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On functional dimensions of group representations II
Case of compact semi-simple Lie groups

By

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§0. Introduction.

Let E be a σ-Hilbert space topologized by a sequence of norms $\{\|\cdot\|_n, n=1,\ldots\}$. A. N. Kolmogorov has defined the functional dimension $d_f(E)$ of E as follows:

$$d_f(E)=\sup_n \inf_m \lim_{\varepsilon\to 0} \frac{\log H(\varepsilon U_n, U_m)}{\log\log 1/\varepsilon}-1,$$

where $H(\varepsilon U_n, U_m)$ is the ε-entropy of U_m with respect to the norm $\|\cdot\|_n$. If, in particular, E is a space of functions on a compact manifold M, of a certain type, it is known that the functional dimension $d_f(E)$ of E is in close connection with the dimension of M. (Y. Kômura [9] and S. Tanaka [16])

Give a Lie group G acting on M as a group of differentiable transformation, then we can define the representation $\mathfrak{T}=(T_g; g\in G, L^2(M))$ by means of these transformations. Throughout this paper the general σ-Hilbert space is taken to be the space $\mathscr{B}(\mathfrak{T})$ of analytic functionals of the representation $\mathfrak{T}$ in the sense of E. Nelson [12], and calculate the functional dimension of this space. Let $\mathfrak{T}$ be one of the following representation:

1) The regular representation of a connected compact semi-simple Lie group G (in this case the manifold M is the group G itself);

2) The regular representation of the n-dimensional torus $\boldsymbol{T}^n$ (M coincides with $\boldsymbol{T}^n$);

3) The quasi-regular representation of $SO(n)$ on the $(n-1)$-dimensional sphere $\boldsymbol{S}^{n-1}$;

4) The class 1 irreducible unitary representations of the group $M(n)$ of n-dimensional Euclidean motions (in this case $M=\boldsymbol{S}^{n-1}$).

In the above cases, we shall prove the following main theorem.

Theorem. $d_f(\mathscr{B}(\mathfrak{T}))=\dim M$.

To prove the theorem we shall proceed as follows: In §1 we shall discuss the functional dimension $d_f(E)$, in the case where the countable norms $\|\cdot\|_n$ of E are defined as $\|\cdot\|_n=\|A^n\cdot\|$ by using a positive definite operator A. In this case we can calculate $d_f(E)$ with the help of the spectrum of the operator A. In §2 and §3 we shall consider the space $\mathscr{B}(\mathfrak{T})$ of analytic functionals of a representation of Lie group and prove that $\mathscr{B}(\mathfrak{T})$ is an example of the space E discussed in §1. In §4 we shall show that the operator A may be taken to be the Casimir operator $\mathscr{C}$ for a compact semi-simple Lie group. In §5 we shall calculate the spectrum of the Casimir operator $\mathscr{C}$ to prove the main theorem in the case of a compact semi-simple Lie group. The last section will be devoted to some other applications and to the proof of the theorem in the remaining cases.

A characterization of $\mathscr{B}(\mathfrak{T})$ and Lemma 2 in this paper have also been given, independently of the present work, in a rescent work of K. Okamoto and others [5], a part of which the author was able to know by private communication. The author expresses his thanks to the members of the Seminar on Probablity in Nagoya University and the members of the Seminar on Representations of Groups in Kyoto University, and especially to Professor H. Yoshizawa.

§1. Preliminaries.

Let $\{E_n, \|\cdot\|_n\}$, $(n=1,\dots)$, be a sequence of Hilbert spaces which satisfies the following conditions

$$E_n\supset E_{n+1} \quad \text{and} \quad \|\cdot\|_n\leq\|\cdot\|_{n+1} .$$

Then we define the σ-Hilbert space E as the projective limit of the sequence, $E=\varprojlim_n E_n$. Now, let us introduce the functional dimension $d_f(E)$ of E following A. N. Kolmogorov [8] and I. M. Gelfand [3]. Let U_n be the unit ball of E_n and

$$H(\varepsilon U_n, U_m)=\inf\{\log_2(\# N);\ N\subset E_m,\ \forall x\in U_m,\ \exists y\in N,\ x\in y+\varepsilon U_n\}.$$

Definition 1. *We call*

$$(1)\qquad d_f(E)=\sup_n \inf_m \lim_{\varepsilon\to 0}\frac{\log_2 H(\varepsilon U_n, U_m)}{\log_2\log_2 1/\varepsilon}-1,$$

the ***functional dimension*** *of the σ-Hilbert space E.*

To calculate the functional dimension of E, we can avail ourselves of the following results. Assume that we are given a self-adjoint compact operator T on a Hilbert space $\mathfrak{H}$. Let S be the unit ball of $\mathfrak{H}$ and U be the image of S by T, $U=T(S)$. Let $m_T(t)$ be the number of eigenvalues of T, taking multiplicities into account, greater than $1/t$. Then we have the following theorem:

Theorem 1. (B.S. Mityagin [11] and S. Takenaka [14])

If the limit

$$\gamma=\lim_{\varepsilon\to 0}\frac{\log_2 H(\varepsilon S, U)}{\log_2\log_2 1/\varepsilon}-1$$

exists, then it holds that,

$$\log_2 m_T(t)=\gamma\log_2\log_2 t+o(\log_2\log_2 t).$$

The converse is also true.

Let T be a positive self-adjoint compact operator on a Hilbert space $\mathfrak{H}$. Define $E_n=T^n\mathfrak{H}$ and the norm of E_n by the form $\|\cdot\|_n=\|T^{-n}\cdot\|_{\mathfrak{H}}$. Then the sequence $\{E_n, \|\cdot\|_n\}$ defines the σ-Hilbert space E. The functional dimension $d_f(E)$ of E can be calculated from the spectrum of T as follows:

Corollary 1. *Let $m_{T^n}(t)$ be the function of eigenvalue distribution* *of* T^n. *If*

$$\lim_n \log_2(m_{T^n}(t)) = \gamma \log_2\log_2 t + o(\log_2\log_2 t),$$

then $d_f(E) = \gamma$.

§2. Analytic vectors of representations.

Let G be a real Lie group and $\mathfrak{g}$ be its Lie algebra with a base $\{x_i;\ i=1,\dots, d\}$. Let $\mathfrak{T}=(T, \mathfrak{H})$ be an unitary representation of the group G and ∂T be the representation of $\mathfrak{g}$ derived from the operators $\{T_g;\ g \in G\}$. Following E. Nelson [12] we define the analytic vectors of $\mathfrak{T}$ as follows:

Definition 2. *An element* f *of* $\mathfrak{H}$ *is called an analytic vector of the representation* $\mathfrak{T}$ *if the condition*

$$\sum_{n=0}^{\infty} \sum_{\sigma \in G(n)} \left\| \frac{\prod_{i=1}^{n} t_{\sigma(i)} \partial T(X_{\sigma(i)})}{n!} f \right\| < \infty \tag{2}$$

holds for all multi-index $t=(t_1, t_2,\dots, t_d)$ *with* $|t_i| < t_0$ *for some positive constant* t_0, *where* $G(n)$ *denote the totality of mappings from* $\{1,\dots, n\}$ *into* $\{1,\dots, d\}$.

Set

$$\Delta = -\sum_{i=1}^{d} (\partial T(X_i))^2 \quad \text{and} \quad A = \sqrt{1+\Delta}. \tag{3}$$

Then, using the method of analytic domination, E. Nelson [12] and R. Goodman [4] have obtained the following theorem:

Theorem 2. *An element* f *of* $\mathfrak{H}$ *is an analytic vector if and only if the condition*

$$\sum_{n=0}^{\infty} \left\| \frac{t^n A^n}{n!} f \right\| < \infty \tag{4}$$

holds for all positive t *less than some positive constant* t_0.

Let B be a positive self-adjoint operator acting on the Hilbert space $\mathscr{H}$. Then we shall define two kinds of exponentials of B as follows:

Definition 3.

$$\text{(4)} \qquad 1)\quad \operatorname{Exp}(tB) \equiv \sum_{n=0}^{\infty} \frac{t^n B^n}{n!},$$

$$D(\operatorname{Exp}(tB)) \equiv \left\{ f \in \mathscr{H};\ \sum_{n=0}^{\infty} \left\| \frac{t^n B^n}{n!} f \right\|_{\mathscr{H}} < \infty \right\};$$

$$\text{(5)} \qquad 2)\quad e^{tB} \equiv \int_0^{\infty} e^{t\lambda} dE_\lambda,$$

$$D(e^{tB}) \equiv \left\{ f \in \mathscr{H};\ \int_0^{\infty} e^{2(\Re e\, t)} d(E_\lambda f, f) < \infty \right\},$$

where $\{E_\lambda;\ \lambda \leq 0\}$ *is the system of projections in the spectral decomposition of the operator* B.

From the definitions it is easily seen that

$$\text{(6)} \qquad D(\operatorname{Exp}(tB)) \subset D(e^{tB}).$$

Here arises a question asking whether the converse inclusion holds. To discuss the problem we prepare the following two lemmas:

Lemma 1. *An element f of $\mathscr{H}$ belongs to the domain $D(\operatorname{Exp}(t_0 B))$ for $t_0 \neq 0$, if and only if there exist positive constants M and C such that*

$$\text{(7)} \qquad \|B^n f\| \leq C M^n n!.$$

Proof. Let f be in $D(\operatorname{Exp}(t_0 B))$ and let g be in $\mathscr{H}$. Then the function $(\operatorname{Exp}(tB)f, g)$ of t is holomorphic in the disk $\mathscr{D}_{t_0} = \{t \in \boldsymbol{C}, |t| < t_0\}$. Therefore, $F_f(t) = \operatorname{Exp}(tB)f$ is a $\mathscr{H}$-valued holomorphic function in the disk $\mathscr{D}_{t_0}$. Thus we can apply the Cauchy's estimation of the coefficients to prove the only if part. The if part is clear.

Q.E.D.

Lemma 2. (Fundamental lemma). *If f belongs to the domain*

$D(e^{t_0 B})$, for some positive constant t_0 then f belongs to the domain $D(\mathrm{Exp}(tB))$ for all t in the disk $\mathscr{D}_{t_0}$.

Proof. Since f belongs to $D(e^{t_0 B})$, we can define the function $F_f(t)=e^{tB}f$ in the disk $\mathscr{D}_{t_0}$. By Lemma 1, to prove our lemma it is enough to show that $F_f(t)$ is holomorphic in $\mathscr{D}_{t_0}$. Take t and t' in $\mathscr{D}_{t_0}$ such $\mathfrak{Re}(t)\geq\mathfrak{Re}(t')$. Then, we have following estimation;

$$\|F_f(t)-F_f(t')\|^2=\int_0^\infty |e^{t\lambda}-e^{t'\lambda}|^2 d(E_\lambda f, f)$$

$$\leq\int_0^\infty e^{2|t|\lambda}|1-e^{-(t-t')\lambda}|^2 d(E_\lambda f, f)$$

$$\leq\int_0^\infty e^{2|t|\lambda}(1+|e^{-2(t-t')\lambda}|-2\mathfrak{Re}(e^{-2(t-t')\lambda}))d(E_\lambda f, f)$$

$$\leq 4\int_0^\infty e^{2|t|\lambda}d(E_\lambda f, f)=4\|e^{tB}f\|^2<\infty\,.$$

Therefore by Lebesgue's theorem, $\|F_f(t)-F_f(t')\|\to 0$ as $|t-t'|\to 0$. That is, $F_f(t)$ is a continuous function. Let $\mathscr{C}_t$ be a circle included in $\mathscr{D}_{t_0}$ with center t and put

$$I_f(t)=\frac{1}{2\pi i}\oint_{\mathscr{C}_t}\frac{F_f(z)}{t-z}dz\,.$$

For the any element g of $\mathscr{H}$, we consider the inner product $(I_f(t), g)$:

$$|(I_f(t), g)|=\frac{1}{2\pi}\left|\oint_{\mathscr{C}_t}\int_0^\infty e^{z\lambda}d(E_\lambda f, g)\frac{dz}{t-z}\right|$$

$$\leq\frac{1}{2\pi}\left|\oint_{\mathscr{C}_t}\int_0^\infty e^{t_0\lambda}d(E_\lambda f, g)\frac{dz}{t-z}\right|$$

$$\leq\|F_f(t_0)\|\cdot\|g\|\,.$$

Therefore, by Fubini's theorem, we can exchange the order of integrations:

$$(I_f(t), g)=\int_0^\infty\left[\frac{1}{2\pi i}\oint_{\mathscr{C}_t}\frac{e^{z\lambda}}{t-z}dz\right]d(E_\lambda f, g)$$

$$=\int_0^\infty e^{t\lambda}d(E_\lambda f, g)=(F_f(t), g)\,.$$

This shows the analyticity of $F_f(t)$. **Q.E.D.**

§3. σ-norms of the space of analytic vectors.

We introduce norms $\|\cdot\|_{E(t)}$ and $\|\cdot\|_{e^{tB}}$ on the spaces $D(\mathrm{Exp}(tB))$ and $D(e^{tB})$ respectively, for a given positive self-adjoint operator B.

Definition 4.

(8) 1) $\|f\|_{E(t)}\equiv \sum_{n=0}^{\infty}\left\|\frac{t^nB^n}{n!}f\right\|_{\mathscr{H}},$ for $f\in D(\mathrm{Exp}(tB))$;

(9) 2) $\|f\|_{e^{tB}}\equiv\|e^{tB}f\|_{\mathscr{H}},$ for $f\in D(e^{tB})$.

With those norms $D(\mathrm{Exp}(tB))$, $D(e^{tB})$ become a Banach space and a Hilbert space respectively.

Proposition 1. *For any positive t and t' such that $t'<t$, the inclusions.*

(10) $$D(\mathrm{Exp}(tB))\subset D(e^{tB})\subset D(\mathrm{Exp}(t'B))$$

are continuous with respect to the norms (8) *and* (9).

Proof. The continuity of the first injection is clear, so we shall prove the continuity of the second one only. Let f be in $D(e^{tB})$, by the analyticity of $e^{tB}f$ (see Lemma 2) we have

$$\left\|\frac{B^n}{n!}f\right\|\le\frac{\|f\|_{e^{tB}}}{t^n}\,.$$

Therefore,

$$\|f\|_{E(t')}\le\|f\|_{e^{tB}}\sum_{n=0}^{\infty}\left(\frac{t'}{t}\right)^n=\left(\frac{t}{t-t'}\right)\|f\|_{e^{tB}}\,.$$

Q.E.D.

Now we define two spaces of (real-) analytic vectors as follows:

Definition 4.

(11) 1) $\mathfrak{H}^{\omega}=\varinjlim_{t\downarrow 0} D(\mathrm{Exp}\,(tB))$;

(12) 2) $\mathfrak{H}'^{\omega}=\varinjlim_{t\downarrow 0} D(e^{tB})$.

The topologies of these spaces are the inductive limit topologies derived from the norms (8) and (9). Thus these spaces are dual-Fréchet spaces.

The set theoretical equation $\mathfrak{H}^{\omega}=\mathfrak{H}'^{\omega}$ is derived from (10) and Lemma 2. Furtheremore by Proposition 1 we know that $\mathfrak{H}^{\omega}=\mathfrak{H}'^{\omega}$ as linear topological spaces. Thus we have

Theorem 3. *$\mathfrak{H}^{\omega}$ is isomorphic to $\mathfrak{H}'^{\omega}$.*

Our purpose of this section is to define the spaces by which we can consider the "size" of the group representations. By Theorem 2 we know the spaces of analytic vectors in the sense of Nelson is equal to $\mathfrak{H}^{\omega}$ with respect to the positive self-adjoint operator $A=\sqrt{1+\Delta}$ (see (3)). Then we can take the space $\mathfrak{H}^{\omega}$ as a candidate to measure the "size". By Theorem 3 we know the dual-σ-Hilbert structure of $\mathfrak{H}^{\omega}$. (see I. M. Gelfand [3]) Since the functional dimension is defined only to the σ-Hilbert spaces, we will take the dual space $\mathfrak{H}^{\omega *}$ of $\mathfrak{H}^{\omega}$ and calculate its functional dimension $d_f(\mathfrak{H}^{\omega *})$. The σ-Hilbert structure of the space $\mathfrak{H}^{\omega *}$ is derived from the projective limit of norms $\{\|e^{-tA}(\cdot)\|_{\mathfrak{H}};\, t>0\}$. And the space $\mathfrak{H}^{\omega *}$ seems to play a certain role in the theory of group representations. (S. Helgason [7] and K. Okamoto and others [5])

§4. Analytic vectors on a compact semi-simple lie group.

Let G be a compact semi-simple Lie group and $\mathfrak{D}_r=(T, L^2(G))$ be the regular representation of G. We denote the space of analytic vectors $\mathfrak{H}^{\omega}$ of $\mathfrak{D}_r$ by $\mathscr{A}(G)$ and the space of analytic functionals $\mathfrak{H}^{\omega *}$ by $\mathscr{B}(G)$. The space $\mathscr{B}(G)$ may be called the space of Hyperfunctions on the group G. In this section we characterized the spaces $\mathscr{A}(G)$ and $\mathscr{B}(G)$ by the spectrum of the Casimir operator of G.

Let $\mathfrak{g}$ be the Lie algebra of G and $\{X_i;\, i=1,\dots, d\}$ be a base of this algebra. Then following lemma holds:

Lemma 3. *In the case of the regular representation of a compact semi-simple Lie group, the operator defined in* § 1, (3) *can be taken as the Casimir operator* $\mathscr{C}$.

Proof. Using the base $\{X_i\}$, the Killing form $(\cdot, \cdot)$ can be realized by a symmetric matrix M as $(X, Y)=XM^tY$. Since G is semi-simple, M is negative-definite and nondegenerate, and so is the inverse matrix M^{-1}. And the Casimir operator can be expressed by the following form:

$$(13) \qquad \mathscr{C}=(\partial T(X_1),\dots, \partial T(X_d))M^{-1t}(\partial T(X_1),\dots, \partial T(X_d)).$$

Let a_{ij} be the i,j element of $-M^{-1}$. As we have $-X_1M^{-1t}X_1=a_{11}>0$, so for $\partial T(X)=(x_1\partial T(X_1),\dots, x_d\partial T(X_d))$,

$$\begin{aligned} -XM^tX &= \sum_{i,j}^{d} a_{ij}x_ix_j\partial T(X_i)\partial T(X_j) \\ &= a_{11}x_1^2\partial T(X_1)^2+\sum_{i=2}^{d}(a_{ij}x_ix_1)\{\partial T(X_1)\partial T(X_i) \\ &\quad +\partial T(X_i)\partial T(X_1)\}+P_1 \\ &= a_{11}\{x_1\partial T(X_1)^2+\sum_{2=i}^{d}(a_{i1}/a_{11})x_i\partial T(X_i)\}^2+P_2 \\ &= Y_1^2+P_2, \end{aligned}$$

where P_1 and P_2 are polynomials of variables $X_2,\dots, X_d$ and $Y_1=\sum_{i=1}^{d}(a_{i1}/\sqrt{a_{11}})X_i$. Since M^{-1} is negative-definite nondegenerate, P_2 is a nondegenerate positive-definite Quadratic form. Thus by induction, there exists a new base $\{Y_i;\ _i=1,\dots, d\}$, and

$$-\mathscr{C}=\sum_{i=1}^{d}\partial T(Y_i)^2. \qquad \textbf{Q.E.D.}$$

Corollary 1 shows that in order to calculate $d_f(\mathscr{B}(G))$ it is enough to know the spectrum of $\mathscr{C}$.

§5. Spectrum of $\mathscr{C}$ and $d_f(\mathscr{B}(G))$.

In this section we shall prove the following theorem:

Theorem 4. *Let G be a connected compact semi-simple Lie group and $\mathscr{B}(G)$ be the space of analytic functionals on G. Then,*

$$d_f(\mathscr{B}(G)) = \dim G.$$

In §4, we have obtained the σ-Hilbert structure of $\mathscr{B}(G)$ by the Casimir operator $\mathscr{C}$. The theory of representations of compact connected semi-simple Lie groups tells us about the spectrum of the Casimir operator $\mathscr{C}$. In what follows we shall count the eigenvalues of $\mathscr{C}$ of the given Lie group G, and we prove our theorem in 2 steps.

Step I. Compact connected simply connected semi-simple Lie group G. Let $\mathfrak{h}$ be the Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G, and $\mathfrak{h}_c$, $\mathfrak{g}_c$ be the complexification of $\mathfrak{h}$, $\mathfrak{g}$. Let $\alpha_1,\dots,\alpha_k$ be a system of positive roots of $\mathfrak{g}_c$ with respect to $\mathfrak{h}_c$. Set $\dim \mathfrak{h}_c = l$, we take $\alpha_1,\dots,\alpha_l$ a system of simple roots. Let $\lambda_1,\dots,\lambda_l$ be a dual base of $\alpha_1,\dots,\alpha_l$ with respect to the Killing form $(\cdot,\cdot)$, that is,

$$2(\lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}. \tag{14}$$

The set Λ of all the highest weight is defined as that

$$\Lambda = \{m_1\lambda_1 + \cdots + m_l\lambda_l;\ m_i \text{ is a non-negative integer}\} - \{\mathbf{0}\}. \tag{15}$$

Then we can get all eigenvalues of $\mathscr{C}$ from Λ as follows: Let $e(\lambda)$ and $d(\lambda)$ be the eigenvalue and its multiplicity. Then,

$$e(\lambda) = -(\lambda + 2\delta, \lambda), \tag{16}$$

$$d(\lambda) = \Big(\prod_{i=1}^{k} (\lambda+\delta, \alpha_i)/(\delta, \alpha_i)\Big)^2, \tag{17}$$

where $$\delta = \frac{1}{2}\sum_{i=1}^{k} \alpha_i.$$

1°) To simplify the following calculations we introduce some notations. For monotonically increasing positive functions $f(x)$ and $g(x)$, we write

$$f \succcurlyeq g, \quad \text{if} \quad g(x) = o(f(x)) \quad \text{and}$$

$$f \asymp g, \quad \text{if} \quad f \succcurlyeq g \quad \text{and} \quad f \preccurlyeq g.$$

Now we set

(18) $$b(t) = \#\{\lambda \in \Delta;\, (\lambda + 2\delta, \lambda) + 1 < t^2\},$$

(the symbol # denotes the cardinality of a set). Then we have

(19) $$b(t) \asymp t^l.$$

In fact

$$(\lambda + 2\delta, \lambda) = \sum_{i,j=1}^{d} m_i m_j (\lambda_i, \lambda_j) + \sum_{i=1}^{d} m_i (\lambda_i, 2\delta),$$

where $\lambda = m_1 \lambda_1 + \cdots + m_l \lambda_l$. We can have

$$(\lambda, \lambda) \leq \max_{i,j} \{(\lambda_i, \lambda_j)\} \sum_{i,j}^{d} m_i m_j = S_1 \Big(\sum_{i=1}^{d} m_i\Big)^2 \leq d S_1 \Big(\sum_{i=1}^{d} m_i^2\Big)$$

for some positive constant S_1. Let $K = \inf d^2(\lambda, X_i)$, where X_i is the linear space spaned by $\{\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_l\}$ and let $d^2(\lambda, X_i) = \inf_{\mu \in X_i} (\lambda + \mu, \lambda + \mu)$. Since $\{\lambda_i\}$ is linearly independent system, K has to be positive. Then $(\lambda, \lambda) \geq (\max_i m_i^2) k$. Therefore there exist two positive constants S_2 and S_3 such that

(20) $$S_2 \max_i m_i^2 \leq (\lambda + 2\delta, \lambda) + 1 \leq S_3 \sum_{i=1}^{d} m_i^2.$$

Set

(21) $$a(t) = \#\{\lambda \in \Lambda;\, S_3 \sum_{i=1}^{d} m_i^2 \leq t^2\} \quad \text{and}$$

(22) $$c(t) = \#\{\lambda \in \Lambda;\, S_2 \max_i m_i^2 \leq t^2\}.$$

Then it is easy to see that $a(t) \leq b(t) \leq c(t)$ and $a(t) \underset{\frown}{\smile} c(t) \underset{\frown}{\smile} t^l$. Hence we have (19).

2°) Let $n(t)$ be the number of eigenvalues $A = \sqrt{-\mathscr{C}+1}$ (see (3)) being less that t taking multiplicities into account, then

$$n(t) \underset{\frown}{\smile} t^{2k+l}. \tag{23}$$

In fact, let

$$\Lambda_t = \left\{ \lambda \in \Lambda ; \frac{t^2}{2l} \leq S_3 m_i^2 \leq \frac{t^2}{l} \right\}.$$

Then by the same way as in 1°), we have $\# \Lambda_t \underset{\frown}{\smile} t^l$. For any element λ in Λ_t

$$\begin{aligned} d(\lambda) &= \prod_1^k \left(\sum_j m_j \lambda_j + \delta, \alpha_i \right)^2 / (\delta, \alpha_i)^2 \\ &\geq S_4 \frac{t^{2k}}{(2l)} k \prod_{l+1}^k (\lambda_1 + \cdots + \lambda_l, \alpha_j)^2 \\ &= S_5 t^{2k} \end{aligned}$$

where S_4 and S_5 are some positive constants. This proves that $n(t) \succcurlyeq t^{2k+l}$. The upper estimation of $n(t)$ is easily obtained. Thus, we have (23).

3°) Compare two norms $\|\cdot\|_{-\sigma}$ and $\|\cdot\|_{-\sigma'}$ introduced in $\mathscr{B}(G)$ for $\sigma' > \sigma > 0$. We know that $e^{(\sigma'-\sigma)A}$ is the injection from the space $(\mathscr{B}(G), \|\cdot\|_{-\sigma'})$ into the space $(\mathscr{B}(G), \|\cdot\|_{-\sigma})$. Hence $m(t)$ in Theorem 1 is estimated as

$$n(\log_2 t) = m(t) \underset{\frown}{\smile} (\log_2 t)^{2k+l}. \tag{23'}$$

Here $2k+l$ is the dimension of G. With this and Corollary 1, we have

$$d_f(\mathscr{B}(G)) = 2k + l = \dim G. \tag{24}$$

Step II. Compact connected semi-simple Lie group G. Let $\tilde{G}$ be the universal covering group of G. Then there exists a finite subgroup Z of the center of G such that $\tilde{G}/Z = G$. Let $\tilde{H}$ and H be the Cartan subgroup of $\tilde{G}$ and G respectively, corresponding to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We note that any element $\lambda \in \Lambda$ can be viewed as a character of $\tilde{H}$. Set $\Lambda_0=\{\lambda \in \Lambda; \lambda$ is trivial on $Z\}$. The eigenvalues $e(\lambda)$ and the multiplicities $d(\lambda)$ are given as follows:

(16) $$e(\lambda)=(\lambda+2\delta, \lambda)$$

(17) $$d(\lambda)=(\prod_{i=1}^{k}(\lambda+\delta, \alpha_i)/(\delta, \alpha_i))^2, \qquad \lambda \in \Lambda_0.$$

Let m be a positive integer such that $z^m=e$ for any element z in Z. Then, $\exp m\lambda(Y)=\exp \lambda(mY)=\lambda(z^m)=1$, for any element z of Z such that $z=\exp Y$. These equalities imply the relation

(25) $$m\Lambda \subset \Lambda_0.$$

By a similar arguments to that in the step I we can complete the proof of our Theorem.

§6. Other applications of the results in §3.

In this section we shall apply the characterization in §3 of analytic functionals $\mathscr{B}(\mathfrak{D})$ of a group representation $\mathfrak{D}$ to some other cases to calculate the functional dimension $d_f(\mathscr{B}(\mathfrak{D}))$.

a) The regular representation $\mathfrak{D}_r$ of the n-dimensional torus $\boldsymbol{T}^n$. In this case the operator Δ_1 can be expressed as $\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$. A system of eigenvectors is given as follows:

$\{e^{2\pi i(m_1x_1+\cdots+m_nx_n)}; m_i$ integer, $1\leq i\leq n\}$, and its eigenvalues is

$$\{\sum_{i=1}^{n} m_i^2\}.$$

In this case it holds that $m(e^t)=n(t)\asymp t^n$. Therefore we have

(26) $$d_f(\mathscr{B}(G))=n.$$

b) The quasi-regular representation $\mathfrak{D}$ of $SO(n)$ on the $(n-1)$-dimensional sphere S^{n-1}. The quasi-regular representation $\mathfrak{D}=(T, L^2(S^{n-1}))$ of $SO(n)$ is realized on the Hilbert space $L^2(S^{n-1})$, as follows:

$$T_g f(x) = f(gx),$$

where $f(x) \in L^2(S^{n-1})$ and where g acts $x \in S^{n-1}$ as a rotation.

Let $X_{ij} = (a_{kl})$ for $i > j$, where $a_{kl} = \delta_{ki}\delta_{jl} - \delta_{il}\delta_{kj}$, that is, let X_{ij} be the infinitesimal rotation of i,j plane, then $\{X_{ij}\}$ forms a base of the Lie algebra $\mathfrak{so}(n)$. The element $Z = -\sum_{i>j} X_{ij}^2$ is a nontrivial element of the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{so}(n))$. Then $\Delta_2 = \partial T(Z)$ is essentially the spherical Laplacian Λ_{n-1} on S^{n+1}. By the classical results of the spherical functions (P. Appell and J. Kempé de Fériet [1]), we know that the set of eiginvalues of Λ_{n-1} is $\{\lambda_l = -l(l+n-2);\ l$ is a positive integer $> n-2\}$, and that its multiplicity is l^{n-2}. Thus we have

$$d_f(\mathscr{B}(\mathfrak{D})) = n - 1. \tag{27}$$

From the above results (26) and (27), we have

Theorem 5. *Let $\mathfrak{D}$ be the regular representation of T^n or the quasi-regular representation of $SO(n+1)$ then*

$$d_f(\mathscr{B}(\mathfrak{D})) = n.$$

c) The class 1 irreducible unitary representation $\mathfrak{D}_\rho$ of the n-dimensional Euclidian motion group $M(n)$. Let $M(n)$ be expressed as the semi-direct product: $M(n) = SO(n) \times R^n$. For a non-negative number ρ, the class 1 irreducible unitary representation $\mathfrak{D}_\rho$ is realized on $L^2(S^{n-1})$ as follows:

$$T_\rho(h, x) f(y) = e^{i\rho(x,y)} f(hy),$$

where $x \in R^n$, $h \in SO(n)$ and $f(y) \in L^2(S^{n-1})$. (see N. Ya. Vilenkin [17]) Let us decompose the operator Δ into two part $\Delta = \Delta_1 + \Delta_2$. The restriction of T_ρ to $SO(n)$ is equal to the quasi-regular representation of $SO(n)$, therefore Δ_1 is equal to the operator Λ_{n-1}. The restriction to R^n is the multipliers by variables, which implies $\Delta_2 = -\rho^2$.

Therefore we can use the result in b) and we have

Theorem 6. *Let $\mathfrak{D}_\rho$ be the class 1 irreducible unitary representation of the n-dimensional Euclidean motion group $M(n)$. Then*

$$d_f(\mathscr{B}(\mathfrak{D}_\rho))=n-1.$$

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