J. Math. Kyoto Univ. (JMKYAZ)
16-1 (1976) 137-141

An example of indecomposable vector bundle of rank $n-1$ on P^n.

By

Hiroshi TANGO

(Communicated by Prof. Nagata, Dec. 20, 1974)

Introduction and notation

It is well known that there exists a vector bundle of rank $n-1$ on $\boldsymbol{P}^n$ for n odd, which is not direct sums of line bundles cf. [1]. In this paper we shall give an example of indecomposable vector bundle of rank $n-1$ on $\boldsymbol{P}^n$ for each $n\geqq 3$.

In this paper we shall use the following notation: $\mathcal{O}_{\boldsymbol{P}^n}$ is the structure sheaf of n-dimensional projective space $\boldsymbol{P}^n$ defined over an algebraically closed field k of an arbitrary characteristic; $\mathcal{O}_{\boldsymbol{P}^n}(1)$ is the line bundle associated with a hyperplane of $\boldsymbol{P}^n$; $\Omega^1_{\boldsymbol{P}^n}$ is the sheaf of germs of regular differential 1-forms; $T_{\boldsymbol{P}^n}$ is the tangent bundle on $\boldsymbol{P}^n$; $\check{E}$ is the dual vector bundle of a vector bundle E; $E(m)$ is the vector bundle $E\otimes\mathcal{O}_{\boldsymbol{P}_n}(1)^{\otimes m}$; $\boldsymbol{c}_i(E)$ is the i-th Chern class of E; $\boldsymbol{c}(E)=1+\boldsymbol{c}_1(E)+\boldsymbol{c}_2(E)+\cdots$ is the Chern polynomial of E; $\boldsymbol{h}=\boldsymbol{c}_1(\mathcal{O}_{\boldsymbol{P}^n}(1))$ i.e. the first Chern class of a hyperplane; $H^i(E)=H^i(X,E)$ and $h^i(E)=\dim_k H^i(X,E)$ for a vector bundle E on a complete nonsingular variety X defined over k; $\boldsymbol{Gr}(n,d)$ is the Grassmann variety which parametrizes d-dimensional linear subspaces of $\boldsymbol{P}^n$; $\boldsymbol{Q}(n,d)$ is the universal quotient bundle of $\boldsymbol{Gr}(n,d)$; L_x is the d-dimensional linear subspace of $\boldsymbol{P}^n$ which is represented by a point x of $\boldsymbol{Gr}(n,d)$; $\omega_{s,0,\dots,0}(A)=\{x\in\boldsymbol{Gr}(n,d)\,|\,L_x\cap A\neq\phi\}$ is the special Schubert variety for an $n-d-s$ dimensional linear subspace A of $\boldsymbol{P}^n$; and $\omega_{s,0,\dots,0}$ is the Schubert cycle associated with a $\omega_{s,0,0,\dots,0}(A)$.

Construction of the example

Lemma 1. *$\Omega^1_{\boldsymbol{P}^n}(2)$ is generated by its global sections.*

Proof. Consider the following commutative diagram with exact rows and exact colums.

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \text{Kernel } g_1 & \xrightarrow{\ f'\ } & \text{Kernel } g_2 \\
 & & & & \downarrow & & \downarrow \\
0\to & H^0(\Omega^1_{\boldsymbol{P}^n}(2))\otimes_k\mathcal{O}_{\boldsymbol{P}^n} & \to & H^0(\overset{n+1}{\bigoplus}\mathcal{O}_{\boldsymbol{P}^n}(1))\otimes_k\mathcal{O}_{\boldsymbol{P}^n} & & \xrightarrow{\ f\ } & H^0(\mathcal{O}_{\boldsymbol{P}^n}(2))\otimes_k\mathcal{O}_{\boldsymbol{P}^n} \\
 & \downarrow g_0 & & & \downarrow g_1 & & \downarrow g_2 \\
0 \longrightarrow & \Omega^1_{\boldsymbol{P}^n}(2) & \longrightarrow & & \overset{n+1}{\bigoplus}\mathcal{O}_{\boldsymbol{P}^n}(1) & \longrightarrow & \mathcal{O}_{\boldsymbol{P}^n}(2)\longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
\end{array}$$

It is easy to see that f and f' are surjections. Hence, the Snake lemma shows that g_0 is surjective. q.e.d.

By virtue of the proof of Lemma 1, we have

$$h^0(\Omega^1_{\boldsymbol{P}^n}(2)) = (n+1)h^0(\mathcal{O}_{\boldsymbol{P}^n}(1)) - h^0(\mathcal{O}_{\boldsymbol{P}^n}(2)) = \tfrac{1}{2}n(n+1).$$

We denote Kernel g_0 by $\check{E}_n$. Then, we have the following exact sequence of vector bundles

$$\text{(1)} \qquad 0\to T_{\boldsymbol{P}^n}(-2)\to\overset{N_n}{\bigoplus}\mathcal{O}_{\boldsymbol{P}^n}\to E_n\to 0$$

where $N_n=\frac{1}{2}n(n+1)$ and rank $E_n=N_n-n=\frac{1}{2}n(n-1)$. Using the long exact sequences of cohomology groups

$$0\to H^0(T_{\boldsymbol{P}^n}(-2))\to H^0(\overset{N_n}{\bigoplus}\mathcal{O}_{\boldsymbol{P}^n})\to H^0(E_n)\to H^1(T_{\boldsymbol{P}^n}(-2))$$

$$0=H^0(\overset{n+1}{\bigoplus}\mathcal{O}_{\boldsymbol{P}^n}(-1))\to H^0(T_{\boldsymbol{P}^n}(-2))\to H^1(\mathcal{O}_{\boldsymbol{P}^n}(-2))=0$$

$$0=H^1(\overset{n+1}{\bigoplus}\mathcal{O}_{\boldsymbol{P}^n}(-1))\to H^1(T_{\boldsymbol{P}^n}(-2))\to H^2(\mathcal{O}_{\boldsymbol{P}^n}(-2))=0$$

we obtain $h^0(T_{\boldsymbol{P}^n}(-2))=h^1(T_{\boldsymbol{P}^n}(-2))=0$ and $h^0(E_n)=N_n$.

Theorem 2. *E_n has an indecomposable quotient bundle E_n' of* *rank* $n-1$.

In order to prove the Theorem, we need the following four lemmas.

Lemma 3. $\boldsymbol{c}_n(E_n)=0$ *and* $\boldsymbol{c}_{n-1}(E_n)\neq 0$.

Proof. Indeed the exact sequences

$$0\to T_{\boldsymbol{P}^n}(-2)\to \bigoplus^{N_n}\mathcal{O}_{\boldsymbol{P}^n}\to E_n\to 0$$

$$0\to \mathcal{O}_{\boldsymbol{P}^n}(-2)\to \bigoplus^{n+1}\mathcal{O}_{\boldsymbol{P}^n}(-1)\to T_{\boldsymbol{P}^n}(-2)\to 0$$

shows that $\boldsymbol{c}(E_n)\cdot\boldsymbol{c}(T_{\boldsymbol{P}^n}(-2))=1$ and

$$\boldsymbol{c}(T_{\boldsymbol{P}^n}(-2))\cdot\boldsymbol{c}(\mathcal{O}_{\boldsymbol{P}^n}(-2))=\boldsymbol{c}(\bigoplus^{n+1}\mathcal{O}_{\boldsymbol{P}^n}(-1)).$$

Hence, we have

$$\boldsymbol{c}(E_n)=\boldsymbol{c}(T_{\boldsymbol{P}^n}(-2))^{-1}=(1-2h)(1-h)^{-n-1}=\left(\sum_{i=0}^{n}\binom{n+i}{i}h^i\right)(1-2h).$$

Therefore, $\boldsymbol{c}_n(E_n)=\left(\binom{2n}{n}-2\binom{2n-1}{n-1}\right)h^n=0$ and

$$\boldsymbol{c}_{n-1}(E_n)=\left(\binom{2n-1}{n-1}-2\binom{2n-2}{n-2}\right)h^{n-1}\neq 0\,. \qquad \text{q.e.d.}$$

Lemma 4. *Let E be a vector bundle of rank r on a complete nonsingular variety X. Suppose that E is generated by its global sections and $\boldsymbol{c}_s(E)=0$ for a positive integer $s\leq r$. Then E has a trivial vector bundle of rank $r-s+1$ as a subbundle.*

Proof. Since E is generated by its global sections, there exists an exact sequence of vector bundles

$$\bigoplus^{m+1}\mathcal{O}_X\to E\to 0$$

where $m+1=h^0(E)$. Then, there is a canonical morphism $f\colon X\to \boldsymbol{Gr}(m, m-r)$ such that $E=f^*\boldsymbol{Q}(m, m-r)$. Since $0=\boldsymbol{c}_s(E)$ $=f^*\boldsymbol{c}_s(\boldsymbol{Q}(m, m-r))=f^*\omega_{s,0,\dots,0}$, we see that $f(X)\cdot\omega_{s,0,\dots,0}=0$. Hence, there exists a linear subspace A of dimension $r-s$ of $\boldsymbol{P}^n$ such that

$L_{f(x)} \cap A = \phi$ for any point x of X (cf. [2]). This shows that E has a trivial vector bundle of rank $r-s+1$ as a subbundle. q.e.d.

Lemma. 5. *Let $n>s>d\geqq 0$ and let f be a morphism from $\boldsymbol{P}^n$ to $\boldsymbol{Gr}(s, d)$, then $f(\boldsymbol{P}^n)$ consists only of one point. cf.* [2].

Lemma 6. *(i) Let E be a nontrivial vector bundle of rank r on $\boldsymbol{P}^n$. If E is generated by its global sections, then $h^0(E)\geqq n+1$.*
(ii) Let E be a vector bundle which has no trivial vector bundle as a direct summand. Assume that E is generated by its global sections and that $h^0(E)\leqq 2n+1$. Then, E is indecomposable.

Proof. (i). Since E is generated by its global sections, there exists an exact sequence of vector bundles

$$\overset{m+1}{\bigoplus} \mathcal{O}_{\boldsymbol{P}^n} \to E \to 0$$

where $m+1=h^0(E)$. Then, there exists a canonical morphism f: $\boldsymbol{P}^n \to \boldsymbol{Gr}(m, m-r)$ such that $E=f^*\boldsymbol{Q}(m, m-r)$. Since E is nontrivial vector bundle, we see that $f(\boldsymbol{P}^n)$ is not one point. Hence, we have $m\geqq n$, by virtue of Lemma 5.
(ii). (ii) follows from (i). q.e.d.

Proof of Theorem 2. Since E_n is generated by its global sections and $\boldsymbol{c}_n(E_n)=0$, we have the exact sequence of vector bundles

$$0\to F\to E_n\to E_n'\to 0$$

where F is a trivial vector bundle of rank $\frac{1}{2}n(n-1)-n+1$ and E_n' is the quotient bundle of rank $n-1$, by virtue of Lemma 4. From the exact sequence of cohomology groups

$$0\to H^0(F)\to H^0(E_n)\to H^0(E_n')\to H^1(F)=0$$

we obtain that $h^0(E_n')=h^0(E_n)-h^0(F)=2n-1$. The fact that $\boldsymbol{c}_{n-1}(E_n')=\boldsymbol{c}_{n-1}(E_n)\neq 0$ shows that E_n' has no trivial vector bunble as a direct summand. Since E_n is generated by its global sections, so is E_n'. These results shows that E_n' is indecomposable, by virtue of Lemma 6 (ii). q.e.d.

Remark. Canonically $\boldsymbol{Gr}(n, 1)$ is embedded in $\boldsymbol{P}^{N_n-1}$. By this embedding $\omega_{n-1,0}(P) = \{x \in \boldsymbol{Gr}(n, 1) L_x \ni P\}$ is $n-1$ dimensional linear subspace of $\boldsymbol{P}^{N_n-1}$. Hence, we have a map φ: $\boldsymbol{P}^n \to \boldsymbol{Gr}(N_n-1, n-1)$. On the other hand, by virtue of the exact sequence (1), we have a morphism Ψ: $\boldsymbol{P}^n \to \boldsymbol{Gr}(N_n-1, n-1)$. In this senes φ and Ψ are projectively equivalent, i.e. there exists a collineation f: $\boldsymbol{P}^n \to \boldsymbol{P}^n$ such that $\varphi = \Psi \circ f$.

KYOTO UNIVERSITY OF EDUCATION

Bibliography

[1] Hartshorne, R. Varieties of small codimension in projective space. to appear.
[2] Tango, H. On (n-1)-dimensional projective spaces contained in the Grassmann variety **Gr**(n, 1). J. Math. Kyoto Univ. Vol. 14-3 (1974).