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On Eakin-Nagata's theorem

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The following is well known:

Let S be a ring which is finitely generated over a subring R as a left R-module. If R is left noetherian, so is S.

In case that S is commutative, the converse of the above result is proved by P. M. Eakin [3] and M. Nagata [5]. In this note we shall give an alternative proof of their theorem using the characterization of noetherian rings by means of injective modules [1, Th. 1.1] and give a noncommutative generalization, that is the case that there exists a finite set of generators of S as a left R-module whose elements commute with all elements of R. Furthermore, we shall give an example which shows that the above assumption on generators can not be omitted.

Throughout this note, we shall assume that every ring has an identity, a subring of a ring always contains the identity and every module is unitary.

We begin with an easy

Proposition 1. *Let R be a ring and $\{M_i\}$ $(i \in I)$ a family of left R-modules. For any finitely generated left R-module A, we have a canonical isomorphism*

$$\mathrm{Hom}_R(A, \bigoplus_{i\in I} M_i) \cong \bigoplus_{i\in I} \mathrm{Hom}_R(A, M_i).$$

Proof. Put $M = \bigoplus_{i\in I} M_i$. Let u_i be canonical monomorphisms from $\mathrm{Hom}_R(A, M_i)$ to $\mathrm{Hom}_R(A, M)$ which are induced by inclusion maps from

M_i to M. Then $u=\bigoplus_{i\in I} u_i$ is the desired isomorphism. For, u is obviously a monomorphism and that u is an epimorphism follows easily from the fact that $f(A)$ is contained in a finite direct sum of M_i's for any $f\in \mathrm{Hom}_R(A, M)$, since A is finitely generated.

Proposition 2. *Let S be a ring and R a subring of S. If E is an injective left R-module, then* $\mathrm{Hom}_R(S, E)$ *is an injective left S-module.*

Proof. [2, II, Prop. 6. 1a)].

Let R be a ring and M a left R-module. We denote by $E_R(M)$ the injective envelope of M.

Proposition 3. *Let S be a ring, R a subring of S and M be a left R-module. If there exists a finite set of generators of S as a left R-module whose elements commute with all elements of R, then the inclusion map from M to $E_R(M)$ induces an isomorphism*

$$E_S(\mathrm{Hom}_R(S, M)) \cong \mathrm{Hom}_R(S, E_R(M)).$$

Proof. By Prop. 2 $\mathrm{Hom}_R(S, E_R(M))$ is an injective left S-module. Therefore it suffices to show that $\mathrm{Hom}_R(S, M) \subseteq \mathrm{Hom}_R(S, E_R(M))$ is an essential extension [4, III, 11. 2]. Let $\{x_1,\dots, x_n\}$ be a system of generators of S as a left R-module such that x_i $(i=1,\dots, n)$ commutes with all elements of R. Let $0\neq f\in \mathrm{Hom}_R(S, E_R(M))$. If $f(x_1)\neq 0$, there exists a non zero element r_1 of R such that $0\neq r_1(f(x_1))\in M$. If $r_1(f(x_2))\notin M$, there exists a non zero element r_2 of R such that $0\neq r_2(r_1 f(x_1))\in M$. Going on with such a work, we obtain a non-zero element r of R such that $r(f(x_i))\in M$ $(i=1,\dots, n)$ and $r(f(x_i))$'s are not all zero. Let x be an arbitrary element of S. Put $x=r_1x_1+\dots+r_nx_n$ $(r_1,\dots, r_n\in R)$. Since $(rf)(x)=f(r_1x_1r+\dots+r_nx_nr)=f(r_1rx_1+\dots+r_nrx_n)=r_1(rf(x_1))+\dots+r_n(rf(x_n))$, rf belongs to $\mathrm{Hom}_R(S, M)$. Clearly rf is not zero. Therefore, $\mathrm{Hom}_R(S, M) \subseteq \mathrm{Hom}_R(S, E_R(M))$ is an essential extension.

Proposition 4. *Let S be a ring and R a subring of S satisfies the conditions in Proposition 3. If M is a left R-module such that* $\mathrm{Hom}_R(S, M)$ *is an injective left S-module, then M is also R-injective.*

Proof. It suffices to prove $E_R(M) \subseteq M$. Let x be any element of $E_R(M)$ and define an R-homomorphism f from R to $E_R(M)$ by $f(r)=rx$ $(r \in R)$. Since $E_R(M)$ is R-injective, f can be extended to an R-homomorphism from S to $E_R(M)$. On the other hand, by Prop. 3 we have $\mathrm{Hom}_R(S, E_R(M)) = \mathrm{Hom}_R(S, M)$. Therefore f is contained in $\mathrm{Hom}_R(S, M)$ and hence $x=f(1) \in M$. This completes the proof.

Theorem. *Let S be a ring and R a subring which satisfies the conditions in Proposition 3. If S is left noetherian, so is R.*

Proof. Let $\{E_i\}$ $(i \in I)$ be a family of injective left R-modules. By [1, Th. 1. 1] it suffices to show that $E = \bigoplus_{i \in I} E_i$ is R-injective. By Prop. 1 we have an isomorphism $\mathrm{Hom}_R(S, E) \cong \bigoplus_{i \in I} \mathrm{Hom}_R(S, E_i)$. Applying successively Prop. 2, [1, Th. 1. 1] and Prop. 4, it follows that E is R-injective.

Corollary. *Let R be a commutative ring and S an R-algebra which contains R as a subring. Furthermore, assume that S is finitely generated as an R-module. If S is left (or right) noetherian, R is also noetherian.*

Finally, we give an example which shows that the assumption in Theorem, that the elements of the set of generators commute with all elements of R, can not be dropped.

Let T be a non noetherian integral domain which is not a field and K its quotient field. Let S be the ring of all 2×2 matrices with entries in K. Put $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a \in T, b, c \in K \right\}$. It is easily seen that R is a subring of S and S is generated by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as a left R-module. It is clear that S is left noetherian, but R is not left noetherian. For, if I is an ideal in T which is not finitely generated, $I^* = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| a \in I \right\}$ is a non finitely generated left ideal in R.

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