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A note on Kronecker's "Randwertsatz"

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Kronecker brought to light the determinative properties of certain sets of prime numbers for algebraic number fields and their invariants in his "Ueber die irreductibilität von Gleichungen" which was dedicated to Kummer on his 70th birthday celebration. Kronecker's assertion was called "Randwertsatz" of algebraic number theory by M. Bauer because of Kronecker's statement——

"es ist also (in ähnlicher Weise, wie nach dem Cauchy'schen Satze einen Function durch ihre Randwerte bestimmt wird) mit blossen Congruenzbestimmungen der ganze Inbegriff der durch die Gleichung definierten algebrasschen Irrationalitäten bestimmt".

Since then the base of Kronecker's assertion has been amplified into Frobenius-Tschebotareff's dencity theorem, Bauer's theorem and Gaßmann's theorem, and furthermore his plan has been realized as class field theory in the case of relative abelian number fields. However, in this note, we shall mainly discuss Kronecker-Bauer's "Randwertsatz".

1. Let $\boldsymbol{k}$ be a finite number field, $\Omega/\boldsymbol{k}$ a finite extension. Let $\boldsymbol{K}/\boldsymbol{k}$ be the minimal normal extension containing $\Omega/\boldsymbol{k}$, and let $\boldsymbol{K}'/\boldsymbol{k}$ be the maximal normal extension contained in $\Omega/\boldsymbol{k}$, namely

$$\boldsymbol{K}=\Omega^{(0)}\Omega^{(1)}\cdots\cdots\Omega^{(m-1)},$$

$$\boldsymbol{K}'=\Omega^{(0)}\cap\Omega^{(1)}\cap\cdots\cdots\cap\Omega^{(m-1)},$$

where $\Omega^{(0)}=\Omega$, $m=[\Omega:\boldsymbol{k}]$ and $\{\Omega^{(i)};\ i=0,1,\cdots,m-1\}$ are all the conjugates of Ω over $\boldsymbol{k}$. We define $P(\Omega/\boldsymbol{k})$ and $Q(\Omega/\boldsymbol{k})$ by setting

$P(\Omega/k)=\{$prime ideals $\mathfrak{p}$ of k; $^{\exists}\mathfrak{P}$ in Ω, $N_{\Omega/k}\mathfrak{P}=\mathfrak{p}$, $\mathfrak{p}\nmid D(\Omega/k)\}$,
$Q(\Omega/k)=\{$prime ideals $\mathfrak{p}$ of k; $\mathfrak{p}=\mathfrak{P}_1\cdots\mathfrak{P}_m$ in Ω, $\mathfrak{p}\nmid D(\Omega/k)\}$.
$P(\Omega/k)$ is called the regular domain of Ω/k.

In the following propositions **I—XV**, we shall observe that, if a relation containing notations $P(\Omega/k)$ or $Q(\Omega/k)$ is a conclusion, then the relation holds without any exceptional prime ideals, while if a relation containing notations $P(\Omega/k)$ or $Q(\Omega/k)$ is premise, then we can leave a set of prime ideals of density 0 out of account. First we enumerate Bauer's results **I—VI**:

I. $Q(\Omega/k)=Q(K/k)$.

II. $Q(\Omega/k)=P(\Omega/k)\Leftrightarrow K=\Omega(=K')$.

III. $Q(\Omega_1/k)=Q(\Omega_2/k)\Leftrightarrow K_1=K_2$.

IV. $Q(\Omega_1/k)\subseteq Q(\Omega_2/k)\Leftrightarrow K_1\supseteq K_2$.

V. $P(\Omega_1/k)\subseteq Q(\Omega_2/k)\Leftrightarrow \Omega_1\supseteq K_2$.

VI. $Q(\Omega_1/k)\cap Q(\Omega_2/k)=Q(\Omega_1\Omega_2/k)$.

From **III, IV** and **VI**, we obtain immediately

$$\text{VII.}\begin{cases} Q(\Omega_1/k)\cap\cdots\cap Q(\Omega_s/k)\subset Q(\Omega/k)\Leftrightarrow K\cdots K_s\supset K,\\ Q(\Omega_1/k)\cap\cdots\cap Q(\Omega_s/k)=Q(\Omega/k)\Leftrightarrow K_1\cdots K_s=K,\\ Q(\Omega_1/k)\cap\cdots\cap Q(\Omega_s/k)\supset Q(\Omega/k)\Leftrightarrow K_1\cdots K_s\subset K.\end{cases}$$

The above theorems are related with $\{Q(\Omega/k)\}$ except **II** and **V**. Now we shall show analogaus matters related with $\{P(\Omega/k)\}$ in the following. For this purpose the fact **V** gives the starting point; namely, corresponding to **III**, we obtain from **V** the following fact:

VIII. $P(\Omega_1/k)=P(\Omega_2/k)\Rightarrow K_1'=K_2'$.

Based on this fact, we obtain, corresponding to **IV**,

IX. $P(\Omega_1/k)\subseteq P(\Omega_2/k)\Rightarrow K_1'\supseteq K_2'$.

(Obviously $\Omega_1\supseteq\Omega_2\Rightarrow P(\Omega_1/k)\subseteq P(\Omega_2/k)$.)

The following fact means a generalization of **VI**.

X. $P(\Omega_1/k)\cap Q(\Omega_2/k)\,(=P(\Omega_1/k)\cap P(K_2/k))=P(\Omega_1K_2/k)$.

Corresponding to **VII**, we get

XI. $P(\Omega_1\cdots\Omega_2/k)\subseteq P(\Omega_1/k)\cap\cdots\cap P(\Omega_s/k)\subseteq P(K_1'\cdots K_s'/k)$.

Therefore,

XII. *If* $P(\Omega_0/\boldsymbol{k})\subseteq P(\Omega_1/\boldsymbol{k})\cap\cdots\cap P(\Omega_s/\boldsymbol{k})\subseteq P(\Omega_{s+1}/\boldsymbol{k})$,
then $\boldsymbol{K}_0'\supseteq\boldsymbol{K}_1'\cdots\boldsymbol{K}_s'$ *and* $\boldsymbol{K}_{1\ldots s}'\supseteq\boldsymbol{K}_{s+1}'$,
where $\boldsymbol{K}_{1\ldots s}'/\boldsymbol{k}$ *is the maximal normal extension contained in* $\Omega_1\cdots\Omega_s/\boldsymbol{k}$.

In particular,

XIII. *If* $P(\Omega_1/\boldsymbol{k})\cap\cdots\cap P(\Omega_s/\boldsymbol{k})=P(\Omega/\boldsymbol{k})$,
then $\boldsymbol{K}_{1\ldots s}'\supseteq\boldsymbol{K}'\supseteq\boldsymbol{K}_1'\cdots\boldsymbol{K}_s'$.

To prove **IX**, **X** and **XI** would be enough for us. For this purpose, let $\boldsymbol{N}/\boldsymbol{k}$ be an arbitrary finite normal extension which contains all the extensions $\{\Omega/\boldsymbol{k}\}$ in present discussion, let $\mathfrak{G}$ be its Galois group, and let $\{\mathfrak{H}\}$ be the subgroups of $\mathfrak{G}$ corresponding with $\{\Omega\}$ respectivery. Then we have the well-known——

(**A**) $\mathfrak{p}\in P(\Omega/\boldsymbol{k})\Leftrightarrow\left(\frac{\boldsymbol{N}/\boldsymbol{k}}{\mathfrak{p}}\right)\subseteq\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}\sigma$,

(**B**) $\mathfrak{p}\in Q(\Omega/\boldsymbol{k})\Leftrightarrow\left(\frac{\boldsymbol{N}/\boldsymbol{k}}{\mathfrak{p}}\right)\subseteq\bigcap_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}\sigma$,

for the prime ideals $\mathfrak{p}$ *of* $\boldsymbol{k}$ *such that* $\mathfrak{p}\nmid D(\boldsymbol{N}/\boldsymbol{k})$.

Further, the normal subgroup $\mathfrak{K}$ of $\mathfrak{G}$ correponding with $\boldsymbol{K}$ coincides with the intersection of all the conjugates of $\mathfrak{H}$, and the normal subgroup $\mathfrak{K}'$ of $\mathfrak{G}$ corresponding with $\boldsymbol{K}'$ is generated by the union of all the conjugates of $\mathfrak{H}$, namely

$$\mathfrak{K}=\bigcap_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}\sigma,\qquad \mathfrak{K}'=\left\langle\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}\sigma\right\rangle.$$

Proof of IX. Let $\mathfrak{K}_1'$ and $\mathfrak{K}_2'$ be the normal subgroups of $\mathfrak{G}$ corresponding with $\boldsymbol{K}_1'$ and $\boldsymbol{K}_2'$ respectively, i.e. $\mathfrak{K}_1'=\left\langle\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_1\sigma\right\rangle$, $\mathfrak{K}_2'=\left\langle\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_2\sigma\right\rangle$. Then, from the lemma (**A**), we have

$$P(\Omega_1/\boldsymbol{k})\subseteq P(\Omega_2/\boldsymbol{k})\Rightarrow\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_1\sigma\subseteq\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_2\sigma$$
$$\Rightarrow\mathfrak{K}_1'\subseteq\mathfrak{K}_2'\Rightarrow\boldsymbol{K}_1'\supseteq\boldsymbol{K}_2'.$$

This completes the proof.

Proof of X. Let $\mathfrak{K}_2(=\bigcap_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_2\sigma)$ be the normal subgroup of $\mathfrak{G}$ corresponding with $\boldsymbol{K}_2$. Then $\mathfrak{H}_1\cap\mathfrak{K}_2$ is the subgroup of $\mathfrak{G}$ which corresponds with $\Omega_1\boldsymbol{K}_2$. Now we see

$$\Big(\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_1\sigma\Big)\cap\mathfrak{K}_2=\bigcup_{\sigma\in\mathfrak{G}}(\sigma^{-1}\mathfrak{H}_1\sigma\cap\mathfrak{K}_2)=\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}(\mathfrak{H}_1\cap\mathfrak{K}_2)\sigma.$$

Then, by the above lemmas (**A**) and (**B**), it follows that the relation $\boldsymbol{X}$ holds except for prime factors of $D(\boldsymbol{N}/\boldsymbol{k})$. We shall show, however, that the relation $\boldsymbol{X}$ holds without any exceptional prime ideals. To do this, let us consider the differents $\{\mathfrak{D}(\Omega/\boldsymbol{k})\}$. Obviously

$$\mathfrak{D}(\Omega_1\Omega_2/\boldsymbol{k})=\mathfrak{D}(\Omega_1\Omega_2/\Omega_i)\mathfrak{D}(\Omega_i/\boldsymbol{k}),\ i=1,2.$$

Further we have

$$\mathfrak{D}(\Omega_1\Omega_2/\Omega_2)\supseteq\mathfrak{D}(\Omega_1/\boldsymbol{k}),$$

because $\mathfrak{D}(\Omega_1\Omega_2/\Omega_2)$ is the greatest common divisor of all the relative differents $\mathfrak{D}_{\Omega_1\Omega_2/\Omega_2}(\theta)$ of integers θ of $\Omega_1\Omega_2$, and so $\mathfrak{D}(\Omega_1\Omega_2/\Omega_2)$ contains all the differents $\mathfrak{D}_{\Omega_1/\boldsymbol{k}}(\theta)$ of integers of Ω_1. Therefore

$$\mathfrak{D}(\Omega_1\Omega_2/\boldsymbol{k})\supseteq\mathfrak{D}(\Omega_1/\boldsymbol{k})\mathfrak{D}(\Omega_2/\boldsymbol{k}),$$

hence

$$\{\mathfrak{p};\mathfrak{p}\,|\,D(\Omega_1\Omega_2/\boldsymbol{k})\}\subseteq\{\mathfrak{p};\mathfrak{p}\,|\,D(\Omega_1/\boldsymbol{k})\}\cup\{\mathfrak{p};\mathfrak{p}\,|\,D(\Omega_2/\boldsymbol{k})\}.$$

On the other hand, since

$$D(\Omega_1\Omega_2/\boldsymbol{k})=N_{\Omega_i/\boldsymbol{k}}(D(\Omega_1\Omega_2/\Omega_i))\cdot D(\Omega_i/\boldsymbol{k})^{n_i},\ n_i=[\Omega_1\Omega_2:\Omega_i],\ i=1,2,$$

we have

$$\{\mathfrak{p};\mathfrak{p}\,|\,D(\Omega_1\Omega_2/\boldsymbol{k})\}\supseteq\{\mathfrak{p};\mathfrak{p}\,|\,D(\Omega_1/\boldsymbol{k})\}\cup\{\mathfrak{p};\mathfrak{p}\,|\,D(\Omega_2/\boldsymbol{k})\}.$$

Thus we get

$$\{\mathfrak{p};\mathfrak{p}\,|\,D(\Omega_1\Omega_2/\boldsymbol{k})\}=\{\mathfrak{p};\mathfrak{p}\,|\,D(\Omega_1/\boldsymbol{k})\}\cup\{\mathfrak{p};\mathfrak{p}\,|\,D(\Omega_2/\boldsymbol{k})\},$$

namely

$$\{\mathfrak{p};\mathfrak{p}\nmid D(\Omega_1\Omega_2/\boldsymbol{k})\}=\{\mathfrak{p};\mathfrak{p}\nmid D(\Omega_1/\boldsymbol{k})\}\cap\{\mathfrak{p};\mathfrak{p}\nmid D(\Omega_2/\boldsymbol{k})\}.$$

In this way, we can see that the relation $\boldsymbol{X}$ holds without any exception. (See also the proof of **XV** given below.) Thus proposition $\boldsymbol{X}$ is established.

Proof of XI. Using the lemma (**A**) and what has been mentioned in the proof of **X**, we need only to observe the next group theoretical relation:

$$\begin{aligned}&\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}(\mathfrak{H}_1\cap\cdots\cap\mathfrak{H}_s)\sigma\\ &\subseteq(\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_1\sigma)\cap\cdots\cap(\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_s\sigma)\\ &\subseteq\langle\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_1\sigma\rangle\cap\cdots\cap\langle\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_s\sigma\rangle.\end{aligned}$$

This relation is evident. Thus the proposition **XI** is proved.

Remark. Since $(\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_1\sigma)\cap\cdots\cap(\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_s\sigma)$ is a union of some compartments (i. e. "Abteilungen") in $\mathfrak{G}$, by Frobenius's density theorem we see that

$$\boldsymbol{M}=P(\Omega_1/\boldsymbol{k})\cap\cdots\cap P(\Omega_s/\boldsymbol{k})$$

has a definite Dirichlet-Kronecker's density $d(\boldsymbol{M})$, that is

$$d(\boldsymbol{M})=\frac{(\text{number of elements of }(\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_1\sigma)\cap\cdots\cap(\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}_s\sigma))}{(\text{order of }\mathfrak{G})}.$$

Examples of XIII. Let $\boldsymbol{R}$ be the rational number field.

(i) Put $\Omega_1=\boldsymbol{R}(\sqrt[3]{2})$, $\Omega_2=\boldsymbol{R}(\omega)$, where $\omega=\dfrac{-1+\sqrt{-3}}{2}$, and $\Omega=\boldsymbol{R}(\sqrt[3]{2},\omega)$, then $\boldsymbol{K}_1'=\boldsymbol{R}$, $\boldsymbol{K}_2'=\boldsymbol{K}_2=\Omega_2$, and we have

$$P(\Omega_1)\cap P(\Omega_2)=P(\Omega)$$

and

$$\boldsymbol{K}'=\boldsymbol{K}_{12}'\neq\boldsymbol{K}_1'\boldsymbol{K}_2'(=\boldsymbol{R}(\omega)).$$

(ii) Put $\Omega_1=\boldsymbol{R}(\sqrt[3]{2})$, $\Omega_2=\boldsymbol{R}(\sqrt[3]{3})$, $\Omega_3=\boldsymbol{R}(\omega)$ where ω is the same as the above, and $\Omega=\Omega_1\Omega_2\Omega_3$, then we have

$$P(\Omega_1)\cap P(\Omega_2)\cap P(\Omega_3)=P(\Omega)$$

and

$$\boldsymbol{K}'=\Omega_1\Omega_2\Omega_3\neq\boldsymbol{K}_1'\boldsymbol{K}_2'\boldsymbol{K}_3'(=\boldsymbol{R}(\sqrt{-3})).$$

(iii) Put $\Omega_1=\boldsymbol{R}(\sqrt[3]{2})$, $\Omega_2=\boldsymbol{R}(\sqrt[3]{2}\,\omega)$ where ω is the same as the above, and $\Omega=\Omega_1$, then we have

$$P(\Omega_1)\cap P(\Omega_2)=P(\Omega)$$

and

$$\boldsymbol{K}'=\boldsymbol{K}_1'\boldsymbol{K}_2'(=\boldsymbol{R})\neq\boldsymbol{K}_{12}'(=\Omega_1\Omega_2).$$

2. It seems to me that the following problem has not been solved jet.

Problem *Is the number of distinct extensions Ω/k corresponding to one and the same regular domain $P(\Omega/k)$ always finite ?*

The difficulty of this problem comes from group theoretical reasons, because we can get very little information about those conditions on which all conjugates of a subgroup $\mathfrak{H}$ of a finite group $\mathfrak{G}$ unite themselves into a normal subgroup of $\mathfrak{G}$.

On the other hand, it is natural for us to associate the above problem with Hermite-Minkowski's theorem on discriminants. Indeed we have a few results based on this association, for example——

XIV. *If $P(\Omega_1/k)=P(\Omega_2/k)$, then, for any finite normal extension N/k containing both of Ω_1/k and Ω_2/k, we have*

$$\{\textit{prime ideals } \mathfrak{p} \textit{ of } k;\ \mathfrak{p}\,|\,N_{\Omega_1/k}(D(N/\Omega_1))\}$$
$$=\{\textit{prime ideals } \mathfrak{p} \textit{ of } k;\ \mathfrak{p}\,|\,N_{\Omega_2/k}(D(N/\Omega_2))\}.$$

XV. *If $Q(\Omega_1/k)=Q(\Omega_2/k)$, then we have*

$$\{\textit{prime ideals } \mathfrak{p} \textit{ of } k;\ \mathfrak{p}\,|\,D(\Omega_1/k)\}$$
$$=\{\textit{prime ideals } \mathfrak{p} \textit{ of } k;\ \mathfrak{p}\,|\,D(\Omega_2/k)\}.$$

Though the essential point of **XV** has been proved in the proof of **X**, we shall give another proof as follows.

Let $\mathfrak{p}$ be an arbitrary prime ideal of k and $\mathfrak{P}$ its arbitrary prime factor in N. Let $\mathfrak{G}$ be the Galois group of N/k, $\mathfrak{H}$ the subgroup of $\mathfrak{G}$ corresponding with Ω, and $\mathfrak{T}$ and $\mathfrak{T}'$ the inertia groups of $\mathfrak{P}$ concerned with N/k and N/Ω respectively. As is well known, $\mathfrak{T}'=\mathfrak{T}\cap\mathfrak{H}$. Then the propositions **XIV** and **XV** are immediate consequences of the following lemmas——

(**C**) $$\mathfrak{p}\,|\,N_{\Omega/k}(D(N|\Omega))\Longleftrightarrow\mathfrak{T}\cap\Big(\bigcup_{\sigma\in\mathfrak{G}}\sigma\mathfrak{H}\sigma^{-1}\Big)\neq 1.$$

(**D**) $$\mathfrak{p}\,|\,D(\Omega/k)\Longleftrightarrow\mathfrak{T}\not\subseteq\bigcap_{\sigma\in\mathfrak{G}}\sigma\mathfrak{H}\sigma^{-1}.$$

Now we need only to prove (**C**) the (**D**).

Proof of (C) and (D). Let $\mathfrak{D}(N/k)$, $\mathfrak{D}(N/\Omega)$ and $\mathfrak{D}(\Omega/k)$ be the differents of N/k, N/Ω and Ω/k respectively, and let $\mathfrak{E}_\sigma$ be the Hilbert's element-ideal of N corresponding to $\sigma\in\mathfrak{G}-1$. Then we have

$$\mathfrak{D}(\boldsymbol{N}/\boldsymbol{k})=\mathfrak{D}(\boldsymbol{N}/\Omega)\mathfrak{D}(\Omega/\boldsymbol{k}),$$

and

$$\mathfrak{D}(\boldsymbol{N}/\boldsymbol{k})=\prod_{\sigma\in\mathfrak{G}-1}\mathfrak{E}_\sigma,\ \mathfrak{D}(\boldsymbol{N}/\Omega)=\prod_{\sigma\in\mathfrak{H}-1}\mathfrak{E}_\sigma,\ \mathfrak{D}(\Omega/\boldsymbol{k})=\prod_{\sigma\in\mathfrak{G}-\mathfrak{H}}\mathfrak{E}_\sigma.$$

Let us denote the $\mathfrak{P}$-component of an ideal $\mathfrak{A}$ of $\boldsymbol{N}$ by $[\mathfrak{A}]_\mathfrak{P}$, then we have

$$[\mathfrak{D}(\boldsymbol{N}/\boldsymbol{k})]_\mathfrak{P}=\prod_{\sigma\in\mathfrak{T}-1}[\mathfrak{E}_\tau]_\mathfrak{P},\ [\mathfrak{D}(\boldsymbol{N}/\Omega)]_\mathfrak{P}=\prod_{\tau\in\mathfrak{T}'-1}[\mathfrak{E}_\tau]_\mathfrak{P},$$

and $$[\mathfrak{D}(\boldsymbol{N}/\boldsymbol{k})]_{\mathfrak{P}^\sigma}=\Pi_\tau[\mathfrak{E}_\tau]_{\mathfrak{P}^\sigma}$$

where τ ranges over all elements of $\sigma^{-1}\mathfrak{T}\sigma-1$,

furtheremore $$[\mathfrak{D}(\boldsymbol{N}/\Omega)]_{\mathfrak{P}^\sigma}=\Pi_\tau[\mathfrak{E}_\tau]_{\mathfrak{P}^\sigma},$$

where τ ranges over all elements of $(\sigma^{-1}\mathfrak{T}\sigma\cap\mathfrak{H})-1$.

Therefore we have

$$\prod_{\sigma\in\mathfrak{G}}[\mathfrak{D}(\boldsymbol{N}/\Omega)]_{\mathfrak{P}^\sigma}\neq1\Longleftrightarrow\Big(\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{T}\sigma\Big)\cap\mathfrak{H}\neq1$$
$$\Longleftrightarrow\mathfrak{T}\cap\Big(\bigcup_{\sigma\in\mathfrak{G}}\sigma\mathfrak{H}\sigma^{-1}\Big)\neq1.$$

i. e. $$\mathfrak{p}\mid N_{\Omega/k}(D(\boldsymbol{N}/\Omega))\Longleftrightarrow\mathfrak{T}\cap\Big(\bigcup_{\sigma\in\mathfrak{G}}\sigma\mathfrak{H}\sigma^{-1}\Big)\neq1.$$

On the other hand, we obtain

$$[\mathfrak{D}(\Omega/\boldsymbol{k})]_\mathfrak{P}=\prod_\tau[\mathfrak{E}_\tau]_\mathfrak{P},$$

where τ ranges over all elements of $\mathfrak{T}-\mathfrak{T}\cap\mathfrak{H}(=\mathfrak{T}-\mathfrak{T}')$, and

$$[\mathfrak{D}(\Omega/\boldsymbol{k})]_{\mathfrak{P}^\sigma}=\prod_\tau[\mathfrak{E}_\tau]_{\mathfrak{P}^\sigma},$$

where τ ranges over all elements of $\sigma^{-1}\mathfrak{T}\sigma-(\sigma^{-1}\mathfrak{T}\sigma\cap\mathfrak{H})$, therefore

$$\mathfrak{p}\mid N_{\Omega/k}(\mathfrak{D}(\Omega/\boldsymbol{k}))\Longleftrightarrow{}^\exists\sigma^{-1}\mathfrak{T}\sigma\not\subseteq\mathfrak{H}$$
$$\Longleftrightarrow\mathfrak{T}\not\subseteq{}^\exists\sigma\mathfrak{H}\sigma^{-1}$$
$$\Longleftrightarrow\mathfrak{T}\not\subseteq\bigcap_{\sigma\in\mathfrak{G}}\sigma\mathfrak{H}\sigma^{-1},$$

i. e. $$\mathfrak{p}\mid D(\Omega/\boldsymbol{k})\Longleftrightarrow\mathfrak{T}\not\subseteq\bigcap_{\sigma\in\mathfrak{G}}\sigma\mathfrak{H}\sigma^{-1}.$$

3. Kronecker-Gaßmann's "Randwertsatz" shows the more exact determinative property of subdivided regular domains

$$P(\Omega/\boldsymbol{k})=\boldsymbol{M}(1)\cup\boldsymbol{M}(2)\cup\cdots\cdots\cup\boldsymbol{M}(m),$$

where $m=[\Omega:\boldsymbol{k}]$ and

$$\boldsymbol{M}(i)=\{\text{prime ideals } \mathfrak{p} \text{ of } \boldsymbol{k};\ \nu_{\Omega/\boldsymbol{k}}(\mathfrak{p})=i,\ \mathfrak{p}\nmid D(\Omega/\boldsymbol{k})\}.$$

($\nu_{\Omega/\boldsymbol{k}}(\mathfrak{p})$ means the Kronecker's character number of $\mathfrak{p}$ related to $\Omega/\boldsymbol{k}$, i.e. the number of the distinct prime factors of $\mathfrak{p}$ of the relative degree 1 in Ω.)

Indeed we can express Kronecker-Gaßmann's "Randwertsatz" as follows:

XIV. *Two finite extensions $\Omega_1/\boldsymbol{k}$ and $\Omega_2/\boldsymbol{k}$ are quasi-conjugates, that is, we have for each subgroup $\mathfrak{H}$ of $\mathfrak{G}$,*

$$(\textit{number of elements contained in } \mathfrak{H}_1\cap(\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}\sigma))$$
$$=(\textit{number of elements contained in } \mathfrak{H}_2\cap(\bigcup_{\sigma\in\mathfrak{G}}\sigma^{-1}\mathfrak{H}\sigma)),$$

(or, for each conjugate class $\mathfrak{C}$ of $\mathfrak{G}$,

$$(\textit{number of elements contained in } \mathfrak{H}_1\cap\mathfrak{C})$$
$$=(\textit{number of elements contained in } \mathfrak{H}_2\cap\mathfrak{C}),)$$

if and only if

$$\nu_{\Omega_1/\boldsymbol{k}}(\mathfrak{p})=\nu_{\Omega_2/\boldsymbol{k}}(\mathfrak{p})$$

for almost every prime ideal $\mathfrak{p}$ of **k**.

As for the proof of this theorem, see Nakatsuchi [14]. Further we can find many interesting results concerned with this theorem in Cassels-Fröhlich [5].

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