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Homology submanifolds and homology classes of a homology manifold

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This note is concerned with the problem of the realisation of homology classes of a homology manifold by homology submanifolds. First the C^∞-case of this problem was studied by R. Thom [6]. Next the *PL*-case and *TOP*-case were studied in [1], [2], [3].

The present study is founded on the Williamson's transversality theorem [7]. We shall apply R. Thom's method [6] to homology manifolds.

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1. Statement of the results

We shall obtain the following results:

Theorem 1. *Let V^n be a homology manifold of dimension n ($n\geqq 2$). For $1\leqq k\leqq n/2$, all homology classes of $H_k(V^n, \mathbf{Z}_2)$ can be realized by homology submanifolds which have normal PL-microbundles.*

Theorem 2. *Let V^n be a homology manifold of dimension n ($n\geqq 2$). All homology classes of $H_{n-1}(V^n, \mathbf{Z}_2)$ can be realized by homology submanifolds which have normal PL-microbundles.*

These results are quite in parallel with those of *PL*-case in [2].

2. Preliminaries

A compact polyhedron M is a *homology n-manifold*, if there exists a triangulation K *of* M such that for all $x \in |K|$, and for all r, $H_r(Lk(x, K), Z)$ is isomorphic to $H_r(S^{n-1}, Z)$. Here $Lk(x,K)$ is the boundary of the star $St(x, K)$ of x in K.

It can be seen that this definition is independent of the triangulation chosen.

We know that homology n-manifolds are Poincaré complexes of formal dimension n (cf. Maunder [5]).

Let (M, K) be a homology n-manifold. Then for $n \geqq 2$, any $x \in |K|$, $Lk(x, K)$ is a homology $(n-1)$-manifold (cf. Alexander [4]).

Let M be an homology m-manifold, properly embedded in a homology q-manifold Q. Then we shall say M is a *homology submanifold* of Q.

Let V^n be a homology n-manifold and W^p be a homology submanifold of dimension p of V^n. The inclusion map $i: W^p \to V^n$ induces the homomorphism

$$i_*: H_p(W^p, \mathbf{Z}_2) \to H_p(V^n, \mathbf{Z}_2).$$

Let $z \in H_p(V^n, \mathbf{Z}_2)$ be the image by i_* of the fundamental class w of the homology p-manifold W^p. Then we say that the homology class z is *realized* by the homology sub-manifold W^p.

Here the following question is considered : Let a homology class z mod 2 of a homology manifold V^n be given. Is it realizable by a homology submaniford ?

3. Williamson's transversality theorem

In this section we shall recall Williamson's transversality theorem (cf. Williamson [7]).

Let ξ be a PL-microbundle:

$$\xi: B(\xi) \xrightarrow{i} E(\xi) \xrightarrow{j} B(\xi),$$

X be a complex, and suppose $E(\xi)$ is contained in X so that $B(\xi)$ is a closed PL-subspace of X. Then we say X *contains the PL-*

microbundle ξ. If $E(\xi)$ is a neighborhood of $B(\xi)$, then we say ξ is a *normal PL-microbundle* for $B(\xi)$ in X.

Definition. Let S and T be locally finite simplicial complexes and ξ be a normal PL-microbundle for $B=B(\xi)$ in T. Let $f: S\to T$ be a PL-map. If $A=f^{-1}(B)$ has a normal PL-microbundle η in S such that η is isomorphic to $(f/A)^{*}\xi$, then we shall say f is *transverse regular* for (η, ξ), or briefly, f is *t-regular*.

R. Williamson Jr. obtained the following theorem.

Theorem 3. *Let S and T be locally finite simplicial complexes and let $f: S\to T$ be a PL-map. Suppose that T contain a PL-microbundle ξ. Then there is a PL-homotopy H_t of f such that H_1 is t-regular for (η,ξ).*

4. A lemma on homology manifolds

Lemma. *Suppose V is a homology $(n+q)$-manifold and M is a PL-subspace of V which has a normal PL-microbundle of dimension q in V $(n,q\geqq 1)$. Then M is a homology n-manifold.*

Proof. Given any $x\in M$ there is an open neighborhood U of x in M and a neighborhood W of x in V, also open, such that $U\times \boldsymbol{R}^q$ is PL-homeomorphic to W, by the definition of normal PL-microbundles. So it suffices to prove the lemma for the special case $M=U$, $V=W$, and W itself is $U\times \boldsymbol{R}^q$. If the lemma is true for $q=1$, it follows that $U\times \boldsymbol{R}^{q-1}$ is a homology $(n+q-1)$-manifold, then by induction that U is a homology n-manifold. So it suffices to consider $q=1$.

We triangulate $U\times \boldsymbol{R}$ by the convex product cells of U and a simplicial subdivision of $\boldsymbol{R}$, and we suppose x is a vertex of U and O is a vertex of $\boldsymbol{R}$. The link of x relative to $U\times \boldsymbol{R}$, that is the unique cell complex $Lk(x, W)$ such that the closed star $St(x, W)$ is the join $Lk(x, W)*x$ is the same, up to x, PL-homeomorphism, for any two convex cell subdivision of $U\times \boldsymbol{R}$.

In the product cell triangulation of $U\times \boldsymbol{R}$.

$$St((x,O),W)=St(x,U)\times St(O,\boldsymbol{R})$$

and

$$Lk((x,O),W)=Lk(x,U)\times St(O,\boldsymbol{R})\cup St(x,U)\times Lk(O,\boldsymbol{R}).$$

Now $Lk(O,\boldsymbol{R})$ is just two points, say 1 and -1, while in $Lk((x,O),W)$

$$St((x,1),\ Lk((x,O),W))=St(x,U)\times 1.$$

It follows that

$$Lk((x,1),\ Lk((x,O),W))=Lk(x,U)\times 1.$$

However, $Lk((x,O),W)$ is a homology n-manifold. Therefore, $Lk(x,U)$ has the same homology group as the $(n-1)$-sphere. Thus we have obtained the lemma.

5. Fundamental thenrem.

Definition. We say that a cohomology class $u\in H^k(A,\boldsymbol{Z}_2)$ of a space A is *PL_k-realizable*, if there exists a mapping $f:A\rightarrow MPL_k$ such that u is the image, for the homomorphism f^* induced by f, of the fundamental class U_k of the Thom complex MPL_k of the universal PL-microbundle $\gamma(PL_k)$ of dimension k.

Then we have the following fundamental theorem.

Theorem 4. *Let V^n be a homology manifold of dimension n $(n\geqq 2)$. Then, in order that a homology class $z\in H_{n-k}(V^n,\boldsymbol{Z}_2)$, $k>0$, can be realized by a homology submanifold W^{n-k} of dimension $n-k$ which has a normal PL-microbundle in V^n, it is necessary and sufficient that the cohomology class $u\in H^k(V^n,\boldsymbol{Z}_2)$, corresponding to z by the Poincaré duality, is PL_k-realizable.*

Proof. i) *Necessity.* Homology manifolds are Poincaré complexes. Therefore. the proof of the necessity is the same as that of PL-case in [1].

ii) *Sufficiency.* Let

$$\gamma(PL_k): B(PL_k) \xrightarrow{i_k} E(PL_k) \xrightarrow{j_k} B(PL_k)$$

be the universal PL-microbundle of dimension k. Suppose that there exists a mapping f of V^n into $M(PL_k)$ such that $f^*(U_k)=u$. Then the Thom complex $M(PL_k)$, deprived the point $*$ at infinity, is considered as a locally finite simplicial complex, and PL-subspace $B(PL_k)$ has the normal PL-microbundle $\gamma(PL_k)$ in $M(PL_k)-*$. By the Williamson's transversality theorem, we have a mapping f_1, homotopic to f, t-regular for $(\nu,\gamma(PL_k))$, where ν is a normal PL-microbundle of $f_1^{-1}(B(PF_k))$ in V^n. However, by the lemma in §3, $f_1^{-1}(B(PL_k))$ is a homology submanifold W^{n-k} of dimension $(n-k)$. Moreover, by the definition of t-regularity, the induced PL-microbundle $f_1^*\gamma(PL_k)$ is isomorphic to ν. We know $f_1^*(U_k)=f^*(U_k)=u$. Then, as in the proof of Theorem in [1], we can see that the homology submaniford W^{n-k} realizes the homology class z, correspoding to u by the Poincaré duality. Thus we have obtained the theorem.

6. Proof of Theorem 1 and 2.

As in §3 of [2], Theorem 1 follows easily the fundamental theorem and Proposition 4 in [2]. Theorem 2 follows also the fundamental theorem and § 2, d) in [2].

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