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On the cohomology ring of some homogeneous spaces.

By

Hirosi TODA

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Introduction.

Let G be a compact connected Lie group and let U be its torsion free connected subgroup of maximal rank. The purpose of the present paper is to establish a method to describe the integral cohomology ring $H^*(G/U)$, by a minimum system of generators and relations, from the results of the mod p cohomology ring $H^*(G; Z_p)$ of G and the rational cohomology ring $H^*(G/U; Q)$ of G/U.

The homogeneous space G/U is equivalent to the total space of a principal G-bundle over the classifying space BU of U. In §1, we shall discuss mod p cohomology of principal G-bundles of such type, and the result will be stated in Theorem 1.1. A description of the integral cohomology ring $H^*(G/U)$ will be given in Theorem 2.1 of §2, and will be exhibited for simple G and $U=T$, a maximal torus of G, in §3 as applications. Another application to the cohomology structure of G will be seen in forthcoming papers.

§1. Mod p cohomology of some principal bundles.

Let G be a compact connected Lie group and consider a principal G-bundle

(1.1) $$G \xrightarrow{i} X \xrightarrow{\pi} B.$$

We always assume that the base space B is arcwise connected and its cohomology groups are finitely generated for each dimension. So, the same holds for the total space X.

Let p be a prime and consider the following three hypotheses:

(1.2) *The cup-product gives an isomorphism*

$$\Delta(x_1, \ldots, x_m) \otimes M \cong H^*(G; Z_p)$$

for a graded submodule $M = \sum M^i$ *and homogeneous elements* x_i $(i=1, \ldots, m)$, *where* $\Delta(x_1, \ldots, x_m)$ *indicates the submodule spanned by the monomials* $\{x_1^{\varepsilon_1} \ldots x_m^{\varepsilon_m} \mid \varepsilon_i = 0$ *or* $1\}$.

(1.3) $M \subset \operatorname{Im} i^*$ *for the induced homomorphism* $i^* : H^*(X; Z_p) \longrightarrow H^*(G; Z_p)$.

(1.4) $P(H^*(X; Z_p), t) = P(M, t) \cdot P(H^*(B; Z_p), t) \cdot \prod_{i=1}^{m} (1 - t^{\deg x_i + 1})$, *where* P *indicates the Poincaré series*: $P(\sum V^j, t) = \sum (\dim V^j) t^j$.

The purpose of this section is to prove the following

Theorem 1.1. *Let* N *be a positive integer and assume that the principal bundle* (1.1) *satisfies* (1.2), (1.3) *and* (1.4) *for degree* $\leq N$. *Then, for a suitable choice of the elements* x_i, *the followings hold*:

(i) *For degree* $\leq N-1$, $M = \operatorname{Im} i^*$ *and the set of the transgressive elements is spanned by* M^+ *and* $\{x_i \mid \deg x_i \leq N-1\}$.

(ii) $H^*(X; Z_p)$ *is isomorphic to* $M \otimes \operatorname{Im} \pi^*$ *as* $\operatorname{Im} \pi^*$*-modules for degree* $\leq N$.

(iii) *For transgression images* $\{r_i\}$ *of* $\{x_i \mid \deg x_i \leq N-1\}$, *we have a natural isomorphism* $H^*(B; Z_p)/(r_i) \cong \operatorname{Im} \pi^*$ *for degree* $\leq N$.

(iv) *The elements* $\{r_i\}$ *are of no relation in* $H^*(B; Z_p)$ *up to degree* N.

Here we call that homogeneous elements $\{r_i\}$ of a graded commutative algebra A over Z_p *are of no relation in* A *up to degree* N if one of the following equivalent conditions holds (cf. [6]):

(1.5), (i). *The multiplication by* r_i *is an injection of* $A/(r_1, \ldots, r_{i-1})$

in itself for degree $\leq N$.

(ii). *There exists a submodule* B *of* A *such that the natural map of* $Z_p[r_1, r_2, \ldots]\otimes B$ *into* A *is bijective for degree* $\leq N$.

(iii). $P(A/(r_1, r_2, \ldots), t) = P(A, t) \cdot \prod_i (1-t^{\deg r_i})$ *for degree* $\leq N$.

Let $(E_r^{s,q})$ be the mod p cohomology spectral sequence associated with the principal bundle (1.1), then

$E_2^{*,*} = H^*(B; Z_p)\otimes H^*(G; Z_p)$ converging to $H^*(X; Z_p)$,

$\operatorname{Im} i^* = E_\infty^{0,*} \subset E_2^{0,*} = H^*(G; Z_p)$ and $\operatorname{Im} \pi^* = E_\infty^{*,0} \subset H^*(X; Z_p)$.

Lemma 1.1. (i). *The multiplication gives an injection of* $E_r^{*,0}\otimes E_r^{0,*}$ *into* $E_r^{*,*}$.

(ii). *Let* $\tilde{M}$ *be a graded submodule of* $H^*(X; Z_p)$ *which is injectively mapped into* $H^*(G; Z_p)$ *under* i^*, *then the cup-product gives an injection of* $\tilde{M}\otimes \operatorname{Im} \pi^*$ *into* $H^*(X; Z_p)$.

Proof. The right translation μ gives a commutative diagram

$$\begin{array}{ccccc} G\times G & \xrightarrow{i\times 1} & X\times G & \xrightarrow{\pi\times 0} & B\times * \quad (*: \text{a point}) \\ \downarrow{\mu} & & \downarrow{\mu} & & \downarrow \\ G & \xrightarrow{i} & X & \xrightarrow{\pi} & B \end{array}$$

which is a map of fiberings, and induces a map of spectral sequences

$$\mu^* : E_r^{*,*} \longrightarrow E_r^{*,*}\otimes H^*(G; Z_p)$$

such that $\mu^*(b) = b\otimes 1$ for $b\in E_r^{*,0}$ and $\mu^*(z) = 1\otimes z + \sum z_i'\otimes z_i''$ (deg $z_i'>0$) for $z\in E_r^{0,q}\subset E_2^{0,q} = H^q(G; Z_p)$. Then $\mu^*(b\cdot z) = b\otimes z + \sum b\cdot z_i' \otimes z_i''$, and the assertion (i) is proved. (ii) is proved similarly by considering $\mu^* : H^*(X; Z_p) \longrightarrow H^*(X; Z_p)\otimes H^*(G; Z_p)$ in which $\mu^*(x) = x\otimes 1$ holds for $x\in \operatorname{Im}\pi^*$. q. e. d.

We assume the following inductive hypothesis for $n<N(\varepsilon=0, 1)$.

(1.6). *The elements* x_i *of* $\deg x_i \leq n-1$ *are transgressive and transgression images* $\{r_i\}$ *are of no relation up to degree* $n+\varepsilon$.

Put $M_{n+1} = \sum_{j=0}^{n+1} M^j$ then $M_{n+1}\subset \operatorname{Im} i^*$ by the assumption (1.3) of the theorem, and the differential d_r of the spectral sequence satisfies

$$d_r(b\otimes 1)=d_r(1\otimes m)=0 \qquad \text{for } b\in H^*(B;\ Z_p) \text{ and } m\in M_{n+1}$$

and $\quad d_r(1\otimes x_i)=0 \ (r\leq \deg x_i), \quad d_r(1\otimes x_i)=r_i\otimes 1 \ (r=\deg x_i+1)$

for $\deg x_i\leq n-1$. We put

$\Delta_r=\Delta(x_i;\ r-1\leq \deg x_i\leq n-1) \qquad (\Delta(\phi)=Z_p)$

$J_r=$the ideal of $H^*(B;\ Z_p)$ generated by $\{r_i \mid \deg r_i < \operatorname{Min}(r,\ n+1)\}$

and $\quad \bar{E}_r^{*,*}=H^*(B;\ Z_p)/J_r\otimes(\Delta_r\otimes M_{n+1})$.

A differential d_r in $\bar{E}_r^{*,*}$ is defined by the derivativity and the above equalities for d_r. Then using (1.5) we have easily

(1.7). $\bar{E}_{r+1}^{*,*}\subset H(\bar{E}_r^{*,*})$ *and the equality* $\bar{E}_{r+1}^{s,q}=H(\bar{E}_r^{s,q})$ *holds if* $q < r-1$ *or* $r\geq n+1$ *or if* $s+r\leq n+\varepsilon$. *Also* $\{r_i\}$ *are of no relation up to degree* k *if and only if the equality* $\bar{E}_{r+1}^{s,r-1}=H(\bar{E}_r^{s,r-1})$ *holds for* $s+r\leq k$.

Now we have natural maps

$$f_r^{s,q}\ :\ \bar{E}_r^{s,q}\longrightarrow E_r^{s,q}$$

which commutes with the differential d_r and induces

$\bar{f}_{r+1}^{s,q}\ :\ H(\bar{E}_r^{s,q})\longrightarrow E_{r+1}^{s,q}=H(E_r^{s,q}) \quad$ such that $f_{r+1}^{s,q}=\bar{f}_{r+1}^{s,q}|\bar{E}_{r+1}^{s,q}$.

Lemma 1.2. (1.6). *implies the followings:*

(i). $f_r^{s,q}$ *is injective if* $s\leq n+1+\varepsilon$ *and* $r\leq n+1$ *or if* $s\leq n$.

(ii). $f_r^{s,q}$ *is surjective if* $q=0$, *if* $s+q\leq n-1+\varepsilon$ *and* $q\leq n-1$ *or if* $s\leq n+1+\varepsilon-r$ *and* $q\leq n-1$.

(iii) *The natural map of* Coker $f_{r+1}^{0,n}$ *into* Coker $f_r^{0,n}$ *is injective for* $2\leq r\leq n$.

(iv) *Let* $\varepsilon=0$ *and* $1\leq q\leq n-1$, *then* $H(\bar{E}_{q+1}^{n-q,q})/\bar{E}_{q+2}^{n-q,q}$ *is isomorphic to* Coker $f_r^{n-q,q}$ *for* $r\geq q+2$.

Proof. (i). and (ii). are obvious for $r=2$ and also for $s<0$ or $q<0$. $\bar{f}_{r+1}^{s,q}$ is injective (resp. surjective) if $f_r^{s,q}$ is injective (resp. surjective) and $f_r^{s-r,q+r-1}$ is surjective (resp. $f_r^{s+r,q-r+1}$ is injective). By induction on $r\geq 2$, the assertion (ii). by use of (1.7), the assertion (i)$_\varepsilon$ for $s=0$ or $q=0$, and then the assertion (i)$_\varepsilon$, by Lemma 1.1 since $\bar{E}_r^{s,q}=\bar{E}_r^{s,0}\otimes\bar{E}_r^{0,q}$, are proved. Next we have the following commutative exact diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H(\bar{E}_r^{0,n}) & \longrightarrow & \bar{E}_r^{0,n} & \xrightarrow{d_r} & \bar{E}_r^{r,n-r+1} \\
 & & \downarrow \bar{f}_{r+1}^{0,n} & & \downarrow f_r^{0,n} & & \downarrow f_r^{r,n-r+1} \\
0 & \longrightarrow & E_{r+1}^{0,n} & \longrightarrow & E_r^{0,n} & \xrightarrow{d_r} & E_r^{r,n-r+1}.
\end{array}$$

For $2\leq r\leq n$, $H(\bar{E}_r^{0,n})=\bar{E}_{r+1}^{0,n}$ by (1.7) and $f_r^{r,n-r+1}$ is injective by (i)$_\varepsilon$. Then $\operatorname{Im} f_r^{0,n}\cap E_{r+1}^{0,n}\subset \operatorname{Im}\bar{f}_{r+1}^{0,n}=\operatorname{Im} f_{r+1}^{0,n}$, and (iii) follows.

Let $\varepsilon=0$ and $1\leq q\leq n-1$. $\bar{f}_{q+2}^{n-q,q}$ is bijective since $f_{q+1}^{n-q,q}$ is bijective $f_{q+1}^{n-2q-1,2q}$ is surjective and $f_{q+1}^{n+1,0}$ is injective. Thus (iv) is true for $r=q+2$. Let $r>q+2$, then $H(\bar{E}_{r-1}^{n-q,q})=\bar{E}_r^{n-q,q}$ by (1.7) and we have an exact sequence: $\xrightarrow{d_{r-1}}\bar{E}_{r-1}^{n-q,q}\longrightarrow\bar{E}_r^{n-q,q}\longrightarrow 0$. The same holds for $\{E_r\}$. By the compatibility of $\{f_r\}$ we have an exact sequence

$$\operatorname{Coker} f_{r-1}^{n-q-r+1,q+r-2}\xrightarrow{d_{r-1}}\operatorname{Coker} f_{r-1}^{n-q,q}\longrightarrow\operatorname{Coker} f_r^{n-q,q}\longrightarrow 0.$$

The first cokernel is trivial by (ii). Therefore (iv) is proved by induction on $r\geq q+2$.

Lemma 1.3. (1.6)$_0$ *implies* (1.6)$_1$ *and that* $\operatorname{Coker} f_{n+1}^{0,n}$ *is naturally isomorphic to* $\operatorname{Coker} f_2^{0,n}$ *and it is mapped isomorphically onto* $d_{n+1}(E_{n+1}^{0,n})\subset E_{n+1}^{n+1,0}$ *under* d_{n+1}.

Proof. We assume (1.6)$_\varepsilon$. For degree$\leq n+\varepsilon$, $P(M,t)=P(M_{n+1},t)=P(\bar{E}_\infty^{*,0},t)$ and $P(H^*(B;Z_p),t)\cdot\prod_{i=1}^{m}(1-t^{\deg x_i+1})=P(H^*(B;Z_p)/(r_i),t)\cdot(1-a_nt^{n+1})=P(\bar{E}_\infty^{*,0},t)\cdot(1-a_nt^{n+1})$ by (1.5), where $a_n=\dim\operatorname{Coker} f_2^{0,n}=$number of $\{x_i\mid\deg x_i=n\}$. Since $\bar{E}_\infty^{*,*}=\bar{E}_\infty^{*,0}\otimes\bar{E}_\infty^{0,*}$, it follows from (1.4)

(1.8)$_\varepsilon$ $P(E_\infty^{*,*},t)=P(H^*(X;Z_p),t)=P(\bar{E}_\infty^{*,*}\ t)\cdot(1-a_nt^{n+1})$
for degree $\leq n+\varepsilon$.

By (i)$_0$ of Lemma 1.2, $f_\infty^{s,q}$ is injective, and $\dim E_\infty^{s,q}\geq\dim\bar{E}_\infty^{s,q}$ for $s\leq n$. By (1.8)$_0$, $\sum_{s+q=n}\dim E_\infty^{s,q}=\sum_{s+q=n}\dim\bar{E}_\infty^{s,q}$. Thus $f_\infty^{s,q}$ is bijective for $s+q=n$. By (iv) of Lemma 1.2, we have $H(\bar{E}^{s,q}_{q+1})=\bar{E}^{s,q}_{q+2}$ for $s+q=n$ ($\bar{E}^{0,n}_{n+1}=\bar{E}^{0,n}_{n+2}=M^n$). Thus $(1.6)_0$ implies $(1.6)_1$ by use of (1.7) for $k=n+1$.

Next, assuming $(1.6)_1$ it follows from $(1.8)_1$

$$a_n=\sum_{s+q=n+1}(\dim\bar{E}^{s,q}_{\infty}-\dim E^{s,q}_{\infty})\leq\dim\bar{E}^{n+1,0}_{n+2}-\dim E^{n+1,0}_{n+2}.$$

$\bar{E}^{n+1,0}_{n+2}=\bar{E}^{n+1,0}_{n+1}$ by definition. $\bar{E}^{n+1,0}_{n+1}$ is isomorphic to $E^{n+1,0}_{n+1}$ by $(\text{i})_1$ and $(\text{ii})_1$ of Lemma 1.2. Since $d_{n+1}=0$ in $\bar{E}^{*,*}_{n+1}$, $d_{n+1}(\operatorname{Im}f^{0,n}_{n+1})=f^{n+1,0}_{n+1}(d_{n+1}\bar{E}^{0,n}_{n+1})=0$ and d_{n+1} induces a surjection of $\operatorname{Coker}f^{0,n}_{n+1}$ onto $\operatorname{Im}d_{n+1}\subset E^{n+1,0}_{n+1}$. We have also $E^{n+1,0}_{n+2}=E^{n+1,0}_{n+1}/\operatorname{Im}d_{n+1}$. Thus

$$\dim\ \operatorname{Coker}f^{0,n}_2=a_n\leq\dim E^{n+1,0}_{n+1}-\dim E^{n+1,0}_{n+2}$$
$$=\dim\ \operatorname{Im}d_{n+1}\leq\dim\ \operatorname{Coker}f^{0,n}_{n+1}.$$

By (iii) of Lemma 1.2, the equality $\dim\ \operatorname{Coker}f^{0,n}_2=\dim\ \operatorname{Im}d_{n+1}=\dim\ \operatorname{Coker}f^{0,n}_{n+1}$ holds and the second half of the lemma is proved.

Proof of Theorem 1.1.

The theorem is obvious for $N=1$. By induction on N, we may assume $(1.6)_0$ for $n=N-1$. Let $\{x_k\}$ be the set of x_i with $\deg x_i=n$. By definition $\operatorname{Coker}f^{0,n}_2$ has a basis $\{x_k\}$ mod $\bar{E}^{0,*}_2=\Delta_2\otimes M$, $\Delta_2=\Delta(x_i;\ \deg x_i<n)$. By Lemma 1.3 there exist elements $x'_k\equiv x_k$ mod $\Delta_2\otimes M$ such that $x'_k(=1\otimes x'_k)\in E^{0,n}_{n+1}$ and that $\{x'_k\}$ and $M^n=\bar{E}^{0,n}_{n+1}$ span $E^{0,n}_{n+1}$. Changing x'_k modulo M^n if it is necessary, we may choose x'_k such that $x'_k\equiv x_k$ mod decomposables. Then replacing x_k by x'_k we obtain new generators $\{x_i\}$ satisfying (1.2) and $x_k\in E^{0,n}_{n+1}$. Since $E^{0,n}_{n+1}$ coincides with the set of the transgressive elements of degree n, (i) of Theorem 1.1 is proved.

Let r_k's be transgression images of the x_k's. Lemma 1.3 shows that $(1.6)_1$ holds and that $\{r_k\}$ are linearly independent in $E^{n+1,0}_{n+1}\cong\bar{E}^{n+1,0}_{n+1}\subset H^*(B;\ Z_p)/(r_i;\ \deg r_i\leq n)$. Thus (iv) of Theorem 1.1 is proved. Again by Lemma 1.3

$$\pi^*H^{n+1}(B;\ Z_p)=E^{n+1,0}_{\infty}=E^{n+1,0}_{n+2}\cong E^{n+1,0}_{n+1}/\{r_k\},$$

and (iii) follows. In Lemma 1.1, (ii), take $\tilde{M}$ such that it is mapped isomorphically onto M_{n+1}, then $\tilde{M}\otimes \operatorname{Im}\pi^*$ is mapped injetaively into $H^*(X; Z_p)$. The Poincaré series of $H^*(X; Z_p)$ and $M\otimes \operatorname{Im}\pi^*$ are given both sides of (1.4) for degree$\leq n+1$. Thus (ii) of the theorem is proved. q. e. d.

§2. Cohomology of some homogeneous spaces.

Let U be a connected subgroup of G and let $EG\longrightarrow BU=EG/U$ be a universal U-bundle. In the principal G-bundle

$$G\xrightarrow{i} EG\times_U G\xrightarrow{\pi} BU \tag{2.1}$$

the projection $EG\times_U G\longrightarrow G/U=*\times_U G$ is a homotopy equivalence. So we have a fibering

$$G\xrightarrow{\pi_0} G/U\xrightarrow{i_0} BU \tag{2.2}$$

equivalent to (2.1), where i_0 is a map classifying the U-bundle: $G\longrightarrow G/U$.

By Hopf-Borel theorem [6], we have for each prime p

$$H^*(G; Z_p)=\Lambda(x'_1,\dots, x'_{r'})\otimes Z_p[y'_1,\dots, y'_{s'}]/(y'^{h_1}_1,\dots, y'^{h_{s'}}_{s'}) \tag{2.3}$$

where h_j is a power of p ($h_j\geq 4$ if $p=2$) and if $p>2$ then $\deg x'_i$ *is odd and* $\deg y'_j$ *is even,*

and for the rational coefficient

$$H^*(G; Q)=\Lambda(z_1,\dots, z_\ell),\qquad \deg z_i: \textit{odd}. \tag{2.4}$$

Let M be the subalgebra generated by $\{y'_j \mid \deg y'_j$ *even*$\}$ and additionally $\{y'^2_j\}$ and $\{x'_i \mid \deg x'_i$ *even*$\}$ *if* $p=2$, and let $\{x_i \mid 1\leq i\leq r\}$ be the union of $\{x'_i \mid \deg x'_i$ *odd*$\}$ and $\{y'_j \mid \deg y'_j$ *odd*$(p=2)\}$. Then (1.2) is satisfied:

$$\Delta(x_1,\dots, x_r)\otimes M \cong H^*(G; Z_p) \tag{2.5}$$

where $M=Z_p[y_1,\dots,y_s]/(y^{k_1}_1,\dots, y^{k_s}_s)$ *with* k_j*: power of* p.

Now we consider the following hypothesis

(2.6). $r=\ell$ *and for each* j, $1\leq j\leq s$, *there corresponds an* $i=i(j)$ *such* *that* $x_{i(j)}$ *is transgressive with respect to* (2.2) *and that* $\beta(x_{i(j)})=y_j$, *where* β *indicates the Bockstein homomorphism assoicated with the exact sequence* $0\longrightarrow Z_p\longrightarrow Z_{p^2}\longrightarrow Z_p\longrightarrow 0$.

We shall see in §3 that the simply connected simple Lie groups G enjoy (2.6) if $U=T$ or if $(G, p)\neq(E_8, 2)$.

We shall consider the case that U is torsion free and of maximal rank. According to Borel [6],

$$H^*(U)=\Lambda(u_1,\dots, u_\ell), \qquad \deg u_i: \text{odd},$$

and $$H^*(BU)=Z[t_1,\dots, t_\ell], \qquad \deg t_i=\deg u_i+1: \text{even},$$

where t_i is a transgression image of u_i. We shall denote the i_0^*-image of t_i by the same symbol

$$t_i=i_0^*(t_i)\ni H^*(G/U).$$

Assuming (2.6) for all prime p, we denote $\{y_1,\dots, y_m\}$ the collection of the y_j in (2.6) for all possible prime, and by p_j the prime corresponding to y_j.

Then we have the following description of $H^*(G/U)$.

Theorem 2.1. *Let* U *be a torsion free connected subgroup of maximal rank in* G. *Assume* (2.6) *for all prime, and let* δ_j *and* σ_i *be homogeneous elements of* $Z[t_1,\dots, t_\ell]$ *such that* $\delta_j \pmod{p_j}$ *is a transgression image of* $x_{i(j)}$ *and that* $H^*(G/U; Q)=Q[t_1,\dots, t_\ell]/(\sigma_1,\dots, \sigma_\ell)$, $\deg\sigma_1\leq \dots\dots \leq \deg\sigma_\ell$.

Then there exist generators $\gamma_j\in H^*(G/U)$ *and relations* ρ_i, $\rho'_j\in Z[t_1,\dots, t_\ell, \gamma_1,\dots, \gamma_m]$ *such that* ($\deg\rho_i=\deg\sigma_i$, $\deg\gamma_j=\deg\rho'_j=\deg\delta_j$)

$$H^*(G/U)=Z[t_1,\dots, t_\ell, \gamma_1,\dots, \gamma_m]/(\rho_1,\dots, \rho_\ell, \rho'_1,\dots, \rho'_m),$$

$$\pi_0^*(\gamma_j)\equiv y_j \pmod{p_j}$$

and $$\rho'_j=p_j\cdot\gamma_j+\delta_j,$$

where the relation ρ_i *is determined by the maximality of the integer* n *in*

$$n\cdot\rho_i\equiv\sigma_i \qquad \mathrm{mod}\ (\rho_1,\dots,\rho_{i-1},\rho'_1,\dots,\rho'_m).$$

In order to prove the theorem we prepare two lemmas. The first lemma is well-known and proved by checking integral cochains.

Lemma 2.1. *Let* $F\xrightarrow{i}X\xrightarrow{\pi}B$ *be a fibering,* x *a transgressive element of* $H^n(F;Z_p)$ *and let* δ *be an element of* $H^{n+1}(B)$ *such that* $\delta\ (\mathrm{mod}\,p)$ *is a transgression image of* x*. Then there exists an element* γ *of* $H^{n+1}(X)$ *such that*

$$i^*(\gamma)\equiv\beta(x)\ (\mathrm{mod}\,p)\ \text{ and }\ p\cdot\gamma=-\pi^*(\delta).$$

The rational cohomology ring $H^*(G/U;Q)$ is determined by the action of the Weyl groups $\Phi(G)$ and $\Phi(U)$ on a maximal torus $T\subset U\subset G$ [6: *Ch. VI*]:

(2.7) $\quad H^*(BG;Q)=H^*(BT;Q)^{\Phi(G)}\subset H^*(BU;Q)=H^*(BT;Q)^{\Phi(U)}$

and $\quad H^*(G/U;Q)=H^*(BU;Q)/(H^+(BG;Q))$

$$=Q[t_1,\dots,t_\ell]/(\sigma_1,\dots,\sigma_\ell)$$

where σ_i is a transgression image of z_i and $\{\sigma_i\}$ are of no relation in $H^*(BU;Q)=Q[t_1,\dots,t_\ell]$. By [10], G/T and U/T are torsion free. Since U is torsion free, so is G/U by Proposition 30.1 of [6]. Thus

(2.8) $\quad P(H^*(G/U;Z_p),t)=P(H^*(BU;Z_p),t)\cdot\prod_{i=1}^{\ell}(1-t^{\deg z_i+1}).$

Lemma 2.2. *The assumption of Theorem* 2.1 *implies the assumptions* (1.2), (1.3), (1.4) *of Theorem* 1.1 *for all prime and for arbitrary* N.

Proof. (1.2) is already satisfied by (2.5). Since $x_{i(j)}$ is transgressive, so is $y_j=\beta(x_{i(j)})$. Since $H^n(BU;Z_p)=0$ for odd n, the transgression image of y_j is trivial, that is, y_j is an i_0^*-image. It follows (1.3): $M\subset\mathrm{Im}\,i_0^*$. Consider the mod p Bockstein spectral sequence (E_r) for G: $E_1=H^*(G;Z_p)$, $E_2=H(E_1$ w.r.t. $\beta)$ converging to $E_\infty=(H^*(G)/\mathrm{torsion})\otimes Z_p=\Lambda(z_1,\dots,z_\ell)$. From (2.6) we have E_2 as a cohomology (subquotient) of

$$E_2'=\Delta(x_i\ \ (i\neq i(j)\ \text{for any}\ j),\ x_{i(j)}y_j^{k_j-1}).$$

Then $\dim E_2\leq\dim E_2'=2^{\ell}$, but $2^{\ell}=\dim E_\infty\leq\dim E_2$. Thus we have $E_2'=E_\infty$ and

(2.9). *The set* $\{\deg z_i\}$ *coincides with the set*

$$\{\deg x_i\ \ (i\neq i(j)),\ k_j\cdot\deg y_j-1\}.$$

That is, $\prod_{i=1}^{\ell}(1-t^{\deg z_i+1})=\prod_{i\neq i(j)}(1-t^{\deg x_i+1})\cdot\prod_{j=1}^{s}(1-t^{k_j\cdot\deg y_j})=\prod_{i=1}^{\ell}(1-t^{\deg x_i+1})\cdot P(M,t)$. Then it follows from (2.8) that (1.4) holds:

$$P(H^*(G/U;\ Z_p),\ t)=P(H^*(BU;\ Z_p),\ t)\cdot P(M,\ t)\cdot\prod_{i=1}^{\ell}(1-t^{\deg x_i+1}).$$

Proof of Theorem 2.1. Apply Lemma 2.1 to $x=x_{i(j)}$, $p=p_j$ and $\delta=\delta_j$, then we have the existence of γ_j such that $\pi_0^*(\gamma_j)\equiv y_j \pmod{p_j}$ and that $\rho_j'=p_j\cdot\gamma_j+\delta_j$ vanishes in $H^*(G/U)$. Put

$$R=Z[t_1,\ldots,t_\ell,\ \gamma_1,\ldots,\gamma_m]\ \text{and}\ I_i=(\rho_1,\ldots,\rho_i,\ \rho_1',\ldots,\rho_m')\subset R.$$

Since $\sigma_i\neq 0$ in $(R/I_{i-1})\otimes Q=H^*(BU;\ Q)/(\sigma_1,\ldots,\sigma_{i-1})$, $\sigma_i \pmod{I_{i-1}}$ is of infinite order. Since R/I_{i-1} is finitely generated for each degree, there exists the maximum of the integers n such that $n\cdot x\equiv\sigma_i \pmod{I_{i-1}}$ for some x. So, the existence of the relation ρ_i with the required property is proved inductiely. We have obtained a natural homomorphism $\eta:\ R/I_\ell\longrightarrow H^*(G/U)$, and by tensoring Z_p, $\eta_p:(R/I_\ell)\otimes Z_p\longrightarrow H^*(G/U;\ Z_p)$. Then it is sufficient to prove that η_p is bijective for each prime p.

By Lemma 2.2 we apply Theorem 1.1 to (2.2), and obtain

$$H^*(G/U;\ Z_p)=R_p/(r_1,\ldots,r_\ell,r_1',\ldots,r_s')\quad\text{for}\ R_p=Z_p[t_1,\ldots,t_\ell,\gamma_1,\ldots,\gamma_s],$$

where r_j' is a relation satisfying $r_j'\equiv\gamma_j^{k_j}\ \mathrm{mod}(t_1,\ldots,t_\ell)$. On the other hand, in $(R/I_\ell)\otimes Z_p$, γ_j and δ_j are cancelled to each other if $p_j\neq p$, and ρ_j' is replaced by $\delta_j=r_{i(j)}$ if $p_j=p$. Thus η_p is equivalent to the natural map

$$R_p/(\rho_1,\ldots,\rho_\ell,\ \delta_1,\ldots,\delta_s)\longrightarrow R_p/(r_1'',\ldots,r_\ell'',\ \delta_1,\ldots,\delta_s),$$

where $\{r_i''\}=\{r_i\mid i\neq i(j)\}\cup\{r_1',\ldots,r_s'\}$. By (2.9), we may assume that $\deg \rho_i = \deg r''_i$. Put

$$J_i = (\rho_1, \ldots, \rho_i, \delta_1, \ldots, \delta_s) \text{ and } J''_i = (r''_1, \ldots, r''_i, \delta_1, \ldots, \delta_s).$$

The property of ρ_i shows that $\{\rho_{i+1}, \ldots, \rho_\ell\}$ are linearly independent mod J_i, and the same is true for $\{r''_{i+1}, \ldots, r''_\ell\}$ mod J''_i. By induction on the degree of ρ_i we assume that $J_i = J''_i$ for $\deg \rho_i < \deg \rho_{i+1} = \ldots = \deg \rho_k < \deg \rho_{k+1}$. Then the equality $J_k = J''_k$ is proved at first for degree $\leq \deg \rho_k$ and then for all degree. So, we obtain $J_\ell = J''_\ell$, that is, η_p is bijective and so is η. q. e. d.

Corollary 2.2. *Theorem* 2.1 *gives a minimum system of generators and relations for* $H^*(G/U)$ *if there is no pair* (i, j) *with* $\deg t_i = \deg \rho_j$, $\deg t_i = \deg \rho'_j$ *or with* $\deg \rho'_i = \deg \rho'_j$ *and* $p_i \neq p_j$.

§3. $H^*(G/T)$ for simple Lie group G.

Let G be a compact connected semi-simple Lie group and let T be its maximal torus, then the universal covering $\tilde{G}$ of G is compact and

$$\tilde{G}/\tilde{T} = G/T \tag{3.1}$$

for the inverse image $\tilde{T}$ of T which is a maximal torus of $\tilde{G}$. By Corollary 2.2,

(3.2) *if a simply connected compact* G *satisfies* (2.6) *and if there is no pair* $(y_j (\text{mod}\, p), y_k (\text{mod}\, p'))$ *with* $\deg y_j = \deg y_k$ *and* $p \neq p'$, *then Theorem* 2.1 *gives a minimum system of generators and relations for* $H^*(G/T)$.

For classical cases we have

Proposition 3.1. *For* $G = SU(\ell+1)$, $Sp(\ell)$ *and* $SO(n)$, $Spin(n)$, $\ell = \left[\frac{n}{2}\right]$, (2.6) *is satisfied for arbitrary* U, *and we have the following description of* $H^*(G/U)$ *by minimum systems of generators and relations :*

$$H^*(SU(\ell+1)/T) = Z[t_1, \ldots, t_\ell]/(\rho_2, \rho_3, \ldots, \rho_{\ell+1}),$$

$$H^*(Sp(\ell)/T)=Z[t_1,\ldots,t_\ell]/(\rho_2,\rho_4,\ldots,\rho_{2\ell}),$$

and $$H^*(SO(n)/T)=H^*(Spin(n)/\tilde{T})$$

$$=Z[t_1,\ldots,t_\ell,\gamma_1,\ldots,\gamma_m]/(\rho_2,\rho_3,\ldots,\rho_\ell,\rho'_{2m+2},\rho_{2m+4},\ldots,\rho_{2s}),$$

where $\deg t_i=2$, $\deg\rho_k=\deg\rho'_k=2k$, $\deg\gamma_j=4j+2$, $m=\left[\frac{n-3}{4}\right]$ *and* $s=\left[\frac{n-1}{2}\right]$. *(Explicit forms of the relations may be obtained from the results in* §2 *of* [12].)

Proof. Except the case $G=SO(n)$, $Spin(n)$ and $p=2$, $H^*(G;Z_p)=\Lambda(x_1,\ldots,x_\ell)$ and (2.6) is satisfied. Let $z_i\in H^i(SO(n);Z_2)$ be the suspension image of the $(i+1)$-th Stiefel-Whitney class $w_{i+1}\in H^{i+1}(BSO(n);Z_2)$. Then z_i is universally transgressive and we have $z_i^2=z_{2i}(=0$ if $2i\geq n)$ and $\beta(z_{2j-1})=Sq^1(z_{2j-1})=z_{2j}$ by Wu formula. Thus (2.5) and (2.6) are satisfied for $x_i=z_{2i-1}$, $y_j=z_{4j-2}$ and for powers h_j of 2 such that $n\leq h_j(4j-2)<2n$:

$$H^*(SO(n);Z_2)=\Delta(x_1,\ldots,x_\ell)\otimes M,\quad M=Z_2[y_1,\ldots,y_s]/(y_1^{h_1},\ldots,y_s^{h_s})$$

for $s=\left[\frac{n+1}{4}\right]$. The covering homomorphism $p^*:H^*(SO(n);Z_2)\longrightarrow H^*(Spin(n);Z_2)$ has the kernel $(x_1,y_1)=(x_1)$ and $H^*(Spin(n);Z_2)=\Delta(t)\otimes\operatorname{Im}p^*$, $\deg t=2\cdot h_1-1$. Thus (2.6) is satisfied by omitting y_1 and by replacing x_1 by t. Then the above descriptions are obtained directly from Theorem 2.1, (2.9) and (3.2).

q. e. d.

The simply connected exceptional Lie groups have p-torsions only in the cases listed below [2], [3], [4], [5], [8], $(\deg x_i=i)$:

(3.3) $$H^*(G_2;Z_2)=\Delta(x_3,x_5)\otimes Z_2[x_6]/(x_6^2),$$

$$H^*(F_4;Z_2)=\Delta(x_3,x_5,x_{15},x_{23})\otimes Z_2[x_6]/(x_6^2),$$

$$H^*(E_6;Z_2)=\Delta(x_3,x_5,x_9,x_{15},x_{17},x_{23})\otimes Z_2[x_6]/(x_6^2),$$

$$H^*(E_7;Z_2)=\Delta(x_3,x_5,x_9,x_{15},x_{17},x_{23},x_{27})\otimes Z_2[x_6,x_{10},x_{18}]/(x_6^2,x_{10}^2,x_{18}^2),$$

and $$H^*(E_8;Z_2)=\Delta(x_3,x_5,x_9,x_{15},x_{17},x_{23},x_{27},x_{29})\otimes Z_2[x_6,x_{10},x_{18},x_{30}]/(x_6^8,x_{10}^4,x_{18}^2,x_{30}^2),$$

where $x_{i+2}=Sq^2x_i$ *for* $i=3, 27$; $x_{i+4}=Sq^4x_i$ *for* $i=5, 25$; $x_{i+8}=Sq^8x_i$ *for* $i=9, 15$ and $x_{2i}=x_i^2=\beta x_{2i-1}$ *for* $i=3, 5, 9, 15$.

(3.4) $$H^*(F_4; Z_3)=\Lambda(x_3, x_7, x_{11}, x_{15})\otimes Z_3[x_8]/(x_8^3),$$

$$H^*(E_6; Z_3)=\Lambda(x_3, x_7, x_9, x_{11}, x_{15}, x_{17})\otimes Z_3[x_8]/(x_8^3),$$

$$H^*(E_7; Z_3)=\Lambda(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35})\otimes Z_3[x_8]/(x_8^3),$$

and $$H^*(E_8; Z_3)=\Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47})\otimes Z_3[x_8, x_{20}]/(x_8^3, x_{20}^3),$$

where $x_7=\mathscr{P}^1x_3$, $x_8=\beta x_7$ and $x_{20}=\beta x_{19}=\beta\mathscr{P}^3x_{11}$.

(3.5) $$H^*(E_8; Z_5)=\Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47})\otimes Z_5[x_{12}]/(x_{12}^5),$$

where $x_{11}=\mathscr{P}^1x_3$ *and* $x_{12}=\beta x_{11}$.

Proposition 3.2. *Let p be a prime and G a simply connected exceptional Lie group. Let U be a connected subgroup of maximal rank in G which is torsion free if $G=E_8$ and $p=2$. Then* (2.6) *is satisfied. $H^*(G/T)$ has the following minimum systems of generators and relations*:

$$H^*(G_2/T)=Z[t_1, t_2, \gamma_3]/(\rho_2, \rho_3, \rho_6),$$

$$H^*(F_4/T)=Z[t_1, t_2, t_3, t_4, \gamma_3, \gamma_4]/(\rho_2, \rho_3, \rho_4, \rho_6, \rho_8, \rho_{12})$$

$$H^*(E_6/T)=Z[t_1,\dots, t_6, \gamma_3, \gamma_4]/(\rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{12}),$$

$$H^*(E_7/T)=Z[t_1,\dots, t_7, \gamma_3, \gamma_4, \gamma_5, \gamma_9]/(\rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{18})$$

and $$H^*(E_8/T)=Z[t_1,\dots, t_8, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_9, \gamma_{10}, \gamma_{15}]/(\rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{15}, \rho_{18}, \rho_{20}, \rho_{24}, \rho_{30}),$$

where $\deg t_i=2$, $\deg\gamma_j=2j$, $\deg\rho_k=2k$,

and $\rho_i=2\cdot\gamma_i+\delta_i$ $(i=3, 5, 9, 15$; $\delta_3\equiv Sq^2\rho_2$, $\delta_5\equiv Sq^4\delta_3$, $\delta_9\equiv Sq^8\delta_5$, $\delta_{15}\equiv Sq^{14}\rho_8 (\mathrm{mod}\ 2))$, $\rho_i=3\cdot\gamma_i+\delta_i$ $(i=4, 10$; $\delta_4\equiv\mathscr{P}^1\rho_2$, $\delta_{10}\equiv\mathscr{P}^3\delta_4(\mathrm{mod}\ 3))$ *and* $\rho_6=5\cdot\gamma_6+\delta_6$ $(\delta_6\equiv\mathscr{P}^1\rho_2(\mathrm{mod}\ 5))$.

(For $G=G_2$, F_4, E_6 explicit forms of the relations may be obtained from the results of [12].)

Proof. Except $x_{30}(p=2)$, each generators x_{2i} of even degree satisfy $x_{2i}=\beta x_{2i-1}$ and $x_{2i-1}=\alpha\cdot x_3$ for a cohomology operation α. Since x_3 is universally transgressive, so is x_{2i-1} if $i\neq 15$. Thus (2.6) holds for $(G, p)\neq(E_8, 2)$. Now let $(G, p)=(E_8, 2)$ and U be torsion free. Then Lemma 2.2 and thus Theorem 2.1 are valid for degree≤ 29. It follows from Theorem 1.1, (i) that there exists a transgressive element $x'_{15}\equiv x_{15}$ mod decomposables. Putting $x'_{15}=x_{15}+a\cdot x_9x_3^2+b\cdot x_5^3+c\cdot x_3^5$ (a, b, c$\in Z_2$) we have $(x'_{15})^4=x_{15}^4+a\cdot x_{18}^2x_6^4+b\cdot x_{10}^6+c\cdot x_6^{10}=0$. Thus we may replace x_{15} by x'_{15} (so x_{23} by $Sq^8x'_{15}$ and so on) in the last formula of (3.3). Then (2.6) is satisfied for torsion free U. By Theorem 2.1 and (3.2), we have the above descriptions of $H^*(G/T)$. q. e. d.

Corollay 3.3. *For a suitable choice of the generators in* (3.3), (3.4), (3.5), *the generators are transgressive and satitfy* (i) *of Theorem* 1.1 *with respect to the fibering* (2.2) *for torsion free connected subgroup* U *of maximal rank.*

DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY

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