tag{3.8}$$

$$=\int_{E_B^g\cap CE_Q^a}k(x-y)[\alpha^2 u_\mu(y)+a(y)(e^{-u_\mu}-1)]dy+$$

$$+\int_{R^n-(E_B^g\cap CE_Q^a)}k(x-y)[\alpha^2 u_\mu(y)+a(y)(e^{-u_\mu}-1)]dy$$

$$=\frac{1}{A+B}\{\alpha^2(A+B)+Q(e^{-A-B}-1)\}\int_{E_B^g\cap CE_Q^a}k(x-y)u_\mu(y)dy+$$

$$+\int_{R^n-(E_B^g\cap CE_Q^a)}k(x-y)[\alpha^2 u_\mu(y)+a(y)(e^{-u_\mu}-1)]dy.$$

Observing (3.7) and (3.8), we have

$$\int_{E_B^g\cap CE_Q^a} u_\mu(x)dx \leqq$$

$$\leqq \frac{1}{\alpha^2}\int_{R^n} f(x)dx + \frac{1}{\alpha^2(A+B)}\{(A+B)+Q(e^{-A-B}-1)\}\int_{E_B^g\cap CE_Q^a} u_\mu(x)dx$$

$$+\int_{R^n-(E_B^g\cap CE_Q^a)} u_\mu(x)dx$$

and hence we have

$$\text{(3.9)} \qquad \frac{Q(1-e^{-A-B})}{\alpha^2(A+B)}\int_{E_B^g\cap CE_Q^a} u_\mu(x)dx \leqq \frac{1}{\alpha^2}\int_{R^n} f(x)dx$$

$$+\int_{R^n-(E_B^g\cap E_Q^a)} u_\mu(x)dx.$$

On the other hand, since

$$\text{(3.10)} \qquad m\{R^n-(E_B^g\cap CE_Q^a)\} = m(CE_B^g\cup E_Q^a) < +\infty,$$

it follows

$$\int_{R^n-(E_B^g\cap CE_Q^g)} u_\mu dx \leqq Am(CE_B^g\cup E_Q^a) + \int_{R^n} g(x)dx$$

from Proposition 3.2. Together with (3.9) and (3.10), we have

$$\int_{R^n} u_\mu(x)dx \leqq M(A, B, Q, f),$$

where M is a constant independent of μ. Because of Theorem 3.1, Theorem 3.2 is proved.

As for the uniqueness of the solution, we have

Theorem 3.3 *If the problem* (3.3) *has a solution, then it is determined uniquely.*

Proof. Let $v(x)$ be an arbitrary solution of (3.3) and $u(x)$ be the solution obtained as a limit function of $u_\mu(x)$ in our scheme (3.4), then we have first

$$0 \leqq w \leqq K(\alpha^2 w) \tag{3.11}$$

with $w=v-u$. In fact, since we can easily $v-u_\mu \geqq 0$ by the same technique used in the proof of Proposition 3.1, we have immediately $w \geqq 0$. Next we note that w satisfies

$$\Delta w-\alpha^2 w=(a(x)e^{-u-\theta w}-\alpha^2)w \tag{3.12}$$

where θ satisfies $0<\theta<1$. Thus we have

$$w=K((\alpha^2-a(x)e^{-u-\theta w})w) \leqq K(\alpha^2 w). \tag{3.13}$$

From (3.11) it follows that for any positive integer k,

$$0 \leqq w \leqq \alpha^{2k} K^k(w). \tag{3.14}$$

if we note

$$\mathscr{F}[\alpha^{2k}K^k(w)]=\left(\frac{\alpha^2}{\alpha^2+4\pi^2|\xi|^2}\right)^k \mathscr{F}[w],$$

then, for sufficiently large k, we have

$$\begin{aligned} \alpha^{2k}K^k(w) &\leqq \int_{R^n} |\mathscr{F}[\alpha^{2k}K^k)w)]|\, d\xi \\ &\leqq \int_{R^n}\left[\frac{\alpha^2}{\alpha^2+4\pi^2|\xi|^2}\right]^k d\xi \times \|w\|_{L^1}. \end{aligned} \tag{3.15}$$

Hence we obtain

$$0 \leqq w \leqq \int_{R^n}\left[\frac{\alpha^2}{\alpha^2+4\pi^2|\xi|^2}\right]^k d\xi \times \|w\|_{L^1} \tag{3.16}$$

and letting $k \to \infty$, we see $w \equiv 0$ by the well-known Lebegue theorem.

This shows that our problem (3.3) can not have any solution more than one.

As for the non-existence of solution we have the following;

Theorem 3.4 *Let* $a(x)$ $(0 \leqq a(x) \leqq \alpha^2)$ *be of class* $L^1(R^n)$ *and*

$$\|a\|_{L^1(R^n)} \leqq \|f\|_{L^1(R^n)},$$

then there exists no solution of (3.3).

Proof. Using

$$u_\mu = Kf + K(\alpha^2 u_{\mu-1} + a(e^{-u_{\mu-1}} - 1)),$$

we have by Lemma 3.2

$$\int_{R^n} u_\mu(x)\,dx - \int_{R^n} u_{\mu-1}(x)\,dx = \frac{1}{\alpha^2}\int_{R^n}(f(x) - a(x))dx + \tag{3.17}$$

$$+\frac{1}{\alpha^2}\int_{R^n} a(x)e^{-u_{\mu-1}}dx.$$

Hence

$$\int_{R^n}(u_\mu(x) - u_{\mu-1}(x))dx \geqq \frac{1}{\alpha^2}\int_{R^n} a(x)e^{-u_{\mu-1}}dx. \tag{3.18}$$

Suppose there exists a solution of (3.3), then it follows

$$\int_{R^n}(u_\mu(x) - u_{\mu-1}(x))dx \to 0 \qquad \text{as } \mu \to \infty,$$

because $\int_{R^n} u_\mu(x)dx$ has to converge as $\mu \to \infty$. Thus then it follows

$$\int_{R^n} a(x)e^{-u}dx = 0 \tag{3.19}$$

with $u(x) = \lim_{\mu\to\infty} u_\mu(x)$.

If $m(\text{supp. } a(x)) < 0$, then by (3.18) $u(x) = +\infty$ on supp. $a(x)$ except a null set and this contradicts to that $u(x)$ is of class $L^1(R_n)$. If $m(\text{supp. } a(x)) = 0$, by (3.17) we have

$$\|u_\mu\|_{L^1(R^n)} = \frac{\mu}{\alpha^2}\|f\|_{L^1(R^n)},$$

and then

$$\|u_\mu\|_{L^1(R^n)} \to \infty \qquad \text{as } \mu \to \infty$$

since $\|f\|_{L^1(R^n)} \neq 0$. Thus according to Theorem 3.1, there can not exist any solution of (3.3).

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