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On a ring with a plenty of high order derivations

By

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1. Let k and A be commutative rings with 1 and assume that A is a k-algebra. we shall say that the k-algebra A has a plenty of high order derivations over k if the ring of endomorphisms of A over k is filled up with the derivation algebra of A over k, or equivalently any k-linear endomorphism f of A such that $f(1)=0$ is a high order derivation.[1] Such a ring A will be referred to as a P. H. D. ring in the sequel. In the case where both of A and k are fields it was proved in [2] and [4] that A is a P. H. D. ring over k if and only if A is a purely inseparable finite extension of k. The purpose of the present paper is to generalize this result by deleting the assumption that A is a field. The final result is the following

Theorem. *Let k be a field and let A be a commutative k-algebra. Then A is a P. H. D. ring if and only if A satisfies the following three conditions*:

(1) *A is a quasi-local ring.*

(2) *The maximal ideal M of A is nilpotent.*

(3) *The residue field A/M is either k or a purely inseparable finite extension of k.*

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1) Cf [3] for the definition and main properties of high order derivations.

2. The proof of the "if" part of the Theorem

First we shall take up the case where the natural injection $k \to A/M$ is surjective. For the proof of the theorem it suffices to show that the ideal $I_A = \mathrm{Ker}(A \otimes A \to A)$ is nilpotent (cf. [2]). Let $\omega_i (i \in I)$ be a basis of k-module M. Then I_A is generated by $\tau(\omega_i) = 1 \otimes \omega_i - \omega_1 \otimes 1 (i \in I)$ as a left A-module. Hence the nilpotency of I_A follows immediately from the nilpotency of M and the following

Lemma 1. *Let us set $\tau(x) = 1 \otimes x - x \otimes 1$ for $x \in A$. Then we have*

$$(1)_q \qquad \tau(x_0)\tau(x_1)\cdots\tau(x_q) = \sum_{s=0}^{q}(-1)^s \sum_{i_1<\cdots<i_s} x_{i_1}\cdots x_{i_s}\tau(x_0\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_q).$$

Proof. The proof will be carried out by induction on q. The case $q=1$ is immediate since we have $\tau(x_0)\tau(x_1) = \tau(x_0x_1) - x_0\tau(x_1) - x_1\tau(x_0)$. Assuming $(1)_q$ we have

$$\begin{aligned}
&\tau(x_0)\tau(x_1)\cdots\tau(x_q)\tau(x_{q+1}) \\
&= \sum_{s=0}^{q}(-1)^s \sum_{i_1<\cdots<i_s\le q} x_{i_1}\cdots x_{i_s}\tau(x_0\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_q)\tau(x_{q+1}) \\
&= \sum_{s=0}^{q}(-1)^s \sum_{i_1<\cdots<i_s\le q} \{x_{i_1}\cdots x_{i_s}\tau(x_0\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_q x_{q+1}) \\
&\qquad - x_{i_1}\cdots x_{i_s}x_{q+1}\tau(x_0\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_q) - (\prod_{i=0}^{q} x_i)\tau(x_{q+1})\} \\
&= \sum_{s=0}^{q}(-1)^s \sum_{i_1<\cdots<i_s\le q+1} x_{i_1}\cdots x_{i_s}\tau(x_0\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_{q+1}) \\
&\qquad + \sum_{i=0}^{q}(-1)^{q+1}x_0\cdots\hat{x}_i\cdots x_{q+1}\tau(x_i) + \sum_{s=0}^{q}(-1)^{s+1}\binom{q+1}{s}x_0\cdots x_q\tau(x_{q+1}) \\
&= \sum_{s=0}^{q+1}(-1)^s \sum_{i_1<\cdots<i_s\le q+1} x_{i_1}\cdots x_{i_s}\tau(x_0\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_{q+1})
\end{aligned}$$

since we have

$$\sum_{s=0}^{q}(-1)^{s+1}\binom{q+1}{s} = (-1)^{q+1}.$$

Next we shall take up the case where A/M is a purely inseparable extension of degree m over k. Then the characteristic p of k is a positive prime number and m is of the form p^f. Let $y_j (j=1, \cdots, m)$ be elements of A whose residue classes form a basis of A/M

over k and let us set $E_1=\sum_{j=1}^{m} A\tau(y_j)A$. Let $\omega_i (i\in I)$ be a basis of M over k, and let us set $E_2=\sum_{i\in I} A\tau(\omega_i)A$. Then we have $I_A=E_1+E_2$ and the Lemma 1 implies that there exists an integer n such that $E_2^n=(0)$. On the other hand we see easily that $E_1^{m^2}\subseteq E_2$ because $\tau(y_i)^m\in E_2$. From these considerations we easily arrive at the conclusion that I_A is nilpotent.

3. The proof of "only if" part of the Theorem

Lemma 2. *Let A be a k-algebra and let B be a sub-k-algebra. Assume that B is a direct summand of A as a k-module. Then if A is a P. H. D. ring, B is also a P. H. D. ring.*

Proof. Let j be the projection of A onto B and let i be the injection of B into A. Let f be an element of $\mathrm{Hom}_k(B, B)$ such that $f(1)=0$. Then $ifj\in\mathrm{Hom}_k(A, A)$ and $(ifj)(1)=0$. The assumption implies that $I_A^{q+1}(ifj)=0$ for some q where $I_A=\mathrm{Ker}(A\otimes A\to A)$ (cf [1]). In palticular $I_B^{q+1}(ifj)=0$. Since j is identity on B we have $I_B^{q+1}if=iI_B^{q+1}f=0$. Since i is injective we see that $I_B^{q+1}f=0$, i. e., f is also a q-th order derivation.

This is a generalization of proposition 6 of Kikuchi [2].

Lemma 3. *Let k be a field and let A and B be k-algebras. Assume that there exists a k-algebra homomorphism π of A onto B and A is a P. H. D. ring over k. Then B is also a P. H. D. ring over k.*

Proof. Let f be an endomorphism of B over k such that $f(1)=0$. We shall show that f is a high order derivation. Since k is a field there is a k-linear map F of A over k such that $f\pi=\pi F$. Since $0=f(1_B)=f\pi(1_A)$ we can choose F so as $F(1_A)=0$. Then by assumption F is a high order derivation of order, say, q. We shall show that f is also a q-th order derivation of B. In fact let $b_0, b_1, \cdots, b_q$ be arbitrary $(q+1)$-elements of B and let $a_0, a_1, \cdots, a_q$ be elements of A such that $\pi(a_i)=b_i$, $0\leq i\leq b$. Then we have

$$\begin{aligned} f(b_0b_1\cdots b_q) &= f\pi(a_0a_1\cdots a_q) \\ &= \pi F(a_0a_1\cdots a_q) \\ &= \pi\{\sum_{s=0}^{q}(-1)^s \sum_{i_1<\cdots<i_s} a_{i_1}\cdots a_{i_s}F(a_0\cdots\hat{a}_{i_1}\cdots\hat{a}_{i_s}\cdots a_q)\} \\ &= \sum_{s=0}^{q}(-1)^s \sum_{i_1<\cdots<i_s} b_{i_1}\cdots b_{i_s} f(b_0\cdots\hat{b}_{i_s}\cdots\hat{b}_{i_1}\cdots b_q). \end{aligned}$$

Lemma 4. *Let k be a field and let A and B be two k-algebras and let $C=A\times B$ be the direct product of A and B. Then C is not a P. H. D. ring*

Proof. We shall denote the element of C by (a, b) $(a\in A, b\in B)$. Then the sums and products are defined by componentwise operations and the structure homomorphism h of k into C is given by

$$h(x)=(f(x),\ g(x))\ (x\in k),$$

where f and g are structure homomorphisms of A and B respectively. Let us set $e_1=(1,0)$, $e_2=(0,1)$. Let ϕ be an element of $\mathrm{Hom}_k(C, C)$ such that $\phi(e_1)=e_2$ and $\phi(e_2)=-e_2$. Then $\phi(1)=\phi(e_1+e_2)=0$. We shall show that ϕ is not a high order derivation. In fact if ϕ is a high order derivation of order, say, q, then we have

$$\begin{aligned} e_2=\phi(e_1)=\phi(e_1^{q+1}) &= \sum_{s=1}^{q}(-1)^{s-1}\binom{q+1}{s}e_1^{q+1-s}\phi(e_1^s) \\ &= \sum_{s=1}^{q}(-1)^{s-1}\binom{q+1}{s}e_1e_2=0. \end{aligned}$$

This is contradiction.

Lemma 5. *Assume that a k-algebra A is a P. H. D. ring and let α be an element of A. Then there exist integers m and n and an element a of k such that $(\alpha^n-a)^m=0$.*

Proof. Let α be an arbitrary elment of A and let $I(\alpha)$ be the ideal of $k[x]$ consisting of elements $f(x)$ such that $f(\alpha)=0$. Let $f_\alpha(x)$ be a generator of $I(\alpha)$.

(1) The case where $I(\alpha)$ is a prime ideal. If $I(\alpha)=0$ $k[\alpha]$ is isomorphic to a polynomial ring and is contained in A. By

Lemma 2 $k[\alpha]$ must be a P. H. D. ring. This is impossible as we can see immediately (cf. also [2]). Similarly $f_\alpha(x)$ cannot be a separable polynomial of degree >1. Hence $f_\alpha(x)$ is linear or an inseparable polynomial. Let e be the reduced degree of $f_\alpha(x)$. Then a suitable power of α, say $\alpha^{p^f}=\beta$, is a separable element of degree e over k. Hence by the same reasoning as above we must have $e=1$, i. e., $f_\alpha(x)$ has the from $x^{p^f}-a$, $a\in k$, $a\notin k^p$.

(2) The case where $I(\alpha)$ is a primary ideal. In this case $f_\alpha(x)=g_\alpha(x)^n$ for a suitable irreducible polynomial $g_\alpha(x)$ over k. Since $k[\alpha]=k[x]/I(\alpha)$ is a subring of A, and k is a field, $k[\alpha]$ is a P. H. D. ring. Then the Lemma 3 implies $k[x]/g_\alpha(x)$ is also a P. H. D. ring. So the above consideration implies that $g_\alpha(x)$ should be an irreducible polynomial of reduced degree 1.

(3) The case $I(\alpha)$ is not primary. Let

$$f_\alpha(x)=\prod_{i=1}^{s} g_i(x)^{m_i}$$

be a decomposition of $f_\alpha(x)$ into relatively prime factors. Since $k[\alpha]$ is a P. H. D. ring the residue class ring of $k[\alpha]$ is also a P. H. D. ring (Lemma 3). Hence every factor $g_i(x)$ is of reduced degree 1, i. e., $g_i(x)$ is of the from x^n-a. we shall show that the number s of the factors cannot be >1. In fact if $s\geqslant 2$ then $g_1(x)g_2(x)$ has one of the forms, $x(x^n-a)$, $(x^n-a)(x^m-b)$ where a and b are non-zero elements of k and x^n-a and x^m-b are relatively prime. Then $k[x]/g_1(x)g_2(x)$ is isomorphic to the direct product of two k-algebras and is not a P. H. D. ring by Lemma 4. On the other hand this is a residue class ring of $k[\alpha]$ and hence should be a P. H. D. ring by Lemma 3. This is a contradiction. Thus we have seen that the case (3) can not occur, and the proof of Lemma 5 is complete.

From Lemma 5 we can see that every non-unit of A is nilpotent. In fact if α is non-unit of A, then the element a such that $(\alpha^n-a)^m=0$ cannot be a non-zero element. Hence the corresponding a is zero and α is seen to be nilpotent. Since A is assumed to be commutative the non-units of A form an ideal. Thus A is a quasi-local ring. The assertion (2) on residue field is the consequence of Lemma 3 and Theorem 2 of [2].

we shall show next that M is nilpotent. Let us set $B=k+M$. Then B is a sub k-algebra of A and hence a P. H. D. ring by Lemma 2. Now assume that M is not nilpotent and let f be a k-linear endomorphism of B such that $f(1)=0$ and $f(x)=x$ for $x\in M$. We shall show that f is not a high order derivation. In fact if f were a high order derivation there is an even integer q such that f is a q-th order derivation. By assumption $M^{q+1}\neq(0)$ and hence there exist elements $x_0, x_1, \cdots x_q$ in M such that $x_0 x_1\cdots x_q\neq 0$. On the other hand the difining equation for high order derivations implies that

$$\prod_{i=0}^{q} x_i=f(\prod_{i=0}^{q} x_i)=\sum_{s=1}^{q}(-1)^{s-1}\sum_{i_1<\cdots<i_s} x_{i_1}\cdots x_{i_s} f(x_0\cdots \hat{x}_{i_1}\cdots \hat{x}_{i_s}\cdots x_q)$$
$$=\{\sum_{s=1}^{q}(-1)^{s-1}\tbinom{q+1}{s}\}\prod_{i=0}^{q} x_i=\{1+(-1)^{q+1}\}\prod_{i=0}^{q} x_i=0.$$

This is a contradiction !

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