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Some questions on Cohen-Macaulay rings

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In the present paper, we shall deal with the following type of problem on Cohen-Macaulay rings.

Let $R=\sum_{i=0}^{\infty}R_i$ be a graded noetherian ring, R_i being the module of homogeneous elements of degree i. If R_0 is a field, then it is easy to see that R is Cohen-Macaulay if and only if R_M is Cohen-Macaulay, M being the irrelevant prime ideal (i.e., $M=\sum_{i\geq 1}R_i$).

Our problem is to generalize this characterization.

For instance, one can ask the following questions:

Question 1. Assume that R_M is Cohen-Macaulay for every maximal ideal M such that $M\supseteq\sum_{i\geq 1}R_i$. Does it follow that R is Cohen-Macaulay ?

Question 2. Let K be the total quotient ring of R_0. Then, does the condition that both $R\otimes_{R_0}K$ and R_0 are Cohen-Macaulay imply that R is Cohen-Macaulay ?

To the writer's knowledge, Question 1 is unsolved yet and Question 2 has a negative answer. In the present paper we discuss some facts related to these questions, and main results relate to the case where R is a projective module over R_0.

In connection with these question we ask the following

Question 3. Assume that R is Cohen-Macaulay. Does it follow that R_0 is Cohen-Macaulay ?

We have really a negative answer to this question, and on the other hand, we have an affirmative answer under an additional assumption that R is a projective R_0-module.

We add here a remark that the condition that R is a projective R_0-module is equivalent to that R is a flat R_0-module. This fact follows from that each R_i is a finite R_0-module (for, R is noetherian and therefore R_0 is noetherian and R is finitely generated over R_0).

As for the term "Cohen-Macaulay ring", we understand it as the same as "locally Macaulay ring", namely, it is a noetherian ring T such that for every maximal ideal M of T, the local ring T_M has a system of parameters which is a regular sequence.

Throughout this article, we maintain the notation that $R=\sum_{i=0}^{\infty}R_i$ is a graded noetherian ring, R_i being the module of homogeneous elements of degree i.

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1. The case where R_0 is an Artin ring.

Theorem 1.1. *Assume that R_0 is an Artin ring. If R_M is Cohen-Macaulay for every maximal ideal M containing $\sum_{i\geq 1}R_i$, then R is a Cohen-Macaulay ring.*

Proof. R_0 is a direct sum of Artin local rings: $R_0=R'_1\oplus\cdots\oplus R'_s$ Let e_i be the identity of R'_i. Then R is the direct sum of Re_i $(i=1,\cdots,s)$; M contains all e_j except for only one e_i and $R_M=(Re_i)_{Me_i}$. Therefore, it is sufficient to prove the assertion under the assumption that R_0 is an Artin local ring. Let m be the maximal ideal of R_0, then $M=m+\sum_{i\geq 1}R$. Now, if R_0/m contains only a finite number of elements, we consider $R_0(x)$ instead of R_0.

Thus we can reduce to the case where R_0/m contains infinitely many elements. Since R is noetherian, R is finitely generated over R_0. Therefore, there is an R_i, say R_d such that R is integral over $R_0[R_d]$. Then, by the normarization theorem, there are $z_1,\cdots,z_t\in R_d$ such that (i) $z_1,\cdots,z_t$ are algebraically independent over R_0 and (ii) $R_0[R_d]$ is integral over $F=R_0[z_1,\cdots,z_t]$. Then R is integral over F. Then $M\cap F=m+\sum z_iF$. Since R_M is Cohen-Macaulay and since $z_1,\cdots,z_t$ is a system of parameters of R_M, we see that length $R_M/(z_1,\cdots,z_t)^n=$

$$(\text{length } R_M/(z_1,\cdots,z_t))(\text{length } (R_0/m)[z_1,\cdots,z_t]/(z_1,\cdots,z_t)^n).$$

Therefore the theorem of transition holds for the extension $R_M / R_0[z_1,\cdots,z_t]_{(z_1,\cdots,z_t,m)}$.

Let $f_1,\cdots,f_u$ be a set of homogeneous generators for R over $R_0[z_1,\cdots,z_t]$ and let N be the kernel of the natural surjection of a free module $F=\sum_{i=1}^{u}R_0[z_1,\cdots,z_t]X_i\to R$. Let $\{g_{ij}$ $(j=1,\cdots, n(i);$ $i=1,\cdots, e$ with e such that $m^e=0)\}$ be a set of homogeneous elements of N (with a gradation defined by deg $X_i=\deg f_i$) such that $g_{i1},\cdots,g_{i,n(i)}$ modulo $m^iN+\sum z_iF$ form a linearly independent basis for $m^{i-1}N/(m^iN+(\sum z_jF\cap N))$ over the field R_0/m. Then, letting $\{U_{ij}(j=1,\cdots,h(i);\, i=1,2,\cdots)\}$ be the set of all monomials in $z_1,\cdots,z_t$ with deg $U_{ij}=i$, we see that $G_i=\{g_{ij}U_{\alpha\beta}$ (for all possible j, α, β)$\}$ form a linearly independent basis for $m^{i-1}N/m^iN$ over the field R_0/m.

This property and the fact that $R_0[z_1,\cdots,z_t]$ is a Cohen-Macaulay ring imply that R is Cohen-Macaulay.

2. Question 3.

We begin with an affirmative case:

Theorem 2.1. *Assume that R is a projective module over R_0 and that R_M is Cohen-Macaulay for every maximal ideal M containing $\sum_{i\geq 1}R_i$. Then R_0 is Cohen-Macaulay.*

Proof. We may assume that $M\cap R_0=m_0$ is the unique maximal ideal of R_0. If m_0 contains a non-zero-divisor a, then considering R/aR, we can proceed with our proof by induction on Krull dim R_0.

Therefore we assume that m_0 consists only of zero-divisors. Let $x_1,\cdots,x_n$ be homogeneous elements of R which generate R over R_0. We consider a polynomial ring $P=R_0[X_1,\cdots,X_n]$ with a gradation such that deg $X_i=$deg x_i. Then we have an R_0-homomorphism φ: $P\to R$ so that $\varphi X_i=x_i$. Let I be the kernel φ. I is homogeneous, and we have an irredundant expression $I=Q_1\cap\cdots\cap Q_s$ with homogeneous primary ideals $Q_1,\cdots,Q_s$. Now, our assumption that m_0 consists only of zero-divisors means that some of these Q_i, say $Q_1,\cdots,Q_t$ $(t\geq 1)$ contain some power of m_0. If all of these Q_i contain a power of m_0, then m_0 must be nilpotent and R_0 is an Artin ring, Assume the contrary, i.e., $\sqrt{Q_i}\not\supseteq m_0$ for $i=t+1,\cdots,s$; $t<s$. Since $R=P/I$ is Cohen-Macaulay, I has no imbedded prime divisor. Therefore, for each $i>t$, there is a homogeneous element $y_i\in Q_i$ which is not in any $\sqrt{Q_1},\cdots,\sqrt{Q_t}$. Note that deg $y_i>0$ because $\sqrt{Q_j}\supseteq m_0$ for every $j\leq t$. Set $y=y_{t+1}\cdots y_s$ and let m'' be a power of m_0 such that $m''\subseteq\bigcap_{j\leq t}Q_j$. Then $m''y\subseteq I$. Since R is a free R_0-module and since $m''\neq 0$, we see that $y\in\ker\varphi=I$, which contradicts our assumption that $y_i\notin\sqrt{Q_j}$ for $j\leq t$. q.e.d.

Next we give an example which shows that Question 3 has a negative answer.

Let (R_0, m_0) be a (noetherian) local ring of Krull dimension one such that m_0 consists only of zero-divisors. Then

$$0=q_1\cap\cdots\cap q_u$$

with primary ideals $q_1,\cdots,q_u$ such that $\sqrt{q_1}=m_0$ and $\sqrt{q_i}\neq m_0$ for every $i\geq 2$. Consider a polynomial ring $P=R_0[X]$ whih gradation such that deg $X=1$. Set $Q_i=q_iP+XP$ for $i\geq 2$ and $Q_1=q_1P$. Then

Example 2.2. *$R=P/(Q_1\cap\cdots\cap Q_u)$ is the required example.*

Proof. Each Q_i is obviously a primary ideal of P, and Krull dim $P/Q_i=1$. Therefore R is of Krull dimension one, and the zero ideal of R has no imbedded prime divisor. Thus R is Cohen-Macaulay. $Q_i\cap R_0=q_i$ and therefore $R_0\cap Q_1\cap\cdots\cap Q_u=0$. Therefore under the natural gradation, the degree zero part of R is R_0 which is not Cohen-Macaulay because m_0 consists only of zero-divisors.

We add here one obvious remark: It is not true that "R is Cohen-Macaulay $\Rightarrow R$ is a projective R_0-module". The example above is one of such examples. There are many other much easier examples to this. For instance, we get such an example among $R=R_0[x_1,\cdots,x_n]/(l)$ with (R_0, m) a Cohen-Macaulay local ring of Krull dim≥ 1 and with $l=a_1x_1+\cdots+a_nx_n$; $a_i\in m$.

Therefore, for further investigation of these questions, it is necessary to find out a good method to deal with the case where (R_0, m) is Cohen-Macaulay local ring of Krull dim≥ 1 and every elements of m is a zero divisor in R.

3. Questions 1, 2 in case *R* is a projective R_0-module.

Theorem 3.1. *Assume that R is a projective module over R_0. If R_M is Cohen-Macaulay for every maximal ideal M containing $\sum_{i\geq 1}R_i$, then R is Cohen-Macaulay.*

Proof. Let $d=$Krull dim R_0. Let N be an arbitrary maximal ideal of R and we have only to show that R_N is Cohen-Macaulay. Let $S=R_0-(N\cap R)$. Then considering R_S instead of R, we may assume that R_0 is a local ring and that $m=N\cap R_0$ is maximal. If $d=$height $m=0$, then R is Cohen-Macaulay by Theorem 1.1. If $d\geq 1$, then let $a_1,\cdots,a_d$ be a system of parameters of R_0. By virtue of Theorem 2.1, R_0 is Cohen-Macaulay, hence a_1 is not a zero-divisor in R_0. Since R is a projective module over the local ring R_0, a_1 is not a zero-divisor in R. R/a_1R is a projective module over R_0/a_1R_0, and therefore by an induction on d, we see that R/a_1R is Cohen-Macaulay. Since a_1 is not a zero-divisor, It follows that R_N is Cohen-Macaulay. q. e. d.

It is nearly obvious that Question 2 has a negative answer. But we give hare a concrete example.

Let (R_0, m_0) be a Cohen-Macaulay local integral demain of Krull dimension at least one. Let P be the polynomial ring $R_0[X_1,\cdots, X_n]$ with $n\geq 2$. Let Q be the prime ideal of P generated by $X_1,\cdots, X_r$, where the integer r is chosen so that $r\geq 1$ and that $r\neq$Krull dim R_0. Let Q' be the prime ideal m_0P. Considering the natural gradation, we have

Example 3. 2. *$R=P/(Q\cap Q')$ is the required example.*

Proof. Since $Q\cap R_0=0$, we see that the degree zero part of R *is* R_0 and is Cohen-Macaulay by our assumption. Since Krull dim $P/Q'=n$ and since Krull dim $P/Q=n-r+$Krull dim $R_0\neq n$, we see that R is not Cohen-Macaulay. Let K be the field of quotients of R_0. Then $R\otimes K\cong(P\otimes K)/(Q\otimes K\cap Q'\otimes K)$. Since $Q'\supseteq m_0$, we see that $Q'\otimes K\ni 1$, whence $R\otimes K\cong K[X_1,\cdots, X_n]/(X_1,\cdots, X_r)\cong K[X_{r+1},\cdots,X_n]$,

Even if we add an assumption that R is a projective module over R_0, we still have a negative answer of Question 2.

Example 3. 3. *Let R_0 be a discrete valuation ring with a prime element p. Let u, v be algebraically independent elements over R_0. Let R be $R_0[u^3, pu^2v, uv^2, v^3]$ with* $\deg u=\deg v=1/3$. *Then R is the required example.*

Proof. Consider the natural homomorphirm $\varphi: R_0[x, p, z, w]\to R$. The kernel I of φ is generated by $y^2-p^2zx, y^3-p^3wx^2, z^3-w^2x, yz-pwx, yw-pz^2$. Hence $\varphi_p: (R_0/pR_0)[x, y, z, w]\to R/pR$ has kernel I_p generated by $y^2, y^3, yz, yw, z^3-w^2x$. Therefore p, x form a maximal R-sequence in the homogeneous maximal ideal, hence R is not Cohen-Macaulay. On the other hand, $R\otimes K$ is nothing but $K[u^3, u^2v, uv^2, v^3]$, which is clearly Cohen-Macaulay. Since R_0 is a discrete valuation ring, we see easilly that R is R_0-free. q. e. d.

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