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V-Transformations of Finsler spaces I.

Definition, infinitesimal transformations and isometries

Dedicated to Professor T. Nakae on his 60th birthday

By

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The theory of transformations of Finsler spaces has been studied by several authors and many results have been obtained. (See, for example, [6],[1] p. 199; [23], p. 172; [27], p. 181.) Almost all of the authors are concerned with the so-called *extended point transformation* composed of a point transformation $\bar{x}^i = x^i + X^i(x)dt$ of the manifold M and $\bar{y}^i = y^i + \partial_j X^i(x) y^j dt$, where $X^i(x)$ are components of a tangent vector field X on M and $y^i = \dot{x}^i$. When the extended point transformation is treated in the tangent bundle $T(M)$, we have the tangent vector field

$$\bar{X} = X^i(x)\partial/\partial x^i + (\partial_j X^i(x)) y^j \partial/\partial y^i$$

on $T(M)$, which is called the complete lift of X [28] or the derived vector field from X [16], p. 187.

There are, however, some authors who are concerned with the generalizations of the extended point transformation. For instance, the present author introduces the notion of *linear transformation* of the tangent bundle [11, 12, 14, 16,]. It is shown that the tangent vector field on the tangent bundle appearing in the case of the linear transformation is written as the sum of tangent vector fields $\bar{X}$ and

1) Numbers in brackets refer to the references at the end of the paper.

$$X^* = X^i{}_j(x) y^j \partial/\partial y^i$$

where $X^i{}_j(x)$ are components of a tenso e ld of $(1,1)$-type on M.

Moreover there are some authors [17, 18, 19] who consider an infinitesimal transformation $\bar{x}^i = x^i + X^i(x,y)dt$, where $X^i(x,y)$ are components of a vector field X in a sense of the differential geometry of spaces of line-elements. A reasonable formulation of such a transformation in a global viewpoint is the starting point of the contents of the present paper.

After the basic concepts of Finsler connections are given in the first section, we shall define the concept of a V-transformation in the second section. The definition is quite similar to the one of a V-connection introduced by the present author, which was called an F-connection at first [15]. In the third section, the theory of infinitesimal V-transformations are developed by using a general Finsler connection. The fundamental theorem of infinitesimal V-transformations is proved in the fourth section: The tangent vector field on the tangent bundle appearing in the theory of infinitesimal V-transformations is written as the sum of the tangent vector fields

$$\underline{X}_v = X^i(x,y)\partial/\partial x^i + (\delta_j X^i(x,y)) y^j \partial/\partial y^i,$$

and

$$\underline{X}_\rho = X^i{}_j(x,y) y^j \partial/\partial y^i,$$

where $X^i(x,y)$ (resp. $X^i{}_j(x,y)$) are components of a vector field (resp. tensor field of $(1,1)$-type) in a sense of the differential geometry of spaces of line-elements and $\delta_j = \partial_j - N^k{}_j(x,y)\dot{\partial}_k$ are the partial differential operators with respect to a non-linear connection with the connection parameters $N^k{}_j(x,y)$. Therefore it may be said that a V-transformation is a natural generalization of an extended point transformation and a linear transformation, because the $\underline{X}_v$ (resp. $\underline{X}_\rho$) is of the similar form to the $\bar{X}$ (resp. X^*). Since the components of tensor fields appearing in the differential geometry of spaces of line-elements are functions of variables x^i and y^i in general, it seems proper to deal with the V-transformations in the theory of transformations of such a space.

It is quite interesting and important problem to study the behavior of some geometric objects under V-transformations. The isometric V-transformations of Finsler metrics will be discussed in the final section of the Part I of the present paper. The Part II will be concerned with the behavior of some special Finsler connections under V-transformations.

§1. Introduction to Finsler connections

Let $L(M)=(L, \pi_L, M, G)$ be the bundle of linear frames over a differentiable n-manifold M, where $\pi_L : L \rightarrow M$ is the projection from the total space L on the base space M and $G=GL(n, R)$ is the structure group. The tangent bundle $T(M)=(T, \pi_T, M, V, G)$ over M is the associated bundle with $L(M)$ having the real vector n-space V with a fixed base $(e_1, \cdots, e_n)$ as the standard fibre on which G operates, where $\pi_T : T \rightarrow M$ is the projection from the total space T on M. Then the projection π_T induces the principal bundle $\pi_T^{-1}L(M)$ over T, called the *Finsler bundle* of M and denoted by $F(M)=(F, \pi_1, T, G)$, where $\pi_1 : F \rightarrow T$ is the projection from the total space F on T. The total space F is given by $F=\{(y, z) \in T \times L \mid \pi_T(y) =\pi_L(z)\}$ and diffeomorphic to the product manifold $L \times V$ by the mapping

$$\iota : (y, z) \in F \mapsto (z, z^{-1}y) \in L \times V, \tag{1.1}$$

where $z^{-1} : T \rightarrow V$ is the inverse of the admissible mapping ${}_z\alpha : v \in V \mapsto zv \in T$. The diffeomorphism ι is composed of two mappings $\pi_2 : (y, z) \in F \mapsto z \in L$ and $\varepsilon : (y, z) \in F \mapsto z^{-1}y \in V$. The former π_2 is the so-called induced mapping, while the latter ε is called the *element of support* which plays an important role in the Finsler geometry.

The present author has written the monograph "The theory of Finsler connections" [16] in 1970. Several concepts and results given in the monograph will be used in the present paper without special reference. We shall, however, sketch the outline of the theory for the latter convenience.

(1.2) **Definition.** A *Finsler connection* $F\Gamma$ on a differentiable manifold M is the pair (Γ, N) of a connection Γ in the Finsler bundle $F(M)$ and a non-linear connection N in the tangent bundle $T(M)$.

There are other two definitions of a Finsler connection $F\Gamma$ equivalent to (1.2); the one is a Finsler pair (Γ^h, Γ^v) and the other is a Finsler triad (Γ_V, N, Γ^v). In order to give them, we shall introduce the necessary concepts previously.

(1.3) **Definition.** A *vertical connection* Γ^v in $F(M)$ is a distribution $u \in F \rightarrow \Gamma_u^v \subset F_u$ (the tangent space to F at a point u) such that for any point $x \in M$ its restriction to the subbundle $F(x)$ of $F(M)$ over the fibre $\pi_T^{-1}(x)$ is a connection in $F(x)$.

The subspace $F_u^i = \{X \in F_u \mid \pi_2' X = 0\} \subset F_u$ is called the *induced vertical subspace* and the distribution $F^i : u \in F \rightarrow F_u^i$ is a kind of a vertical connection, which is called the *vertical flat connection*. The vertical subspace F_u^v is spanned by the fundamental vector fields $Z(A)$, $A \in L(G)$ (the Lie algebra of the G), while the induced vertical subspace F_u^i is spanned by the *induced fundamental vector fields* $Y(v)$, $v \in V$, defined by $\pi_2' Y(v) = 0$ and $\varepsilon' Y(v) = S(v)$, where $S(v)$ is the tangent vector field on V, corresponding to $v = v^a e_a \in V$ and having the constant components v^a. The name "vertical flat connection" of F^i is justified by the equation

$$[Y(v_1), Y(v_2)] = 0, \tag{1.4}$$

for any $v_1, v_2 \in V$. On the other hand, a general vertical connection Γ^v is spanned by the *v-basic vector fields* $B^v(v)$, corresponding to $v \in V$, defined by $B^v(v)_u = l_u \cdot {}_z\alpha'(S(v)_{\varepsilon(u)})$ at $u = (y, z) \in F$, where l_u denotes the lift to the point u with respect to the connection Γ^v. Then the *Cartan tensor field* C of Γ^v is introduced by the equation

(1.5) $$Y(v)=B^v(v)+Z(C(v)).$$

The Cartan tensor of the flat F^i vanishes obviously.

Now we are in a position to give the second definition of a Finsler connection as follows.

(1.6) **Definition.** [21] A *Finsler pair* (Γ^h, Γ^v) on F is a pair of two distributions $u\in F\to\Gamma_u^h$ and $u\in F\to\Gamma_u^v$ such that the latter Γ^v is a vertical connection in $F(M)$ and

(1) $F_u=\Gamma_u^h\oplus\Gamma_u^v\oplus F_u^v$, (2) $\tau_g'\Gamma_u^h=\Gamma_{ug}^h$,

where F_u^v is the vertical subspace of F_u and τ_g the right translation of F by $g\in G$.

It is shown that $\Gamma_u=\Gamma_u^h\oplus\Gamma_u^v$ gives a connection in F and $N_y=\pi_1'\Gamma_u^h$, $u=(y, z)$, does a non-linear connection N. Thus we obtain the Finsler connection $F\Gamma=(\Gamma, N)$ from the Finsler pair (Γ^h, Γ^v). Conversely, consider a Finsler connection $F\Gamma=(\Gamma, N)$ and denote by l_u the lift to a point $u\in F$ with respect to the Γ. If we put $\Gamma_u^h=l_uN_y$ and $\Gamma_u^v=l_uT_y^v$, where $u=(y, z)$ and T_y^v is the vertical subspace of the tangent space T_y, then the pair (Γ^h, Γ^v) is the Finsler pair.

To state the third definition of a Finsler connection we need another concept as follows.

(1.7) **Definition.** [15] A *V-connection* Γ_V in $L(M)$ or on M is a family $\{\Gamma_{(v)}|v\in V\}$ of distributions $\Gamma_{(v)}$ on L corresponding to every $v\in V$, such that

(1) $L_z=\Gamma_{(v)z}\oplus L_z^v$, (2) $\underline{\tau}_g'\Gamma_{(v)z}=\Gamma_{(g^{-1}v)zg}$,

where L_z^v is the vertical subspace of the tangent space L_z and $\underline{\tau}_g$ the right translation of L by $g\in G$.

Then the third definition of a Finsler connection is given as follows.

(1.8) **Definition.** [15] A *Finsler triad* (Γ_V, N, Γ^v) on M is a triad of a V-connection Γ_V in $L(M)$, a non-linear connection N in $T(M)$ and a vertical connection Γ^v in $F(M)$.

Consider a Finsler triad (Γ_V, N, Γ^v). Then $\Gamma_u^h=\{X\in F_u \mid \pi_1' X \in N_y, \pi_2' X\in \Gamma_{(v)z}, u=(y,z), y=zv\}$ together with the Γ^v gives a Finsler pair (Γ^h, Γ^v). Conversely, if a Finsler connection $F\Gamma=(\Gamma, N)=(\Gamma^h, \Gamma^v)$ is given, the family Γ_V of $\Gamma_{(v)}$ defined by $\Gamma_{(v)z}=\pi_2'\Gamma_u^h$, $u=(zv, z)$, is a V-connection and thus we obtain a Finsler triad (Γ_V, N, Γ^v).

As we have seen, the v-part Γ^v of a Finsler connection $F\Gamma=(\Gamma^h, \Gamma^v)$ is spanned by the v-basic vector fields $B^v(v)$ of the vertical connection Γ^v, while its h-part Γ^h is now spanned by the *h-basic vector fields* $B^h(v)$, corresponding to $v\in V$, defined by $B^h(v)_u=l_u\cdot l_y(zv)$, where $u=(y,z)$ and l_u (resp. l_y) denotes the lift to u (resp. y) with respect to Γ (resp. N).

We shall describe the above contents in terms of coordinates. Let U be a coordinate neighborhood of a coordinate (x^i) of the base manifold M. The $\pi_T^{-1}(U)$ is the coordinate neighborhood of the total space T of $T(M)$ in which the coordinate (x^i, y^i) of a point y is such that $y=y^i\partial/\partial x^i$. Next $\pi_L^{-1}(U)$ is the coordinate neighborhood of the total space L of $L(M)$ in which the coordinate (x^i, z_a^i) of a point z is such that $z=(z_a^i\partial/\partial x^i)$, $a=1, \cdots, n$. Then $\pi_1^{-1}\cdot\pi_T^{-1}(U)=\pi_2^{-1}\cdot\pi_L^{-1}(U)$ is the coordinate neighborhood of the total space F of $F(M)$ in which the coordinate (x^i, y^i, z_a^i) of a point $u=(y,z)$ is such that $y=(x^i, y^i)$ and $z=(x^i, z_a^i)$.

Now the v-basic vector field $B^v(v)$, corresponding to $v=v^a e_a\in V$, is written in the from

$$B^v(v)=z_a^i v^a(\partial/\partial y^i - z_b^k C_k{}^j{}_i\partial/\partial z_b^j),$$

and $C_k{}^j{}_i(x,y)$ in the above are the components of the Cartan tensor field C and also the connection parameters of Γ^v. The h-basic vector field $B^h(v)$, corresponding to $v=v^a e_a\in V$, is written in the form

$$B^h(v)=z_a^i v^a(\partial/\partial x^i - N^j{}_i\partial/\partial y^j - z_b^k F_k{}^j{}_i\partial/\partial z_b^j).$$

If we consider a Finsler connection $F\Gamma$ as the Finsler triad (Γ_V, N, Γ^v), then $F_k{}^j{}_i(x, y)$ in the above are the connection parameters of Γ_V and $N^j{}_i(x, y)$ the ones of N.

The components of the element of support ε are equal to y^i at a point $u=(x^i, y^i, z_a{}^i)$. In general a tensor field K of (r, s)-type in a sense of the differential geometry of spaces of line-elements is the entity having n^{r+s} components $K^{i_1\cdots i_r}_{j_1\cdots j_s}(x, y)$ and thought of as a mapping $K: F\to V_s^r$ (the tensor space of (r, s)-type constructed from V) satisfying the equation $K\cdot\tau_g=g^{-1}K$. Such a tensor field K will be called the *Finsler tensor field* of (r, s)-type. The element of support ε is a Finsler tensor field of $(1, 0)$-type. Let us consider a Finsler tensor field K of $(1, 0)$-type, for brevity. There are well-known three kinds of covariant derivatives given by

$$(1.9)' \qquad \begin{aligned} K^i{}_{|j} &= \partial_j K^i - N^k{}_j \dot{\partial}_k K^i + K^k F_k{}^i{}_j, \\ K^i|_j &= \dot{\partial}_j K^i + K^k C_k{}^i{}_j, \\ K^i{}_{\|j} &= \dot{\partial}_j K^i. \end{aligned}$$

These are the components of the respective Finsler tensor fields $\varDelta^h K$, $\varDelta^v K$ and $\varDelta^0 K$ defined by

$$(1.9) \qquad \varDelta^h K(v) = B^h(v)K, \quad \varDelta^v K(v) = B^v(v)K, \quad \varDelta^0 K(v) = Y(v)K.$$

As to the v-covariant derivative $\varDelta^v\varepsilon$ of the element of support ε, it is shown that

$$(1.10) \qquad \varDelta^v\varepsilon(v) = v + C(\varepsilon, v),$$

while the Finsler tensor field D of $(1, 1)$-type arises from the h-covariant derivative

$$(1.11) \qquad \varDelta^h\varepsilon(v) = D(v),$$

and called the *deflection tensor field*. The components $D^i{}_j$ of D are given by $D^i{}_j = y^k F_k{}^i{}_j - N^i{}_j$.

The structure equations of a Finsler connection are written

$$
\begin{aligned}
&[B^h(1), B^h(2)] = B^h(T(1,2)) + B^v(R^1(1,2)) + Z(R^2(1,2)),\\
(1.12)\quad &[B^h(1), B^v(2)] = B^h(C(1,2)) + B^v(P^1(1,2)) + Z(P^2(1,2)),\\
&[B^v(1), B^v(2)] = \qquad\qquad\quad B^v(S^1(1,2)) + Z(S^2(1,2)),
\end{aligned}
$$

where we wrote the letters 1, 2 merely instead of $v_1, v_2 \in V$. In the structure equations, T, C (the Cartan tensor field), R^1, P^1 and S^1 are called the *torsion tensor fields*, while R^2, P^2 and S^2 are the *curvature tensor fields*. If the Finsler connection is such that $D=0$ and $C(\varepsilon, v)=0$ for any $v \in V$, then

$$
\begin{aligned}
(1.13)\quad &R^1(1,2) = R^2(\varepsilon, 1, 2), \quad P^1(1,2) = P^2(\varepsilon, 1, 2),\\
&S^1(1,2) = S^2(\varepsilon, 1, 2).
\end{aligned}
$$

Finally, we shall refer to Cartan's Finsler connection $C\Gamma$ [4, 7, 13]. Let $L(x, y)$ be a Finsler fundamental function. Then we construct the so-called fundamental tensor field $g = \Delta^0\Delta^0 L^2/2$; the $C\Gamma$ is uniquely determined by the four axioms

$$
(1.14)\quad
\begin{aligned}
&(1)\quad \Delta^h g = \Delta^v g = 0, &&(2)\quad T=0,\\
&(3)\quad S^1 = 0, &&(4)\quad D=0.
\end{aligned}
$$

The Cartan tensor $C = \Delta^0 g/2$ of $C\Gamma$ satisfies $C(\varepsilon, v) = C(v, \varepsilon) = 0$ for any $v \in V$ and the connection parameters $F_j{}^i{}_k$, $N^i{}_j$ and $C_j{}^i{}_k$ of $C\Gamma$ are used to be written as $\Gamma^*{}_j{}^i{}_k$, $y^k\Gamma^*{}_k{}^i{}_j$ and $C_j{}^i{}_k$ respectively.

§2. Theory of V-transformations

It was seen in the first section that the connection parameters $F_j{}^i{}_k$ of a V-connection Γ_V defined in (1.7) were functions of (x^h) and (y^h). The condition (2) in (1.7) is a little different from the well-known condition $\tau'_g \Gamma_z = \Gamma_{zg}$ of a linear connection Γ, while the condition (1) in (1.7) is the same with the one of the Γ [10, 20]. It may be said that $F_j{}^i{}_k$ turn out to be the functions of not only (x^h) but also (y^h) as a consequence of the condition (2) in (1.7). We now lay down the following definition which is closely related to the condition (2) in (1.7) as a matter of form.

(2.1) **Definition.** A *V-transformation* μ_V of a differentiable n-manifold M is the family $\{\mu_{(v)} | v \in V\}$ of transformations $\mu_{(v)}$ of the total space L of the bundle of linear frames $L(M)$ over M, corresponding to every element v of the standard fibre V of the tangent bundle $T(M)$ over M, such that the relation

$$\underline{\tau}_g \cdot \mu_{(v)} = \mu_{(g^{-1}v)} \cdot \underline{\tau}_g$$

holds for any right translation $\underline{\tau}_g$ of L by $g \in G$.

The property "positively homogeneous" is essential for geometric objects in spaces of line-elements. Although we shall not confine ourselves in the present paper to deal with the geometry of such a space, we have to pay attention to this property of our V-transformations throughout the present paper.

(2.2) **Definition.** A V-transformation μ_V of M is called *positively homogeneous* if $\mu_{(rv)} = \mu_{(v)}$ holds for any $v \in V$ and any positive number $r \in R^+$.

From a given V-connection Γ_V, the unique non-linear connection N is derived by the equation $N_y = \alpha'_v \Gamma_{(v)z}$, $y = zv$, where $\alpha_v : z \in L \mapsto zv \in T$ is the associated mapping [15]. Such an N is called the *associated non-linear connection* with Γ_V. The deflection tensor field D of a Finsler connection $F\Gamma = (\Gamma_V, N, \Gamma^v)$ vanishes if and only if its non-linear connection N is associated one with Γ_V. In like manner we shall obtain the unique transformation of T from a V-transformation as follow.

(2.3) **Definition.** Let $\mu_V = \{\mu_{(v)} | v \in V\}$ be a V-transformation of M. Then the transformation μ of the total space T of the tangent bundle $T(M)$ is derived from μ_V by the equation

$$\mu(y) = \alpha_v \cdot \mu_{(v)}(z) = (\mu_{(v)}(z))v, \qquad y = zv \in T.$$

This μ is called the *associated transformation* with μ_V.

Owing to the condition in (2.1), it will be easy to show that the above μ is well-defined by the equation in (2.3), independent of the choice of the point $(z, v) \in L \times V$ such that $y = zv$.

(2.4) **Proposition.** *If a V-transformation μ_V is positively homogeneous, the associated transformation μ with μ_V satisfies the equation $\mu \cdot {}_r\mathrm{h} = {}_r\mathrm{h} \cdot \mu$, where ${}_r\mathrm{h} : y \in T \mapsto ry \in T$ is a homogeneous transformation of T by $r \in R^+$.*

It is here remarked that when we are concerned with the homogeneity property of geometric objects, the space T should mean the set of all the non-zero tangent vectors to M throughout the present paper. The proof of (2.4) will be easy.

In order to derive from μ_V a transformation $\bar{\mu}$ of the total space F of the Finsler bundle $F(M)$, it is convenient to do a transformation μ^* of the product manifold $L \times V$ beforehand.

(2.5) **Definition.** The *lift* μ^* of a V-transformation μ_V of M to the product manifold $L \times V$ is the transformation of $L \times V$ defined by

$$\mu^*(z, v) = (\mu_{(v)}(z), v), \qquad (z, v) \in L \times V.$$

It is noted that the first component z of the point (z, v) is transformed in connection with the second one v, while v itself is fixed. The following proposition will be easily proved.

(2.6) **Proposition.** *If a V-transformation μ_V is positively homogeneous, the lift μ^* of μ_V to $L \times V$ satisfies $\mu^* \cdot {}_r\mathrm{h}^* = {}_r\mathrm{h}^* \cdot \mu^*$, where ${}_r\mathrm{h}^* : (z, v) \in L \times V \mapsto (z, rv) \in L \times V$ is a homogeneous transformation of $L \times V$ by $r \in R^+$.*

The notion of our V-transformation may be rather complicated, because it is not a transformation but a family of transformations. The following proposition will give it a clear character.

(2. 7) **Proposition.** *A necessary and sufficient condition for a transformation* μ^* *of* $L\times V$ *to be the lift of a V-transformation* μ_V *of* M *to* $L\times V$ *is that* μ^* *satisfies*

$$(1) \qquad \tau_g^*\cdot\mu^*=\mu^*\cdot\tau_g^*, \qquad\qquad (2) \qquad \varepsilon^*\cdot\mu^*=\varepsilon^*,$$

where $\tau_g^*:(z,v)\in L\times V\mapsto(zg,g^{-1}v)\in L\times V$ *is a right translation of* $L\times V$ *by* $g\in G$ *and* $\varepsilon^*:(z,v)\in L\times V\mapsto v\in V$ *is a canonical projection.*

Proof. If μ^* is the lift of μ_V, then (2) in (2. 7) is obviously a consequence of (2. 5). While (1) in (2. 7) will be easily verified by (2. 1) and (2. 5). Conversely, if μ^* satisfies the conditions (1) and (2) in (2. 7), we shall first construct the transformation $\mu_{(v)}$ of L, corresponding to $v\in V$, by

$$(2.8) \qquad \mu_{(v)}=\pi_2^*\cdot\mu^*\cdot\beta_v^*,$$

where we used the two mappings

$$\pi_2^*:L\times V\to L, \qquad (z,v)\mapsto z,$$
$$\beta_v^*:L\to L\times V, \qquad z\mapsto(z,v).$$

The family $\mu_V=\{\mu_{(v)}\mid v\in V\}$ as thus obtained is certainly a V-transformation of M, because

$$\underline{\tau}_g\cdot\mu_{(v)}=\underline{\tau}_g\cdot\pi_2^*\cdot\mu^*\cdot\beta_v^*=\pi_2^*\cdot\mu^*\cdot\tau_g^*\cdot\beta_v^*$$
$$=\pi_2^*\cdot\mu^*\cdot\beta_{g^{-1}v}^*\cdot\underline{\tau}_g=\mu_{(g^{-1}v)}\cdot\underline{\tau}_g\,.$$

The lift of this μ_V coincides with the original μ^*, because

$$(\mu_{(v)}(z),v)=(\pi_2^*\cdot\mu^*\cdot\beta_v^*(z),v)=(\pi_2^*\cdot\mu^*(z,v),\varepsilon^*(z,v))$$
$$=(\pi_2^*\cdot\mu^*(z,v),\varepsilon^*\cdot\mu^*(z,v))=\mu^*(z,v).$$

This completes the proof.

We are now in a position to derive the transformation $\bar{\mu}$ of the total space F of $F(M)$ from a V-tansformation μ_V of M as follows.

(2. 9) **Definition.** The *lift* $\bar{\mu}$ of a V-transformation μ_V of M to the total space F of the Finsler bundle $F(M)$ of M is the trans-

formation of F, defined by the equation $\bar{\mu}=\iota^{-1}\cdot\mu^{*}\cdot\iota$, where ι is the diffeomorphism $F\rightarrow L\times V$ given by (1.1) and μ^{*} is the lift of μ_V to $L\times V$.

Then we shall translate the condition (1) and (2) in (2.7) into the languages of the Finsler bundle $F(M)$ by means of the diffeomorphism ι and obtain

(2.10) **Theorem.** *A necessary and sufficient condition for a transformation $\bar{\mu}$ of F to be the lift of a V-transformation μ_V of M to F is that $\bar{\mu}$ satisfies*

(1) $\tau_g\cdot\bar{\mu}=\bar{\mu}\cdot\tau_g$, (2) $\varepsilon\cdot\bar{\mu}=\varepsilon$,

where τ_g is the right translation of F by $g\in G$ and $\varepsilon: F\rightarrow V$ is the element of support.

The theorem will immediately follow from the facts that $\varepsilon=\varepsilon^{*}\cdot\iota$, $\pi_2=\pi_2^{*}\cdot\iota$ and $\tau_g=\iota^{-1}\cdot\tau_g^{*}\cdot\iota$. Similarly (2.8) will be translated into

$$\mu_{(v)}=\pi_2\cdot\bar{\mu}\cdot\beta_v\,, \tag{2.11}$$

where $\beta_v=\iota^{-1}\cdot\beta_v^{*}: z\in L\mapsto(zv, z)\in F$. Moreover, in virtue of the transformation ${}_r\mathrm{H}=\iota^{-1}\cdot{}_r\mathrm{h}^{*}\cdot\iota: (y, z)\in F\mapsto(ry, z)\in F$ by $r\in R^{+}$, Proposition (2.6) is translated into

(2.12) **Proposition.** *If a V-transformation μ_V is positively homogeneous, the lift $\bar{\mu}$ of μ_V to F satisfies $\bar{\mu}\cdot{}_r\mathrm{H}={}_r\mathrm{H}\cdot\bar{\mu}$.*

The lift $\bar{\mu}$ to F will be given by an alternative and more convenient expression without use of the product manifold $L\times V$ as follows.

(2.13) **Proposition.** *The lift $\bar{\mu}$ of a V-transformation μ_V to F is written as $\bar{\mu}=(\mu\cdot\pi_1, \mu_{(\varepsilon)}\cdot\pi_2)$, where μ is the associated transformation of T with μ_V and $\mu_{(\varepsilon)}$ is the transformation $\mu_{(v)}\in\mu_V$, corresponding to the value $v=\varepsilon(u)$ of the element of support ε at a point $u\in F$.*

Proof. At a point $(zv, z)=\iota^{-1}(z, v)\in F$, we have from (2.9)

$$\pi_1\cdot\bar{\mu}(zv, z)=\pi_1\cdot\iota^{-1}(\mu_{(v)}(z), v)=(\mu_{(v)}(z))v=\mu(zv),$$
$$\pi_2\cdot\bar{\mu}(zv, z)=\pi_2\cdot\iota^{-1}(\mu_{(v)}(z), v)=\mu_{(v)}(z).$$

If we pay attention to $\varepsilon(zv, v)=v$, the proof is complete.

We now return to the consideration of more simple transformations than V-transformation. First, a necessary and sufficient condition for a transformation $\bar{\mu}$ of F to be the one *derived from* a transformation of the base manifold M is that $\bar{\mu}$ satisfies the following four conditions [16], p. 198:

$$(2.14)\quad \begin{array}{lll} (1) & \tau_g\cdot\bar{\mu}=\bar{\mu}\cdot\tau_g, & g\in G,\\ (2) & \varepsilon\cdot\bar{\mu}=\varepsilon, & \\ (3) & s_v\cdot\bar{\mu}=\bar{\mu}\cdot s_v, & v\in V,\\ (4) & \theta^h\cdot\bar{\mu}'=\theta^h. & \end{array}$$

Here $s_v:(y, z)\in F\mapsto(y+zv, z)\in F$ is a transformation of F corresponding to $v\in V$ and θ^h is a V-valued 1-form on F which is called the h-basic form and defined by $\theta^h=\theta\cdot\pi_2'$, where θ is the well-known basic form on L [20]. Secondly a theory of *linear transformations* has been developed by the present author [11, 12]; [16], Ch. VI. A linear transformation μ is by definition a transformation of T such that μ is fibre-preserving and linear on every fibre. Then the transformation $\bar{\mu}$ of F is derived from μ. Such a transformation $\bar{\mu}$ of F is characterized by the first three conditions of (2.14). Therefore the concept of a linear transformation seems to be a natural generalization of the one of a transformation of the base manifold M. Moreover the concept of a V-transformation seems to be a natural generalization of the one of a linear transformation, because the conditions in (2.10) are nothing but the first two of (2.14).

In the case of a linear transformation μ, the transformation $\underline{\mu}$ of the base manifold M is induced such that $\pi_T\cdot\mu=\underline{\mu}\cdot\pi_T$. Then, comparing the transformation μ with the one derived from $\underline{\mu}$, we obtain the concept of a special linear transformation called the rotation [11].

On the other hand, in the case of our V-transformations, if the associated transformation μ with a V-transformation μ_V is fibre-preserving, then a generalization of the rotation will be obtained in the following.

(2. 15) **Proposition.** *A necessary and sufficient condition for the associated transformation μ of T with a V-transformation μ_V to be fibre-preserving is that $\pi_L \cdot \mu_{(v_1)} = \pi_L \cdot \mu_{(v_2)}$ holds for any $v_1, v_2 \in V$.*

Proof. Assume that the above condition be satisfied. If we take two points y_i, $i=1, 2$, of a fibre $\pi_T^{-1}(x)$ and put $v_i = z^{-1} y_i \in V$ for a point $z \in \pi_L^{-1}(x)$, we see $\pi_L \cdot \mu_{(v_1)}(z) = \pi_L \cdot \mu_{(v_2)}(z)$; it then follows from (2. 3) that

$$\pi_T \cdot \mu(y_1) = \pi_T(\mu_{(v_1)}(z) v_1) = \pi_L(\mu_{(v_1)}(z)) = \pi_L(\mu_{(v_2)}(z)),$$

so that

$$(2.16) \qquad \pi_T \cdot \mu(y_1) = \pi_T \cdot \mu(y_2), \qquad \text{if } \pi_T(y_1) = \pi_T(y_2).$$

This shows that μ is fibre-preserving. The converse will be easily proved.

From (2. 16) we see that the μ induces the transformation $\underline{\mu}$ of the base manifold M such that

$$(2.17) \qquad \underline{\mu} \cdot \pi_T = \pi_T \cdot \mu.$$

Consequently we obtain the differential $\underline{\mu}'$ of $\underline{\mu}$, a transformation of T, and the n-ple differential $\underline{\mu}''$ of $\underline{\mu}$, a transformation of L [16], p. 187. It is seen from (2. 3) and (2. 17) that

$$\pi_L(\mu_{(v)}(z)) = \pi_T(\mu(zv)) = \underline{\mu} \cdot \pi_L(z),$$

which implies that two points $\mu_{(v)}(z)$ and $\underline{\mu}''(z)$ of L are on the same fibre, so that there exists a unique $g \in G$ such that $\mu_{(v)}(z) = \underline{\mu}''(z) g$. From this fact we are led to

(2. 18) **Definition.** If a V-transformation μ_V satisfies the condition in (2. 15), we obtain the family $\alpha_V = \{\alpha_{(v)} \mid v \in V\}$ of the mappings

$\alpha_{(v)}: L \rightarrow G$, corresponding to every $v \in V$, such that

$$\mu_{(v)}(z) = \underline{\mu}^n(z)\alpha_{(v)}(z).$$

The α_V is called the *deviation* of such a V-transformation μ_V.

In the case of a linear transformation, we have obtained the important property of the deviation, that is (44.3) of [16]. More generally, we shall show

(2.19) **Proposition.** *The deviation α_V satisfies*

$$\alpha_{(v)} \cdot \underline{\tau}_g = \mathrm{i}_{g^{-1}} \cdot \alpha_{(gv)}, \qquad g \in G,$$

where $\mathrm{i}_{g^{-1}}$ denotes the inner automorphism of G by g^{-1}.

Proof. It follows from (2.1) that $\underline{\tau}_g \cdot \mu_{(gv)}(z) = \mu_{(v)} \cdot \underline{\tau}_g(z)$ for $z \in L$, which, owing to (2.18), is written in the form

$$\underline{\tau}_g(\underline{\mu}^n(z)\alpha_{(gv)}(z)) = \underline{\mu}^n(zg)\alpha_{(v)}(zg).$$

Since $\underline{\mu}^n$ commutes with the right translation $\underline{\tau}_g$, this proves the proposition immediately.

The equation in (2.18) is rewritten in the form $\mu_{(v)}(z) = \underline{\mu}^n(z\alpha_{(v)}(z))$, which leads us to

(2.20) **Definition.** The family of transformations $\rho_V = \{\rho_{(v)} | v \in V\}$ of L given by $\rho_{(v)}(z) = z\alpha_{(v)}(z)$ for $z \in L$, is called the *V-rotation* of M, provided that the family $\alpha_V = \{\alpha_{(v)} | v \in V\}$ of the mappings $\alpha_{(v)}: L \rightarrow G$ satisfies the condition in (2.19).

The V-rotation ρ_V is certainly a kind of a V-transformation, because it follows from (2.19) that

$$\underline{\tau}_g \cdot \rho_{(v)}(z) = z(\alpha_{(v)}(z)g) = ((zg)\alpha_{(g^{-1}v)}(zg)) = \rho_{(g^{-1}v)} \cdot \underline{\tau}_g(z).$$

It has been shown [16] that *any* linear transformation is the composition of the derived one from a transformation of the base manifold M and a rotation. On the other hand, in the case of our V-transformation, this situation has been shown only when the condition in (2.15) holds. If we consider a general V-transformation, the similar situation will occur when we shall treat the infinitesimal V-transformation, which will be proved in (4. 5).

§3. Infinitesimal V-transformations

We shall be concerned with a one-parameter group of V-transformations $\mu_{Vt}=\{\mu_{(v)t}\mid v\in V\}$ of M with the parameter t. Then the tangent vector field X_μ of the total space F of the Finsler bundle $F(M)$ is induced by the lift $\bar{\mu}_t$ of μ_{Vt} to F. $\bar{\mu}_t$ is the one-parameter group of transformations of F. The X_μ will be called the *infinitesimal V-transformation.* It follows immediately from (2. 10) that

(3. 1) **Theorem.** *An infinitesimal V-transformation X_μ is characterized by the two equations*

(1) $\tau'_g X_\mu = X_\mu$, $g\in G$, (2) $X_\mu(\varepsilon)=0$.

The equation (1) *is also written in the form*

(1)′ $[X_\mu, Z(A)]=0$,

where $Z(A)$ is a fundamental vector field on F.

If X_μ is a infinitesimal *linear* transformation, it follows from (2. 14) (3) that X_μ satisfies the above (1), (2) and moreover

(3. 1) (3) $$[X_\mu, Y(v)]=0,$$

where $Y(v)$ is the induced fundamental vector field [16], p. 201. Next, if X_μ is a infinitesimal transformation *derived* from a one-parameter group of transformations of the base manifold M, it follows from (2. 14) (4) that X_μ satisfies the above (1), (2), (3) and moreover

(3. 1) (4) $$\mathcal{L}_\mu \theta^h = 0.$$

Here and throughout the present paper the notation $\mathcal{L}_\mu$ will denote the Lie derivative with respect to X_μ.

Assume that a general Finsler connection $F\Gamma$ be given on M. Then with respect to $F\Gamma$ the X_μ is written in the form

(3.2) $$X_\mu = B^h(v_\mu) + B^v(w_\mu) + Z(A_\mu),$$

where v_μ and w_μ are V-valued functions on F and A_μ is an $L(G)$-valued function on F, which are determined by X_μ.

(3.3) **Theorem.** *In the equation* (3.2), v_μ *and* w_μ *are Finsler tensor fields of* $(1,0)$*-type and* A_μ *is a Finsler tensor field of* $(1,1)$*-type. Among them there is the relation*

$$A_\mu(\varepsilon) = D(v_\mu) + w_\mu + C(\varepsilon, w_\mu),$$

where D (*resp.* C) *is the deflection tensor* (*resp. Cartan tensor*) *of the Finsler connection under consideration.*

Proof. At a point $u \in F$ we have from (3.2)

$$\tau'_g(X_\mu)_u = B^h(g^{-1}(v_\mu)_u) + B^v(g^{-1}(w_\mu)_u) + Z(\mathrm{ad}(g^{-1})(A_\mu)_u).$$

Then (3.1) (1) gives $(v_\mu)_{ug} = g^{-1}(v_\mu)_u$, $(w_\mu)_{ug} = g^{-1}(w_\mu)_u$ and $(A_\mu)_{ug} = \mathrm{ad}(g^{-1})(A_\mu)_u$, which prove the first half of (3.3). Next (3.2), (3.1) (2), (1.10), (1.11) and $Z(A)\varepsilon = -A\varepsilon$ lead us immediately to the relation mentioned in (3.3).

(3.4) **Definition.** If every V-transformation μ_{Vt} (t being fixed) of a one-parameter group of V-transformations is positively homogeneous, the infinitesimal V-transformation X_μ induced by μ_{Vt} is called *positively homogeneous.*

(3.5) **Proposition.** *If an infinitesimal V-transformation* X_μ *is positively homogeneous, then* X_μ *is* (0)*p.-h. in the sense of* (22.6) *of* [16].

Proof. If we consider a real-valued function f on F, it follows from (2.12) that

$$({}_r\mathrm{H}'X_\mu)f = \lim_{t\to 0}(f\cdot{}_r\mathrm{H}\cdot\bar{\mu}_t - f\cdot{}_r\mathrm{H})/t = (\lim_{t\to 0}(f\cdot\bar{\mu}_t - f)/t)\cdot{}_r\mathrm{H}\,,$$

which implies ${}_r\mathrm{H}'X_\mu = X_\mu$ and the proof is complete.

(3.6) **Proposition.** *Assume that an infinitesimal V-transformation and a Finsler connection $F\Gamma$ be positively homogeneous. Then, in the equation* (3.2), v_μ *and* A_μ *are* $(0)p.$*-h., while* w_μ *is* $(1)p.$*-h.*

Proof. The Finsler connection $F\Gamma$ is positively homogeneous if and only if the h- (resp. v-)basic vector field $B^h(v)$ (resp. $B^v(v)$) is $(0)p.$-$h.$ (resp. $(1)p.$-$h.$) [16], p. 95, namely, ${}_r\mathrm{H}'B^h(v)=B^h(v)$ and ${}_r\mathrm{H}'B^v(v)=B^v(rv)$ for $r\in R^+$. Moreover ${}_r\mathrm{H}'Z(A)=Z(A)$ is well-known. According to these facts we obtain from (3.2) $v_\mu\cdot{}_r\mathrm{H}=v_\mu$, $w_\mu\cdot{}_r\mathrm{H}=rw_\mu$ and $A_\mu\cdot{}_r\mathrm{H}=A_\mu$, which are required results.

Now we shall be concerned with the Lie derivative $\mathcal{L}_\mu$ with respect to an infinitesimal V-transformation X_μ. In terms of $\mathcal{L}_\mu$, the properties stated in (3.1) are expressed by saying

(3.7) **Theorem.** *The Lie derivative* $\mathcal{L}_\mu$ *with respect to an infinitesimal V-transformation* X_μ *is characterized by the equations*

(1) $\mathcal{L}_\mu Z(A)=0$, (2) $\mathcal{L}_\mu\varepsilon=0$.

Let us express the Lie derivatives of the h- and v-basic vector fields of a Finsler connection with respect to an infinitesimal V-transformation X_μ by writing

$$(3.8)\quad \mathcal{L}_\mu B^h(v)=[X_\mu, B^h(v)]=B^h(\alpha(v))+B^v(\alpha^1(v))+Z(\alpha^2(v)),$$

$$(3.9)\quad \mathcal{L}_\mu B^v(v)=[X_\mu, B^v(v)]=B^h(\beta(v))+B^v(\beta^1(v))+Z(\beta^2(v)),$$

where α, α^1, β and β^1 are Finsler tensor fields of (1, 1)-type, while α^2 and β^2 are the ones of (1, 2)-type, as will be easily verified. Substituting from (3.2) in the above equations and using the structure equations (1.12) of the Finsler connnection, these Finsler tensor fields are written in terms of the h-, v-covariant derivatives, torsion tensors and curvature tensors of the $F\Gamma$ as follows:

$$(3.10)\quad \begin{aligned} &(1)\quad -\alpha(v)=\Delta^h v_\mu+T(v, v_\mu)+C(v, w_\mu)-A_\mu(v),\\ &(2)\quad -\alpha^1(v)=\Delta^h w_\mu(v)+R^1(v, v_\mu)+P^1(v, w_\mu),\\ &(3)\quad -\alpha^2(v)=\Delta^h A_\mu(v)+R^2(v, v_\mu)+P^2(v, w_\mu),\end{aligned}$$

and

$$(3.11)\quad \begin{aligned} &(1)\quad -\beta(v)=\Delta^{v}v_{\mu}(v)-C(v_{\mu},v)=\Delta^{0}v_{\mu}(v),\\ &(2)\quad -\beta^{1}(v)=\Delta^{v}w_{\mu}(v)-P^{1}(v_{\mu},v)+S^{1}(v,w_{\mu})-A_{\mu}(v),\\ &(3)\quad -\beta^{2}(v)=\Delta^{v}A_{\mu}(v)-P^{2}(v_{\mu},v)+S^{2}(v,w_{\mu}). \end{aligned}$$

We shall consider $(\mathcal{L}_{\mu}B^{h}(v))\varepsilon$ and $(\mathcal{L}_{\mu}B^{v}(v))\varepsilon$. For example, it follows from (3.8) that

$$X_{\mu}(B^{h}(v)\varepsilon)-B^{h}(v)(X_{\mu}\varepsilon)=B^{h}(\alpha(v))\varepsilon+B^{v}(\alpha^{1}(v))\varepsilon+Z(\alpha^{2}(v))\varepsilon,$$

Then (1.10), (1.11) and (3.1) (2) lead us to the first equation of the following (3.12). In like manner the second of (3.12) will be obtained from (3.9).

$$(3.12)\quad \begin{aligned} &(1)\quad \mathcal{L}_{\mu}D(v)=D(\alpha(v))+C(\varepsilon,\alpha^{1}(v))+\alpha^{1}(v)-\alpha^{2}(\varepsilon,v),\\ &(2)\quad \mathcal{L}_{\mu}C(\varepsilon,v)=D(\beta(v))+C(\varepsilon,\beta^{1}(v))+\beta^{1}(v)-\beta^{2}(\varepsilon,v). \end{aligned}$$

If we restrict ourselves to deal with some special Finsler connections, these equations together with (3.3) give a clear and interesting result as follows.

(3.13) **Proposition.** *If the Finsler connection $F\Gamma$ is such that $D=0$ and $C(\varepsilon,v)=0$ for any $v\in V$, then*

(1) $w_{\mu}=A_{\mu}\varepsilon$, (2) $\alpha^{1}(v)=\alpha^{2}(\varepsilon,v)$, (3) $\beta^{1}(v)=\beta^{2}(\varepsilon,v)$,

for any $v\in V$.

In some sense this may be thought of as the analogue of (1.13). The condition for $F\Gamma$ mentioned in (3.13) is not a strong restriction, because the well-known Finsler connections due to Berwald, Cartan and Rund satisfy the condition. Its geometric meaning was made clear in [16], Ch. IV.

We shall turn to treat the so-called Jacobi identities satisfied by Lie products of the X_{μ} and two basic vector field:

$$[X_{\mu},[B^{h}(1),B^{h}(2)]]+S_{12}\{[B^{h}(1),[B^{h}(2),X_{\mu}]]\}=0,$$

$$[X_\mu, [B^h(1), B^v(2)]] + [B^h(1), [B^v(2), X_\mu]] + [B^v(2), [X_\mu, B^h(1)]] = 0,$$

$$[X_\mu, [B^v(1), B^v(2)]] + S_{12}\{[B^v(1), [B^v(2), X_\mu]]\} = 0,$$

where and in the following we shall often use the notation $S_{12}\{\cdots\}$, for brevity, which denotes the interchange of the indices 1 and 2 in the parenthesis and subtraction, for instance,

$$S_{12}\{K(\varepsilon, 1, 2)\} = K(\varepsilon, 1, 2) - K(\varepsilon, 2, 1).$$

The h-, v-horizontal parts and the vertical part of the above three Jacobi identities are found easily, according to (1.12), (3.8) and (3.9). Then, equating each part to zero, we obtain the following eight relations and a trivial relation:

$$\text{(3.14)}\quad \mathcal{L}_\mu T(1,2) + \alpha(T(1,2)) + \beta(R^1(1,2)) + S_{12}\{\Delta^h\alpha(1,2) - T(1,\alpha(2)) + C(1,\alpha^1(2)) + \alpha^2(1,2)\} = 0,$$

$$\text{(3.15)}\quad \mathcal{L}_\mu R^1(1,2) + \alpha^1(T(1,2)) + \beta^1(R^1(1,2)) + S_{12}\{\Delta^h\alpha^1(1,2) - R^1(1,\alpha(2)) - P^1(1,\alpha^1(2))\} = 0,$$

$$\text{(3.16)}\quad \mathcal{L}_\mu R^2(1,2) + \alpha^2(T(1,2)) + \beta^2(R^1(1,2)) + S_{12}\{\Delta^h\alpha^2(1,2) - R^2(1,\alpha(2)) - P^2(1,\alpha^1(2))\} = 0,$$

$$\text{(3.17)}\quad \mathcal{L}_\mu C(1,2) + \alpha(C(1,2)) + \beta(P^1(1,2)) - \Delta^h\beta(2,1) + \Delta^v\alpha(1,2) - C(\alpha(1),2) - T(1,\beta(2)) - C(1,\beta^1(2)) + \beta^2(1,2) = 0,$$

$$\text{(3.18)}\quad \mathcal{L}_\mu P^1(1,2) + \alpha(C(1,2)) + \beta^1(P^1(1,2)) - \Delta^h\beta^1(2,1) + \Delta^v\alpha^1(1,2) - P^1(\alpha(1),2) + S^1(2,\alpha^1(1)) - R^1(1,\beta(2)) - P^1(1,\beta^1(2)) - \alpha^2(2,1) = 0,$$

$$\text{(3.19)}\quad \mathcal{L}_\mu P^2(1,2) + \alpha^2(C(1,2)) + \beta^2(P^1(1,2)) - \Delta^h\beta^2(2,1) + \Delta^v\alpha^2(1,2) - P^2(\alpha(1),2) + S^2(2,\alpha^1(1)) - R^2(1,\beta(2)) - P^2(1,\beta^1(2)) = 0,$$

$$\text{(3.20)}\quad \mathcal{L}_\mu S^1(1,2) + \beta^1(S^1(1,2)) + S_{12}\{\Delta^v\beta^1(1,2) - P^1(\beta(1),2) - S^1(1,\beta^1(2)) + \beta^2(1,2)\} = 0,$$

$$\text{(3.21)}\quad \mathcal{L}_\mu S^2(1,2) + \beta^2(S^1(1,2)) + S_{12}\{\Delta^v\beta^2(1,2) - P^2(\beta(1),2) - S^2(1,\beta^1(2))\} = 0.$$

These equations give, of course, the Lie derivatives of the torsion and curvature tensors. Moreover these will be used such that, for instance, if X_μ satisfies the differential equations $\alpha=\alpha^1=0$ and $\alpha^2=\beta^2=0$, then (3.16) show that $\mathcal{L}_\mu R^2=0$ is regarded as one of the integrability conditions for $\alpha^2=0$, that is $\Delta^h\alpha^2(1,2)-\Delta^h\alpha^2(2,1)=0$ (cf. [8, 24, 25, 26]).

Finally it follows from (3.9) and (1.5) that

$$(3.22)\qquad \mathcal{L}_\mu Y(v)=[X_\mu, Y(v)]=B^h(\beta(v))+Y(\beta^1(v)) \\ +Z(\beta^2(v)-C(\beta^1(v))+\mathcal{L}_\mu C(v)).$$

In the case of infinitesimal linear transformations, it follows from (3.1) (3) that $\mathcal{L}_\mu Y(v)=0$, so that (3.22) gives $\beta=\beta^1=0$ and β^2 is equal to $-\mathcal{L}_\mu C$.

§4. Decomposition of infinitesimal V-transformations

In the first place we shall be concerned with the tangent vector field X_ρ induced by a one-parameter group of V-rotations ρ_{Vt} defined by (2.20), which will be called the *infinitesimal V-rotation.* Let $\underline{X}_{\rho(v)}$ be the tangent vector field on L induced by a one-parameter group of transformations $\rho_{(v)t}$ (v being fixed) belonging to ρ_{Vt}. If we take a real-valued function f on L and a point $z\in L$, then (2.20) gives

$$(\underline{X}_{\rho(v)}f)_z=\mathrm{d}/\mathrm{d}t(f(\rho_{(v)t}(z)))\,|_{t=0}=\mathrm{d}/\mathrm{d}t(f(z(\alpha_{(v)t}(z)))\,|_{t=0}\,.$$

Therefore, if $A_{(v)\rho}(z)$ denotes the tangent vector to the curve $t\mapsto\alpha_{(v)t}(z)$ on G at the unit $e\in G$, we obtain $(\underline{X}_{\rho(v)})_z=\underline{Z}(A_{(v)\rho}(z))_z$, the fundamental vector at $z\in L$ corresponding to $A_{(v)\rho}(z)$. It is observed from (2.9) and (2.5) that

$$(X_\rho)_{(zv,z)}=(\iota^{-1})'((\underline{X}_{\rho(v)})_z, 0_v)=(\iota^{-1})'(\underline{Z}(A_{(v)\rho}(z))_z, 0_v),$$

where $(zv, z)=\iota^{-1}(z, v)\in F$ and 0_v denotes the zero-vector at $v\in V$. Consequently the equation (10.6) (3) of [16] gives immediately

$$(X_\rho)_{(zv,z)}=Y(A_{(v)\rho}(z)\varepsilon)+Z(A_{(v)\rho}(z)).$$

In order to obtain the simple form of X_ρ, we shall introduce the mapping $A_\rho: (zv, z) \in F \mapsto A_{(v)\rho}(z) \in L(G)$; it is concluded that

(4.1) **Theorem.** *An infinitesimal V-rotation X_ρ is written as*

$$X_\rho = Y(A_\rho(\varepsilon)) + Z(A_\rho),$$

where A_ρ is a Finsler tensor field of $(1,1)$*-type.*

It remains to prove that the above A_ρ is a Finsler tensor field. In fact, this has been proved in (3.3); the direct proof, however, is as follows. It is seen from (2.19) that $\alpha_{(g^{-1}v)t}(zg) = g^{-1}\alpha_{(v)t}(z)g$, so that $A_\rho \cdot \tau_g(zv, z) = \mathrm{ad}(g^{-1})A_\rho(zv, z)$, which shows the tensor property of A_ρ.

Next, we shall introduce a special infinitesimal V-transformation X_ν, which will play an important role in future.

(4.2) **Definition.** An infinitesimal V-transformation X_ν is called *N-natural* with respect to a given non-linear connection N if X_ν satisfies $\pi'_T \cdot \pi'_1 \mathcal{L}_\nu B^h(v) = 0$, where $\mathcal{L}_\nu$ denotes the Lie derivative with respect to X_ν and $B^h(v)$ is the h-basic vector field of any Finsler connection $F\Gamma = (\Gamma, N)$ having the N as its non-linear connection.

In order to justify the name "N-natural", it must be shown that this definition is independent of the choice of the connection Γ together with which the given N consists of the Finsler connection $F\Gamma = (\Gamma, N)$ under consideration. To do so we shall refer to the contents of [16], §25. If we consider two Finsler connections $F\Gamma = (\Gamma, N)$ and $^*F\Gamma = (^*\Gamma, N)$ with the same non-linear connection N, the h-basic vector fields are in the relation

$$^*B^h(v) = B^h(v) + Z(A^h(v)),$$

where A^h is a Finsler tensor field of (1,2)-type. Hence (3.7) (1) gives $\mathcal{L}_\nu {}^*B^h(v) = \mathcal{L}_\nu B^h(v) + Z(\mathcal{L}_\nu A^h(v))$, so that $\pi'_T \cdot \pi'_1 \mathcal{L}_\nu {}^*B^h(v) = \pi'_T \cdot \pi'_1 \mathcal{L}_\nu B^h(v)$, which proves the assertion. It is noted that the condition in (4.2) means that the h-horizontal part of $\mathcal{L}_\nu B^h(v)$

vanishes.

(4.3) **Theorem.** *We shall consider a Finsler connection* $F\Gamma=(\Gamma, N)$. *The N-natural infinitesimal V-transformation* X_ν *with respect to the N of* $F\Gamma$ *is written as* $X_\nu=B^h(v_\nu)+B^v(w_\nu)+Z(A_\nu)$ *in terms of the* $F\Gamma$, *where*

(1) $$w_\nu=\Delta^h v_\nu(\varepsilon)+T(\varepsilon, v_\nu)-D(v_\nu),$$

(2) $$A_\nu(v)=\Delta^h v_\nu(v)+C(v, w_\nu)+T(v, v_\nu).$$

Proof. It is observed from (4.2) that X_ν is characterized by $\alpha=0$ in (3.8), hence (3.10) (1) gives the equation (2). Next (3.3) and (2) give the equation (1) immediately.

As a consequence of (4.3), it is noteworthy that X_ν is determined by v_ν only.

(4.4) **Corollary.** *If a Finsler connection* $F\Gamma=(\Gamma, N)$ *satisfies* $T=D=0$, *the N-natural infinitesimal V-transformation* X_ν *with respect to N is written in terms of* $F\Gamma$ *as*

$$X_\nu=B^h(v_\nu)+Y(\Delta^h v_\nu(\varepsilon))+Z(\Delta^h v_\nu),$$

where v_ν *is a Finsler tensor field of* (1, 0)*-type.*

The proof of (4.4) will be easily obtained from (4.3) and (1.5). It is remarked that the conditions $T=D=0$ in (4.4) are satisfied by the well-known Finsler connections due to Berwald, Cartan and Rund.

We are now in the position to mention the fundamental property of infinitesimal V-transformations announced in the introduction.

(4.5) **Fundamental decomposition theorem.** *Let* X_μ *be an arbitrary infinitesimal V-transformation and N be a given non-linear connection. Then* X_μ *is expressed uniquely as* $X_\mu=X_\nu+X_\rho$, *where* X_ν *is the N-natural infinitesimal V-transformation with respect to N and* X_ρ *is the infinitesimal V-rotation.*

Proof. We shall refer to the Finsler N-connection ${}^{0}F\Gamma=(\Gamma, N)$ (cf. [16], p. 112), which is constructed from the N only; the Cartan tensor C of ${}^{0}FD$ vanishes. It follows from (3.2) and (1.5) that

$$\begin{aligned} X_\mu &= B^h(v_\mu) + Y(w_\mu) + Z(A_\mu) \\ &= \{B^h(v_\mu) + Y(\Delta^h v_\mu(\varepsilon) + T(\varepsilon, v_\mu) - D(v_\mu)) + Z(\Delta^h v_\mu + T(\delta, v_\mu))\} \\ &+ \{Y(w_\mu - \Delta^h v_\mu(\varepsilon) - T(\varepsilon, v_\mu) + D(v_\mu)) + Z(A_\mu - \Delta^h v_\mu - T(\delta, v_\mu))\}, \end{aligned}$$

where δ is the Kronecker's delta, a Finsler tensor field of (1, 1)-type. Let us denote the term in the first (resp. second) parenthesis of the above equation by X_ν (resp. X_ρ). Then X_ν is N-natural in virtue of (4.3) and $C=0$. While X_ρ is an infinitesimal V-rotation, because it is observed from (3.3) that

$$(A_\mu - \Delta^h v_\mu - T(\delta, v_\mu))\varepsilon = w_\mu - \Delta^h v_\mu(\varepsilon) - T(\varepsilon, v_\mu) + D(v_\mu),$$

hence (4.1) proves the assertion. The uniqueness of the decomposition will be easily shown.

For the sake of clarity we shall refer now to the induced coordinate $(x^i, y^i, z_a{}^i)$ of F and write down the explicit expressions of X_ρ and X_ν. First, $Y(v)$ and $Z(A)$ are written as

$$Y(v)_u = z_a{}^i v^a \partial/\partial y^i, \qquad Z(A)_u = A^a{}_b z_a{}^i \partial/\partial z_b{}^i,$$

where $u=(x^i, y^i, z_a{}^i)$, $v=(v^a)$ and $A=(A^a{}_b)$. Then, if we denote by $X^i{}_j(x, y)$ the components of the Finsler tensor field A_ρ in (4.1), then (4.1) gives the expression of X_ρ:

(4.1)′ $$X_\rho = X^i{}_j(x, y) y^j \partial/\partial y^i + X^i{}_j(x, y) z_a{}^j \partial/\partial z_a{}^i.$$

Next, let us treat the N-natural infinitesimal V-transformation X_ν. If we denote by $X^i(x, y)$ the components of the Finsler tensor field v_ν in (4.3), it follows from (1) and (2) in (4.3) that the components $X^{(i)}$ (resp. $X^i{}_j$) of w_ν (resp. A_ν) are written in the form

$$\begin{aligned} X^{(i)} &= (\delta_j X^i) y^j + N^i{}_j X^j, \\ X^i{}_j &= \delta_j X^i + C_j{}^i{}_k X^{(k)} + F_j{}^i{}_k X^k, \end{aligned}$$

where $\delta_j = \partial_j - N^k{}_j \dot{\partial}_k$ is a kind of partial differential operator with respect to the non-linear connection N as has been announced in the introduction. Hence (4.3) leads us to the expression of X_ν:

$$(4.3)' \qquad X_\nu = X^i(x,y)\partial/\partial x^i + (\delta_j X^i(x,y))y^j\partial/\partial y^i + (\delta_j X^i(x,y))z_a{}^j\partial/\partial z_a{}^i.$$

The tangent vector field $\underline{X}_\nu$ (resp. $\underline{X}_\rho$) on the total space T of the tangent bundle $T(M)$ given in the introduction is nothing but the projection of the X_ν (resp. X_ρ) on T, the tangent vector field induced by the one-parameter group of the associated transformations in virtue of (2.13). If we interprete (4.3)′ in the classical sense, *the N-natural infinitesimal V-transformation is thought of as*

$$(4.3)'' \qquad \bar{x}^i = x^i + X^i(x,\dot{x})\mathrm{d}t, \qquad \dot{\bar{x}}^i = \dot{x}^i + (\delta_j X^i(x,\dot{x}))\dot{x}^j\mathrm{d}t.$$

Roughly speaking, this is generated by the Finsler vector field $X^i(x,\dot{x})$. On the other hand, the interpretation of (4.1)′ in the classical sense is that *the infinitesimal V-rotation is regarded as*

$$(4.1)'' \qquad \bar{x}^i = x^i, \qquad \dot{\bar{x}}^i = X^i{}_j(x,\dot{x})\dot{x}^j\mathrm{d}t,$$

that is the infinitesimal non-linear rotation of the element of support.

We shall finally introduce a special infinitesimal V-transformation which satisfies a weaker condition than (3.1) (3) for an infinitesimal linear transformation.

(4.6) **Definition.** A infinitesimal V-transformation X_μ is called *semi-linear*, if $\mathcal{L}_\mu Y(v)$ is vertical.

We shall here deal only with a semi-linear infinitesimal V-rotation X_σ. Since the property "semi-linear" is independent of any Finsler connection, we shall refer to the Finsler connection $F\Gamma$ such that its Cartan tensor C vanishes. It then follows from (3.22) that $\mathcal{L}_\sigma Y(v) = B^h(\beta(v)) + Y(\beta^1(v)) + Z(\beta^2(v))$, hence $\beta = \beta^1 = 0$ for the semi-linear X_σ. Thus (3.11) and (4.1) lead us to $\Delta^0 w_\sigma(v) = A_\sigma(v)$. It is remarked that this is consistent with (3.3) in virtue of (3.6), provided that X_σ be positively homogeneous. Hence we obtain

(4.7) **Proposition.** *A positively homogeneous and semi-linear infinitesimal V-rotation X_σ is written in the form*

$$X_\sigma = Y(w_\sigma) + Z(\Delta^0 w_\sigma),$$

where w_σ is a Finsler (1)*p.-h. tensor field of* (1, 0)*-type.*

In terms of the induced coordinate $(x^i, y^i, z_a{}^i)$ of F, the X_σ is written as

$$(4.7)' \qquad X_\sigma = X^{(i)}(x, y)\partial/\partial y^i + (\dot{\partial}_j X^{(i)}(x, y))z_a{}^j \partial/\partial z_a{}^i,$$

where $X^{(i)}(x, y)$ are the components of w_σ and (1)*p.-h.* If w_σ in (4.7) is the element of support ε, then $X_\sigma = Y(\varepsilon) + Z(\delta)$. This special X_σ has been treated in detail in [14] and [16], p. 203.

According to (3.6) it should be remarked that, if we deal with the positively homogeneous case, then $X^i{}_j(x, y)$ in (4.1)′ are (0)*p.-h.* and $X^i(x, y)$ in (4.3)′ are (0)*p.-h.*

§5. Isometric *V*-transformations of Finsler metrics

Let $L(x, y)$ be a Finsler fundamental function of an n-dimensional Finsler space M, by which the length s of a curve $t \in [a, b] \mapsto x(t)$ of M is defined by the integral

$$s = \int_a^b L(x, \mathrm{d}x/\mathrm{d}t)\,\mathrm{d}t.$$

In the following let us assume that L satisfies all the usual conditions [23]. The fundamental tensor field g of (0, 2)-type is given by

$$(5.1) \qquad g = \Delta^0 \Delta^0 L^2/2,$$

which is regarded as a V^0_2-valued function on the total space F of the Finsler bundle $F(M)$. The relation

$$(5.2) \qquad L^2 = g(\varepsilon, \varepsilon)$$

holds as a consequence of the homogeneity property of $L(x, y)$ with respect to $y = (y^i)$.

(5.3) **Proposition.** *Let $\mathcal{L}_\mu$ be the Lie derivative with respect to an infinitesimal V-transformation X_μ. The equation $\mathcal{L}_\mu L(x, y)=0$ is a consequence of the equation $\mathcal{L}_\mu g(x, y)=0$.*

This will be proved immediately from (3.1) (2) and (5.2). If X_μ is an infinitesimal linear transformation, then (3.1) (3) means that the operators $\mathcal{L}_\mu$ and Δ^0 commute with each other according to (1.9); it follows from (5.1) that the equation $\mathcal{L}_\mu g(x, y)=0$ is derived from the equation $\mathcal{L}_\mu L(x, y)=0$. Therefore $\mathcal{L}_\mu L=0$ is equivalent to $\mathcal{L}_\mu g=0$, the usual situation in the case of the ordinary theory of transformations [27]. This is no longer the case of our general V-transformation. For this reason we are led to the following definition of the concept of isometry.

(5.4) **Definition.** A V-transformation μ_V is called an *isometry* of a Finsler metric $L(x, y)$, if the lift $\bar{\mu}$ of μ_V to F preserves the fundamental tensor g derived from L. An infinitesimal V-transformation X_μ is called an *isometry* of $L(x, y)$, if the equation $\mathcal{L}_\mu g=0$ is satisfied, where $\mathcal{L}_\mu$ is the Lie derivative with respect to X_μ.

Proposition (5.3) shows that the above definition for X_μ to be an isometry is generally stronger than the one given by the equation $\mathcal{L}_\mu L=0$.

In this section we shall only refer to the Finsler connection $C\Gamma$ due to Cartan [4], because $C\Gamma$ is metric ($\Delta^h g=\Delta^v g=0$) and will be convenient for dealing with an isometry. Then (3.2) gives a condition for an isometry:

$$(5.5)\qquad \mathcal{L}_\mu g(v_1, v_2)=g(A_\mu(v_1), v_2)+g(A_\mu(v_2), v_1)=0,$$

for any $v_1, v_2 \in V$.

In the first place we shall be concerned with an N-natural infinitesimal V-transformation X_ν with respect to the non-linear connection N of $C\Gamma$, which is given by the connection parameters $N^i{}_j(x, y) = y^k \Gamma^*{}_k{}^i{}_j(x, y)$ in the notation of Cartan. It is well-known that this non-linear connection N has been treated by several authors and coincides with the ones of the Finsler connections due to L. Berwald ($y^k G_k{}^i{}_j$) [1] and H. Rund ($y^k P^*{}_k{}^i{}_j = y^k P_k{}^i{}_j$) [22, 23]. It follows from (4.4) and (1.5) that

$$(5.6) \qquad A_\nu(v) = \Delta^h v_\nu(v) + C(v, \Delta^h v_\nu(\varepsilon)), \qquad v \in V,$$

hence (5.5) is of the form

$$(5.7) \qquad g(\Delta^h v_\nu(v_1), v_2) + g(\Delta^h v_\nu(v_2), v_1) + 2C_*(v_1, v_2, \Delta^h v_\nu(\varepsilon)) = 0,$$

for any $v_1, v_2 \in V$, where the Finsler tensor field C_* of (0, 3)-type is defined by $C_*(v_1, v_2, v_3) = g(C(v_1, v_3), v_2)$, $v_1, v_2, v_3 \in V$. In terms of the conponents (5.7) is written down in the form

$$(5.7)' \qquad X_{i|j} + X_{j|i} + 2C_i{}^r{}_j X_{r|0} = 0,$$

in the notation of (1.9)′ and (4.3)′, where $X_i = g_{ij} X^j$ and $X_{r|0} = X_{r|j}(x, y) y^j$. Therefore the condition for X_ν to be an isometry coincides formally with the one of the ordinary case [27], p. 180.

(5.8) **Definition.** A Finsler tensor field v of (1, 0)-type with the components $X^i(x, y)$ is called a *Killing vector field*, if (5.7)′ is satisfied with respect to the Finsler connection $C\Gamma$ of Cartan.

Next we shall be concerned with an infinitesimal V-rotation X_ρ. It then follows from (4.1) that the A_μ in (5.5) is equal to $A_\rho + C(A_\rho(\varepsilon))$, so that (5.5) is written in the from

$$(5.9) \qquad g(A_\rho(v_1), v_2) + g(A_\rho(v_2), v_1) + 2C_*(v_1, v_2, A_\rho(\varepsilon)) = 0,$$

for any $v_1, v_2 \in V$. If we use the notation in (4.1)′, then (5.9) is written down as

$$(5.9)' \qquad X_{ij} + X_{ji} + 2C_i{}^r{}_j X_{r0} = 0,$$

where we put $X_{ij} = g_{ik} X^k{}_j$ and $X_{r0} = X_{rj}(x, y) y^j$. It is interested to observe that (5.9)′ coincides with the equation (2.6) of [9], which was derived by A. Kawaguchi from a different and geometric standpoint.

Let us treat (5.9)′ in detail. If we denote by Y_{ij} (resp. Z_{ij}) the symmetric (resp. skew-symmetric) part of X_{ij}, then (5.9)′ is rewritten as $Y_{ij}+C_i{}^r{}_j(Y_{r0}+Z_{r0})=0$, which implies $Y_{i0}=0$ at once. Hence we obtain

(5.10) $$X_{ij}=-C_i{}^r{}_j Z_{r0}+Z_{ij}.$$

As a consequence we have

(5.11) **Theorem.** *There are always isometric V-rotations of the maximal order $n(n-1)/2$ in the n-dimensional Finsler space. The isometric infinitesimal V-rotation is given by the equation* (4.1)′ *and* (5.10), *where $Z_{ij}(x, y)$ are the components of arbitrary skew-symmetric Finsler tensor field of* $(0, 2)$*-type.*

If an infinitesimal *linear* transformation X_μ is isometric, i.e. $\mathcal{L}_\mu L=0$ or $\mathcal{L}_\mu g=0$, the equation $\mathcal{L}_\mu C=0$ holds automatically, because $\Delta^0\mathcal{L}_\mu g=\mathcal{L}_\mu\Delta^0 g=2\mathcal{L}_\mu C_*$. This is, however, no longer the case of our V-transformation in general, because Δ^0 does not commute with $\mathcal{L}_\mu$. Consequently it seems natural to introduce stronger condition than the isometry as follows:

(5.12) **Definition.** A V-transformation μ_V is called a *strict isometry* of a Finsler metric $L(x, y)$, if the lift $\bar{\mu}$ of μ_V to F preserves the fundamental tensor g and the Cartan tensor C. An infinitesimal V-transformation X_μ is called a *strict isometry* of L, if the equations $\mathcal{L}_\mu g=0$ and $\mathcal{L}_\mu C=0$ hold, where $\mathcal{L}_\mu$ is the Lie derivative with respect to X_μ.

It follows from (3.2) that the equation $\mathcal{L}_\mu C=0$ is written as

(5.13) $$\begin{aligned}\mathcal{L}_\mu C(v_1, v_2) &= \Delta^h C(v_1, v_2, v_\mu)+\Delta^v C(v_1, v_2, w_\mu)\\ &\quad -A_\mu(C(v_1, v_2))+C(A_\mu(v_1), v_2)+C(A_\mu(v_2), v_1)=0,\end{aligned}$$

for any $v_1, v_2\in V$.

In the first place we shall consider an N-natural infinitesimal V-transformation X_ν with respect to the non-linear connection N of the Finsler connection $C\Gamma$ of Cartan. It then follows from (5.13) that $\mathcal{L}_\mu C=0$ is written

$$(5.14)\qquad \Delta^h C(v_1, v_2, v_\nu)+\Delta^v C(v_1, v_2, \Delta^h v_\nu(\varepsilon))-\Delta^h v_\nu(C(v_1, v_2)) -C(C(v_1, v_2), \Delta^h v_\nu(\varepsilon))+C(\Delta^h v_\nu(v_1)+C(v_1, \Delta^h v_\nu(\varepsilon)), v_2) +C(\Delta^h v_\nu(v_2)+C(v_2, \Delta^h v_\nu(\varepsilon)), v_1)=0,$$

for any $v_1, v_2 \in V$. In terms of the components (5.14) is written down in the form

$$(5.14)'\qquad C_j{}^i{}_{k|r}X^r+C_j{}^i{}_k|_r X^r{}_{|0}-(X^i{}_{|r}+C_r{}^i{}_s X^s{}_{|0})C_j{}^r{}_k +(X^r{}_{|j}+C_j{}^r{}_s X^s{}_{|0})C_r{}^i{}_k+(X^r{}_{|k}+C_k{}^r{}_s X^s{}_{|0})C_r{}^i{}_j=0.$$

If we denote by U_{ij} (resp. V_{ij}) the symmetric (resp. skew-symmetric) part of $X_{i|j}$, the process by which (5.10) was derived from (5.9)′ is applied also to (5.7)′ and we obtain

$$(5.15)\qquad X_{i|j}=-C_i{}^r{}_j V_{r0}+V_{ij}.$$

Moreover (5.14)′ will be written in terms of X^i and V_{ij} as

$$(6.16)\qquad C_{ijk|r}X^r+C_i{}^r{}_j|_k V_{r0}+C_i{}^r{}_k V_{rj}+C_i{}^r{}_j V_{rk}=0.$$

Summarizing up the above results, we obtain

(5.17) **Definition.** A Finsler tensor field v_ν of (1, 0)-type is called a *strict-Killing vector field*, if v_ν satisfies the equations (5.7) and (5.14).

(5.18) **Theorem.** *A Finsler tensor field v_ν of* (1, 0)*-type with the components X^i is a strict-Killing vector field, if and only if there exists a skew-symmetric Finsler tensor field V of* (0, 2)*-type with the components V_{ij} satisfying the equations* (5.15) *and* (5.16).

We shall turn to the consideration of an infinitesimal V-rotation X_ρ. It then follows from (5.13) that $\mathcal{L}_\rho C=0$ is written

$$(5.19)\qquad \Delta^v C(v_1, v_2, A_\rho(\varepsilon))-A_\rho(C(v_1, v_2))-C(C(v_1, v_2), A_\rho(\varepsilon)) +C(A_\rho(v_1)+C(v_1, A_\rho(\varepsilon)), v_2) +C(A_\rho(v_2)+C(v_2, A_\rho(\varepsilon)), v_1)=0,$$

for any $v_1, v_2 \in V$. In terms of the components (5.19) is written down in the form

$$(5.19)' \qquad C_j{}^i{}_k|_r X^r{}_0 - (X^i{}_r + C_r{}^i{}_s X^s{}_0) C_j{}^r{}_k + (X^r{}_j + C_j{}^r{}_s X^s{}_0) C_r{}^i{}_k + (X^r{}_k + C_k{}^r{}_s X^s{}_0) C_j{}^i{}_r = 0.$$

If we pay attention to (5.10), then (5.19)′ is rewritten in the form

$$(5.20) \qquad C_i{}^r{}_j|_k Z_{r0} + C_j{}^r{}_k Z_{ri} + C_i{}^r{}_k Z_{rj} + C_i{}^r{}_j Z_{rk} = 0.$$

Summarizing up the above results we obtain

(5.21) **Theorem.** *A strictly isometric infinitesimal V-rotation* $X_\rho = Y(A_\rho(\varepsilon)) + Z(A_\rho)$ *is given by* (5.10) *and* (5.20), *where* $X^i{}_j$ $(=g^{ik}X_{kj})$ *are the components of the Finsler tensor field* A_ρ *of* (1, 1)*-type and* Z_{ij} *are the components of a skew-symmetric Finsler tensor field of* (0, 2)*-type.*

We shall be concerned with the case where the Finsler n-space M admits the strictly isometric V-rotations of the maximal order $n(n-1)/2$. In this case (5.20) must be identically satisfied for any skew-symmetric tensor with the components Z_{ij}; this is expressed by

$$(5.22) \qquad C_i{}^r{}_j|_k y^s + C_j{}^r{}_k \delta_i^s + C_k{}^r{}_i \delta_j^s + C_i{}^r{}_j \delta_k^s = C_i{}^s{}_j|_k y^r + C_j{}^s{}_k \delta_i^r + C_k{}^s{}_i \delta_j^r + C_i{}^s{}_j \delta_k^r.$$

Putting $s=k$ in (5.22) and summing with respect to s, we obtain

$$nC_i{}^r{}_j = C_i|_j y^r + C_j \delta_i^r + C_i \delta_j^r,$$

where we put $C_i = C_i{}^r{}_r$ and made use of the complete symmetry property of $C_j{}^i{}_k|_h$ with respect to the lower indices. Contraction of the above by g^{ij} gives at once $(n-2)\ C^r = g^{ij}C_i|_j y^r$; it then follows from $C_r y^r = 0$ that $g^{ij}C_i|_j = 0$, hence $C^r = 0$ if $n \geqq 3$. Therefore the well-known Deicke's theorem [3, 5] leads us to

(5.23) **Theorem.** *If a Finsler space* M^n *of dimension* $n(\geqq 3)$ *admits the strictly isometric V-rotations of the maximal order* $n(n-1)/2$, *then* M^n *is Riemannian.*

Let us deal finally with the exceptional case of the dimension 2. As above obtained, $g^{ij}C_i|_j=0$ holds. As is well-known [2]; [23], p. 253, the components C_{ijk} of the Cartan tensor C_* of a 2-dimensional Finsler space is of the simple form

$$C_{ijk}=Jm_i m_j m_k,$$

where J is a scalar and $m^i(=g^{ij}m_j)$ are the components of the unit vector orthogonal to the element of support. It follows from $m_i|_j=-y_i m_j/L^2$ $(y_i=g_{ij}y^j)$ that

$$g^{ij}C_i|_j=g^{ij}(Jm_i)|_j=J|_j m^j=0,$$

which implies that there exists a scalar K such that $J|_j=Ky_j$. According to the complete symmetry property of $C_{hij}|_k$, we obtain $K=-J/L^2$ easily and

$$\text{(5.24)}\qquad C_{hij}|_k=-(J/L^2)(m_h m_i m_j y_k+m_h m_i y_j m_k +m_h y_i m_j m_k+y_h m_i m_j m_k).$$

Substitute from (5.24) in (5.22) to get

$$\begin{aligned}&-(J/L^2)(m_i m_j y_k+m_i y_j m_k+y_i m_j m_k)(m^r y^s-m^s y^r)\\ &\quad+Jm_i m_j(m^r\delta^s_k-m^s\delta^r_k)+Jm_j m_k(m^r\delta^s_i-m^s\delta^r_i)\\ &\quad+Jm_k m_i(m^r\delta^s_j-m^s\delta^r_j)=0.\end{aligned}$$

It is easy to show that this equation is solely reduced to the trivial equation $g_{ij}=y_i y_j/L^2+m_i m_j$. As a conclusion we obtain from (5.24)

(5.25) **Theorem.** *A necessary and sufficient condition for a Finsler space M^2 of 2 dimensions to admit the strictly isometric V-rotation of the maximal order 1 is that the components C_{ijk} of the Cartan tensor field C_* satisfy the equation*

$$LC_{hij}|_k+C_{hij}l_k+C_{hik}l_j+C_{hkj}l_i+C_{kij}l_h=0,$$

where $l^i(=g^{ij}l_j)$ are the components of the unit vector having the direction of the element of support.

It seems interested to study the Finsler space of general dimension such that the equation in (5.25) holds.

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References

[1] Berwald, L.: Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus. *Math. Z.* **25** (1926), 40-73. Correction: *Math. Z.* **26** (1927), 176.

[2] Berwald, L.: Über zwei-dimensionale allgemeine metrische Räume. I, II. *J. reine angew. Math.* **156** (1927), 191-210 and 211-222.

[3] Brickell, F.: A new proof of Deicke's theorem on homogeneous functions. *Proc. Amer. Math. Soc.* **16** (1965), 190-191.

[4] Cartan, E.: Les espaces de Finsler. Actualités 79, Paris 1934. 2nd ed. 1971.

[5] Deicke, A.: Über die Finsler-Räume mit $A_i=0$. *Arch. Math.* **4** (1953), 45-51.

[6] Egorov. I. P.: Motions in generalized differential-geometric spaces. *Progress in Math.* **6** (1970), 171-227. Translated from *Algebra, Topology, Geometry* 1965 (Russian). *Akad. Nauk SSSR Inst. Naučn. Tehn. Infomacii, Moscow,* 1967.

[7] Hashiguchi, M.: On determinations of Finsler connections by deflection tensor fields. *Fac. Sci. Kagoshima Univ.* **2** (1969), 29-39.

[8] Hiramatu, H.: Groups of homothetic transformations in a Finsler space. *Tensor, N. S.* **3** (1954), 131-143.

[9] Kawaguchi, A.: On the theory of non-linear connections II. Theory of Minkowski spaces and of non-linear connections in a Finsler space. *Tensor, N. S.* **6** (1956), 165-199.

[10] Kobayashi, S. and K. Nomizu: Foundations of differential geometry I. Inters. Publ., New York 1963.

[11] Matsumoto, M.: Affine transformations of Finsler spaces. *J. Math. Kyoto Univ.* **3** (1963), 1-35.

[12] Matsumoto, M.: Linear transformations of Finsler connections. *J. Math. Kyoto Univ.* **3** (1964), 145-167.

[13] Matsumoto, M.: A Finsler connection with many torsions. *Tensor, N. S.* **17** (1966), 217-226.

[14] Matsumoto, M.: Intrinsic transformations of Finsler metrics and connections. *Tensor, N. S.* **19** (1968), 303-313.

[15] Matsumoto, M.: On *F*-connections and associated non-linear connections. *J. Math. Kyoto Univ.* **9** (1969), 25-40.

[16] Matsumoto, M.: The theory of Finsler connections. Publ. of Study Group of Geometry, Vol. 5. Okayama Univ. 1970.

[17] Misra, R. B.: The generalised Killing equation in Finsler space. *Rend. Circ. Mat. Palermo* (2) **18** (1969), 99-102.

[18] Misra, R. B. and R. S. Mishra: Lie-derivatives of various geometric entities in Finsler space. *Revue Fac. Sci. Univ. Istanbul, Sér.* A **30** (1965), 77-82.

[19] Misra, R. B. and R. S. Mishra: The Killing vector and the generalised Killing equation in Finsler space. *Rend. Circ. Mat. Palermo* (2) **15** (1966), 216-222.

[20] Nomizu, K.: Lie groups and differential geometry. Publ. of Math. Soc. of Japan, Vol. 2. Tokyo 1956.

[21] Okada, T.: Theory of pair-connection. (Japanese. English summary) *Sci. Eng. Rev. Doshisha Univ.* **5** (1964), 133-152.

[22] Rund, H.: On the analytic properties of curvature tensors in Finsler spaces. *Math. Ann.* **127** (1954), 82-104.

[23] Rund. H.: The differential geometry of Finsler spaces. Springer, Berlin 1959.

[24] Soós, Gy.: Über Gruppen von Affinitäten und Bewegungen in Finslerschen Räumen. *Acta Math. Hungar.* **5** (1954), 73-83.

[25] Soós, Gy.: Über Gruppen von Automorphismen in affinzusammenhängenden Räumen von Linienelementen. *Publ. Math. Debrecen* **4** (1956), 294-302.

[26] Soós, Gy.: Über eine spezielle Klasse von Finslerschen Räumen. *Publ. Math. Debrecen* **5** (1957), 150-153.

[27] Yano, K.: The theory of Lie derivatives and its applications. North-Holland Publ., Amsterdam 1955.

[28] Yano, K. and S. Kobayashi: Prolongations of tensor fields and connections to tangent bundles I. *J. Math. Soc. Japan* **18** (1966), 194-210.