On the nilpotence of the hypergeometric equation

By

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Introduction

Let \( T \) be an arbitrary scheme, \( S \) a smooth \( T \)-scheme and \( \mathcal{M} \) a quasi-coherent \( \mathcal{O}_S \)-module. A \( T \)-connection on \( \mathcal{M} \) is by definition a homomorphism of \( \mathcal{O}_S \)-modules:

\[
\nabla : \mathcal{D}et_{\mathcal{T}}(\mathcal{O}_S, \mathcal{O}_S) \rightarrow \mathcal{E}_{nd\mathcal{T}}(\mathcal{M})
\]

which satisfies the "product formula":

\[
\nabla(D)(sm) = s\nabla(D)(m) + D(s)m
\]

for sections \( D \) of \( \mathcal{D}et_{\mathcal{T}}(\mathcal{O}_S, \mathcal{O}_S) \), \( s \) of \( \mathcal{O}_S \) and \( m \) of \( \mathcal{M} \) over an open subset \( U \subseteq S \). A section \( m \) of \( \mathcal{M} \) over \( U \) is called horizontal if \( \nabla(D)(m) = 0 \) for all \( D \)'s, derivations on open subsets of \( U \). Both \( \mathcal{D}et_{\mathcal{T}}(\mathcal{O}_S, \mathcal{O}_S) \) and \( \mathcal{E}_{nd\mathcal{T}}(\mathcal{M}) \) are \( \mathcal{O}_T \)-Lie-algebras via the commutator bracket. The connection is called integrable if it is a Lie-algebra homomorphism. The obstruction to a connection being integrable is the curvature homomorphism \( K : \bigwedge^2 \mathcal{D}et_{\mathcal{T}}(\mathcal{O}_S, \mathcal{O}_S) \rightarrow \mathcal{E}_{nd\mathcal{O}_S}(\mathcal{M}) \) defined by \( K(D \wedge D') = [\nabla(D), \nabla(D')] - \nabla([D, D']) \). Henceforth we will deal only with integrable connections.

A horizontal morphism \( \phi \) between modules with connection
\( \phi : (\mathcal{M}, \mathcal{F}) \to (\mathcal{M}', \mathcal{F}') \) is by definition an \( \mathcal{O}_S \)-linear mapping satisfying
\[
\phi \circ \mathcal{F}(D) = \mathcal{F}'(D) \circ \phi.
\]
Taking as objects quasi-coherent \( \mathcal{O}_S \)-modules with \( T \)-connections \((\mathcal{M}, \mathcal{F})\) and as morphisms the horizontal morphisms
we obtain an abelian category. This category has a partially defined internal Hom obtained by defining Hom \((\langle \mathcal{M}, \mathcal{F} \rangle, (\mathcal{M}', \mathcal{F}'))\) as
\[
\text{Hom}(\langle \mathcal{M}, \mathcal{F} \rangle, \langle \mathcal{M}', \mathcal{F}' \rangle) = \text{Hom}(\langle \mathcal{M}, \mathcal{F} \rangle, \langle \mathcal{M}', \mathcal{F}' \rangle).
\]
In particular \( \mathcal{H} = \text{Hom}(\langle \mathcal{M}, \mathcal{F} \rangle, \langle \mathcal{O}_S, \text{standard} \rangle) \) is the underlying module of
\( \text{Hom}(\langle \mathcal{M}, \mathcal{F} \rangle, \langle \mathcal{O}_S, \text{standard} \rangle) \) and hence has a “dual” connection \( \bar{\mathcal{F}} \)
which satisfies the “product formula”
\[
< \bar{\mathcal{F}}(D)(\phi), m > + < \phi, \mathcal{F}(D)(m) > = D < \phi, m >
\]
where \( \phi \) is a local section of \( \mathcal{H}, m \) of \( \mathcal{M} \) and \( D \) of \( \mathcal{D}_\mathcal{O}_T(\mathcal{O}_S, \mathcal{O}_S) \).
The category also has an internal tensor product \((\mathcal{M}, \mathcal{F}) \otimes (\mathcal{M}', \mathcal{F}')\) which by definition is
\((\mathcal{M} \otimes \mathcal{M}', \mathcal{F})\) where \( \mathcal{F} \) is defined by
\[
\mathcal{F}(D)(m \otimes m') = \mathcal{F}(D)(m) \otimes m' + m \otimes \mathcal{F}(D)(m').
\]
As a result, we can define “induced” connections on the exterior powers of a module with
connection and hence can speak of the determinant \( \text{det}(\langle \mathcal{M}, \mathcal{F} \rangle) \)
provided \( \mathcal{H} \) is locally free of constant (finite) rank.

If \( T \) is a scheme of characteristic \( p \) then both \( \mathcal{D}_\mathcal{O}_T(\mathcal{O}_S, \mathcal{O}_S) \)
and \( \mathcal{E}_\mathcal{D}_\mathcal{O}_T(\mathcal{M}) \) are \( p \)-\( \mathcal{O}_T \)-Lie-algebras (by \( D \mapsto D^p, \phi \mapsto \phi^p \)). We
can then ask if \( \mathcal{F} \) is a homomorphism of \( p \)-Lie-algebras, i.e., if
\( \mathcal{F}(D^p) = (\mathcal{F}(D))^p \).

The \( "p\text{-curvature}" \) (introduced by Deligne) is the mapping
\( \Psi : \mathcal{D}_\mathcal{O}_T(\mathcal{O}_S, \mathcal{O}_S) \to \mathcal{E}_\mathcal{D}_\mathcal{O}_T(\mathcal{M}) \) defined by
\( \Psi(D) = (\mathcal{F}(D))^p - \mathcal{F}'(D^p) \).

It is known, [3], that the \( p \)-curvature \( \Psi \) has the following properties:
1) \( \Psi \) is additive
2) \( \Psi \) is \( p \)-linear i.e. \( \Psi(sD) = s^p \Psi(D) \)
3) for each \( D \), a section of \( \mathcal{D}_\mathcal{O}_T(\mathcal{O}_S, \mathcal{O}_S) \) over \( U \), \( \Psi(D) \) is a horizontal endomorphism of \( (\mathcal{M}, \mathcal{F})|U \) (in particular \( \Psi(D) \) is \( \mathcal{O}_U \)-linear).

If for every section \( D \) of \( \mathcal{D}_\mathcal{O}_T(\mathcal{O}_S, \mathcal{O}_S) \) (over an open set \( U \)),
\( \Psi(D) \) is a nilpotent endomorphism, then we say the connection is nilpotent (a notion introduced by Berthelot [2], in the context of crystalline cohomology).
We observe that there is defined a notion of “inverse image” for modules with connection. Namely, if $T$, $S$, $(\mathcal{M}, \mathcal{F})$ are given as above and if we are given a base change $T' \to T$, then there is associated with $\mathcal{F}$ a $T'$-connection, $\mathcal{F}'$, on the $S' = S \times T'$ module $\mathcal{M}' = \mathcal{M} \otimes_{\mathcal{O}_T} \mathcal{O}_{S'}$.

Locally we can give an explicit description of $\mathcal{F}'$:

If we choose affine open sets $\text{Spec}(A)$, $\text{Spec}(A')$, $\text{Spec}(B)$ of $T$ (resp. $T'$, resp. $S$) so as to obtain a commutative diagram

$$
\begin{array}{ccc}
B & \to & B' = B \otimes_A A' \\
\uparrow & & \uparrow \\
A & \to & A'
\end{array}
$$

and if $M$ is a $B$-module with connection $\mathcal{F} : \text{Der}_A(B, B) \to \text{End}_A(M)$ then the connection $\mathcal{F}'$ on the module $M' = M \otimes_A A'$ is defined as the canonical mapping $\mathcal{F} \otimes 1 : \text{Der}_A(B', B') = \text{Der}_A(B, B) \otimes_A A' \to \text{End}_A(M) \otimes_A A' \to \text{End}_A(M')$.

Now let $T = \text{Spec}(A)$, where $A$ is a ring of finite type over $\mathbb{Z}$ and $S = \text{Spec}(B)$ when $B$ is a smooth $A$-algebra. If $M$ is an $S$-module with connection, we say $M$ is globally nilpotent if for each closed point $p$ of $T$ the induced connection on the module $M_k(p)$ is nilpotent.

Let us recall that if $X$ is a smooth $S$-scheme $\pi : X \to S$, then the De-Rham cohomology $\mathcal{H}_{D.R.}(X/S) \overset{\text{def.}}{=} R\pi_* (\Omega^\cdot_X(S))$ has a “canonical” integrable connection: the Gauss-Manin connection [3, 4]. If $T$ is of characteristic $p$, Katz and Berthelot [2, 3] proved that the Gauss-Manin connection is nilpotent. Using this result Katz [3], gave a beautiful arithmetic proof of the local monodromy theorem.

Let $a, b, c \in \mathbb{Q}$, $n$ be a common denominator, $T = \text{Spec}(\mathbb{Z}[\frac{1}{n}])$, $S = \text{Spec} \mathbb{Z}\left[\lambda, \frac{1}{n\lambda(1-\lambda)}\right]$ where $\lambda$ is an indeterminate. Associated to the hypergeometric differential equation

$$
\lambda(1-\lambda) \frac{d^2 u}{d\lambda^2} + [c-(a+b+1)\lambda] \frac{du}{d\lambda} - abu = 0
$$
is an $S$-module, $M_{a,b,c}$, with integrable $T$-connection: It is the free rank 2 module with base \{e_1, e_2\} where

$$
\left( \begin{array}{c} 
\mathcal{P} \left( \frac{d}{d\lambda} \right) (e_1) \\
\mathcal{P} \left( \frac{d}{d\lambda} \right) (e_2) 
\end{array} \right) = 
\left( \begin{array}{c}
0 \\
\frac{ab}{\lambda(1-\lambda)} \quad \frac{(a+b+1)\lambda-c}{\lambda(1-\lambda)}
\end{array} \right) \left( \begin{array}{c} e_1 \\
e_2 \end{array} \right)
$$

We refer to $M_{a,b,c}$ as the hypergeometric module.

Katz has conjectured that the hypergeometric module, $M_{a,b,c}$, is globally nilpotent. In the first section we prove that for a "large class" of \{a,b,c\} $M_{a,b,c}$ occurs as a direct factor (as module with connection) in the De Rham cohomology of a suitable family of curves. As a corollary, each of these hypergeometric modules reduces (for almost all primes $p$) modulo $p$ to a nilpotent module. In the second section we prove the conjecture. The proof is based on the observation that in characteristic $p$, any hypergeometric equation has a nontrivial polynomial solution.

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**Relation to De Rham Cohomology**

Let $n$ be a positive integer, $\zeta_n$ a primitive $n^{th}$ root of 1 and $\lambda$ an indeterminate. Assume $a,b,c$ are positive integers such that $(n, a) = (n, b) = (n, c) = (n, a+b+c) = 1$ and $n > a+b+c$. Let $X$ be the curve defined over $\mathbb{Q}(\zeta_n, \lambda)$ which is the normalization of the projective closure of the affine curve $y^n = x^a(x-1)^b(x-\lambda)^c$. The group $\mu_n$ of $n^{th}$ roots of 1 operates on $X$. Explicitly $\mu_n$ operates on the function field $\mathbb{Q}(\zeta_n, \lambda)(x, y)$ via $\sigma(x, y) = (x, \sigma y)$ where $\sigma \in \mu_n$ because $(\sigma y)^n = a^n y^n = y^n = x^a(x-1)^b(x-\lambda)^c$. Thus $\mu_n$ operates by functoriality on $H^1_{D,R}(X)$, the De Rham cohomology of $X$. Since we are in characteristic zero we may calculate $H^1_{D,R}(X)$ as the factor space of differentials of
the second kind modulo exact differentials. If we extend the action of \( \mu_n \) to \( \Omega_X^{rat} \) by defining \( \sigma(udx) = (\sigma \cdot u)dx \), then this mapping preserves both differentials of the second kind and exact differentials, and hence by passage to the quotient gives the action of \( \mu_n \) on \( H^1_{D,R}(X) \).

Let us explicitly construct the Gauss-Manin connection on \( H^1_{D,R}(X) \). Let \( D \) denote the unique derivation of the function field of \( X \) which extends the action of \( \frac{d}{d\lambda} \) on \( Q(\xi_n, \lambda) \) and kills \( x \). Extend \( D \) to a derivation of \( \Omega_X^{rat} \) by defining \( D(f \cdot dg) = D(f) \cdot dg + f \cdot D(dg) \).

Under this derivation the differentials of the second kind and exact differentials are stable. The induced action of \( D \) on \( H^1_{D,R}(X) \)

\[
\text{d.s.k./exact is } P \left( \frac{d}{d\lambda} \right).
\]

We observe that for \( \sigma \in \mu_n \), \( D \cdot \sigma - \sigma \cdot D \) is a derivation of the function field of \( X \). Since it kills \( \lambda \) and \( x \), it is zero. This means that \( \mu_n \) actually operates via horizontal automorphisms on \( H^1_{D,R}(X) \). Let us denote by \( \chi \) the inverse of the principal character of \( \mu_n \) and by \( H^1_{D,R}(X)^\chi \) the sub-module consisting of elements which transform according to \( \chi \).

**Proposition** The module \( M_{\frac{a+b+c}{n}, \frac{a+c}{n}, \frac{a+c}{n}} \) is isomorphic (as module with connection) to \( H^1_{D,R}(X)^\chi \), and hence is a direct factor of \( H^1_{D,R}(X) \).

**Proof:** Consider \( X \) as lying over \( P^1 \) via the morphism induced by the inclusion made on function fields \( Q(\xi_n, \lambda) \langle x \rangle \rightarrow Q(\xi_n, \lambda) \langle x, y \rangle \). The assumptions made on the four integers \( n, a, b, c \) imply that lying over each of the four points 0, 1, \( \lambda, \infty \) of \( P^1 \) there is exactly one point of \( X \)(denoted respectively \( p_0, p_1, p_\lambda, p_\infty \)). We have \( \text{ord}_{p_0}(x) = n \), \( \text{ord}_{p_0}(y) = a \); \( \text{ord}_{p_1}(x) = n \), \( \text{ord}_{p_1}(y) = b \); \( \text{ord}_{p_\lambda}(x) = n \), \( \text{ord}_{p_\lambda}(y) = c \); \( \text{ord}_{p_\infty}(x) = -n \), \( \text{ord}_{p_\infty}(y) = -(a + b + c) \). This implies that both \( \frac{dx}{y} \) and \( \frac{x \cdot dx}{y} \) have poles only at \( p_\infty \) and hence are differentials of the second kind (because the sum of the residues of any differential is zero). Let \( \omega_1 \)
and $\omega_2$ denote the classes of $\frac{dx}{y}$ and $\frac{xdx}{y}$ in $H^1_{D,R}(X)$.

The proof breaks up into three parts;

1) We show $\omega_1$ and $\omega_2$ span $H^1_{D,R}(X)$

2) We define a surjective horizontal homomorphism

\[ M_{\frac{a+b+c}{n}} - \frac{a+c}{n} \rightarrow H^1_{D,R}(X) \]

3) We prove this horizontal morphism is injective.

1) Represent $H^1_{D,R}(X)$ as a factor space of the space of differentials having poles only at $p$, and of some bounded order $\leq N$ (by Riemann-Roch Theorem this is possible). Then $\mu_n$ operates on this space in a manner compatible with its action on $H^1_{D,R}(X)$. Both this space of differentials and $H^1_{D,R}(X)$ decompose into direct sums where the summands are the spaces of differentials (resp. cohomology classes) which transform according to a given character of $\mu_n$. Thus any cohomology class which transforms according to $\chi$ is represented by a differential, regular except at $p$, which transforms according to $\chi$.

Since $\text{Spec} Q(\xi_n, \lambda)[x, y, \frac{1}{y}]$ (where $y^n = x^a(x-1)^b(x-\lambda)^c$) is nonsingular any differential regular except at $p$ can be written $\frac{R(x, y)}{y^{\text{some power}}} dx$, where $R(x, y) \in (\xi_n, \lambda)[x, y]$. By the division algorithm we can write it as $\left( R_0(x) + \frac{R_1(x)}{y} + \ldots + \frac{R_{n-1}(x)}{y^{n-1}} \right) dx$ where the $R_i \in Q(\xi_n, \lambda)(x)$. It can transform according to $\chi$ if and only if it is $\frac{R_1(x)}{y} dx$. Because this differential is regular except at $p$, $R_1(x)$ must be a polynomial. To conclude the first part, it remains to prove the following lemma.

**Lemma:** The differentials $x^l \frac{dx}{y}$ ($l \geq 2$) are linearly dependent on $\frac{dx}{y}$ and $\frac{xdx}{y}$ modulo exact differentials.

**Proof:** (By induction on $l$). We have

\[ d\left( \frac{x^{l-1}(x-1)(x-\lambda)}{y} \right) = (l+1)x^l \frac{dx}{y} - l(1+\lambda)x^{l-1} \frac{dx}{y} + (l-1)\lambda x^{l-2} \frac{dx}{y} \]
Hypergeometric equation

\[ + x^{l-1}(x-1)(x-\lambda) \left( \frac{-c}{n(x-\lambda)} + \frac{-b}{n(x-1)} + \frac{-a}{nx} \right) \frac{dx}{y} \]

\[ = \left( l+1 - \frac{a+b+c}{n} \right) x^l \frac{dx}{y} + P(x) \frac{dx}{y} \]

where \( P(x) \) is a polynomial of degree \( \leq l-1 \). As \( l+1 - \frac{a+b+c}{n} \neq 0 \) we are done.

2) The existence and the surjectivity of a horizontal morphism \( M_c, \frac{a+b+c}{n}, \frac{a+c}{n} \rightarrow H^{D,K} (X)^t \) will follow immediately from the following three lemmas. Explicitly the mapping will be defined by \( e_1 \rightarrow \omega_1, e_2 \rightarrow \omega_1^t \) where \( \cdot^n \) stands for the action of \( P \left( \frac{d}{d\lambda} \right) \).

Let us write \( \cdot^n \equiv \) to denote congruence modulo exact.

**Lemma:** \( D \left( \frac{x dx}{y} - \frac{dx}{y} \right) = \left[ \left( 1 - \frac{a+b}{n} \right) + \frac{1}{\lambda} \left( \frac{\lambda c + a(1+b) - n(1+b)}{n} \right) \right] \frac{dx}{y} + \frac{1}{\lambda} \left( \frac{2n-(a+b+c)}{n} \right) x dx \frac{1}{y} \)

**Proof:** We compute:

\[ D( y^n) = -cx^a(x-1)b(x-\lambda)^{c-1} \]

\[ ny^{n-1} D(y) = -cx^a(x-1)b(x-\lambda)^{c-1} \]

\[ D(y) = -\frac{c}{n} \left( x^a(x-1)b(x-\lambda)^{c-1} \right) \frac{y^{n-1}}{y} \]

\[ D \left( \frac{1}{y} \right) = \frac{c}{n} \left( x^a(x-1)b(x-\lambda)^{c-1} \right) \frac{y^{n+1}}{y^{n+1}} = \frac{c}{n} \left( x^a(x-1)b(x-\lambda)^{c-1} \right) \frac{y^{n+1}}{ny^{n+1}(x-\lambda)} = \frac{c}{n} \cdot \frac{1}{(x-\lambda)y} \]

Therefore \( D \left( \frac{dx}{y} \right) = \frac{c}{n} \left( \frac{1}{x-\lambda} \right) \frac{dx}{y} \), \( D \left( \frac{x dx}{y} \right) = \frac{c}{n} \left( \frac{x}{x-\lambda} \right) \frac{dx}{y} \) and hence \( D \left( \frac{x dx}{y} - \frac{dx}{y} \right) = \frac{c}{n} \left( \frac{x-1}{x-\lambda} \right) \frac{dx}{y} \)
Now writing \( f(x)=x^a(x-1)^b(x-\lambda)^c \) we have:

\[
d(y^n) = f'(x)dx = [cx^a(x-1)^b(x-\lambda)^{c-1} + bx^a(x-1)^{b-1}(x-\lambda)^c + ax^{a-1}(x-1)^b(x-\lambda)^c]dx
\]

\[
d\left(\frac{1}{y}\right) = -\frac{d(y)}{y^2} = -\frac{f'(x)dx}{ny^{n+1}} = -\frac{f'(x)}{ny^n} \cdot \frac{dx}{y}
\]

\[
d\left(\frac{x-1}{y}\right) = \frac{dx}{y} + (x-1)\left(\frac{-c}{n(x-\lambda)} - \frac{b}{n(x-1)} - \frac{a}{nx}\right)\frac{dx}{y}
\]

\[
d\left(\frac{(x-1)(x-\lambda)}{y}\right) = \left[2x-(1+\lambda)\right]\frac{dx}{y} - (x-1)(x-\lambda)
\]

\[
\times\left(\frac{c}{n(x-\lambda)} + \frac{b}{n(x-1)} + \frac{a}{nx}\right)\frac{dx}{y}
\]

\[
= \left[2x-(1+\lambda)\right]\frac{dx}{y} - \frac{c}{n} (x-1)\frac{dx}{y} - \frac{b}{n} (x-\lambda)\frac{dx}{y}
\]

\[- \frac{(x-1)(x-\lambda)a}{nx} \frac{dx}{y}
\]

\[
= \left[2x-(1+\lambda) - \frac{(x-1)c}{n} - \frac{(x-\lambda)b}{n} \right] \frac{dx}{y}
\]

\[- \frac{a}{n} (x^2-(1+\lambda)x+\lambda) \frac{dx}{y}
\]

\[
= \left[ 2n-(a+b+c) \right] \frac{x dx}{y} + \left[ \frac{c+\lambda b+a(1+\lambda)-n(1+\lambda)}{n} \right] \frac{dx}{y} - \frac{a}{n} \lambda \frac{dx}{xy}
\]
Hypergeometric equation

Therefore
\[ \frac{c}{n} \frac{x-1}{x-\lambda} \frac{dx}{y} = \left(1 - \frac{a+b}{n}\right) \frac{dx}{y} + \frac{a}{n} \frac{dx}{xy} - d \left(\frac{x-1}{y}\right) \]

\[ = \left(1 - \frac{a+b}{n}\right) \frac{dx}{y} + \frac{1}{\lambda} \left[ \frac{2n-(a+b+c)}{n} \frac{dx}{y} \right. \]
\[ + \left. \left(\frac{\epsilon+\lambda b+a(1+\lambda)-n(1+\lambda)}{n} \right) \frac{dx}{y} \right. \]
\[ - d \left(\frac{x-1}{y}\right) \frac{(x-1)(x-\lambda)}{y} \]
\[ = \left(\frac{a+\epsilon-n}{n\lambda}\right) \frac{dx}{y} + \left(\frac{2n-(a+b+c)}{n\lambda}\right) \frac{dx}{y} \]

Lemma:
\[ D \left(\frac{dx}{y}\right) = \left(\frac{n-(a+c)+\epsilon\lambda}{n\lambda(1-\lambda)}\right) \frac{dx}{y} + \left(\frac{a+b+c-2n}{n\lambda(1-\lambda)}\right) \frac{dx}{y} \]
\[ D \left(\frac{xdx}{y}\right) = \left(\frac{n-a}{n(1-\lambda)}\right) \frac{dx}{y} + \left(\frac{a+b+c-2n}{n(1-\lambda)}\right) \frac{xdx}{y} \]

Proof:
\[ d \left(\frac{-x}{y}\right) = - \frac{dx}{y} + x \left(\frac{c}{n(x-\lambda)} + \frac{b}{n(x-1)} + \frac{a}{nx}\right) dx \]
\[ = \frac{[a(n-1)]}{n} \frac{dx}{y} + \frac{c(n+\lambda)}{n(x-\lambda)} \frac{dx}{y} + \frac{b(n+1)}{n(x-1)} \frac{dx}{y} \]
\[ = \frac{(a+b+c-1)}{n} \frac{dx}{y} + \frac{c}{n} \left(\frac{\lambda}{x-\lambda}\right) \frac{dx}{y} + \frac{b}{n} \left(\frac{1}{x-1}\right) \frac{dx}{y} \]
\[ d \left(\frac{x(x-\lambda)}{y}\right) = (2x-\lambda) \frac{dx}{y} \]
\[ + x(x-\lambda) \left(\frac{-\frac{c}{n(x-\lambda)} - \frac{b}{n(x-1)} - \frac{a}{nx}}{y}\right) \frac{dx}{y} \]
\[ = (2x-\lambda) \frac{dx}{y} - \frac{c}{n} \frac{dx}{y} - \frac{a}{n} \frac{x(x-\lambda)dx}{y} \]
\[ = \frac{-b}{n} \frac{x(1-\lambda)+x-\lambda}{x-1} \frac{dx}{y} \]
\[ = \left(2 - \frac{a+b+c}{n}\right) \frac{xdx}{y} + \left(\frac{a\lambda}{n} + \frac{b(\lambda-1)}{n} - \lambda\right) \frac{dx}{y} \]
\[ - \frac{b}{n} \left(\frac{1-\lambda}{x-1}\right) \frac{dx}{y} \]
Therefore
\[-\frac{b}{n}\left(\frac{1}{x-1}\right)\frac{dx}{y} = \frac{1}{1-\lambda}\left[\left(\frac{a+b+c}{n} - \frac{2}{n}\right) \frac{xdx}{y} + \left(\lambda - \frac{a\lambda}{n} + \frac{b(1-\lambda)}{n}\right) \frac{dx}{y}\right]\]

But we have
\[D\left(\frac{dx}{y}\right) = \frac{c}{n}\frac{1}{x-\lambda} \frac{dx}{y} = \frac{1}{\lambda}\left[\left(1 - \frac{a+b+c}{n}\right) \frac{dx}{y} - \frac{b}{n} \left(\frac{1}{x-1}\right) \frac{dx}{y}\right] = \frac{1}{\lambda} \left(1 - \frac{a+b+c}{n}\right) + \frac{1}{\lambda(1-\lambda)} \left(\lambda - \frac{a\lambda}{n} + \frac{b(1-\lambda)}{n}\right) \frac{dx}{y} + \frac{1}{\lambda(1-\lambda)} \left(\frac{a+b+c-2n}{n}\right) \frac{xdx}{y}\]

Combining this expression for \(D\left(\frac{dx}{y}\right)\) with the result of the preceding lemma, we find the desired formulae.

Let us denote by "" the action of \(\varphi\left(\frac{d}{dx}\right)\) on \(H_{B,R}(X)\). Then we have the following

**Lemma:** \(\lambda(1-\lambda)\omega_i' + \left[\frac{a+c}{n} - \left(\frac{a+b+2c}{n}\right)\lambda\right] \omega_1 - \left(\frac{a+b+c-n}{n}\right) c \omega_1 = 0\)

**Proof:** Using the previous lemma we find:

\[\omega_2' - \lambda \omega_i' = \left[\frac{n-a}{n(1-\lambda)} - \frac{n-(a+c)+c\lambda}{n(1-\lambda)}\right] \omega_1 = \frac{e}{n} \omega_1\]

\[\lambda \omega_i' + \frac{c}{n} \omega_1 = \left(\frac{n-a}{n(1-\lambda)}\right) \omega_1 + \left(\frac{a+b+c-2n}{n(1-\lambda)}\right) \omega_2\]

\[\frac{n\lambda \omega_1 + c\omega_1}{n} = \left(\frac{(n-a)\lambda}{n(1-\lambda)}\right) \omega_1 + \left(\frac{a+b+c-2n}{n(1-\lambda)}\right) \omega_2\]

\[\lambda - \lambda^2 (n\lambda \omega_1 + c\omega_1) = (n-a)\lambda \omega_1 + (a+b+c-2n)\lambda \omega_2\]

\[(n\lambda^2 - n\lambda^2) \omega_1 = [n-a(1-\lambda)] \omega_1 + (a+b+c-2n) \omega_2\]

\[(n\lambda - n\lambda^2) \omega_1 = [n-a-c(1-\lambda)] \omega_1 + (a+b+c-2n) \omega_2\]
Therefore \( (n-2n\lambda)\omega'_1+(n\lambda-\lambda^2)\omega'_1=e\omega_1+[n-a-c(1-\lambda)]\omega'_1+(a+b+c-2n)\omega'_2 \). But \( \omega'_2=\frac{c}{n}\omega_1+\lambda\omega'_1 \). Therefore we obtain:

\[
[n\lambda(1-\lambda)]\omega'_1+[n-2n\lambda-(n-a-c(1-\lambda))]\omega'_1-c\omega_1
-[(a+b+c-2n)(\lambda\omega'_1+\frac{c}{n}\omega_1)]=0
\]

and hence

\[
\lambda(1-\lambda)\omega'_1+\left[\frac{a+c}{n}-(\frac{a+b+2c}{n})\lambda\right]\omega'_1-\left(\frac{a+b+c-n}{n}\right)\frac{c}{n}\omega_1=0
\]

3) We now show that our mapping is injective.

If not, there exist \( a, \beta \in Q(\xi_n, \lambda) \) such that \( (ax+\beta)\frac{dx}{y} \) is exact. Then at \( p_0 \) \( \text{ord}(ax+\beta)\frac{dx}{y} \geq n-1-a \); at \( p_1 \) \( \text{ord}(ax+\beta)\frac{dx}{y} \geq n-1-b \); at \( p_2 \) \( \text{ord}(ax+\beta)\frac{dx}{y} \geq n-1-c \). But at \( p_\infty \) \( \text{ord}(ax+\beta)\frac{dx}{y}=a+b+c-n-1 \) if \( a=0 \) and \( \text{ord}(ax+\beta)\frac{dx}{y}=a+b+c-2n-1 \) if \( a \neq 0 \). Let \( g \) be a function such that \( dg=(ax+\beta)\frac{dx}{y} \). Because \( (ax+\beta)\frac{dx}{y} \) has a pole at \( p_\infty \) (as \( n>a+b+c \)), either \( \text{ord}_{p_\infty}(g)=a+b+c-n \) or \( \text{ord}_{p_\infty}(g)=a+b+c-2n \) depending on whether \( a=0 \) or \( a \neq 0 \).

Just as in part 1) above we have \( g \in Q(\xi_n, \lambda)[x, y, \frac{1}{y}] \) because \( (ax+\beta)\frac{dx}{y} \) is regular except at \( p_\infty \). Writing \( g=P_0(x)+\frac{P_1(x)}{y}+\ldots+\frac{P_{n-1}(x)}{y^{n-1}} \) with \( P_i(x) \in Q(\xi_n, \lambda, x) \) and using the projection \( \pi_x=\frac{1}{n} \sum \xi(x) \sigma \) on the relation \( dg=(ax+\beta)\frac{dx}{y} \) we find \( d\left(\frac{P_1(x)}{y}\right)=(ax+\beta)\frac{dx}{y} \). Thus we may assume \( g=\frac{P_1(x)}{y} \). As \( (ax+\beta)\frac{dx}{y} \) is regular except at \( p_\infty \), so is \( \frac{P_1(x)}{y} \), hence also \( P_1(x) \) and therefore \( P_1(x) \) is a polynomial.
Let $S$ be a principal open set of $\text{Spec } \mathbb{Z}[\zeta_n, \lambda, \frac{1}{n\lambda(1-\lambda)}]$ over which there is a proper, irreducible, smooth $S$-scheme $\tilde{X}$ with $\tilde{X} \times_S \text{Spec } Q(\zeta_n, \lambda) = X$. We assume that $S$ has been chosen sufficiently small so that $I_{-1} R. (1/S)$ is locally free and commutes with base change. Furthermore we assume the horizontal isomorphism $M_{e, \frac{a+b+c}{n}, \frac{a+c}{n}} \to H_{b, R.}(X)^{\chi}$ extends to $S$. Thus we can state:

**Theorem:** There is a non-empty open set $S$ of $\text{Spec } \mathbb{Z}[\zeta_n, \lambda, \frac{1}{n\lambda(1-\lambda)}]$ and a horizontal isomorphism $M_{e, \frac{a+b+c}{n}, \frac{a+c}{n}}| S \cong H_{b, R.}(\tilde{X}/S)^{\chi}$.

**Corollary:** For all but finitely many primes $p$, $M_{e, \frac{a+b+c}{n}, \frac{a+c}{n}} \otimes F_p$ is nilpotent.

**Proof:** If a prime ideal $(p)(\neq 0)$ of $\mathbb{Z}$ belongs to the image of $S$, then $M_{e, \frac{a+b+c}{n}, \frac{a+c}{n}}| S \otimes F_p$ is a sub-module of $H_{b, R.}(\tilde{X} \otimes F_p)/S \otimes F_p$.

By the theorem of Katz and Berthelot: the Gauss-Manin connection (in characteristic $p$) is nilpotent, we see that $M| S \otimes F_p$ is nilpotent.

This implies $M_{e, \frac{a+b+c}{n}, \frac{a+c}{n}} \otimes F_p$ is nilpotent.

**The Theorem**

Let us return momentarily to the general situation of the introduction; $T$ arbitrary, $S$ a smooth $T$-scheme, $\mathcal{M}$ a quasi-coherent $S$-module with a $T$-connection $\mathcal{F}$, ... We note the following elementary facts:
1) If $(\mathcal{M}, \mathcal{V}_\mathcal{M})$ and $(\mathcal{N}, \mathcal{V}_\mathcal{N})$ are two $S$-modules with connection, $D, m, n$ are sections of $\mathcal{D}_n\mathcal{O}_T(\mathcal{O}_S, \mathcal{O}_S)$, $\mathcal{M}, \mathcal{N}$ over an open subset $U \subseteq S$ and $l$ is a strictly positive integer, then we have the Leibniz rule:
\[
(\mathcal{V}_\mathcal{M}(\mathcal{N})(m \otimes n)) = \sum_{i=0}^l (\mathcal{V}_\mathcal{M}(D))^i(m) \otimes \mathcal{V}_\mathcal{N}(D)^{l-i}(n) \tag{proved as usual by induction on $l$}
\]

2) Suppose $\mathcal{M}$ free of a fixed finite rank $n$, with base $\{e_1, \ldots, e_n\}$. If $D$ is a section of $\mathcal{D}_n\mathcal{O}_T(\mathcal{O}_S, \mathcal{O}_S)$ and if $\mathcal{V}(D)\mathcal{M} = \mathcal{M}$, then we have the Leibniz rule:
\[
\mathcal{V}(D)^i(m \otimes n) = \sum_{i=0}^l (\mathcal{V}_\mathcal{M}(D))^i(m) \otimes \mathcal{V}_\mathcal{N}(D)^{l-i}(n)
\]

3) $\psi_{\mathcal{M} \otimes \mathcal{N}}(D) = \psi_{\mathcal{M}}(D) \otimes \text{id}_{\mathcal{N}} + \text{id}_{\mathcal{M}} \otimes \psi_{\mathcal{N}}(D)$

4) If $\phi: \mathcal{M} \to \mathcal{N}$ is a horizontal morphism, $\psi_{\mathcal{N}}(D) \circ \phi = \phi \circ \psi_{\mathcal{M}}(D)$

5) Suppose $\mathcal{M}$ free of finite rank. A necessary and sufficient condition that $(\mathcal{M}, \mathcal{V})$ be nilpotent is that for every section $D$ of $\mathcal{D}_n\mathcal{O}_T(\mathcal{O}_S, \mathcal{O}_S)$, every coefficient except the leading one of the characteristic polynomial of $\mathcal{V}(D)$ is nilpotent in $\mathcal{O}_S$.

6) If $\mathcal{M}$ is free of finite rank, then $\psi_{\mathcal{M} \otimes \mathcal{N}}(D) = \text{tr}(\psi_{\mathcal{M}}(D))$

Having completed these preliminaries we turn to the main result.

To fix notation again let $a, b, c \in \mathcal{Q}$, $n$ be a common denominator and $S = \text{Spec } \mathcal{Z} \left[ \lambda, \frac{1}{n\lambda(1-\lambda)} \right]$. Let $\mathcal{M}_{a, b, c}$ be the hypergeometric $S$-module defined in the introduction.

**Theorem:** $\mathcal{M}_{a, b, c}$ is globally nilpotent.

**Proof:** Fix once and for all a prime $p$ which does not become invertible in $\mathcal{Z} \left[ \lambda, \frac{1}{n\lambda(1-\lambda)} \right]$. Consider the $\mathcal{F}_p \left[ \lambda, \frac{1}{\lambda(1-\lambda)} \right]$-module (with connection) $\mathcal{M}_{a, b, c} \otimes_{\mathcal{Z}} \mathcal{F}_p$. We must show that it is nilpotent. By
statement 5) above this is equivalent to showing that the determinant and trace of \( \psi \left( \frac{d}{d\lambda} \right) \) are zero. It suffices to show this at the generic point of \( \text{Spec} \ F_p \left[ \lambda, \frac{1}{\lambda(1-\lambda)} \right] \) and therefore we shall work with the module \( M = M_{a,b,c} \otimes F_p(\lambda). \narrative

First we shall deal with the determinant.

Denoting by \( \bar{M} \) the dual module, it is immediately checked that the mapping \( \phi \mapsto \langle \phi, e_i \rangle \) establishes an \( F_p(\lambda^p) \)-linear isomorphism between the horizontal elements of \( \bar{M} \) and the solutions in \( F_p(\lambda) \) of the differential equation:

\[
(*) \quad \lambda(1-\lambda)u'' + [e-(a+b+1)\lambda]u' - abu = 0.
\]

Suppose for the moment that there is a non-zero solution in \( F_p(\lambda) \) of this equation, i.e., that \( \bar{M} \) possesses a non-zero horizontal section. Then \( \psi_{\bar{M}} \left( \frac{d}{d\lambda} \right) \) has determinant \( = 0 \). Applying 3) and 4) above to the canonical horizontal morphism \( \bar{M} \otimes M \to F_p(\lambda) \) we see that \(-\psi_M \left( \frac{d}{d\lambda} \right)\) is the transpose of \( \psi_M \left( \frac{d}{d\lambda} \right) \) and hence that \( \det \left( \psi_M \left( \frac{d}{d\lambda} \right) \right) = 0 \).

In order to find a non-zero solution of \( (*) \) we may assume that \( a, b, c \in \mathbb{Z} \), \(-p-1 \leq a \leq 0\); \( e \leq a \); \( b, c \neq 0 \) (in \( \mathbb{Z} \)). As is “well-known” [1], the differential equation

\[
\lambda(1-\lambda)u'' + [e-(a+b+1)\lambda]u' - abu = 0 \quad \text{over} \quad \mathbb{Z} \left[ \lambda, \frac{1}{\lambda(1-\lambda)} \right]
\]

has a non-zero solution in \( \mathbb{Q}[\lambda] \), namely

\[
F(a, b, c; \lambda) = \sum_{r=0}^{\lambda - a} \frac{(a)_r (b)_r}{(c)_r r!} \lambda^r \quad \text{where} \quad \left\{ \begin{array}{l}
(\theta)_0 = 1 \\
(\theta)_r = (\theta+1) \cdots (\theta+r-1) \quad \text{for} \quad r \neq 0.
\end{array} \right.
\]

By multiplying \( F(a, b, c; \lambda) \) by the least common multiple of the denominators of its coefficients we obtain a primitive polynomial in \( \mathbb{Z}[\lambda] \) which is still a solution of this differential equation. The reduction mod \( p \) of this polynomial is the desired polynomial solution of \( (*) \).
This completes the proof that $\det(\psi\left(\frac{d}{d\lambda}\right))=0$.

In order to show that $\text{tr}(\psi\left(\frac{d}{d\lambda}\right))=0$ we use statement 6) above, $\text{tr}(\psi\left(\frac{d}{d\lambda}\right)) = \psi_{\text{det}(M)}\left(\frac{d}{d\lambda}\right)$. We observe that $\psi_{\text{det}(M)}\left(\frac{d}{d\lambda}\right)=0$ if and only if $\det(M)$ has a non-trivial horizontal section. By 2) above $\psi_{\text{det}(M)}\left(\frac{d}{d\lambda}\right)=\frac{d}{d\lambda}+\left(\frac{a+b+1)\lambda-c}{\lambda(1-\lambda)}\right)$. Thus it suffices to find $g \in F_p(\lambda)$, $g \neq 0$ such that $\frac{dg}{d\lambda}+\left(\frac{(a+b+1)\lambda-c}{\lambda(1-\lambda)}\right)g=0$. But $g=\lambda e(1-\lambda)^{a+b+1-c}$ is a nonzero solution of the equation, whence $\text{tr}(\psi\left(\frac{d}{d\lambda}\right))=0$; which completes the proof of the theorem.

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References


