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Residual intersections in Cohen-Macauley rings

By

M. ARTIN and M. NAGATA*

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We investigate questions of the following kind: Let I be an ideal of a Cohen-Macauley local ring $(R, \mathfrak{m})$. Suppose $I=\mathfrak{a}\cap\mathfrak{b}$, where $\mathfrak{b}$ has height $\geq r$. Find conditions on I and $\mathfrak{a}$ which will imply properties of $\mathfrak{b}$. We prove among other things (2.2) that if the number of generators of I is at most r, and if $\mathfrak{a}$ is a complete intersection, then $\mathfrak{b}$ can be chosen so that $R/\mathfrak{b}$ is Cohen-Macauley. (On the other hand, if $I=\mathfrak{a}_1\cap\mathfrak{a}_2\cap\mathfrak{b}$, where $\mathfrak{a}_1$, $\mathfrak{a}_2$ are complete intersections, depth $R/\mathfrak{b}$ may be too small (cf. (2.5)). This generalizes Macauley's unmixedness theorem, which may be considered the degenerate case $\mathfrak{a}=R$.

Our method, which is quite elementary, was originally developed to answer some questions raised by Mumford about the critical locus of a map of smooth schemes. These applications are in section 5. Then, in analyzing our proof, we found that it was related to a result of Dubreil [1]. We have arranged the presentation around Dubreil's theorem.

Notation and terminology. By a local ring, we mean a noetherian local ring. Dimension (in symbol, dim) for a ring will mean the Krull dimension of the ring. Depth means, not as in [7], the length of a maximal regular sequence. We note here that if M is a finite module

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over a local ring R and if homological dimension of M (in symbol, $\mathrm{hd}_R\, M$) is finite, then $\mathrm{hd}_R\, M+\mathrm{depth}\, M=\mathrm{depth}\, R$.

1. Dubreil's theorem.

We work throughout with Cohen-Macauley local rings R, say of dimension n. Deductions in the non-local case will be clear.

In a generalized form, Dubreil's theorem [1] is the following:

Theorem (1.1): *Consider an ideal* $I=(a_1, \dots, a_r)$ *generated by* r *elements, of the form* $\mathfrak{a}\cap\mathfrak{b}$ *where* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are pure height* r *and have no associated primes in common. Under these circumstances the* depth $R/\mathfrak{b}$ *and homological dimension* $\mathrm{hd}_R\, R/\mathfrak{b}$ *depend only on* $\mathfrak{a}$.

In other words, if $I'=(a'_1, \dots, a'_r)=\mathfrak{a}\cap\mathfrak{b}'$ is another such ideal, then $\mathrm{depth}\,(R/\mathfrak{b}')=\mathrm{depth}\,(R/\mathfrak{b})$. There is a degenerate case to consider, namely that $\mathfrak{b}=R$. In this case the proof shows that for any $\mathfrak{b}'\neq R$ as above, $\mathrm{depth}\,(R/\mathfrak{b}')=n-r$.

Corollary (1.2): *With the notation of theorem* (1.1), *suppose* $\mathfrak{a}$ *is a complete intersection and that* $\mathfrak{b}\neq R$. *Then* $R/\mathfrak{b}$ *is Cohen-Macauley.*

Following Dubreil, we may introduce the notion of a *chain* $\mathfrak{a}_1$, $\mathfrak{a}_2, \dots$ of pure codimension r, proper ideals: In such a chain, $\mathfrak{a}_i\cap\mathfrak{a}_{i+1}$ is a complete intersection, and $\mathfrak{a}_i$, $\mathfrak{a}_{i+1}$ have no common associated prime. We obtain

Corollary (1.3): *Let* $\mathfrak{a}_1, \mathfrak{a}_2, \dots$ *be such a chain. Then* $\mathrm{depth}\,(R/\mathfrak{a}_1)=\mathrm{depth}\,(R/\mathfrak{a}_s)$ *for odd* s. *If* $\mathfrak{a}_1$ *is a complete intersection, then* $R/\mathfrak{a}_s$ *is Cohen-Macauley for every* s.

These assertions are all due to Dubreil [1] in the case of homogeneous ideals of nonsingular varieties.

Let $\mathfrak{a}$, $\mathfrak{b}$ be as in theorem (1.1). We will (tentatively) call the ideal

$\mathfrak{a}$ *relatively Cohen-Macauley* if $R/\mathfrak{b}$ is Cohen-Macauley. Thus a complete intersection is relatively Cohen-Macauley.

Before proving these results, we recall a well known result on homological dimension and depth.

Lemma (1.4): *Let T be a local ring and let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of finite T-modules. Then:

(1) *One of the following* (i)$\sim$(iii) *must hold*:

(i) $\mathrm{hd}_T A \leq \mathrm{hd}_T B = \mathrm{hd}_T C$,

(ii) $\mathrm{hd}_T B \leq \mathrm{hd}_T A = \mathrm{hd}_T C - 1$,

(iii) $\mathrm{hd}_T C < \mathrm{hd}_T A = \mathrm{hd}_T B$.

(2) *One of the following* (i*)$\sim$(iii*) *must hold*:

(i*) depth $A \geq$ depth $B =$ depth C,

(ii*) depth $B \geq$ depth $A =$ depth $C+1$,

(iii*) depth $C >$ depth $A =$ depth B.

(Actually, (1) is very well known. (2) follows from (1) by the following argument: First, we may reduce to the case where T is complete. Then T is a homomorphic image of a regular local ring T^*. Then we have (1) with T^* in place of T. Since T^* is regular, $\mathrm{hd}_{T^*} A$ is finite and depth $A = \dim T^* - \mathrm{hd}_{T^*} A$, etc., and we have (2).)

Proof of (1.1) *and* (1.2): We fix the ideal $\mathfrak{a}$. Note that an ideal $I = (a_1, \ldots, a_r)$ will be of the form $\mathfrak{a} \cap \mathfrak{b}$ as above (possibly with $\mathfrak{b} = R$) if and only if I has height r and the elements $a_1, \ldots, a_r$ form a system of generators for $\mathfrak{a}$ locally at every associated prime $\mathfrak{p}$ of $\mathfrak{a}$. It follows immediately that the "exchange lemma" holds for our situation, i.e., that if $I = (a_1, \ldots, a_r) = \mathfrak{a} \cap \mathfrak{b}$ and $I' = (a_1', \ldots, a_r') = \mathfrak{a} \cap \mathfrak{b}'$ satisfy the hypotheses, then so does $I'' = (a_1, \ldots, a_{r-1}, a_r')$, provided that the generating set $(a_1', \ldots, a_r')$ is adjusted appropriately. On the other hand, if the generating sets of the two ideals I, I' have an element in common, say $a_r=a'_r$ then we can replace R by $R/(a_r)$ and proceed by induction on r to show depth $(R/\mathfrak{b})=$ depth $(R/\mathfrak{b}')$. Thus we are reduced to the extreme cases $r=0$, 1. The case $r=0$, i.e., $I=(0)$, is trivial since then $\mathfrak{b}$ is uniquely determined by $\mathfrak{a}$.

Consider the case $r=1$. We have $I=(a)=\mathfrak{a}\cap\mathfrak{b}$ and $I'=(a')=\mathfrak{a}\cap\mathfrak{b}'$, and we may suppose that $\mathfrak{b}$ and $\mathfrak{b}'$ have no associated prime in common. If they do, we relate the ideals to a third one, $I''=(a'')=\mathfrak{a}\cap\mathfrak{b}''$, where, say $a''\equiv a \pmod{\mathfrak{a}}$ but a'' is not in any associated prime of (0), $\mathfrak{b}$, or $\mathfrak{b}'$.

Lemma (1.5): *Let $I=\mathfrak{a}\cap\mathfrak{b}$ be an ideal, and let $\alpha\in\mathfrak{a}$ be an element which is not in any associated prime of $\mathfrak{b}$. Then*

$$R/\mathfrak{b}\rightleftarrows(\alpha)/I\cap(\alpha).$$

Proof: The isomorphism sends 1 to α. Thus we need to show that $x\alpha\in I$ implies $x\in\mathfrak{b}$ (the converse being clear). Let $\mathfrak{b}=\mathfrak{q}_1\cap\ldots\cap\mathfrak{q}_e$ be a primary decomposition. Then $x\alpha\in\mathfrak{q}_i$, but $\alpha\notin\operatorname{rad}\mathfrak{q}_i$. Thus $x\in\mathfrak{q}_i$, hence $x\in\mathfrak{b}$, as required.

Now consider the exact sequence

$$0\longrightarrow I\longrightarrow I+I'\longrightarrow I/(I\cap I')\longrightarrow 0.$$

We have $I+I'=I+(a')$, and a' is not in any associated prime of $\mathfrak{b}$, by assumption. Thus $I'/(I\cap I')=R/\mathfrak{b}$ by the lemma, and so the sequence is isomorphic to a sequence

$$0\longrightarrow R\longrightarrow(I+I')\longrightarrow R/\mathfrak{b}\longrightarrow 0.$$

Since depth $R=n\geq$ depth M for any finite R-module M, applying Lemma 1.4, we have depth $(R/\mathfrak{b})\leq$ depth $(I+I')$, with equality if depth $(I+I')<n$. The same is true when $\mathfrak{b}'$ replaces $\mathfrak{b}$. Thus depth $(R/\mathfrak{b})$ = depth (R/b') if depth $(I+I')<n$. Suppose depth $(I+I')=n$. Then depth $(R/\mathfrak{b})\geq n-1$. If $\mathfrak{b}\neq R$, dim $(R/\mathfrak{b})=n-1$, hence depth $(R/\mathfrak{b})=n-1$, which handles this case and completes

the proof of the assersion on depth; The assertion on homological dimension is given similarly.

Proposition (1.6): *Let* $\mathfrak{a}$ *be an ideal of height* r *having* $(r+1)$ *generators. Suppose* $R/\mathfrak{a}$ *is Cohen-Macauley and that* $\mathfrak{a}$ *is generated by* r *elements locally at every associated prime* $\mathfrak{p}$. *Then* $\mathfrak{a}$ *is relatively Cohen-Macauley.*

Proof: Since $\mathfrak{a}$ is generated by r elements locally at every associated prime $\mathfrak{p}$, we may choose a generating set $(a_1, \dots, a_{r+1})$ in such a way that $I=(a_1, \dots, a_r)=\mathfrak{a}\cap\mathfrak{b}$ as in (1.1). Then a_{r+1} cannot lie in any associated prime of $\mathfrak{b}$. Thus lemma (1.5) gives us an exact sequence

$$0 \longrightarrow I \longrightarrow \mathfrak{a} \longrightarrow R/\mathfrak{b} \longrightarrow 0.$$

Since I is a complete intersection, both R/I and $R/\mathfrak{a}$ are Cohen-Macauley. Therefore the exact sequence $0\to I\to R\to R/I\to 0$ shows that depth $I=n-r+1$, and, similarly, depth $\mathfrak{a}=n-r+1$. By the above sequence, depth $(R/\mathfrak{b})\geq n-r$, as required.

Example (1.7): Rational triple points of surfaces are defined by 3 equations in 4-space ([9]), and hence are relatively Cohen-Macauley.

2. Residual intersections of lower dimension.

In this section we need a condition asserting that the number of generators of an ideal is not too large, for which we introduce the following terminology. Let $\mathfrak{a}$ be an ideal. The set $C_\nu=C_\nu(\mathfrak{a})\subset$ Spec R of points p such that $\mathfrak{a}$ requires at least r generators locally at p (i.e., that $\mathfrak{a}\otimes k(p)$ is of rank $\geq r$) is a closed set. Define

$$c_1(\mathfrak{a}) = \operatorname{codim}\,(V(\mathfrak{a})\cap C_1(\mathfrak{a}))$$

$$c_i(\mathfrak{a}) = \operatorname{codim}\, C_i(\mathfrak{a}), \text{ if } i>1.$$

Here, we define the codimension of the empty set to be ∞.

Definition: We say that an ideal $\mathfrak{a}$ satisfies G_s if $c_i = c_i(\mathfrak{a}) \geq i$ for $i=1, \ldots, s$.

For example, an ideal $\mathfrak{a}$ of height r satisfies G_r, since $c_i = r$ for $i=1, \ldots, r-1, r$. A complete intersection $\mathfrak{a}$ of height r satisfies G_∞. The values of c_i are $c_1 = \ldots = c_r = r$, and $c_{r+1} = c_{r+2} = \ldots = \infty$. So does an ideal generated by $(r+1)$ elements, as in (1.6).

Theorem (2.1): *Let $\mathfrak{a}$ be a relatively Cohen-Macauley ideal of height r, let s be an integer $>r$, and let $I=(a_1, \ldots, a_s)$ be an ideal satisfying either of the equivalent conditions*

(i) $I=\mathfrak{a} \cap \mathfrak{b}$, *where* height $\mathfrak{b} \geq s$.

(ii) $I=\mathfrak{a}$ *at every point of* Spec R *of height* $<s$.

If $\mathfrak{a}$ satisfies G_{s-1}, then

(a) depth $(R/I) \geq n-s$, *and*

(b) *I can be written in the form $I=\mathfrak{a} \cap \mathfrak{b}^*$, where* height $\mathfrak{b}^* \geq s$, *no primary component of $\mathfrak{b}^*$ contains $\mathfrak{a}$ and $R/\mathfrak{b}^*$ is Cohen-Macauley.*

Note that if height $(\mathfrak{a}+\mathfrak{b})>s$, then the ideal $\mathfrak{b}^*$ for (b) is uniquely determined and $\mathfrak{b}=\mathfrak{b}^*$.

Corollary (2.2): *The conclusions* (a), (b) *are true if $\mathfrak{a}$ is a complete intersection, or is the ideal of* (1.6).

The equivalence of (i) and (ii) is clear. We prove the theorem by induction on s, and to start it we add the trivial case $s=r$, being careful to strengthen (ii) to include the condition $I \subset \mathfrak{a}$. The conclusion (a) in this case is just Macauley's theorem ([6]), and (b) follows because $\mathfrak{a}$ is a relatively Cohen-Macauley ideal.

Lemma (2.3): *Suppose $s>r$. The generators $(a_1, \ldots, a_s)$ of I may be chosen so that $I'=(a_1, \ldots, a_{s-1})$ is of the form $I'=\mathfrak{a} \cap \mathfrak{b}'$,* *where* height $\mathfrak{b}' \geq s-1$ *and* height $(\mathfrak{a}+\mathfrak{b}') \geq s$.

Assume the lemma. We may apply the induction hypothesis to the ideal I'. Thus depth $(R/\mathfrak{b}') \geq n-s+1$ if $\mathfrak{b}'$ is chosen appropriately, and in particular no associated prime of $\mathfrak{b}'$ has height $>s-1$. Thus no associated prime of $\mathfrak{b}'$ contains one of $\mathfrak{a}$. Since height $\mathfrak{b} \geq s$, the element a_s is $R/\mathfrak{b}'$-regular. Put $\mathfrak{b}'' = \mathfrak{b}'+(a_s)$. Then height $\mathfrak{b}'' \geq s$ and depth $(R/\mathfrak{b}'') \geq n-s$. On the other hand, since $a_s \in \mathfrak{a}$ we have

$$I = I'+(a_s) = (\mathfrak{a} \cap \mathfrak{b}')+(a_s) = \mathfrak{a} \cap \mathfrak{b}''.$$

(If $x \in \mathfrak{a} \cap \mathfrak{b}''$, then $x = b'+ya_s$ with $b' \in \mathfrak{b}'$, and $x \in \mathfrak{a}$. Thus $b' \in \mathfrak{a}$, whence $x \in (\mathfrak{a} \cap \mathfrak{b}')+(a_s)$. Conversely, if $x = b'+ya_s$ with $b' \in \mathfrak{a} \cap \mathfrak{b}'$, then $x \in \mathfrak{a}$, hence $x \in \mathfrak{a} \cap \mathfrak{b}''$.) This proves assertion (b).

Assertion (a) follows from the exact sequence

$$(2.4) \qquad 0 \longrightarrow I' \longrightarrow I \longrightarrow (a_s)/I' \cap (a_s) \longrightarrow 0.$$

Since $a_s \in \mathfrak{a}$ but is not in any associated prime of $\mathfrak{b}$, the term on the right in (2.4) is isomorphic to $R/\mathfrak{b}'$ by (1.5). The exact sequence $0 \to I' \to R \to R/I' \to 0$ shows that depth $I' \geq n-s+2$ if $s>1$. Therefore depth $I \geq n-s+1$ by (2.4). If $s=1$ we have $I'=(0)$, hence depth $I \geq n-s+1$ in this case as well. Therefore depth $(R/I) \geq n-s$, as required.

It remains to prove the lemma, which is done by a general position argument. Since $\mathfrak{a}$ satisfies G_{s-1}, so does I, because of (ii). Put $M^\nu = I/(a_1, \dots, a_\nu)I$ $(\nu=0, \dots, s-1)$, and denote by C_i^ν the locus of points of Spec R at which M^ν requires at least i generators. We want to choose $a_1, \dots, a_s$ so that

(a) codim $C_i^\nu \geq \nu+i$ for $\nu+i<s$ and $i>1$,

(b) codim $(C_1^\nu \cap V(I)) \geq \nu+1$ for $\nu<s$,

(c) codim $C_1^\nu \geq \nu$ for $\nu<s$.

These conditions amount to G_{s-1}, if $\nu=0$. Suppose generators chosen so that they hold for all $\nu'<\nu(\leq s-1)$. Then we can adjust a_ν, so that its residue in $M^{\nu-1}$ is nonzero on every component of $C_i^{\nu-1}$ ($i \geq 1$ and $i+\nu-1 < s$) and on every component of $(C_1^{\nu-1} \cap V(I))$. It follows that no component of $C_i^{\nu-1}$ is a component of C_i^ν, hence that

$$\operatorname{codim} C_i^\nu \geq \min((\operatorname{codim} C_i^{\nu-1}+1), \operatorname{codim} C_{i+1}^{\nu-1}, s) \geq \nu+i,$$
$$\text{if } i>1 \text{ and } \nu+i<s.$$

Thus (a) holds for ν. Similarly,

$$\operatorname{codim}(C_1^\nu \cap V(I))$$
$$\geq \min((\operatorname{codim}(C_1^{\nu-1} \cap V(I))+1), \operatorname{codim} C_2^{\nu-1}, s) \geq \nu+1,$$

and $\operatorname{codim} C_1^\nu \geq \nu$. Setting $\nu=s-1$, we obtain $\operatorname{codim} C_1^{s-1} \geq s-1$ and $\operatorname{codim}(C_1^{s-1} \cap V(I)) \geq s$. Since $I=\mathfrak{a}$ in codimension $<s$, this proves the lemma.

Examples (2.5): Let $\mathfrak{a}$ be the ideal of a rational triple point at the origin in 4-space (cf. (1.7)). Then three functions a_1, a_2, a_3 vanishing on $\mathfrak{a}$ will, in general, cut out the surface together with a 1-dimensional residual intersection. When this is so, the ideal $I=(a_1, a_2, a_3)$ cannot have the origin as associated prime. On the other hand, if $\mathfrak{a}$ were not a relatively Cohen-Macauley ideal, then I would always have the origin as associated prime. An example of such an ideal is $\bar{\mathfrak{a}}=\mathfrak{p}_{12} \cap \mathfrak{p}_{34}$, where $\mathfrak{p}_{ij}$ is the plane $x_i=x_j=0$. The elements x_1x_3, x_2x_4, $x_1x_4+x_2x_3$ generate an ideal of the form $\mathfrak{a} \cap \mathfrak{b}$, where $\mathfrak{b}$ is primary to the origin (x_1, x_2, x_3, x_4).

Remarks (2.6): (a) The existence of an ideal I as in theorem (2.1) is equivalent with the assertion that $c_{s+1} \geq s$ $(s>r)$. Thus a natural condition on $\mathfrak{a}$ would be G_s^-: $c_i \geq i-1$ for $i=2, \dots, s$, i.e., that ideals as in the theorem exist for every s' such that $r<s' \leq s$. We do not know what is true under this weaker hypothesis.

(b) Even if $\mathfrak{a}$ is not relatively Cohen-Macauley, one can still draw some conclusions about depth $(R/\mathfrak{b})$ if the condition on the number of generators of $\mathfrak{a}$ is strengthened suitably.

3. A criterion for unmixedness.

We can rule out embedded components for certain ideals by strengthening the conditions on the number of generators. With the notations of the previous section, we will say that an ideal $\mathfrak{a}$ satisfies G_s^+ if $s \geq r = \text{height}\,\mathfrak{a}$ and

(i) $c_r = r$

(ii) $c_i \geq i+1$ for $i = r+1, \ldots, s$.

Clearly, an ideal $\mathfrak{a}$ satisfying G_∞^+ can have no isolated component of codimension $> r$.

Corollary (3.1): *Let $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_e$ be a primary decomposition, and let $\bar{\mathfrak{a}}$ be the intersection of those $\mathfrak{q}_i$ of minimal height r. Assume that $\bar{\mathfrak{a}}$ is relatively Cohen-Macauley, and that $\mathfrak{a}$ satisfies G_s^+. Then every associated prime $\mathfrak{p}$ of $\mathfrak{a}$ of height $> r$ has height $> s$. In particular, if $\mathfrak{a}$ satisfies G_∞^+ then $\mathfrak{a}$ is unmixed (i.e., $\mathfrak{a} = \bar{\mathfrak{a}}$).*

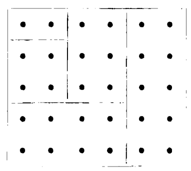
Proof: Assume the contrary, so that there is an associated prime $\mathfrak{p}$ of $\mathfrak{a}$, say of codimension s. Since G_s^+ holds, we may choose elements $a_1, \ldots, a_{s-1} \in \mathfrak{a}$ which generate $\mathfrak{a}$ locally at $\mathfrak{p}$, and we may moreover suppose that $I = (a_1, \ldots, a_{s-1}) = \bar{\mathfrak{a}} \cap \mathfrak{b}$, where codim $\mathfrak{b} \geq s-1$. Thus (2.1) implies depth $(R/I) \geq n-s+1$. Since $\mathfrak{p}$ is an associated prime of $\mathfrak{a}$ and $I = \mathfrak{a}$ locally at $\mathfrak{p}$, it follows that $\mathfrak{p}$ is an associated prime of I. But then depth $(R/I) \leq n - \text{height}\,\mathfrak{p} = n-s$, a contradiction.

Examples (3.2): (a) The ideal $\mathfrak{a} = (x_1x_3, x_2x_4, x_1x_4 + x_2x_3)$ of (2.5) satisfies G_∞^+ but $\bar{\mathfrak{a}}$ is not relatively Cohen-Macauley and $\mathfrak{a}$ has an embedded component. The ideal $\mathfrak{a}' = (x^2, xy) \subset R = k[[x, y]]$ satisfies G_∞, $\bar{\mathfrak{a}}'$ is relatively Cohen-Macauley, and $\mathfrak{a}'$ has an embedded component.

(b) These results seem too weak to apply directly to general determinantal varieties (cf. [3, 5]), but certain cases can be treated. Consider for instance the ideal $\mathfrak{a}_k$ defined by the minors of order k of the generic $k\times(k+1)$ matrix

$$\begin{vmatrix} x_{11} & \cdots\cdots & x_{1k+1} \\ \vdots & & \vdots \\ x_{k1} & \cdots\cdots & x_{kk+1} \end{vmatrix}$$

over a field K. Let I be the ideal of $K[\{x_{ij}\}]=R$ generated by the two minors containing the first $(k-1)$ columns. Then it is easily seen that $I=\mathfrak{a}_k\cap\mathfrak{a}_{k-1}$, where $\mathfrak{a}_{k-1}$ is the ideal generated by the $(k-1)$-rowed minors of the $k\times(k-1)$ matrix (x_{ij}) $(i=1,\dots,k;\ \ j=1,\dots,k-1)$. Proceed inductively to define $\mathfrak{a}_i$ by the figure



where $\mathfrak{a}_i$ is generated by the maximal minors of the boxed i by $(i+1)$ matrix containing the upper left entry x_{11}. Then $\mathfrak{a}_1,\dots,\mathfrak{a}_k$ is a chain in the sense of Dubreil (cf. § 1), and $\mathfrak{a}_1$ is a complete intersection. Therefore ((1.1) and (1.3)) the associated ideal $\bar{\mathfrak{a}}_i$ is relatively Cohen-Macauley for each i. Now suppose that there are m independent columns in a particular matrix. Then $\mathfrak{a}_k$ is generated locally at the corresponding prime of R by the $k-m+1$ minors containing these columns. Since the locus of matrices whose rank is m or less has codimension $(k-m)(k-m+1)$, we have $c_i(\mathfrak{a}_k)=i(i-1)$ if $i=2,\dots,n+1$. Hence G^+_∞ holds for $\mathfrak{a}_k$ (and similarly for all $\mathfrak{a}_i$), and so

Corollary (3.3): *The determinantal ideal $\mathfrak{a}_k$ is relatively Cohen-Macauley, and $R/\mathfrak{a}_k$ is Cohen-Macauley.*

4. Generalization to mixed ideals.

The previous results have natural generalizations to mixed ideals

$\mathfrak{a}$. We will state these without proof. The proofs are routine extensions of the above arguments.

Let $\mathfrak{a}$ be an ideal of R.

Definition (4.1): An $(R, \mathfrak{a})$-*sequence* $(a_1, \ldots, a_r)$ is a sequence of elements of $\mathfrak{a}$ such that for every $i \leq r$ the ideal $(a_1, \ldots, a_i)$ is equal to $\mathfrak{a}$ locally at every point of codimension $< i$ i.e., that $I_i = (a_1, \ldots, a_i) = \mathfrak{a} \cap \mathfrak{b}_i$, where codim $\mathfrak{b}_i \geq i$. An $(R, \mathfrak{a})$-sequence is called *good* if depth $R/I_i \geq n-1$ for $i=1, \ldots, r$.

Thus the existence of an $(R, \mathfrak{a})$-sequence $(a_1, \ldots, a_r)$ is equivalent with the condition G_r^- for $\mathfrak{a}$ (cf. (2.6)).

Theorem (4.2): *Suppose* $\mathfrak{a}$ *satisfies* G_{r-1}. *Then if one* $(R, \mathfrak{a})$ *sequence* $(a_1, \ldots, a_r)$ *is good, so is any other.*

We will call such an ideal *r-good*. Thus $\mathfrak{a}$ is ∞-*good* if and only if it satisfies G_∞ and is "relatively Cohen-Macauley".

Theorem (4.3) : *Suppose that* $\mathfrak{a}$ *is* $(r-1)$*-good. Then* depth (R/I) *is constant for all ideals* I *of the form* $I = (a_1, \ldots, a_r) = \mathfrak{a} \cap \mathfrak{b}$ *with* codim $\mathfrak{b} \geq r$.

Theorem (4.4): *Suppose that* $\mathfrak{a}$ *satisfies the following conditions for* $s \geq r$:

(i) G_{s-1}

(ii) *Every associated prime* $\mathfrak{p}$ *of* $\mathfrak{a}$ *has* codim $\mathfrak{p} \leq r$ *or* codim $\mathfrak{p} \geq s$

(iii) $\mathfrak{a}$ *is r-good.*

Then $\mathfrak{a}$ *is s-good.*

Theorem (4.5): *Suppose that* $\mathfrak{a}$ *satisfies*

(i) G_r^+

(ii) $c_i \geq i+1$ *for* $i = r+1, \ldots, s$ *(notation of* §2).

(iii) $\mathfrak{a}$ *is r-good.*

Then every associated prime $\mathfrak{p}$ *of* $\mathfrak{a}$ *has* height $\mathfrak{p} \leq r$ *or* height $\mathfrak{p} > s$.

5. Applications to critical points of maps.

In this section, we work with maps $f: X \rightarrow Y$ of schemes of finite type over a field K. There are analogous assertions for maps of analytic or algebroid spaces, with similar proofs.

We will call f an *immersion* at $x \in X$ if the fibred product $X \underset{Y}{\times} X$ is isomorphic to the diagonal Δ_X, locally at (x, x) (i.e., if f is unramified at x, in Grothendieck's terminology ([4])). Assume that f is a generic immersion, i.e., that it is an immersion on a dense open set of X. Then we can write the product as a scheme-theoretic union $X \underset{Y}{\times} X = \Delta \cup D$ for some closed subscheme D of $X \underset{Y}{\times} X$, whose associated components, at least, are uniquely determined. The scheme D may be viewed intuitively as the "double locus" of f. Set-theoretically, $D-(D \cap \Delta)$ is the set of pairs of distinct points (x_1, x_2) of X with $f(x_1) = f(x_2)$.

Theorem (5.1): *With the above notation, assume Y smooth of dimension k, and X a local complete intersection of dimension n. Then every isolated component of D has dimension at least $2n-k$.*

Proof: If the component D_0 in question is not embedded in the diagonal Δ, the bound on its dimension follows from standard dimension theory using calculations of the type below. It is the embedded case which requires the results of §1, §2. We choose a point $x \in X$ so that (x, x) is on D_0 but not on any component of higher dimension. Our problem is local at x, so we may assume that X, Y are complete intersections in affine spaces A^m, A^N with coordinates $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_N)$ respectively. We can, moreover, suppose that $y_1, \ldots, y_k$ is a regular system of parameters everywhere in Y (i.e., that the projection $Y \rightarrow A^k$ is etale). Then the map f is determined by equations $f_i(x_1, \ldots, x_m) = y_i$ $(i=1, \ldots, N)$, and the fibred product $X \underset{Y}{\times} X$ is the closed subscheme $\{(x, x, f(x))\}$ of $A^m \times A^m \times A^N = W$ defined by

$$(5.2)\quad \begin{cases} \text{(i)} & \text{the } N-k \text{ equations defining } Y \text{ in } A^N \\ \text{(ii)} & \text{the } 2(m-n) \text{ equations defining } X\times X \text{ in } A^m\times A^m \\ \text{(iii)} & \text{the } 2N \text{ equations } f_i(x')=f_i(x'')=y_i \ (i=1, \dots, N), \end{cases}$$

where x_j', x_j'' are coordinates in the first and second factors of the triple product. But since $y_1, \dots, y_k$ is a regular system of parameters in Y, one sees immediately that the equations (iii) for $i=1, \dots, k$, together with (i), (ii), already cut out $X\underset{Y}{\times}X$ locally at the point in question. Thus $X\underset{Y}{\times}X$ is defined by

$$N-k+2(m-n)+2k=2m+N+k-2n$$

equations in $(2m+N)$-space.

Moreover, the diagonal $\varDelta=\{(x, x, f(x))\}$ is a local complete intersection in W. Therefore theorem (2.1) implies that $X\underset{Y}{\times}X= X\cup D$ for suitable D of depth $\geq(2m+N)-(2m+N+k-2n)= 2n-k$, provided that $\dim D\leq 2n-k$. Thus in any case $\dim D\geq 2n-k$, which completes the proof.

Corollary (5.3): *Let x be an isolated singularity of an n-dimensional variety $\bar{X}$ over an algebraically closed field K. Assume that the normalization X of $\bar{X}$ is a local complete intersection (for instance, that X is smooth), but that $\bar{X}$ is not normal at x. Then the tangent space to $\bar{X}$ at x has dimension at least $2n$.*

For, if k is the dimension of the tangent space to $\bar{X}$ at x, we can find an embedding of $\bar{X}$ into a smooth k-dimensional scheme Y, locally at x. Since x is an isolated singularity, we may suppose $X\approx\bar{X}$ outside of x. Consider the map $f: X\to Y$. This map is finite, and an embedding except above x. Thus $X\underset{Y}{\times}X=\varDelta\cup D$, where D has dimension zero. If $k<2n$, then (5.1) implies that $D=\phi$, hence that f is a 1-1 immersion. Since it is a finite map, it is a closed embedding with image X: in other words, $X=\bar{X}$.

The first interseting case of this corollary is $n=3$: *A 3-fold in 5-space with isolated singularity cannot have a nonsingular normalization.*

With regard to (5.1), we should remark that for a smooth variety X the locus C of points at which f is not an immersion has dimension at least $(2n-k-1)$. This is because C (the "cusp locus") is, in terms of local coordinates $x_1, \ldots, x_n$; $y_1, \ldots, y_k$, set of points of X at which the jacobian $\frac{\partial y}{\partial x}$ has rank $<n$, which is a determinantal variety of codimension $\geq k-n+1$ ([3]). One sees easily that $C=D\cap\Delta$.

Now consider the case that $k=2n$, and that f has isolated critical points, i.e., that $X\underset{Y}{\times}X=\Delta\cup D$ with D of dimension zero. Then, following *Mumford*, we can introduce a measure δ of degeneracy of the map f as follows: The kernel ϵ:

$$0 \longrightarrow \epsilon \longrightarrow \mathcal{O}_{X\underset{Y}{\times}X} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{5.4}$$

is of finite length, and we define

$$\delta(f)=\frac{1}{2}\dim_K \epsilon. \tag{5.5}$$

The point is that this dimension is stable under deformation of f:

Theorem (5.6): *Let S be a noetherian scheme, and let X, Y be schemes of finite type over S, where X is a relative complete intersection of relative dimension n and Y is smooth, of relative dimension $2n$. Let $f: X\to Y$ be an S-map whose critical locus is of relative dimension zero. Then the $\mathcal{O}_{X\underset{Y}{\times}X}$-module ϵ of* (5.4) *commutes with base change $S_1\to S$, and is $\mathcal{O}_S$-flat.*

Proof: The fact that ϵ commutes with base change is clear, since $\mathcal{O}_X$ is $\mathcal{O}_S$-flat. To prove ϵ flat, the key case to consider is that S is the spectrum of a discrete valuation ring. For, by a limit argument we may suppose S affine, of finite type over $\boldsymbol{Z}$, and hence a subscheme of an entire scheme S'. Since X is a local complete intersection, it is obtained locally at any pair of points (x_1, x_2) by base change from a complete intersection X' over S'. Similarly, $Y = Y' \times_{S'} S$ locally for some smooth scheme Y. Then Y' is etale over affine space, and one sees immediately that f extends to an $f'\colon X' \to Y'$ in some etale neighborhood of (x_1, x_2) in X. Of course, the critical locus will be of relative dimension zero near S. Thus we are reduced to the case that $S = S'$ is entire, hence to showing that the "local rank" of ϵ at a point of $X \times_Y X$ does not drop under generalization, hence to the case that S is the spectrum of a discrete valuation ring.

Consider this case. As in the proof of (5.1), we can cut out $X \times_Y X$ in $W = A^m \times_S A^m \times_S A^N$ locally by the equations (5.2) with $i = 1, \ldots, k$ and $k = 2n$, where A^ν now denotes affine S-space. Then $\dim W = 2m + N + 1$. Thus theorem (2.1) shows that, locally at any closed point lying over the closed point of S, the depth of $X \times_Y X$ is at least 1. Therefore since the critical locus is of relative dimension 0, $X \times_Y X$ is S-torsion free. Hence ϵ is torsion free as well, and so is flat.

Corollary (5.7): *The number $\delta(f)$ is an integer.*

To prove, this we may deform f arbitrarily over a henselian base S, because by (5.6) the local contribution to δ in the neighborhood of a point p of $X \times_Y X$ will remain constant. (Since S is henselian, ϵ will split off a finite $\mathcal{O}_S$-module ϵ_0 whose support above the closed point is p, and the local contribution is $\frac{1}{2}(\text{rank } \epsilon_0)$.) Thus we may assume X smooth to begin with. Then f may be deformed into an immersion as is easily seen using the Jacobian matrix of f. Now in this case the product splits: $X \times_Y X = \varDelta \amalg D$, and the symmetry of the product induces a free $\mathbf{Z}/2$ action on D. Therefore $\mathcal{O}_D \approx \epsilon$ and $\dim_K \mathcal{O}_D$ is even.

Actually, it is not difficult to show that f may be deformed into a map whose only critical points are nodes (normal crossings). For such a map, δ is obviously the number of nodes. Thus in general δ measures "the number of nodes equivalent to the singularity of f".

Remark (5.8): (*Mumford*) There is a related invariant, namely the rank of the cokernel C:

$$\mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow C \longrightarrow 0.$$

(In fact $\epsilon \approx \mathcal{O}_X \otimes_{\mathcal{O}_Y} C$.) Suppose for the moment that X is a smooth curve. Then the image $\bar{X}$ of X in Y is of codimension 1, hence is given locally by one equation. It follows easily that for a smooth family of maps $f: X \to Y$, the images $\bar{X} \subset Y$ form a flat family. Thus there is an exact sequence $0 \to \mathcal{O}_{\bar{X}} \to \mathcal{O}_X \to C \to 0$, from which one deduces that C is $\mathcal{O}_S$-flat. Thus $\dim C$ does not change when we deform f into a map having only ordinary nodes, for which direct examination shows that $\dim C = \delta$. However when X has higher dimension, $\dim_K C$ is not invariant under deformation, because the images $\bar{X}$ will not form a flat family! Thus $\delta \neq \dim_K C$ in general. Consider the case that X consists of 3 copies of $\boldsymbol{P}^2$ mapping via f to 3 planes in $Y = \boldsymbol{P}^4$ passing through a given point y. Let $\bar{X}$ be the union of these planes, and let x_1, x_2, x_3 be the inverse image of y. Then the condition that a triple (ϕ_1, ϕ_2, ϕ_3) of functions defined locally at (x_1, x_2, x_3) lie in $\mathcal{O}_{\bar{X}} \subset \mathcal{O}_X$ is that $\phi_1(x_1) = \phi_i(x_i)$ if $i = 2, 3$ (two conditions), and that certain relations among the partial derivatives of ϕ_i at x_i hold. Since the dimensions of the tangent spaces to X at x_i add up to 6, while the tangent space to $\mathcal{O}_{\bar{X}}$ at y has dimension 4, there must be two independent relations. Thus $C = \mathcal{O}_X / \mathcal{O}_{\bar{X}}$ is 4-dimensional. On the other hand, the generic map $X \to Y$ will have 3 normal crossings for which $\dim_K C = \delta = 3$. Hence $\delta(f) = 3$.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
KYOTO UNIVERSITY

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