On amenable transformation semigroups I

By

Kôkichi Sakai

(Communicated by Prof. Yoshizawa, Dec. 8, 1975)

Table of Contents

§ 0. Introduction
§ 1. Transformation semigroups and means
§ 2. Basic properties of amenability
§ 3. Criteria for amenability
§ 4. Følner's conditions for amenability
§ 5. Characterization of extremely amenable algebras
§ 6. Right stationary transformation semigroups

§ 0. Introduction

For a group or semigroup there is the concept of amenability, which has been studied by many authors. The full discussions of the amenability for semigroups and for locally compact groups are found in M. M. Day [1]-[2] and F. P. Greenleaf [11] respectively. Recently the amenability for homogeneous spaces is investigated by P. Eymard [6]. Further F. P. Greenleaf [12], C. O. Wilde et al. [28] and K. Sakai [21]-[25] have discussed amenable transformation groups or semigroups. To unify these investigations concerning the amenability we shall consider amenability, extreme or quasi-extreme amenability associated with any transformation semigroups from the most general standpoint as follows.

Let \( X \) be a space, \( S \) an abstract semigroup and \( X=(S, X) \) a transformation semigroup. Here by a transformation semigroup (denoted by \( \tau \)-semigroup for brevity) we mean that there is given a map
Let $B(X)$ be the commutative $C^*$-algebra of all bounded complex valued functions on $X$ with the supremum norm, and $I_X$ the characteristic function of $X$. For any $f \in B(X)$ and $s \in S$ we define the functions $fs$ and $\bar{f}$ by $sf(x) = f(sx)$ and $\bar{f}(x) = \overline{f(x)}$ (=the complex conjugate of $f(x)$) respectively. A closed linear subspace [resp. subalgebra] $\mathcal{A}$ of $B(X)$ is denoted for the sake of brevity by $X$-space [resp. $X$-algebra] if it is translation invariant (i.e., $f \in \mathcal{A}$ implies $sf \in \mathcal{A}$ for any $s \in S$), self-adjoint (i.e., $f \in \mathcal{A}$ implies $\bar{f} \in \mathcal{A}$) and contains $I_X$. Let $\mathcal{A}$ be an $X$-space [resp. $X$-algebra]. A continuous linear functional $\varphi$ on $\mathcal{A}$ is called a mean [resp. multiplicative mean] on $\mathcal{A}$ if it has the following properties (0.1)—(0.3) [resp. (0.1)—(0.4)]:

\begin{align}
(0.1) & \quad \varphi(I_X) = 1, \\
(0.2) & \quad \varphi(f) = \overline{\varphi(f)} \text{ for any } f \in \mathcal{A}, \\
(0.3) & \quad \varphi(f) \geq 0 \text{ for any nonnegative } f \in \mathcal{A}, \\
(0.4) & \quad \varphi(f \cdot g) = \varphi(f) \varphi(g) \text{ for any } f, g \in \mathcal{A}.
\end{align}

A mean $\varphi$ on $\mathcal{A}$ is said to be invariant if $\varphi(sf) = \varphi(f)$ for any $s \in S$ and $f \in \mathcal{A}$. Then we say that an $X$-space $\mathcal{A}$ is amenable if there exists an invariant mean on it, and that an $X$-algebra $\mathcal{A}$ is extremely amenable if there exists a multiplicative invariant mean on it. Moreover an $X$-algebra $\mathcal{A}$ is said to be quasi-extremely amenable if there exists an invariant mean which is expressed in the convex linear combination of multiplicative means on it. Especially a $\tau$-semigroup $X = (S, X)$ is said to be amenable, extremely or quasi-extremely amenable if so is $B(X)$ respectively.

The purpose of the present paper is to study the above mentioned amenability associated with any $\tau$-semigroup $X$ and especially to consider necessary and sufficient conditions for any $X$-space [resp. $X$-algebra] to be amenable [resp. extremely or quasi-extremely amenable]. The almost all results concerning left amenable semigroups are obtained from our general standpoint in the above, because the left
Semigroups I

amenability theory of any semigroup $S$ is regarded as the amenability theory associated with the $\tau$-semigroup $S_\tau=(S, S)$, where each $s \in S$ acts on itself as a left translation.

§1 is the preliminary of our discussion in which we note the elementary properties of means on any $X$-space. In §2 we shall study the basic properties of amenability associated with any $\tau$-semigroup. In §3 we shall give the fundamental criteria for any amenable $X$-space and extremely or quasi-extremely amenable $X$-algebra respectively. The so-called "Day's weak condition" and "Dixmier's condition" for any left amenable semigroups are extended easily to the case of any amenable $X$-space.

E. Følner [7] has given a necessary and sufficient condition, so-called "Følner's condition", for a group to be left amenable. I. Namioka [19], E. Granirer [8] and A. T. Lau [15] have studied the Følner's conditions for any semigroup to be left amenable, extremely left amenable and quasi-extremely left amenable respectively. Further the Følner's conditions for locally compact groups are discussed by W. R. Emerson et al. [5] and A. Hulanicki [14]. In §4 we shall consider the Følner's conditions and Day's strong ones for any amenable $\tau$-semigroups.

In §5 we shall generalize the investigation by A. T. Lau [16] to the case of any $\tau$-semigroup $X$. It will be shown by the same way as in [16] that any extremely amenable $X$-algebra is contained in an extremely amenable $X$-algebra which is generated by certain family of simple functions. Moreover introducing the concept of $S$-thickness, we shall give a necessary and sufficient condition for any $X$-algebra to be extremely amenable.

T. Mitchell [17] has shown that a semigroup is left amenable if and only if it is right stationary. The analogous result for the case of extremely left amenable semigroup is in E. Granirer [9]. In §6 we shall define the notion of right stationary $X$-spaces and of extremely right stationary $X$-algebras, and consider the relation between right stationarity and amenability.

All discussions in the present paper do not depend on the topolo-
gical structure of underlying $\tau$-semigroup $X$ and do not require any additional conditions for $\mathfrak{A}$ except that $\mathfrak{A}$ is an $X$-space or $X$-algebra. The author's paper [26] is the continuation of the present one in which we shall show that the amenability of any $X$-space is characterized by means of certain properties concerning various actions of $X$. While the amenability associated with any topological transformation semigroups will be discussed in [27].

ACKNOWLEDGMENT. The author wishes to thank Professor H. Yoshizawa, Professor N. Tatsuuma and Professor T. Hirai for their encouragement and kind advices.

NOTATION. Throughout this paper, unless otherwise noted, $S$ is an abstract semigroup, $X$ is a space, $X=(S, X)$ or $X$ is a $\tau$-semigroup, $S_\tau=(S, S)$ or $S_\tau$ is the $\tau$-semigroup defined by the left translations on $S$, and $\mathfrak{A}$ is an $X$-space or $X$-algebra. The following is the list of notations used freely:

- $\mathfrak{A}_r=\{f \in \mathfrak{A}; f$ is real valued$\}$,
- $\mathfrak{A}_+^\ast=\{f \in \mathfrak{A}; f$ is nonnegative valued$\}$,
- $\mathfrak{A}_\ast^\ast$=the dual Banach space of $\mathfrak{A}$,
- $\mathfrak{A}^\ast_\ast$=the dual space of $\mathfrak{A}_\ast$ as a real Banach space,
- $M(\mathfrak{A})$=the space of all means on an $X$-space $\mathfrak{A}$,
- $\mathfrak{M}(\mathfrak{A})$=the space of all multiplicative means on an $X$-algebra $\mathfrak{A}$,
- $M(X)=M(B(X)), \beta X=\mathfrak{M}(B(X))$,
- $\|f\|$=the norm of any element $f$ in a Banach space,
- $l(E, F)$=the space of all continuous linear maps from $E$ to $F$,
- where $E$ and $F$ are Banach spaces,
- $\text{Hom}(E, F)$=the space of all continuous homomorphisms from $E$ to $F$ as $C^*$-algebras, where $E$ and $F$ are $C^*$-algebras,
- $I_A$=the characteristic function of any subset $A$ of $X$,
- $c(F)$=the cardinal number of any finite set $F$,
- $C(I_X)$=the algebra of all constant functions on $X$,
- $C(Z)$=the commutative $C^*$-algebra of all bounded complex valued continuous functions on a topological space $Z$,
- $Co(K)$=the convex hull of any subset $K$ of a linear space,

§ 1. Transformation semigroups and means

In this section, for our later use, we shall note the elementary facts concerning $\tau$-semigroups and means on any $X$-spaces.

1.1. Let $Y$ be a topological space and $Y = (S, Y)$ a $\tau$-semigroup. Then we say briefly that $Y$ is a $\tau(c)$-semigroup if the map $Y \ni y \mapsto sy \in Y$ is continuous for any $s \in S$, and especially that $Y$ is a compact $\tau(c)$-semigroup if $Y$ is compact. Let $(S, X)$ and $(S, X_1)$ be $\tau$-semigroups. A map $\rho$ from $X$ to $X_1$ is called a homomorphism from $(S, X)$ to $(S, X_1)$ if $\rho(sx) = s\rho(x)$ for any $s \in S$ and $x \in X$. Let $Z$ be a subset of $X$ such that $sZ \subseteq Z$ for any $s \in S$. Then restricting the action of $S$ to $Z$, we get a $\tau$-semigroup $(S, Z)$. In this case $(S, Z)$ is said to be contained in $(S, X)$ and denoted by $(S, Z) \subseteq (S, X)$.

1.2. Here we note the elementary properties of means on any $X$-space $\mathcal{A}$. We assume always that $\mathcal{A}^*$ has the $w^*$-topology (i.e., $\sigma(\mathcal{A}^*, \mathcal{A})$-topology).

**Proposition 1.1.** The following conditions for a $\varphi \in \mathcal{A}^*$ are mutually equivalent:

1. $\varphi \in M(\mathcal{A})$,
2. $\|\varphi\| = \varphi(I_x) = 1$ and $\varphi(f) = \varphi(f)$ for any $f \in \mathcal{A}$,
3. $\inf \{f(x); x \in X\} \leq \varphi(f) \leq \sup \{f(x); x \in X\}$ for any $f \in \mathcal{A}$.

For any $x \in X$ we define $\delta(x) \in \mathcal{A}^*$ by $\delta(x)(f) = f(x)$ ($f \in \mathcal{A}$) and put

$$M_\rho(\mathcal{A}) = \{\delta(x); x \in X\}, \quad M_\varphi(\mathcal{A}) = \text{Co}(M_\rho(\mathcal{A})).$$

Each element in $M_\rho(\mathcal{A})$ [resp. $M_\varphi(\mathcal{A})$] is called a point [resp. finite] mean on $\mathcal{A}$.

**Proposition 1.2.** (1) For any $X$-space $\mathcal{A}$, $M(\mathcal{A})$ is the $w^*$-closure of $M_\varphi(\mathcal{A})$ and is a $w^*$-compact convex subset of $\mathcal{A}^*$.

(2) For any $X$-algebra $\mathcal{A}$, $\mathcal{A}(\mathcal{A})$ is the $w^*$-closure of $M_\rho(\mathcal{A})$ and is a $w^*$-compact subset of $\mathcal{A}^*$.
In what follows by $\mathfrak{M}(\mathcal{A})$ we denote the $w^*$-closure of $M_\rho(\mathcal{A})$ even if $\mathcal{A}$ is an $X$-space. For any $s \in S$ and $\varphi \in \mathcal{A}^*$ we define $s\varphi \in \mathcal{A}^*$ by $s\varphi(f) = \varphi(sf)$ ($f \in \mathcal{A}$). Then it is easily seen that $s(t\varphi) = (st)\varphi$ for any $s, t \in S$ and $\varphi \in \mathcal{A}^*$, and that the map $\mathcal{A}^* \ni \varphi \mapsto s\varphi \in \mathcal{A}^*$ is $w^*$-continuous for any $s \in S$. Further $s\varphi \in M(\mathcal{A})$ [resp. $\mathfrak{M}(\mathcal{A})$] for any $s \in S$ if $\varphi \in M(\mathcal{A})$ [resp. $\mathfrak{M}(\mathcal{A})$]. Hence we have

**Proposition 1.3.** $(S, \mathcal{A}^*)$ is a $\tau(c)$-semigroup, $(S, M(\mathcal{A}))$ and $(S, \mathfrak{M}(\mathcal{A}))$ are compact $\tau(c)$-semigroups.

1.3. Let $X_i = (S, X_i)$ be a $\tau$-semigroup and $\mathcal{A}_i$ an $X_i$-space for $i = 1, 2$. Suppose that there is given a $\Phi \in l(\mathcal{A}_2, \mathcal{A}_1)$ with the properties (1.1)–(1.4):

1. $(\mathfrak{1})\quad \Phi(I_{X_2}) = I_{X_1},$
2. $(\mathfrak{2})\quad \Phi(f) = \overline{\Phi(f)}$ for any $f \in \mathcal{A}_2$,
3. $(\mathfrak{3})\quad \Phi(f) \in \mathcal{A}_1^+$ whenever $f \in \mathcal{A}_2^+$,
4. $(\mathfrak{4})\quad \Phi(sf) = s\Phi(f)$ for any $s \in S$ and $f \in \mathcal{A}_2$.

Then we see that $M(\mathcal{A}_1)$ is transformed to $M(\mathcal{A}_2)$ by the adjoint map $\Phi^*$ of $\Phi$ and that $\Phi^*(s\varphi) = s(\Phi^*(\varphi))$ for any $s \in S$ and $\varphi \in M(\mathcal{A}_1)$. Hence under the above assumption we have

**Proposition 1.4.** The map $\Phi^*$ induces a homomorphism from $(S, M(\mathcal{A}_1))$ to $(S, M(\mathcal{A}_2))$. Especially if $\mathcal{A}_i$ is an $X_i$-algebra for $i = 1, 2$ and $\Phi \in \text{Hom}(\mathcal{A}_2, \mathcal{A}_1)$ then $\Phi^*$ induces a homomorphism from $(S, \mathfrak{M}(\mathcal{A}_1))$ to $(S, \mathfrak{M}(\mathcal{A}_2))$.

Let $\rho$ be a homomorphism from $X_1$ to $X_2$ and define the map $\bar{\rho}$ of $B(X_2)$ to $B(X_1)$ by

1. $(\mathfrak{5})\quad \bar{\rho}f(x) = f(\rho(x))$ ($x \in X_1$) for any $f \in B(X_2)$.

Then the map $\Phi = \bar{\rho}$ satisfies (1.1)–(1.4) and $\bar{\rho}(fg) = \bar{\rho}(f)\bar{\rho}(g)$ for any $f, g \in B(X_2)$. Hence the next is an immediate consequence of Prop.
Proposition 1.5. If there is given a homomorphism $\rho$ from $X_1$ to $X_2$, then there corresponds a homomorphism from $(S, M(X_1))$ [resp. $(S, \beta X_1)$] to $(S, M(X_2))$ [resp. $(S, \beta X_2)$]. Moreover if $\bar{\rho}(\mathcal{U}_2) \subseteq \mathcal{U}_1$, there corresponds a homomorphism from $(S, M(\mathcal{U}_1))$ [resp. $(S, \mathcal{M}(\mathcal{U}_1))$] to $(S, M(\mathcal{U}_2))$ [resp. $(S, \mathcal{M}(\mathcal{U}_2))$].

1.4. Let $X = (S, X)$ be a $\tau$-semigroup and $\mathcal{U}$ an $X$-space. For any fixed $\varphi \in M(\mathcal{U})$ we define the map $\varphi \square$ of $\mathcal{U}$ to $B(S)$ by

$$\varphi \square f(s) = \varphi(s)f(s) \quad (s \in S)$$

for any $f \in \mathcal{U}$. Then this is linear and has the following properties (i)–(iv):

(i) $\varphi \square I_X = I_S$, 
(ii) $\varphi \square f = \overline{\varphi \square f}$ for any $f \in \mathcal{U}$, 
(iii) $\varphi \square f \in B(S)^+$ whenever $f \in \mathcal{U}^+$, 
(iv) $\varphi \square f = \varphi(s)f(s)$ for any $s \in S$ and $f \in \mathcal{U}$.

Hence the map $\Phi = \varphi \square$ satisfies (1.1)–(1.4). In particular if $\mathcal{U}$ is an $X$-algebra and $\varphi \in \mathcal{M}(\mathcal{U})$ then $\varphi \square$ is in Hom$(\mathcal{U}, B(S))$. Let $\mathcal{U}(S)$ be an $S_\tau$-space such that $\varphi \square \mathcal{U} \subseteq \mathcal{U}(S)$. For any $\psi \in \mathcal{U}(S)^*$ define $\varphi \bullet \psi \in \mathcal{U}^*$ by

$$\varphi \bullet \psi(f) = \psi(\varphi \square f) \quad (f \in \mathcal{U}).$$

Then the map $\varphi \bullet : \mathcal{U}(S)^* \ni \psi \mapsto \varphi \bullet \psi \in \mathcal{U}^*$ is the adjoint of $\varphi \square$ and satisfies

$$s(\varphi \bullet \psi) = \varphi \bullet s\psi \quad \text{for any } s \in S \text{ and } \psi \in \mathcal{U}(S)^*.$$

Applying Prop. 1.4 to the map $\varphi \square$, we obtain

Proposition 1.6. Let $\mathcal{U}$ be an $X$-space and $\mathcal{U}(S)$ an $S_\tau$-space. Suppose that $\varphi \square \mathcal{U} \subseteq \mathcal{U}(S)$ for some $\varphi \in M(\mathcal{U})$. Then $\varphi \bullet$ induces a homomorphism from $(S, M(\mathcal{U}(S)))$ to $(S, M(\mathcal{U}))$. In particular if $\mathcal{U}$ and $\mathcal{U}(S)$ is an $X$- and $S_\tau$-algebra respectively and $\varphi \in \mathcal{M}(\mathcal{U})$ then $\varphi \bullet$ induces a homomorphism from $(S, \mathcal{M}(\mathcal{U}(S)))$ to $(S, \mathcal{M}(\mathcal{U}))$.  

An \( S \)-space \( \mathfrak{A} \) is said to be left-introverted [resp. left-m-introverted] if \( \varphi(\mathfrak{A}) \leq \mathfrak{A} \) for any \( \varphi \in M(\mathfrak{A}) \) [resp. \( \mathfrak{A}(\mathfrak{A}) \)]. For example \( B(S) \) is left-introverted.

1.5. Let \( X = (S, X) \) be a \( \tau \)-semigroup. A family \( \Sigma \) of subsets of \( X \) is called an \( S \)-invariant ring if it has the following properties (i)--(iii):

(i) \( X \in \Sigma \),

(ii) \( A, B \in \Sigma \) implies \( A \cup B \in \Sigma \) and \( A \sim B \in \Sigma \),

(iii) \( A \in \Sigma \) implies \( s^{-1}A \in \Sigma \) for any \( s \in S \),

where \( A \sim B \) is the complement of \( B \) in \( A \) and \( s^{-1}A = \{ x \in X ; sx \in A \} \). By \( B(X, \Sigma) \) we denote the closed linear subspace of \( B(X) \) generated by \( \{ I_A ; A \in \Sigma \} \). Then \( B(X, \Sigma) \) is an \( X \)-algebra. Let \( \varphi \) be any mean on \( B(X, \Sigma) \) and put \( \tilde{\varphi}(A) = \varphi(\mathfrak{I}_A) (A \in \Sigma) \). The function \( \tilde{\varphi} \) on \( \Sigma \) is a finitely additive probability measure. Further \( \varphi \) is reproduced by the integral with respect to \( \tilde{\varphi} \) as follows (e.g., see [4, p. 258]):

\[
\varphi(f) = \int_X f(x) d\varphi(x) \quad (f \in B(X, \Sigma)).
\]

Therefore \( M(B(X, \Sigma)) \) is identified with the space of all finitely additive probability measures on \( \Sigma \). Especially the measure \( \tilde{\varphi} \) corresponding to any multiplicative mean on \( B(X, \Sigma) \) is characterized by the property:

\[
\tilde{\varphi}(A) = 0 \text{ or } 1 \quad \text{for any } A \in \Sigma.
\]

§ 2. Basic properties of amenability

2.1. Let \( Y = (S, Y) \) be any \( \tau \)-semigroup and \( F \) a subset of \( S \). A point \( y \) in \( Y \) is said to be \( F \)-fixed if \( sy = y \) for all \( s \in F \). A subset \( Y_0 \) of \( Y \) is said to be \( F \)-stable if \( sY_0 = Y_0 \) for all \( s \in F \). We say that \( Y \) has a fixed point [resp. stable set] if there exists an \( S \)-fixed point [resp. nonempty \( S \)-stable subset] in \( Y \). Then by the definition of amenability we have

Proposition 2.1. (1) An \( X \)-space \( \mathfrak{A} \) is amenable if and only
if \((S, M(\mathfrak{U}))\) has a fixed point.

(2) An \(X\)-algebra \(\mathfrak{U}\) is extremely amenable if and only if \((S, \mathfrak{U})\) has a fixed point.

(3) An \(X\)-algebra \(\mathfrak{U}\) is quasi-extremely amenable if and only if \((S, M(\mathfrak{U}))\) has a finite stable set.

**Proof.** (1) [resp. (2)] is obvious from the fact that \(\varphi \in M(\mathfrak{U})\) [resp. \(\mathfrak{U}(\mathfrak{U})\)] is invariant if and only if it is an \(S\)-fixed point in \(M(\mathfrak{U})\) [resp. \(\mathfrak{U}(\mathfrak{U})\)]. If there exists a finite \(S\)-stable subset \(\{\varphi_i; 1 \leq i \leq n\}\) of \(\mathfrak{U}(\mathfrak{U})\) then \(\frac{1}{n} \sum_{i=1}^{n} \varphi_i\) is an invariant mean in \(C(\mathfrak{U}(\mathfrak{U}))\). Conversely let \(\varphi = \sum_{i=1}^{n} \lambda_i \varphi_i\) be an invariant mean in \(C(\mathfrak{U}(\mathfrak{U}))\), where \(\varphi_i\)'s are mutually distinct elements in \(\mathfrak{U}(\mathfrak{U})\) and \(\lambda_i\)'s are positive numbers with \(\sum_{i=1}^{n} \lambda_i = 1\). Then \(\Delta = \{\varphi_i; 1 \leq i \leq n\}\) is an \(S\)-stable set. Indeed, suppose that \(s \varphi_j \subseteq \Delta\) for some \(s \in S\) and \(\varphi_j \in \Delta\). Since any \(X\)-algebra \(\mathfrak{U}\) is isomorphic to \(C(\mathfrak{U}(\mathfrak{U}))\) as \(C^*\)-algebras, there exists an \(f \in \mathfrak{U}^+\) such that \(s \varphi_j(f) > 0\) and \(\varphi_j(f) = 0\) for \(1 \leq i \leq n\). By the invariance of \(\varphi\) we have a contradiction as follows:

\[
0 = \sum_{i=1}^{n} \lambda_i \varphi_i(f) = \sum_{i=1}^{n} \lambda_i \varphi_j(f) \geq \lambda_j s \varphi_j(f) > 0.
\]

Hence \(s \Delta \subseteq \Delta\) for any \(s \in S\). Similarly \(s \Delta \supseteq \Delta\) for any \(s \in S\). Thus (3) is proved.

For any \(X\)-space \(\mathfrak{U}\) we denote by \(IM(\mathfrak{U})\) the space of all invariant means on \(\mathfrak{U}\) and put \(\mathfrak{U}(\mathfrak{U}) = IM(\mathfrak{U}) \cap \mathfrak{U}(\mathfrak{U})\). Moreover we write:

\[
M(\mathfrak{U}_r) = \{\varphi \in \mathfrak{U}^*_r; \varphi(I_\mathfrak{U}) = \|\varphi\| = 1\},
\]

\[
IM(\mathfrak{U}_r) = \{\varphi \in M(\mathfrak{U}_r); \varphi(s) = \varphi(f)\text{ for any } s \in S\text{ and } f \in \mathfrak{U}_r\}.
\]

In particular if \(\mathfrak{U}\) is an \(X\)-algebra we write:

\[
\mathfrak{U}(\mathfrak{U}_r) = \{\varphi \in M(\mathfrak{U}_r); \varphi(fg) = \varphi(f)\varphi(g)\text{ for any } f, g \in \mathfrak{U}_r\},
\]

\[
\mathfrak{U}(\mathfrak{U}) = IM(\mathfrak{U}_r) \cap \mathfrak{U}(\mathfrak{U}_r).
\]

If \(\varphi \in IM(\mathfrak{U})\) [resp. \(\mathfrak{U}(\mathfrak{U})\)] then the restriction \(\varphi|\mathfrak{U}_r\) of \(\varphi\) on \(\mathfrak{U}_r\) belongs to \(IM(\mathfrak{U}_r)\) [resp. \(\mathfrak{U}(\mathfrak{U}_r)\)]. Conversely if \(\varphi \in IM(\mathfrak{U}_r)\) [resp. \(\mathfrak{U}(\mathfrak{U}_r)\)] then it can be extended uniquely to an element in \(IM(\mathfrak{U})\) [resp. \(\mathfrak{U}(\mathfrak{U})\)]. So we get
Proposition 2.2. (1) An $X$-space $\mathcal{A}$ is amenable if and only if $\text{IM}(\mathcal{A})$ is nonempty.

(2) An $X$-algebra $\mathcal{A}$ is extremely amenable if and only if $\text{IM}(\mathcal{A})$ is nonempty.

(3) An $X$-algebra $\mathcal{A}$ is quasi-extremely amenable if and only if $\text{IM}(\mathcal{A}) \cap \text{Co}(\mathcal{M}(\mathcal{A}))$ is nonempty.

2.2. Let $X=(S, X_i)$ be a $\tau$-semigroup for $i=1, 2$. Then we have

Theorem 2.3. Let $\mathcal{A}_i$ be an $X_i$-space [resp. $X_i$-algebra] for $i=1, 2$, and assume that there is a homomorphism $\rho$ from $(S, M(\mathcal{A}_1))$ [resp. $(S, \mathcal{M}(\mathcal{A}_1))$] to $(S, M(\mathcal{A}_2))$ [resp. $(S, \mathcal{M}(\mathcal{A}_2))$]. Then $\mathcal{A}_2$ is amenable [resp. extremely or quasi-extremely amenable] if so is $\mathcal{A}_1$ respectively.

Proof. Let $\varphi \in \text{IM}(\mathcal{A}_1)$ [resp. $\mathcal{M}(\mathcal{A}_1)$]. Then $\rho(s \varphi) = \rho(s \rho) = \rho(\varphi)$ for all $s \in S$, i.e., $\rho(\varphi) \in \text{IM}(\mathcal{A}_2)$ [resp. $\mathcal{M}(\mathcal{A}_2)$]. Hence $\mathcal{A}_2$ is amenable [resp. extremely amenable] if so is $\mathcal{A}_1$ respectively. Similarly we see that $\{\rho(\varphi_i); 1 \leq i \leq n\}$ is a finite $S$-stable subset of $\mathcal{M}(\mathcal{A}_2)$ if $\{\varphi_i; 1 \leq i \leq n\}$ is a finite $S$-stable subset of $\mathcal{M}(\mathcal{A}_1)$. Hence by Prop. 2.1, $\mathcal{A}_2$ is quasi-extremely amenable if so is $\mathcal{A}_1$. q.e.d.

Combining Theorem 2.3 with Props. 1.4 and 1.5 respectively we get

Theorem 2.4. Let $\mathcal{A}_i$ be an $X_i$-space [resp. $X_i$-algebra] for $i=1, 2$ and assume that there exists a $\Phi \in \text{I}(\mathcal{A}_2, \mathcal{A}_1)$ [resp. $\text{Hom}(\mathcal{A}_2, \mathcal{A}_1)$] with the properties (1.1)-(1.4). If $\mathcal{A}_1$ is amenable [resp. extremely or quasi-extremely amenable] then so is $\mathcal{A}_2$ respectively.

Theorem 2.5. Suppose that there is a homomorphism $\rho$ from $X_1$ to $X_2$. If $X_1$ is amenable [resp. extremely or quasi-extremely amenable] then so is $X_2$ respectively. Further let $\mathcal{A}_i$ be an $X_i$-space [resp. $X_i$-algebra] for $i=1, 2$ such that $\rho(\mathcal{A}_2) \subseteq \mathcal{A}_1$ (see (1.5)). If $\mathcal{A}_1$ is amenable [resp. extremely or quasi-extremely amenable] then so is $\mathcal{A}_2$ respectively.
For the case of the \( \tau \)-semigroup \( S = (S, S) \) we use the term “left amenable” instead of “amenable”. We say that a semigroup \( S \) is left amenable, extremely or quasi-extremely left amenable if so is \( B(S) \) respectively. If \( S \) is left amenable, extremely left amenable or quasi-extremely left amenable then any \( \tau \)-semigroup \( (S, X) \) is amenable, extremely amenable or quasi-extremely amenable respectively. This is an immediate consequence of the next theorem.

**Theorem 2.6.** Let \( \mathcal{A} \) be an \( X \)-space [resp. \( X \)-algebra] and \( \mathcal{A}(S) \) an \( S\tau \)-space [resp. \( S\tau \)-algebra]. Suppose that \( \varphi \mathcal{A} \subseteq \mathcal{A}(S) \) for some \( \varphi \in M(\mathcal{A}) \) [resp. \( M(\mathcal{A}) \)]. If \( \mathcal{A}(S) \) is left amenable [resp. extremely or quasi-extremely left amenable] then \( \mathcal{A} \) is amenable [resp. extremely or quasi-extremely amenable].

**Proof.** This follows from Prop. 1.6 and Theorem 2.3. q.e.d.

2.3. If \( X \) is amenable then so is any \( X \)-space. Similarly if \( X \) is extremely or quasi-extremely amenable then so is any \( X \)-algebra respectively. This is obtained from the next

**Theorem 2.7.** Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be \( X \)-spaces [resp. \( X \)-algebras] such that \( \mathcal{A}_2 \subseteq \mathcal{A}_1 \). If \( \mathcal{A}_1 \) is amenable [resp. extremely or quasi-extremely amenable] then so is \( \mathcal{A}_2 \) respectively.

**Proof.** The embedding map \( \Phi \) of \( \mathcal{A}_2 \) into \( \mathcal{A}_1 \) has the properties (1.1)–(1.4). Hence this theorem follows from Theorem 2.4. q.e.d.

Let \( Y = (S, Y) \) be a \( \tau \)-semigroup such that \( Y \subseteq X = (S, X) \) (see § 1.1). Then we have

**Theorem 2.8.** Let \( \mathcal{A}_1 \) be an \( X \)-space [resp. \( X \)-algebra] and \( \mathcal{A}_2 \) an \( Y \)-space [resp. \( Y \)-algebra]. Suppose that for any \( f \in \mathcal{A}_1 \) the restriction \( f|Y \) of \( f \) on \( Y \) belongs to \( \mathcal{A}_2 \). If \( \mathcal{A}_2 \) is amenable [resp. extremely or quasi-extremely amenable] then so is \( \mathcal{A}_1 \) respectively.

**Proof.** The map \( \Phi: \mathcal{A}_1 \ni f \mapsto \Phi(f) = f|Y \in \mathcal{A}_2 \) has the properties (1.1)–
Hence this theorem follows from Theorem 2.4.

Let \( \Sigma \) be an \( S \)-invariant ring of subsets of \( X \) (see \( \S 1.5 \)). A finitely additive measure \( \hat{\varphi} \) on \( \Sigma \) is said to be \( S \)-invariant if \( \varphi(s^{-1}A) = \varphi(A) \) for any \( s \in S \) and \( A \in \Sigma \). Then \( IM(B(X, \Sigma)) \) is identified with the space of all \( S \)-invariant finitely additive probability measures on \( \Sigma \). Because we have \( \hat{\varphi}(s^{-1}A) = \varphi(I_{s^{-1}A}) = \varphi(I_A) = \varphi(s^{-1}_{-1}A) \) for any \( s \in S \) and \( A \in \Sigma \), where \( \varphi \in IM(B(X, \Sigma)) \) and \( \hat{\varphi} \) the measure on \( \Sigma \) corresponding to \( \varphi \). Thus we have

**Theorem 2.9.** \( B(X, \Sigma) \) is amenable [resp. extremely amenable] if and only if there exists an \( S \)-invariant finitely additive probability measure on \( \Sigma \) [with the property (1.10)].

Now let \( Y \subseteq \Sigma \) satisfy \( sY \subseteq Y \) for any \( s \in S \). Then \( Y = (S, Y) \subseteq (S, X) \) and \( \Sigma(Y) = \{ A \cap Y; A \in \Sigma \} \) is an \( S \)-invariant ring of subsets of \( Y \). It is clear that \( f|Y \subseteq B(Y, \Sigma(Y)) \) whenever \( f \in B(X, \Sigma) \). So from Theorem 2.8, \( B(X, \Sigma) \) is amenable [resp. extremely or quasi-extremely amenable] if so is \( B(Y, \Sigma(Y)) \) respectively. The following is the partial converse of this fact.

**Theorem 2.10.** If there exists a \( \varphi \in IM(B(X, \Sigma)) \) [resp. \( IM(B(X, \Sigma)) \)] such that \( \varphi(I_Y) > 0 \), then \( B(Y, \Sigma(Y)) \) is amenable [resp. extremely amenable].

**Proof.** We write \( \varphi(A) = \varphi(I_A) \) for any \( A \in \Sigma \) and put \( k = \varphi(Y)^{-1} \). Then we define the finitely additive probability measure \( \psi \) on \( \Sigma(Y) \) by \( \psi(A) = k \varphi(A) \) for any \( A \in \Sigma(Y) \). Since \( \varphi \) is invariant and \( s^{-1}Y \subseteq Y \), it follows that \( \varphi(s^{-1}Y \sim Y) = 0 \) for any \( s \in S \). Hence for any \( A \in \Sigma \) and \( s \in S \)

\[
\psi(s^{-1}A \cap Y) = k \varphi(s^{-1}A \cap Y) = k \varphi(s^{-1}A \cap s^{-1}Y) = k \varphi(s^{-1}(A \cap Y)) = k \varphi(A \cap Y) = \psi(A \cap Y).
\]

So \( \psi \) is \( S \)-invariant. If \( \varphi \) is multiplicative then \( k = 1 \) and hence \( \psi \) has the property (1.10). Therefore this theorem follows from Theorem
The preceding theorem contains as a special case the following result due to M. M. Day [1, p. 518]: Let $S$ be a semigroup with a left invariant mean $\varphi$ on $B(S)$. Suppose that $T$ is a subsemigroup of $S$ such that $\varphi(I_T)>0$. Then $T$ is also left amenable.

2.4. We note some elementary remarks concerning the results stated in this section.

**Remark 2.11.** (1) Let $X=\{x_i; 0\leq i \leq n\}$ be a finite set with $(n+1)$-elements $(n \geq 2)$ and $Y=\{x_i; 1 \leq i \leq n\} \subset X$. Let $S$ [resp. $G$] be the semigroup [resp. group] of all the maps of $X$ to [resp. onto] itself which fix the point $x_0$. Then we get $\tau$-semigroups $(S, X)$ and $(G, X)$. Moreover $(S, Y)$ and $(G, Y)$ are contained in $(S, X)$ and $(G, X)$ respectively. Since the finite group $G$ is left amenable, $(G, X)$ and $(G, Y)$ are amenable by Theorem 2.6. Precisely speaking, $(G, X)$ is extremely amenable and $(G, Y)$ is quasi-extremely amenable. In fact we define the means $\varphi_1$ and $\varphi_2$ on $B(X)$ as follows:

\begin{equation}
(2.1) \quad \varphi_1(f) = \frac{1}{n} \sum_{i=1}^{n} f(x_i) \quad \text{and} \quad \varphi_2(f) = f(x_0) \quad (f \in B(X)).
\end{equation}

Then they are $G$-invariant and $\varphi_2$ is multiplicative. The restriction $\varphi_1|B(Y)$ of $\varphi_1$ on $B(Y)$ is a unique invariant mean on $B(Y)$. Further it is easily seen that every invariant mean on $B(X)$ is written as a convex linear combination of $\varphi_1$ and $\varphi_2$. Since $\varphi_2$ is also $S$-invariant, $(S, X)$ is extremely amenable. However $(S, Y)$ is not amenable, because $\varphi_1|B(Y)$ is not $S$-invariant. This show that the converse of Theorem 2.8 does not hold in general (cf. Theorem 2.10).

(2) Let $G$ be a group and $H$ its subgroup. Then $G$ acts in natural way on the left cosets space $G/H$. If $G$ is left amenable then the $\tau$-group $(G, G/H)$ is amenable by Theorem 2.6 and $H$ is also left amenable (e.g. see E. Følner [7, Theorem 2]). Conversely if $H$ is left amenable and $(G, G/H)$ is amenable then $G$ is left amenable, which is shown by the same way as in [7, Theorem 4].
(3) Let $SO(2)$ and $SO(3)$ be the 2- and 3-dimensional rotation group respectively. Since $SO(2)$ is abelian, it is left amenable, while $SO(3)$ is not left amenable (e.g., see F. P. Greenleaf [11]). Hence $(SO(3), X)$ is not amenable by the above remark (2), where $X = SO(3)/SO(2)$. However since there exists a bounded $SO(3)$-invariant integral on $C(X), C(X)$ is an amenable $(SO(3), X)$-space. This show that the converse of Theorem 2.7 does not hold in general.

(4) If an $X$-space $\mathcal{A}$ is amenable and $\varphi \in IM(\mathcal{A})$ then $\varphi \mathcal{A}$ is equal to the trivial $S_r$-space $C(I_s)$, which is left amenable. So by Theorem 2.6 we have: An $X$-space [resp. $X$-algebra] $\mathcal{A}$ is amenable [resp. extremely or quasi-extremely amenable] if and only if there exist a left amenable [resp. extremely or quasi-extremely left amenable] $S_r$-space $\mathcal{A}(S)$ and a $\varphi \in M(\mathcal{A})$ [resp. $M(\mathcal{A})$] such that $\varphi \mathcal{A} \subseteq \mathcal{A}(S)$.

§ 3. Criteria for amenability

In this section we shall show various criteria for amenability, extreme or quasi-extreme amenability respectively.

3.1. At first we consider necessary and sufficient conditions for an $X$-space $\mathcal{A}$ to be amenable. To this aim we put:

- $K(\mathcal{A}) =$ the linear span of $\{f - s f; f \in \mathcal{A} \text{ and } s \in S\}$,
- $K(\mathcal{A}) =$ the real linear span of $\{f - s f; f \in \mathcal{A} \text{ and } s \in S\}$,
- $\overline{K}(\mathcal{A})$ [resp. $\overline{K}(\mathcal{A})$] = the norm closure of $K(\mathcal{A})$ [resp. $K(\mathcal{A})$],
- $d(I_X, \overline{K}(\mathcal{A})) = \inf \{\|I_X - h\|; h \in K(\mathcal{A})\}$,
- $d(I_X, \overline{K}(\mathcal{A})) = \inf \{\|I_X - h\|; h \in K(\mathcal{A})\}$.

We say that a net $\{\varphi_s\}$ in $M(\mathcal{A})$ is $w^*$-[resp. norm]-convergent to $S$-invariance if it satisfies the following (3.1) [resp. (3.2)]:

\begin{align}
(3.1) \quad & w^* \text{-lim}_s (s\varphi_s - \varphi_s) = 0 \quad \text{for any} \quad s \in S, \\
(3.2) \quad & \lim_s \|s\varphi_s - \varphi_s\| = 0 \quad \text{for any} \quad s \in S,
\end{align}

where $\|\|$ is the norm in $\mathcal{A}^*$.

**Lemma 3.1.** (1) If a net $\{\varphi_s\}$ in $M(\mathcal{A})$ converges to a $\varphi \in IM(\mathcal{A})$ in the $w^*$-sense then it is $w^*$-convergent to $S$-invariance.
(2) If a net \( \{ \varphi_a \} \) in \( M(\mathfrak{U}) \) is \( w^* \)-convergent to \( S \)-invariance then every \( w^* \)-cluster point of \( \{ \varphi_a \} \) is an invariant mean on \( \mathfrak{U} \).

Proof. (1) Since \((S, M(\mathfrak{U}))\) is a \( \tau(c) \)-semigroup, we have

\[
\text{w}^*\text{-lim}_a(s\varphi_a - \varphi_a) = \text{w}^*\text{-lim}_a s\varphi_a - \text{w}^*\text{-lim}_a \varphi_a = s\varphi - \varphi = 0
\]

for any \( s \in S \).

(2) By the compactness of \( M(\mathfrak{U}) \) there exists at least one \( w^* \)-cluster point of \( \{ \varphi_a \} \). Let \( \varphi \) be any cluster point of \( \{ \varphi_a \} \) and \( \{ \varphi_\beta \} \) be a subnet of \( \{ \varphi_a \} \) such that \( w^*\text{-lim}_\beta \varphi_\beta = \varphi \). Then by (3.1), \( s\varphi - \varphi = w^*\text{-lim}_\beta (s\varphi_\beta - \varphi_\beta) = 0 \) for any \( s \in S \). Hence \( \varphi \in IM(\mathfrak{U}) \). q.e.d.

**Theorem 3.2.** The following conditions for any \( X \)-space \( \mathfrak{U} \) are mutually equivalent:

1. \( \mathfrak{U} \) is amenable,
2. \( \inf \{ h(x); x \in X \} \leq 0 \) for any \( h \in K(\mathfrak{U}) \),
3. \( d(I_X, K(\mathfrak{U})) = 1 \),
4. \( d(I_X, K(\mathfrak{U})) = 1 \),
5. There exists a net \( \{ \varphi_a \} \) in \( M_f(\mathfrak{U}) \) which is \( w^* \)-convergent to \( S \)-invariance;
6. For any finite subset \( F \) of \( S \) there exists a \( \varphi \in M(\mathfrak{U}) \) such that \( s\varphi = \varphi \) for any \( s \in F \),
7. For any finite subset \( \mathfrak{F} \) of \( \mathfrak{U} \) there exists a \( \varphi \in M(\mathfrak{U}) \) such that \( s\varphi(f) = \varphi(f) \) for any \( s \in \mathfrak{S} \) and \( f \in \mathfrak{F} \),
8. For any finite subsets \( F \) of \( S \) and \( \mathfrak{F} \) of \( \mathfrak{U} \) there exists a \( \varphi \in M(\mathfrak{U}) \) such that \( s\varphi(f) = \varphi(f) \) for any \( s \in F \) and \( f \in \mathfrak{F} \).

Proof. Let \( \varphi \in IM(\mathfrak{U}) \). Since \( \varphi(h) = 0 \) for any \( h \in K(\mathfrak{U}) \), it follows from Prop. 1.1 that

\[
\inf \{ h(x); x \in X \} \leq \varphi(h) = 0 \quad \text{for any } h \in K(\mathfrak{U}),
\]

\[
1 = |\varphi(I_X - h)| \leq \|I_X - h\| \quad \text{for any } h \in K(\mathfrak{U}).
\]

Hence \( 1 \leq d(I_X, K(\mathfrak{U})) \leq 1 \). Therefore (1) implies (2) and (4). (2)\( \Rightarrow \) (3) and (4)\( \Rightarrow \) (3) are obvious. If (3) holds, then from Hahn-Banach Theo-
rem there is a \( \varphi_0 \in \mathcal{W}^*_r \) such that \( \| \varphi_0 \| = \varphi_0(I_\mathcal{X}) = 1 \) and \( \varphi_0(\overline{K}(\mathcal{W}_r)) = 0 \), i.e., \( \varphi_0 \in IM(\mathcal{W}_r) \). Hence (1) follows from (3) by Prop. 2.2 (1). Thus all the conditions from (1) to (4) are mutually equivalent. Now recalling Prop. 1.2 (1), we can take a net \( \{ \varphi_n \} \) in \( M_f(\mathcal{W}) \) such that 
\[
\varphi = w^*-\lim_n \varphi_n.
\]
By Lemma 3.1 (1) this net \( \{ \varphi_n \} \) is \( w^* \)-convergent to \( S \)-invariance. Hence (1) implies (5). While (5) \( \Rightarrow \) (1) is shown in Lemma 3.1 (2). Since (1) \( \Rightarrow \) (6) \( \Rightarrow \) (8) and (1) \( \Rightarrow \) (7) \( \Rightarrow \) (8) are trivial, it remains to show that (8) implies (1). Assume that (8) holds. For any \( s \in S \) and \( f \in \mathcal{W} \) we put 
\[
M(s, f) = \{ \varphi \in M(\mathcal{W}); s\varphi(f) = \varphi(f) \}.
\]
Then \( M(s, f) \) is a nonempty \( w^* \)-closed subset of the compact space \( M(\mathcal{W}) \) and the family of these subsets has the finite intersection property. Hence the intersection \( \cap \{ M(s, f); s \in S \) and \( f \in \mathcal{W} \} \) is nonempty and is equal to \( IM(\mathcal{W}) \). Therefore (8) implies (1). q.e.d.

**Theorem 3.3.** An \( \mathcal{X} \)-algebra \( \mathcal{W} \) is amenable if and only if \( \overline{K}(\mathcal{W}) \) is a proper linear subspace of \( \mathcal{W} \).

**Proof.** Suppose \( \overline{K}(\mathcal{W}) \) is proper. Then there exists a nonzero \( \varphi_0 \in \mathcal{W}^*_r \) such that \( \varphi_0(\overline{K}(\mathcal{W}_r)) = 0 \). Since the algebra \( \mathcal{W}_r \) has the structure of lattice, \( \varphi_0 \) is given in the form \( \varphi_0 = \lambda_1 \varphi_1 - \lambda_2 \varphi_2 \) for some \( \varphi_i \in M(\mathcal{W}_r) \) and \( \lambda_i \geq 0 \) \( (i = 1, 2) \), where \( \lambda_1 \varphi_1 = \max(\varphi, 0) \). Since \( \varphi_0 \) is nonzero, we have either \( \lambda_1 > 0 \) or \( \lambda_2 > 0 \). Say \( \lambda_1 > 0 \). Then, as shown in I. Namioka [20, Prop. 3.2], \( \varphi_1 \) is in \( IM(\mathcal{W}_r) \) and hence \( \mathcal{W} \) is amenable. The "only if" part follows from Theorem 3.2 (4). q.e.d.

**Remark 3.4.** (1) For any left-introverted \( S \)-space \( \mathcal{W} \) (see § 1.4), the conditions (7) and (8) in Theorem 3.2 are weakened as follows:

(7)' For any \( f \in \mathcal{W} \) there exists a \( \varphi \in M(\mathcal{W}) \) such that \( s\varphi(f) = \varphi(f) \) for any \( s \in S \),

(8)' For any \( f \in \mathcal{W} \) and finite subset \( F \) of \( S \) there exists a \( \varphi \in M(\mathcal{W}) \) such that \( s\varphi(f) = \varphi(f) \) for any \( s \in F \).

For example we prove that (7)' implies (7). For any \( f \in \mathcal{W} \) we put 
\[
M(f) = \{ \varphi \in M(\mathcal{W}); s\varphi(f) = \varphi(f) \text{ for any } s \in S \}.
\]
Then it suffices to see that the family \( \{ M(f); f \in \mathcal{W} \} \) has the finite intersection property.
For any $f_i \in \mathfrak{A} (1 \leq i \leq n)$ suppose that $\varphi \in \cap_{i=1}^n M(f_i)$. Let $f_0 \in \mathfrak{A}$ and put $\tilde{f} = \varphi \sqcap f_0 \in \mathfrak{A}$ (see (1.6)). We now take a $\psi \in M(\tilde{f})$. Then $\varphi_0 = \varphi \ast \psi \in \cap_{i=0}^n M(f_i)$ (see (1.7)). In fact for any $s \in S$,

$$\varphi_0(s, f_0) = \varphi \ast \psi(s, f_0) = \psi(\varphi \sqcap_s f) = \psi(s, \tilde{f}) = \psi(\tilde{f}) = \varphi_0(f_0),$$

$$\varphi_0(s, f_i) = \varphi \ast \psi(s, f_i) = \psi(\varphi \sqcap_s f_i) = \psi(\varphi \sqcap_s f_i) = \varphi_0(f_i) \quad (1 \leq i \leq n).$$

Similarly (8)** implies (8).

(2) The conditions (2), (3) and (5) in Theorem 3.2 are generalization of the criteria for any left amenable semigroup due to J. Dixmier [3, Theorem 1], E. Hewitt et al. [13, Theorem 17.15 (ii)] and M. M. Day [1, 5 (C)] respectively. The conditions (3) and (4) are regarded as special cases of the so-called "Reiter-Glicksberg inequality", which will be discussed in [26].

3.2. Here we consider some criteria for an $X$-algebra $\mathfrak{A}$ to be extremely amenable. To this aim we put:

$H(\mathfrak{A})$ = the ideal of $\mathfrak{A}$ generated by $\{f - sf; s \in S$ and $f \in \mathfrak{A}\}$,

$H(\mathfrak{A}_r)$ = the ideal of $\mathfrak{A}_r$ generated by $\{f - sf; s \in S$ and $f \in \mathfrak{A}_r\}$,

$\overline{H}(\mathfrak{A})$ = the norm closure of $H(\mathfrak{A})$,

$d(I_X, H(\mathfrak{A})) = \inf \{\|I_X - h\|; h \in H(\mathfrak{A})\};$

$d(I_X, H(\mathfrak{A}_r)) = \inf \{\|I_X - h\|; h \in H(\mathfrak{A}_r)\}.$

**Theorem 3.5.** The following conditions for an $X$-algebra $\mathfrak{A}$ are mutually equivalent:

(1) $\mathfrak{A}$ is extremely amenable,

(2) $\inf \{h(x); x \in X\} \leq 0$ for any $h \in H(\mathfrak{A}_r)$,

(3) $d(I_X, H(\mathfrak{A}_r)) = 1$,

(4) $d(I_X, H(\mathfrak{A})) = 1$,

(5) There exists a net $\{\varphi_\alpha\}$ in $M_\kappa(\mathfrak{A})$ which is w*-convergent to $S$-invariance,

(6) For any finite subset $F$ of $S$ there exists a $\varphi \in \mathfrak{M}(\mathfrak{A})$ such that $s \varphi = \varphi$ for any $s \in F$,

(7) For any finite subset $\bar{F}$ of $\mathfrak{A}$ there exists a $\varphi \in \mathfrak{M}(\mathfrak{A})$ such that $s \varphi(f) = \varphi(f)$ for any $s \in S$ and $f \in \bar{F}$,
(8) For any finite subsets $F$ of $S$ and $\mathfrak{F}$ of $\mathfrak{A}$ there exists a $\varphi \in \mathfrak{M}(\mathfrak{A})$ such that $s\varphi(f) = \varphi(f)$ for any $s \in F$ and $f \in \mathfrak{F}$.

(9) $H(\mathfrak{A})$ is a proper ideal of $\mathfrak{A}$.

(10) There exists a $\varphi \in M(\mathfrak{A})$ such that $\varphi(fg) = \varphi(fg)$ for any $f, g \in \mathfrak{A}$ and $s \in S$.

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$, $(1) \Rightarrow (4)$ and $(8) \Rightarrow (1)$ are shown by the same way as in Theorem 3.2. $(1) \Rightarrow (5)$ follows from Prop. 1.2 (2) and Lemma 3.1. Further $(1) \Rightarrow (6) \Rightarrow (8)$, $(1) \Rightarrow (7) \Rightarrow (8)$, $(3) \Rightarrow (9)$, $(4) \Rightarrow (9)$ and $(5) \Rightarrow (6) \Rightarrow (9)$ are obvious. Thus it remains to see that $(9)$ implies $(1)$. Now if $H(\mathfrak{A})$ is a proper ideal of $\mathfrak{A}$, then there exists a maximal ideal $H_0$ of $\mathfrak{A}$ containing $H(\mathfrak{A})$. The natural homomorphism $\varphi$ from $\mathfrak{A}$ onto the factor algebra $\mathfrak{A}/H_0 = \mathfrak{C}$ is in $\mathfrak{M}(\mathfrak{A})$. Hence $(9)$ implies $(1)$.

By the same reason as in Remark 3.4 (1), the conditions $(7)$ and $(8)$ in the above are weakened for the case of any left-$m$-introverted $S_r$-algebra $\mathfrak{A}$ as follows:

(7)' For any $f \in \mathfrak{A}$ there exists a $\varphi \in \mathfrak{M}(\mathfrak{A})$ such that $s\varphi(f) = \varphi(f)$ for any $s \in S$.

(8)' For any finite subset $F$ of $S$ and $f \in \mathfrak{A}$ there exists a $\varphi \in \mathfrak{M}(\mathfrak{A})$ such that $s\varphi(f) = \varphi(f)$ for any $s \in F$.

If $\varphi \in M(\mathfrak{A})$ satisfy the condition $(10)$ in Theorem 3.5 then it is in $IM(\mathfrak{A})$. But it is not multiplicative in general. However we have

Proposition 3.6. Let $\mathfrak{A}$ be an extremely amenable $X$-algebra. Then every extreme point of the $w^*$-compact convex subset $M_0 = \{\varphi \in M(\mathfrak{A}); \varphi(fg) = \varphi(fg) \text{ for any } f, g \in \mathfrak{A} \text{ and } s \in S\}$ is multiplicative.

Proof. Let $\varphi_0$ be an extreme point of $M_0$. For any fixed $f \in \mathfrak{A}^+$ with $\|f\| \leq 1$, we define a $\psi \in \mathfrak{A}^*$ by $\psi(g) = \varphi_0(fg) - \varphi_0(f)\varphi_0(g)$ ($g \in \mathfrak{A}$). Then $\psi(I_X) = 0$, $\psi(\bar{g}) = \overline{\psi(g)}$ and $\psi(hg) = \psi(hg)$ for any $g, h \in \mathfrak{A}$ and $s \in S$. Moreover for any $g \in \mathfrak{A}^+$,

$$(\varphi_0 + \psi)(g) = \varphi_0(g)(1 - \varphi_0(f)) + \varphi_0(fg) \geq 0.$$
Hence \( \varphi_0 \pm \psi \in M_0 \). Since \( \varphi_0 = (\varphi_0 + \psi)/2 + (\varphi_0 - \psi)/2 \) and \( \varphi_0 \) is extreme, we have \( \psi = 0 \). That is, \( \varphi_0(fg) = \varphi_0(f)\varphi_0(g) \) for any \( g \in A \) and \( f \in A^+ \) with \( \|f\| \leq 1 \). From this relation it is easily concluded that \( \varphi_0 \) is multiplicative.

The following is immediately obtained from the above and Krein-Milman theorem.

**Proposition 3.7.** For any extremely amenable \( X \)-algebra \( A \), every extreme point of \( IM(A) \) is multiplicative if and only if every \( \varphi \in IM(A) \) satisfies \( \varphi(fg) = \varphi(f)\varphi(g) \) for any \( f, g \in A \) and \( s \in S \).

It has been proved by E. Granirer [9, Theorem 6] that for any extremely left amenable semigroup \( S \), every extreme point of \( IM(B(S)) \) is multiplicative. Further the author [21] has shown that for any extremely amenable \( \tau \)-semigroup \((S, X)\), every extreme point of \( IM(B(X)) \) is multiplicative if \( S \) is extremely left amenable. But these facts do not hold in general. For example let \((G, X)\) be as in Remark 2.11 (1). Then the set of extreme points of \( IM(B(X)) \) consists of \( \varphi_1 \) and \( \varphi_2 \) defined by (2.1) and \( \varphi_1 \) is not multiplicative.

3.3. Finally we consider necessary and sufficient conditions for an \( X \)-algebra \( A \) to be quasi-extremely amenable. For any finite subset \( \alpha = \{a_i; 1 \leq i \leq n\} \) of \( S \) and \( \varphi \in M(A) \) we define \( \alpha(\varphi) \in C(A) \) by

\[
(3.3) \quad \alpha(\varphi) = -\frac{1}{n} \sum_{i=1}^{n} a_i \varphi .
\]

Then we have

**Proposition 3.8.** An \( X \)-algebra \( A \) is quasi-extremely amenable if and only if there exist a finite subset \( \alpha \) of \( S \) and \( \varphi \in M(A) \) such that \( \alpha(\varphi) \in IM(A) \).

**Proof.** If \( A \) is quasi-extremely amenable then there exists a finite \( S \)-stable subset \( \{\varphi_i; 1 \leq i \leq n\} \) of \( M(A) \) by Prop. 2.1 (3). We put \( A \)
\[ \{ s \varphi_s ; s \in S \} \text{ and } m = c(\Delta). \] Then we can find a finite subset \( \alpha = \{ a_i ; 1 \leq i \leq m \} \) of \( S \) such that \( \Delta = \{ a_i \varphi_s ; 1 \leq i \leq m \} \). Since \( A \) is also an \( S \)-stable subset of \( M(\mathfrak{A}) \), \( \alpha(\varphi_s) \) is in \( IM(\mathfrak{A}) \cap Co(\mathfrak{A}(\mathfrak{A})) \). Hence the "only if" part is proved. The converse is trivial. q.e.d.

For any finite subset \( \alpha = \{ a_i ; 1 \leq i \leq n \} \) of \( S \) we put:

\[ L(\alpha) = \frac{1}{n} \sum_{i=1}^{n} L_{a_i}, \]
where \( L_{a} \) is the linear operator on \( \mathfrak{A} \) defined by \( L_{a}(f) = af \) for any \( f \in \mathfrak{A} \),

\[ H(\alpha, \mathfrak{A}) = \text{the ideal of } \mathfrak{A} \text{ generated by } \{ L(\alpha)(f-sf) ; s \in S \text{ and } f \in \mathfrak{A} \}, \]

\[ H(\alpha, \mathfrak{A})_r = \text{the ideal of } \mathfrak{A}_r \text{ generated by } \{ L(\alpha)(f-sf) ; s \in S \text{ and } f \in \mathfrak{A} \}, \]

\[ \overline{H(\alpha, \mathfrak{A})} = \text{the norm closure of } H(\alpha, \mathfrak{A}), \]

\[ d(I_X, H(\alpha, \mathfrak{A})) = \inf \{ \| I_X - h \| ; h \in H(\alpha, \mathfrak{A}) \}, \]

\[ d(I_X, H(\alpha, \mathfrak{A}_r)) = \inf \{ \| I_X - h \| ; h \in H(\alpha, \mathfrak{A}_r) \}. \]

**Theorem 3.9.** Let \( \mathfrak{A} \) be any \( X \)-algebra and \( \alpha \) denote a non-empty finite subset of \( S \). Then the following conditions are mutually equivalent:

1. \( \mathfrak{A} \) is quasi-extremely amenable,
2. There exists an \( \alpha \) such that \( \inf \{ h(x) ; x \in X \} \leq 0 \) for any \( h \in H(\alpha, \mathfrak{A})_r \),
3. There exists an \( \alpha \) such that \( d(I_X, H(\alpha, \mathfrak{A})) = 1 \),
4. There exists an \( \alpha \) such that \( d(I_X, H(\alpha, \mathfrak{A})) = 1 \),
5. For some integer \( n \geq 1 \) there exists a net \( \{ \varphi_s = \frac{1}{n} \sum_{i=1}^{n} p_s^{\alpha} ; p_s^{\alpha} \in M_p(\mathfrak{A}) \} \) which is \( w^{*} \)-convergent to \( S \)-invariance,
6. \( H(\alpha, \mathfrak{A}) \) is a proper ideal of \( \mathfrak{A} \) for some \( \alpha \),
7. There exist an \( \alpha \) and \( \varphi \in \mathfrak{A}(\mathfrak{A}) \) such that \( \varphi(fL(\alpha)g) = \varphi(fL(\alpha)g) \) for any \( f, g \in \mathfrak{A} \) and \( s \in S \).

**Proof.** Suppose that \( \mathfrak{A} \) is quasi-extremely amenable. Then by Prop. 3.8, there exist some \( \alpha \) and \( \varphi \in \mathfrak{A}(\mathfrak{A}) \) such that \( \alpha(\varphi) \in IM(\mathfrak{A}) \). Let \( f, g \in \mathfrak{A} \), \( s \in S \) and put \( h = fL(\alpha)(h-s) \). Then

\[ \varphi(h) = \varphi(f)\varphi(L(\alpha)(g-s)) = n\varphi(f)\alpha(\varphi)(g-s) = 0, \]
where \( n = \sigma(\alpha) \). Hence we have

\[
\inf \{ h(x); x \in X \} \leq 0 \quad \text{for any} \quad h \in H(\alpha, \mathfrak{M}),
\]

\[
1 = \phi(L(x) - h) \leq \| L(x) - h \| \quad \text{for any} \quad h \in H(\alpha, \mathfrak{M}),
\]

\[
\phi(fL(x)g) = \phi(fL(x)g) \quad \text{for any} \quad f, g \in \mathfrak{A} \text{ and } s \in S.
\]

Therefore (1) implies (2), (4) and (7). (2) \( \Rightarrow \) (3) \( \Rightarrow \) (6), (4) \( \Rightarrow \) (6) and (7) \( \Rightarrow \) (6) are obvious. Now assume that (6) holds. Then there exists a maximal ideal of \( \mathfrak{A} \) containing \( \mathcal{H}(\alpha, \mathfrak{M}) \), that is, there exists a \( \varphi \in \mathfrak{M}(\mathfrak{A}) \) such that \( \varphi(\mathcal{H}(\alpha, \mathfrak{M})) = 0 \). Hence this \( \varphi \) satisfies \( \varphi(L(\alpha)f) = \varphi(L(\alpha)f) \) for any \( f \in \mathfrak{A} \) and \( s \in S \). This show that \( \alpha(\varphi) \in IM(\mathfrak{A}) \cap Co(\mathfrak{M}(\mathfrak{A})) \). Therefore (6) implies (1). At last we prove that (1) and (5) are equivalent. Let \( \alpha(\varphi) \in IM(\mathfrak{A}) \), where \( \varphi \in \mathfrak{M}(\mathfrak{A}) \) and \( \alpha \) is some finite set of \( S \). By Prop. 1.2 (2) there is a net \( \{ p_\alpha \} \) in \( M_\beta(\mathfrak{A}) \) such that \( \varphi = w^*-\lim \alpha(p_\beta) \). Putting \( \varphi_\alpha = \alpha(p_\alpha) \), we get \( \alpha(\varphi) = w^*-\lim \alpha(p_\alpha) \). Since \( \alpha(\varphi) \) is invariant, it follows from Lemma 3.1 (1) that \( \{ \varphi_\alpha \} \) is \( w^*- \) convergent to \( S \)-invariance. Hence \( \{ \varphi_\alpha \} \) satisfies the condition (5). Conversely let \( \{ \varphi_\alpha \} \) be as in (5). By the compactness of \( \mathfrak{M}(\mathfrak{A}) \), for each \( 1 \leq i \leq n \) we can choose a subnet \( \{ p_\beta \} \) of \( \{ p_\alpha \} \) convergent to some \( \varphi_i \in \mathfrak{M}(\mathfrak{A}) \), where the index set \( \{ \beta \} \) does not depend on \( i \). By Lemma 3.1 (2), \( \frac{1}{n} \sum_{i=1}^{n} \varphi_i \) is in \( IM(\mathfrak{A}) \cap Co(\mathfrak{M}(\mathfrak{A})) \). Thus (1) and (5) are equivalent.

q. e. d.

The criteria for extremely and quasi-extremely left amenable semigroups have been studied by E. Granirer [8]–[9] and A. T. Lau [15] respectively.

§ 4. Følner’s conditions for amenability

In this section we shall give the so-called Følner’s conditions and Day’s strong conditions for any amenable, extremely or quasi-extremely amenable \( \tau \)-semigroups.

4.1. E. Følner has proved in [7] that a group \( G \) is left amenable if and only if, for any finite subset \( F \) of \( G \) and \( \varepsilon > 0 \) there exists a nonempty finite subset \( A \) of \( G \) such that \( c(sA \sim A) < \varepsilon c(A) \) for all
This is the so-called "Følner's condition" for left amenability. On the other hand it is shown at first by M. M. Day [1, p. 524] that a semigroup $S$ is left amenable if and only if there exists a net of finite means on $B(S)$ which is norm-convergent to left invariance (see (3.2)). This is the so-called "Day's strong condition" for left amenability. Further I. Namioka [19] has proved by using Day's strong condition that the "only if" part of the above Følner's result holds also for any left amenable semigroup. Furthermore these conditions are generalized to the case of any amenable $\tau$-semigroup as follows.

**Theorem 4.1.** A $\tau$-semigroup $(S, X)$ is amenable if and only if there exists a net of finite means on $B(X)$ which is norm-convergent to $S$-invariance.

**Theorem 4.2.** If a $\tau$-semigroup $(S, X)$ is amenable, then for any finite subset $F$ of $S$ and $\varepsilon > 0$ there exists a nonempty finite subset $A$ of $X$ such that $c(A\sim sA) < \varepsilon c(A)$ for all $s \in F$.

Theorems 4.1 and 4.2 are shown in the author [25] by slight modifications of the discussions in [19]. The condition in Theorem 4.2 is not sufficient. For example let $(S, Y)$ be the $\tau$-semigroup as in Remark 2.11 (1). In this case we have $c(sY \sim Y) = 0$ for any $s \in S$. Hence $(S, Y)$ satisfies the conditions in Theorem 4.2. But $(S, Y)$ is not amenable. On the other hand we have the following sufficient condition of Følner's type which is also proved in [25].

**Theorem 4.3.** A $\tau$-semigroup $(S, X)$ is amenable if for any finite subset $F$ of $S$ and $\varepsilon > 0$ there exists a nonempty finite subset $A$ of $X$ such that $c(A\sim sA) < \varepsilon c(A)$ for all $s \in F$.

Combining Theorem 4.2 and 4.3 we get

**Theorem 4.4.** Let $(S, X)$ be a $\tau$-semigroup such that every $s \in S$ acts on $X$ as a injective map. Then $(S, X)$ is amenable if and only
if for any finite subset $F$ of $S$ and $\varepsilon > 0$ there exists a nonempty finite subset $A$ of $X$ such that $c(sA \sim A) < \varepsilon c(A)$ for all $s \in F$.

Proof. In this case, $c(sA \sim A) = c(A \sim sA)$ for any $s \in S$ and finite $A \subset X$. So the assertion follows from Theorems 4.2 and 4.3. q.e.d.

4.2. Here we shall consider the condition of Følner’s type for any extremely amenable $\tau$-semigroup. E. Granirer [8] and T. Mitchell [18] have proved that a semigroup $S$ is extremely left amenable if and only if for any $s, t \in S$ there exists a $e \in S$ such that $se = te = e$. This condition is generalized to the case of any extremely amenable $\tau$-semigroup as follows.

Theorem 4.5. A $\tau$-semigroup $(S, X)$ is extremely amenable if and only if for any finite subset $F$ of $S$ there exists an $F$-fixed point in $X$.

The sufficiency is easily seen. Indeed let $F$ be any finite subset of $S$ and $x_0$ an $F$-fixed point in $X$. Then the point mean $\delta(x_0)$ satisfies $s\delta(x_0) = \delta(sx_0) = \delta(x_0)$ for any $s \in F$. So it follows from Theorem 3.5 (6) that $B(X)$ is extremely amenable. To see the necessity we prepare some lemmas. Let $\Sigma$ be the $S$-invariant ring of all subsets of $X$. Then $B(X)$ coincides with $B(X, \Sigma)$ and every $\phi \in M(X)$ is identified with the finitely additive probability measure $\tilde{\phi}$ on $\Sigma$ defined by $\tilde{\phi}(A) = \phi(I_A)$ ($A \in \Sigma$). This measure is also denoted by the same letter $\phi$.

Lemma 4.6. For any $\phi \in M(X)$ the family $\{A \in \Sigma; \phi(A) = 1\}$ has the finite intersection property.

Proof. This follows from the relation that $\phi(A \cap B) = \phi(A) + \phi(B) - \phi(A \cup B)$ for any $A, B \in \Sigma$. q.e.d.

Lemma 4.7. Let $s \in S$ and $\phi \in \beta X$ satisfy $s\phi = \phi$. If $sA \cap A = \emptyset$ for $A \in \Sigma$, then $\phi(A) = 0$.

Proof. Since $\phi$ is multiplicative, $\phi(A) = 1$ or $0$. If $\phi(A) = 1$, then $1 = \phi(A) \leq \phi(s^{-1}(sA)) = \phi(sA) \leq 1$. Hence $\phi(sA \cap A) = 1$ by Lemma 4.6.
But this is impossible. Therefore $\varphi(A)=0$. q.e.d.

Let $s$ be any fixed element in $S$. We define the subsets $X^\infty$ and $X^t_i$ for any integers $0 \leq i < k$ as follows:

$$X^\infty = \{ x \in X; s^n x \neq s^m x \text{ for any distinct integers } n, m \geq 0 \},$$

$$X^t_i = \{ x \in X; x, sx, s^2x, \ldots, s^{k-1}x \text{ are mutually distinct and } s^k x = s^ix \}.$$ Then the family $\{X^\infty, X^t_i; i = 0, 1, 2, \ldots, k-1 \}$ is a partition of $X$ and we have

**Lemma 4.8.** (1) For any $k \geq 2$, $X^t_0$ admits a partition $\{Y^t_j; 1 \leq j \leq k \}$ such that $sY^t_j \subseteq Y^t_1$ and $sY^t_j \subseteq Y^t_{j+1}$ for $1 \leq j < k-1$.

(2) $X^\infty$ admits a partition $\{X^\infty_1, X^\infty_2 \}$ such that $sX^\infty_1 \subseteq X^\infty_2$ and $sX^\infty_2 \subseteq X^\infty_1$.

(3) $sX^t_i \subseteq X^t_{i-1}$ for any $1 \leq i < k$.

**Proof.** To see (1) we assume that $X^t_0$ is nonempty. Let $\mathfrak{P}$ be the family of $k$-tuples $(P_1, P_2, \ldots, P_k)$ of subsets of $X^t_0$ such that:

\[(\ast) \quad P_i \cap P_j = \emptyset \quad \text{if} \quad i \neq j,\]

\[(\ast \ast) \quad sP_k \subseteq P_1, \quad sP_j \subseteq P_{j+1} \quad \text{for} \quad 1 \leq j \leq k-1.\]

For any $x \in X^t_0$, we put $P_j = \{s^{j-1}x\}$ for $1 \leq j \leq k$. Then $(P_1, P_2, \ldots, P_k) \in \mathfrak{P}$. So $\mathfrak{P}$ is nonempty. In $\mathfrak{P}$ we define a partial order "\(\subseteq\)" as follows:

\[(P_1, P_2, \ldots, P_k) \subseteq (P'_1, P'_2, \ldots, P'_k) \Leftrightarrow P_j \subseteq P'_j \quad \text{for all} \quad 1 \leq j \leq k.\]

Let $\mathfrak{P}(A) = \{(P^t_1, P^t_2, \ldots, P^t_k) \in \mathfrak{P}; \lambda \in A \}$ be any linear ordered subset of $\mathfrak{P}$. We put $\bar{P}_j = \cup \{P^t_j; \lambda \in A \}$ for $1 \leq j \leq k$. Then $(\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_k)$ is in $\mathfrak{P}$ and is an upper bound of $\mathfrak{P}(A)$. Hence by Zorn's Lemma there exists a maximal element in $\mathfrak{P}$, say $(Y_1, Y_2, \ldots, Y_k)$. Putting $Y = \cup_{j=1}^k Y_j$, we show that $Y$ coincides with $X^t_0$. Suppose that there is a point $x \in X^t_0 \sim Y$. We set $S(x) = \{s^i x; 0 \leq i \leq k-1 \}$. At first assume that $Y \cap S(x) = \emptyset$. Then, setting $Y'_j = Y \cup \{s^{j-1}x\}$ for $1 \leq j \leq k$, we have
(Y_1, Y_2, \ldots, Y_k) \subseteq (Y'_1, Y'_2, \ldots, Y'_k) \in \mathfrak{B}$. This contradicts to the maximality of \((Y_1, Y_2, \ldots, Y_k)\). Next we assume that \(Y \cap S(x) \neq \emptyset\). Let \(j\) be the smallest integer for which \(s^{j-1} x \in Y\). Then \(j \geq 2\). Let \(s^{j-1} x \in Y_{i_0}\) and put \(z = s^{j-2} x\). If \(i_0 \geq 2\), we set \(Y_{i_0-1} = Y_{i_0-1} \cup \{z\}\) and \(Y'_i = Y_i\) for \(i \neq i_0 - 1\). If \(i_0 = 1\), we set \(Y'_1 = Y_k \cup \{z\}\) and \(Y'_i = Y_i\) for \(1 \leq i \leq k - 1\).

In both cases since \(z \in Y\) and \(sz \in Y\), we have \((Y_1, Y_2, \ldots, Y_k) \subseteq (Y'_1, Y'_2, \ldots, Y'_k) \in \mathfrak{B}\). This is also a contradiction. Hence \(Y = X^*_0\). Therefore \((Y_1, Y_2, \ldots, Y_k)\) is our desired partition of \(X^*_0\). (2) is proved similarly as in (1), and (3) is seen easily. \(\blacksquare\)

**Lemma 4.9.** \(\bigcup_{k=2}^{n} \bigcup_{i=0}^{k-1} X^*_i\) admits a partition \(\{Z_i; 1 \leq i \leq 5\}\) such that \(Z_i \cap sZ_i = \emptyset\) for any \(1 \leq i \leq 5\).

**Proof.** Using the notation in Lemma 4.8, we put:

\[
Z_1 = \bigcup_{k=1}^{n} Y_{3k-1}^0 \cup Y_{3k}^0, \quad Z_2 = \bigcup_{k=1}^{n} Y_{3k}^0 \cup Y_{3k+1}^0, \\
Z_3 = \bigcup_{k=1}^{n} Y_{3k+1}^0, \quad Z_4 = \bigcup_{k=1}^{n} Y_{3k+2}^0 \cup Y_{3k+1}^0, \\
Z_5 = \bigcup_{k=1}^{n} (X_{3k}^0 \cup X_{3k+1}^0) \cup X_{3k+2}^0.
\]

Then by Lemma 4.8 (1, 3), \(\{Z_i; 1 \leq i \leq 5\}\) is our desired partition. \(\blacksquare\)

**Lemma 4.10.** If \(s \varphi = \varphi\) for \(\varphi \in \beta X\) then \(\varphi(X^*_0) = 1\).

**Proof.** By Lemmas 4.8 (2) and 4.9, \(X \sim X^*_0\) admits a partition \(\{Z_i; 1 \leq i \leq 7\}\) such that \(Z_i \cap sZ_i = \emptyset\) for all \(1 \leq i \leq 7\). So it follows from Lemma 4.7 that \(\varphi(Z_i) = 0\) for all \(1 \leq i \leq 7\). Hence \(\varphi(X^*_0) = 1\). \(\blacksquare\)

Now suppose that \(\varphi \in \beta X\) is invariant. We put \(A_s = \{x \in X; sx = x\}\) for any \(s \in S\). Then \(\varphi(A_s) = 1\) for any \(s \in S\) by Lemma 4.10 and hence the family \(\{A_s; s \in S\}\) has the finite intersection property by Lemma 4.6. This mean that for any finite subset \(F\) of \(S\) there exists an \(F\)-fixed point in \(X\). Thus the necessity in Theorem 4.5 is proved.

4.3. By virtue of Theorem 4.5 we can get the following Theorems 4.11 \~ 4.13, which are proved in the author [21].
Theorem 4.11. A $\tau$-semigroup $(S, X)$ is extremely amenable if and only if there exists a net of point means on $B(X)$ which is norm-convergent to $S$-invariance.

The condition in the above is the Day's strong condition for any extremely amenable $\tau$-semigroup.

Theorem 4.12. A $\tau$-semigroup $(S, X)$ is extremely amenable if and only if every function in $H(B(X))$ (see §3.2) has a zero point in $X$.

Theorem 4.13. A $\tau$-semigroup $(S, X)$ is extremely amenable if and only if for any finite partition $\{A_i; 1 \leq i \leq n\}$ of $X$ there exists some integer $i$ such that $A_i$ contains an $s$-fixed point for any $s \in S$.

Now for any subset $A$ of a semigroup $S$ we consider the following condition:

(#) \hspace{1cm} For any $s \in S$ there exists a $c \in A$ such that $sc = c$.

Then Theorem 4.13 is weakened for the case of any semigroup as follows.

Theorem 4.14. A semigroup $S$ is extremely left amenable if and only if, for any subset $A$ of $S$ either $A$ or $S \sim A$ satisfies the condition (#)

Proof. The "only if" part is follows from Theorem 4.13. We prove the "if" part. Let $s, t \in S$ and put $A_s = \{c \in S; sc = c\}$. Then $S \sim A_s$ does not satisfy (#). By the assumption $A_s$ satisfies (#). Hence there exists a $c \in A_s$ such that $tc = c = sc$. So it follows from the result mentioned in the beginning of §4.2 that $S$ is extremely left amenable.

q.e.d.

Remark 4.15. (1) Any $\tau$-semigroup with a fixed point is extremely amenable. Conversely if a $\tau$-semigroup $(S, X)$ is extremely amenable and $S$ is finitely generated, then $(S, X)$ has a fixed point. This follows from Theorem 4.5. Similarly if a compact $\tau(c)$-semigroup $(S, Z)$ is extre-
mely amenable, then it has a fixed point. In fact put $A_s = \{z \in \mathbb{Z}; sz = z\}$ for any $s \in S$. Then by Theorem 4.5, $\{A_s; s \in S\}$ is a family of closed subset of $\mathbb{Z}$ with the finite intersection property. Hence by the compactness of $\mathbb{Z}$, the intersection $\cap \{A_s; s \in S\}$ is nonempty.

(2) Let $X$ be an infinite set and $G$ the group of all bijective maps $g$ of $X$ onto itself such that $\{x \in X; gx \neq x\}$ is finite. Then it is seen easily from Theorem 4.5 that $(G, X)$ is extremely amenable. In this case $(G, X)$ does not have any fixed points.

4.4. Finally we consider the Følner's condition and Day's strong condition for any quasi-extremely amenable $\tau$-semigroup. Now assume that $(S, X)$ is quasi-extremely amenable. Then by Prop. 2.1 (3) there exists a nonempty finite $S$-stable subset of $\beta X$. An $S$-stable subset is said to be minimal if it contains no proper $S$-stable subset. The following is obvious.

**Lemma 4.16.** A finite $S$-stable subset of $\beta X$ is decomposed into minimal $S$-stable subsets.

Now let $A=\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ be a minimal $S$-stable subset of $\beta X$. We define two equivalence relations $(r)$ and $(r')$ as follows: For any $s, t \in S$

- $s(r)t$ if and only if $s\varphi_i = t\varphi_i$,
- $s(r')t$ if and only if $s\varphi_i = t\varphi_i$ for all $1 \leq i \leq n$.

The relation $(r')$ is two-sided stable, i.e., $s(r')t$ implies $sc(r')tc$ and $cs(r')ct$ for any $c \in S$. So factor space $S/(r')$ can be regarded as a semigroup. Moreover since every $s \in S$ acts on $A$ as a permutation, $S/(r')$ becomes a finite group. We put

$$K = \{k \in S; k\varphi_i = \varphi_i \text{ for all } 1 \leq i \leq n\},$$

$$H = \{h \in S; h\varphi_i = \varphi_i\}.$$ 

Then we have

**Lemma 4.17.** If $s(r)t$ for $s, t \in S$, then $ks = th$ for some $k \in K$.
and \( h \in H \).

Proof. As \( S/(r') \) is a group, there exists a \( c \in S \) such that \( K \) contains both \( c't \) and \( tc \). Setting \( k=tc \) and \( h=cs \), we have \( ks=tcsc=th \) and \( h \in H \), because \( h\varphi_1=cs\varphi_1=ct\varphi_1=\varphi_1 \). q.e.d.

For any subset \( \alpha \) of \( S \) and \( x \in X \) we write \( \alpha x=\{ax; \ a \in \alpha \} \). By a coset representative of \( S/(r) \) we mean a finite subset \( \alpha=\{a_i; \ 1 \leq i \leq n \} \) of \( S \) such that \( a_i\varphi_1=\varphi_1 \) for \( 1 \leq i \leq n \).

Lemma 4.18. Let \( \alpha=\{a_i; \ 1 \leq i \leq n \} \) and \( \alpha_j=\{a_{ij}; \ 1 \leq i \leq n \} \) \((1 \leq j \leq m)\) be coset representatives of \( S/(r) \). Then there exists a point \( x_0 \) in \( X \) such that \( \alpha x_0=\alpha_1 x_0=\cdots=\alpha_n x_0 \) and \( c(\alpha x_0)=n \).

Proof. From the relations \( a_i(r)a_{ij} \) \((1 \leq i \leq n, 1 \leq j \leq m)\) and Lemma 4.17 there exist \( h_i^j \in H \) and \( k_i^j \in K \) \((1 \leq i \leq n, 1 \leq j \leq m)\) such that \( k_i^j a_i=a_{ij} h_i^j \). We put

\[
H_0=\{h_i^j; \ 1 \leq i \leq n, 1 \leq j \leq m \}, \quad Y=\{x \in X; \ hx=x \text{ for all } h \in H_0\},
\]

\[
K_i=\{k_i^j; \ 1 \leq j \leq m \}, \quad X_i=\{x \in X; \ kx=x \text{ for all } k \in K_i\} \ (1 \leq i \leq n).
\]

Since \( \varphi_1 \) and \( \varphi_i \) are \( H_0 \)- and \( K_i \)-fixed points in \( \beta X \) respectively, it follows from Lemmas 4.6 and 4.10 that \( \varphi_i(Y)=\varphi_i(X_i)=1 \) for any \( 1 \leq i \leq n \). On the other hand we can take mutually disjoint subsets \( A_1, A_2, \ldots, A_n \) of \( X \) such that \( \varphi_i(A_j)=\delta_{ij} \) \((1 \leq i, j \leq n)\). Setting \( B_i=X_i \cap A_i \), we have also \( \varphi_i(B_j)=\delta_{ij} \) \((1 \leq i, j \leq n)\). Since \( a_i \varphi_1=\varphi_i \), \( \varphi_i(B_i)=\varphi_i(a_i^{-1} B_i)=1 \) for any \( 1 \leq i \leq n \). Hence putting \( B_0=(\cap_{i=1}^n a_i^{-1} B_i) \cap Y \), we have \( \varphi_i(B_0)=1 \) and \( a_i B_0 \subseteq B_i \subseteq X_i \) for \( 1 \leq i \leq n \). Let \( x_0 \in B_0 \). Then

\[
a_i x_0=k_i^j a_i x_0=a_{ij} h_i^j x_0=a_{ij} x_0 \quad (1 \leq i \leq n, 1 \leq j \leq m).
\]

Therefore \( \alpha x_0=\alpha_1 x_0=\cdots=\alpha_n x_0 \). Moreover since \( B_i \)'s are mutually disjoint, \( \alpha x_0 \) consists of \( n \)-elements. q.e.d.

Lemma 4.19. Let \( \alpha \) be a coset representative of \( S/(r) \) and \( F \) a finite subset of \( S \). Then there exists a point \( x_0 \in X \) such that \( \alpha x_0 \).
Proof. Let \( F=\{s_1, s_2, \ldots, s_m\} \). Then \( s_1x, s_2x, \ldots, s_mx \) are coset representatives of \( S/(r) \). So the assertion follows from Lemma 4.18. q.e.d.

**Theorem 4.20.** For a finite subset \( \alpha=\{a_i; 1 \leq i \leq n\} \) of \( S \) and \( \varphi \in \beta X \), assume that \( \alpha(\varphi) \) (see (3.3)) is invariant and \( a_i\varphi \neq a_j\varphi \) for \( 1 \leq i < j \leq n \). Then for any finite subset \( F \) of \( S \) there exists a point \( x_0 \) in \( X \) such that \( s\alpha x_0=\alpha x_0 \) for all \( s \in F \).

**Proof.** Since \( \{a_1\varphi, a_2\varphi, \ldots, a_n\varphi\} \) is a minimal \( S \)-stable subset of \( \beta X \), our assertion follows from Lemma 4.19. q.e.d.

**Theorem 4.21.** Let \( n \) be a positive integer. The following conditions for a \( \tau \)-semigroup \( (S, X) \) are mutually equivalent:

1. There exists a finite \( S \)-stable subset \( \Delta \) of \( \beta X \) with \( c(\Delta)=n \),
2. There exists mutually disjoint subsets \( A_1, A_2, \ldots, A_n \) of \( X \) such that, for any finite subset \( F \) of \( S \) there corresponds a finite subset \( A_F=\{x_i; 1 \leq i \leq n\} \) of \( X \) with the properties:

\[
(\ast) \quad x_i \in A_i \quad \text{for any } 1 \leq i \leq n,
\]

\[
(\ast\ast) \quad sA_F=A_F \quad \text{for all } s \in F.
\]

3. For some mutually disjoint subsets \( A_1, A_2, \ldots, A_n \) of \( X \) there exists a net \( \{\varphi_n=\frac{1}{n} \sum_{i=1}^{n} \delta(x_i); x_i \in A_i (1 \leq i \leq n)\} \) in \( M_F(B(X)) \) which is norm-convergent to \( S \)-invariance.

**Proof.** From Lemma 4.16 and Theorem 4.20 it follows that (1) implies (2). Assume that (2) holds. Let \( \mathcal{F} \) be the family of all finite subsets of \( S \) ordered upward by inclusion. Then it is a directed set. For any \( F \in \mathcal{F} \) we take a finite subset \( A_F=\{x_i \in X; 1 \leq i \leq n\} \) with the properties (\( \ast \)) and (\( \ast\ast \)). Then putting \( \varphi_F=\frac{1}{n} \sum_{i=1}^{n} \delta(x_i) \), we have \( \|s\varphi_F-\varphi_F\|=0 \) for all \( s \in F \). So the net \( \{\varphi_F; F \in \mathcal{F}\} \) is our desired one and hence (2) implies (3). Let \( \{\varphi_n\} \) be a net as in (3). Then it
Kôkichi Sakai

has at least one w*-cluster point \( \varphi \in M(X) \) of the form \( \varphi = \frac{1}{n} \sum_{i=1}^{n} \varphi_i \), where \( \varphi_i \)'s are mutually distinct elements in \( \beta X \). It is obvious from Lemma 3.1 (2) that \( \varphi \) is invariant. So \( \{\varphi_1, \varphi_2, \ldots, \varphi_n\} \) is \( S \)-stable and hence (3) implies (1).

q.e.d.

The next is an immediate consequence of Theorem 4.21.

**Theorem 4.22.** The following conditions for any \( \tau \)-semigroup \((S, X)\) are mutually equivalent:

1. \((S, X)\) is quasi-extremely amenable,
2. There is an integer \( n \geq 1 \) such that for any finite subset \( F \) of \( S \) there exists a finite subset \( A \) of \( X \) with the properties: (i) \( c(A) = n \), and (ii) \( sA = A \) for all \( s \in F \).
3. For some integer \( n \geq 1 \) there exists a net \( \{ \varphi_n = \frac{1}{n} \sum_{i=1}^{n} \delta(x_i) \} \) in \( M(B(X)) \) which is norm-convergent to \( S \)-invariance.

The conditions (2) in Theorems 4.21 and 4.22 are of Følner's types and the conditions (3) in them are of Day's strong types. For the case of any semigroup \( S \) it is shown by A. T. Lau [15] that every finite minimal \( S \)-stable subset of \( \beta S \) has the same cardinal number, say \( n \). In this case \( S \) is said to be \( n \)-extremely left amenable.

§ 5. Characterization of extremely amenable algebras

In this section we shall note that almost all results in A. T. Lau [16] can be generalized to the case of any \( \tau \)-semigroup \( X = (S, X) \).

5.1. For any \( \varphi \in M(X) \) we put:

\[
\Pi(\varphi) = \{ A \subseteq X ; \varphi(s^{-1}A) = \varphi(A) = 1 \text{ for any } s \in S \},
\]

\[
\Sigma(\varphi) = \text{the ring of subsets of } X \text{ generated by } \Pi(\varphi),
\]

\[
B(X, \varphi) = B(X, \Sigma(\varphi)) \quad \text{(see § 1.5)}.
\]

Then \( \Sigma(\varphi) \) is an \( S \)-invariant ring and \( B(X, \varphi) \) is an \( X \)-algebra. For any \( A \subseteq X \) we denote by \( \overline{A} \) the \( w^* \)-closure of \( \{\delta(x); x \in A\} \) in \( \beta X \).

**Theorem 5.1.** For any \( \varphi \in M(X) \), \( B(X, \varphi) \) is extremely amenable.
Proof. Since $\Pi(\varphi)$ has the finite intersection property, we can take an element $\varphi_0$ in $\cap \{ A; A \in \Pi(\varphi) \}$. Then it is seen easily that $\varphi_0(s^{-1}A) = \varphi_0(A)$ for any $s \in S$ and $A \in \Sigma(\varphi)$. So it follows from Theorem 2.9 that $B(X, \varphi)$ is extremely amenable. q.e.d.

Lemma 5.2. Let $f \in B(X)$, and $\varphi \in \beta X$ satisfy $s\varphi(f) = \varphi(f)$ for any $s \in S$. If $\varphi(f) = 0$, then $\{ x \in X; f(x) < \varepsilon \} \in \Pi(\varphi)$ for any $\varepsilon > 0$.

This is proved by the same way as in [16, Lemma 2]. The next is a generalization of Theorem 1 in [16].

Theorem 5.3. An $X$-algebra $\mathfrak{A}$ is extremely amenable if and only if there exists some $\varphi \in M(X)$ such that $\mathfrak{A} \subseteq B(X, \varphi)$.

Proof. The "if" part is obvious from Theorems 5.1 and 2.7. Conversely let $\varphi_0 \in \mathfrak{A}(\mathfrak{A})$. Then there exists an extension $\varphi \in \beta X$ of $\varphi_0$. We show that $\mathfrak{A} \subseteq B(X, \varphi)$. Since $\mathfrak{A}$ is a direct sum of $\mathfrak{A}_0 = \{ f \in \mathfrak{A}; \varphi(f) = 0 \}$ and the the trivial $X$-space $C(I_X)$ as linear spaces, it suffices to see that $\mathfrak{A}_0 \subseteq B(X, \varphi)$. Let $f \in \mathfrak{A}_0$ and $\varepsilon > 0$. For every integer $n \geq 1$ we put:

$$E_n = \{ x \in X; \varepsilon(n + 1) < f(x) \leq \varepsilon(n + 1) \}, \quad E_{-n} = \{ x \in X; -\varepsilon(n + 1) < f(x) \leq -\varepsilon n \}.$$ 

Then by Lemma 5.2, $X \sim E_n$ and $X \sim E_{-n}$ are in $\Pi(\varphi)$ for any $n \geq 1$. Moreover we have $\| f - \sum_{n=-\infty}^\infty E_n \| < \varepsilon$. As $\varepsilon$ is arbitrary, $f \in B(X, \varphi)$ Hence $\mathfrak{A}_0 \subseteq B(X, \varphi)$. q.e.d.

From the above theorem we see that the family of extremely amenable algebras $\{ B(X, \varphi); \varphi \in M(X) \}$ exhausts essentially all extremely amenable $X$-algebras.

5.2. We now introduce the concept of $S$-thickness, which is an analogue of the left-thickness for semigroups due to T. Mitchell [17].

Lemma 5.4. The following conditions for any subset $A$ of $X$ are mutually equivalent:

1. The family $\{ s^{-1}A; s \in S \}$ has the finite intersection property,
(2) For any finite subset \( F \) of \( S \) there exists a point \( x \) in \( X \) such that \( sx \in A \) for any \( s \in F \).

(3) There exists a \( \varphi \in \beta X \) such that \( \varphi(s^{-1}A) = \varphi(A) = 1 \) for any \( s \in S \).

(4) There exists a \( \varphi \in M(X) \) such that \( \varphi(s^{-1}A) = \varphi(A) = 1 \) for any \( s \in S \).

Proof. The implications (1)\( \Rightarrow \)(2) and (3)\( \Rightarrow \)(4) are obvious, and (4)\( \Rightarrow \)(1) follows from Lemma 4.6. So it remains to see that (1) implies (3). Assume that (1) holds. Then we can take an element \( \varphi_0 \in \beta X \) in \( \bigcap \{ s^{-1}A ; s \in S \} \). Let \( \psi \in \beta S \) and put \( \varphi = \varphi_0 \ast \psi \in \beta X \) (see (1.7)). Since \( \varphi_0(s^{-1}A) = 1 \) for any \( s \in S \), it is easily seen that \( \varphi(s^{-1}A) = \varphi(A) = 1 \) for any \( s \in S \). Hence (1) implies (3).

A subset \( A \) of \( X \) is said to be \( S \)-thick if it satisfies any one of the conditions in the above lemma. If \( \varphi \in M(X) \) is invariant and a subset \( A \) of \( X \) satisfies \( \varphi(A) = 1 \), then \( A \) is \( S \)-thick. Further we have

**Theorem 5.5.** Let \( (S, X) \) be a \( \tau \)-semigroup such that \( S \) is left amenable. Then a subset \( A \) of \( X \) is \( S \)-thick if and only if there exists an invariant mean \( \varphi \) on \( B(X) \) such that \( \varphi(A) = 1 \).

Proof. It suffices to see the "only if" part. Let \( \varphi_0 \in M(X) \) satisfy \( \varphi_0(s^{-1}A) = \varphi_0(A) = 1 \) for any \( s \in S \), and \( \psi \) a left invariant mean on \( B(S) \). Then by (1.8), \( \varphi = \varphi_0 \ast \psi \) is invariant and satisfies \( \varphi(s^{-1}A) = \varphi(A) = 1 \) for any \( s \in S \). q.e.d.

**Theorem 5.6.** For any \( \tau \)-semigroup \( X = (S, X) \), there is an extremely amenable \( X \)-algebra containing properly \( C(I_X) \) if and only if there exists a proper \( S \)-thick subset of \( X \).

Proof. Let \( A \) be a proper \( S \)-thick subset of \( X \) and \( \varphi \in M(X) \) satisfy \( \varphi(s^{-1}A) = \varphi(A) = 1 \) for any \( s \in S \). Then \( A \in \Pi(\varphi) \) and \( I_A \in B(X, \varphi) \). Hence the extremely amenable algebra \( B(X, \varphi) \) contains properly \( C(I_X) \). Conversely let \( \mathcal{A} \) be an extremely amenable \( X \)-algebra different from \( C(I_X) \). Then by Theorem 5.3, there exists a \( \varphi \in M(X) \) such that
Semigroups

\[ B(X, \varphi). \] In this case \( \Pi(\varphi) \) contains a proper subset of \( X \). Indeed if \( \Pi(\varphi)=\{X\} \), then \( B(X, \varphi) \) coincides with \( C(I_X) \). Since every member in \( \Pi(\varphi) \) is \( S \)-thick, there is a proper \( S \)-thick subset of \( X \). q.e.d.

Theorems 5.5 and 5.6 are generalizations of Theorem 7 in [17] and Corollary in [16, p. 332] respectively.

5.3. Let \( \Sigma \) be an \( S \)-invariant ring of subsets of \( X \). Using the notion of \( S \)-thickness, we shall show a necessary and sufficient condition for \( B(X, \Sigma) \) to be extremely amenable, which is a generalization of Theorem 2 in [16].

**Theorem 5.7.** The following conditions are equivalent:

1. \( B(X, \Sigma) \) is extremely amenable.
2. For any finite partition \( \{A_i \in \Sigma; 1 \leq i \leq n\} \) of \( X \), there exists some \( i \) for which \( A_i \) is \( S \)-thick.

**Proof.** Let \( \varphi \in IM(B(X, \Sigma)) \) and \( \{A_i \in \Sigma; 1 \leq i \leq n\} \) a partition of \( X \). Then \( \sum_{i=1}^{n} \varphi(A_i)=1 \) and hence \( \varphi(A_i)=1 \) for some \( i \). Since \( \varphi \) is invariant, \( A_i \) is \( S \)-thick. Conversely assume that (2) holds. By Theorem 2.9 it suffices to prove that there exists a \( \varphi \in \beta X \) such that \( \varphi(s^{-1}A) = \varphi(A) \) for any \( s \in S \) and \( A \in \Sigma \). We put \( K_A = \{ \varphi \in \beta X; \varphi(s^{-1}A) = \varphi(A) \} \) for any \( A \in \Sigma \). Then \( K_A \) is a closed subset of \( \beta X \). We show that the family \( \{K_A; A \in \Sigma\} \) has the finite intersection property. Let \( A \in \Sigma \) (\( 1 \leq i \leq n \)). Then we take a partition \( \{E_j \in \Sigma; 1 \leq j \leq m\} \) of \( X \) which is a refinement of the partitions \( \{A_i, X \sim A_i\} \) for all \( 1 \leq i \leq n \). By the assumption there is an \( S \)-thick set \( E_j \) among \( E_i \)'s. We put \( F=E_j \). Let \( \varphi_0 \in \beta X \) satisfy \( \varphi_0(s^{-1}F) = \varphi_0(F) = 1 \) for any \( s \in S \). If \( F \subseteq A_i \), then \( s^{-1}F \subseteq s^{-1}A_i \) and \( \varphi_0(s^{-1}A_i) = \varphi_0(A_i) = 1 \) for any \( s \in S \). Similarly if \( F \subseteq X \sim A_i \), then \( \varphi_0(s^{-1}A_i) = \varphi_0(A_i) = 0 \) for any \( s \in S \). So \( \varphi_0 \in \cap_{i=1}^{n} K_A \). Thus \( \{K_A; A \in \Sigma\} \) has the finite intersection property. By the compactness of \( \beta X \cap \{K_A; A \in \Sigma\} \) is nonempty and is contained in \( \mathfrak{M}(B(X, \Sigma)) \). Hence (2) implies (1). q.e.d.

§ 6. Right stationary transformation semigroups

It is proved by T. Mitchell [17] [resp. E. Granirer [9]] that a
semigroup $S$ is left amenable [resp. extremely left amenable] if and only if it is right [resp. extremely right stationary]. In this section we shall generalize these results to the case of any $X$-space or $X$-algebra.

6.1. We begin with the definitions of right and of extremely right stationary $X$-spaces. Let $\mathcal{A}$ be an $X$-space. For $f \in \mathcal{A}$ and $\varphi \in M(\mathcal{A})$ we define $\varphi \square f \in B(S)$ by $\varphi \square f(s) = \varphi(s_f)$ $(s \in S)$. Now we put:

$$Z(f) = p_c \text{cl}\{\varphi \square f; \varphi \in M(\mathcal{A})\} = p_c \text{cl}\{\text{Co}(\delta(x) \square f; x \in X)\},$$

$$W(f) = p_c \text{cl}\{\delta(x) \square f; x \in X\},$$

where $p_c \text{cl}(A)$ denotes the closure of a subset $A$ of $B(S)$ with respect to the pointwise convergence topology $p_c$ of $B(S)$. Further we put:

$$K(f) = \{c \in C; c I_S \in Z(f)\}, \quad H(f) = \{c \in C; c I_S \in W(f)\}.$$

We say that $\mathcal{A}$ is right stationary [resp. extremely right stationary] if $K(f)$ [resp. $H(f)$] is nonempty for any $f \in \mathcal{A}$.

**Lemma 6.1.** Let $s \in S, f \in \mathcal{A}$ and $h \in \mathcal{A}^+$. Then we have:

1. $K(c I_X) = H(c I_X) = \{c\}$ for any $c \in C$,
2. $K(I_X - f) = \{1 - c; c \in K(f)\}, \quad H(I_X - f) = \{1 - c; c \in H(f)\}$,
3. If $c \in K(f)$ then $|c| \leq \|f\|$,
4. If $c \in K(f - s f)$ then $c = 0$,
5. If $c \in H(h(f - s f))$ then $c = 0$.

**Proof.** (1), (2), and (3) are evident. (4) is proved by the same way as in T. Mitchell [17, Theorem 1]. Let us show (5). We can assume without loss of generality that $f$ is real valued. Let $\{\varphi_\alpha\}$ be a net in $M_\mu(\mathcal{A})$ such that

$$p_c \text{lim}_\alpha \varphi_\alpha \square (h(f - s_f)) = p_c \text{lim}_\alpha (\varphi_\alpha \square h)(\varphi_\alpha \square f - s(\varphi_\alpha \square f)) = c.$$

Since $W(h)$ and $W(f)$ are $p_c$-compact (see [17, Lemmas 2 and 3]), we can take a subnet $\{\varphi_\beta\}$ of $\{\varphi_\alpha\}$ such that $\varphi_\beta \square h$ and $\varphi_\beta \square f$ converge pointwise to $h_0 \in B(S)^+$ and $f_0 \in B(S)_r$ respectively. Hence $h_0(t)(f_0(t)$
Semigroups I

589

for any $t \in S$. Suppose $c \neq 0$. Then for any positive integer $n$,

$$|f_0(s) - f_0(s^{n+1})| = \sum_{k=1}^{n} f_0(s^k) - f_0(s^{k+1}) = |c| \sum_{k=1}^{n} h_0(s^k)^{-1} \geq |c| \|h_0\|^{-1}.$$ 

This relation contradicts the boundedness of $f_0$. So $c = 0$. q.e.d.

**Lemma 6.2.** Let $\mathcal{A}$ be an $X$-space [resp. $X$-algebra] and $\varphi \in IM(\mathcal{A})$ [resp. $\mathcal{A}$-$\mathcal{M}(\mathcal{A})$]. Then $\varphi(f) \in K(f)$ [resp. $H(f)$] for any $f \in \mathcal{A}$.

**Proof.** We take a net $\{\varphi_n\}$ in $M_f(\mathcal{A})$ [resp. $M_p(\mathcal{A})$] such that $\varphi = w^{*}\lim \varphi_n$. Then $\lim \varphi_n(f) = \varphi(f)$ for any $f \in \mathcal{A}$ and $s \in S$. Since $\varphi(f(s)) = \varphi(f(s)) \in Z(f)$ [resp. $W(f)$], it follows that $p_c \lim \varphi_n(f) = \varphi(f)I_S$. Hence $\varphi(f) \in K(f)$ [resp. $H(f)$] for any $f \in \mathcal{A}$. q.e.d.

**Theorem 6.3.** An amenable $X$-space is right stationary and an extremely amenable $X$-algebra is extremely right stationary.

**Proof.** This is obvious from Lemma 6.2. q.e.d.

**Theorem 6.4.** Let $\mathcal{A}$ be an $X$-space [resp. $X$-algebra]. Then the following conditions are mutually equivalent:

1. $\mathcal{A}$ is amenable [resp. extremely amenable],

2. $K(h)$ [resp. $H(h)$] contains 0 for any $h \in K(\mathcal{A})$ [resp. $H(\mathcal{A})$],

3. $K(h)$ [resp. $H(h)$] contains 0 for any $h \in K(\mathcal{A}, \mathcal{A})$ [resp. $H(\mathcal{A}, \mathcal{A})$].

**Proof.** Let $\varphi \in IM(\mathcal{A})$ [resp. $\mathcal{A}$-$\mathcal{M}(\mathcal{A})$] and $h \in K(\mathcal{A})$ [resp. $H(\mathcal{A})$]. Then $\varphi(h) = 0$ and hence 0 is contained in $K(h)$ [resp. $H(h)$] by Lemma 6.2. So (1) implies (2). (2)$\Rightarrow$(3) is clear. Assume that (3) holds. Then it follows from Lemma 6.1 (2, 3) that $1 \leq \|I_\mathcal{A} - h\|$ for any $h \in K(\mathcal{A})$ [resp. $H(\mathcal{A})$]. So we have $d(I_\mathcal{A}, K(\mathcal{A})) = 1$ [resp. $d(I_\mathcal{A}, H(\mathcal{A})) = 1$]. By Theorem 3.2 (3) [resp. 3.5 (3)], $\mathcal{A}$ is amenable [resp. extremely amenable]. Hence (3) implies (1). q.e.d.

Let $\mathcal{A}$ be an $X$-space and $p$ a sublinear real functional on $\mathcal{A}$, i.e., $p(f + g) \leq p(f) + p(g)$ and $p(\lambda f) = \lambda p(f)$ for any $f, g \in \mathcal{A}$, and
nonnegative \( \lambda \in \mathbb{R} \). We say that \( p \) is invariant if \( p(f - s f) = 0 \) for any \( s \in S \) and \( f \in \mathcal{X}_r \). Now assume that \( p \) has the following property:

\[
(6.1) \quad p(\lambda I_X) = \lambda \quad \text{and} \quad p(f) \leq \|f\| \quad \text{for any} \quad \lambda \in \mathbb{R} \quad \text{and} \quad f \in \mathcal{X}_r.
\]

Then we have

**Lemma 6.5.**

1. \( p(f - \lambda I_X) = p(f) - \lambda \) for any \( \lambda \in \mathbb{R} \) and \( f \in \mathcal{X}_r \).
2. If \( f \in \mathcal{X}_r^+ \), then \( 0 \leq -p(-f) \).
3. If \( \varphi \in \mathcal{X}_r^\ast \) is dominated by \( p \) (i.e., \( \varphi(f) \leq p(f) \) for any \( f \in \mathcal{X}_r \)), then \( \varphi \in M(\mathcal{X}_r) \). Moreover if \( p \) is invariant, then so is \( \varphi \).

**Proof.** Let \( f \in \mathcal{X}_r \). By Hahn-Banach theorem there exists a \( \varphi_0 \in \mathcal{X}_r^\ast \) which is dominated by \( p \) and satisfies \( \varphi_0(f) = p(f) \). Then \( \varphi_0(\lambda I_X) = \lambda \) and \( \varphi_0(f - \lambda I_X) \leq p(f - \lambda I_X) \leq p(f) - \lambda = \varphi_0(f - \lambda I_X) \) for any \( \lambda \in \mathbb{R} \). Hence (1) is shown. Let \( f \in \mathcal{X}_r^+ \) and put \( g = \|f\|I_X - f \). Then \( g \in \mathcal{X}_r^+ \) and \( \|g\| \leq \|f\| \). By (1) and (6.1), \( p(g) = \|f\| + p(-f) \leq \|g\| \leq \|f\| \). Hence \( -p(-f) \geq 0 \). So (2) holds. If \( \varphi \in \mathcal{X}_r^\ast \) is dominated by \( p \), then \(-p(-f) \leq \varphi(f) \leq p(f) \) for any \( f \in \mathcal{X}_r \). From this relation, (6.1) and the above (2), the assertion (3) follows immediately.

**Remark 6.6.** A sublinear functional \( p \) on \( \mathcal{X}_r \) satisfies (6.1) if and only if every \( \varphi \in \mathcal{X}_r^\ast \) dominated by \( p \) is in \( M(\mathcal{X}_r) \). In fact the "if" part and the "only if" part follow from Hahn-Banach theorem and Lemma 6.5 (3) respectively.

**Lemma 6.7.** Let \( p \) be an invariant sublinear functional on \( \mathcal{X}_r \) with (6.1). Then for any \( f_0 \in \mathcal{X}_r \) and scalar \( \lambda \) with \( -p(-f_0) \leq \lambda \leq p(f_0) \), there exists an \( \varphi \in IM(\mathcal{X}_r) \) which is dominated by \( p \) and satisfies \( \varphi(f_0) = \lambda \).

**Proof.** This is obvious from Lemma 6.5 (3) and Hahn-Banach theorem.

**Theorem 6.8.** The following conditions for an \( X \)-space \( \mathcal{X} \) are mutually equivalent:
(1) $\mathfrak{A}$ is amenable,
(2) There exists an invariant sublinear functional $p$ on $\mathfrak{A}$, with (6.1),
(3) $\mathfrak{A}$ is right stationary and there exists a sublinear functional $p$ on $\mathfrak{A}$, such that $p(f) \in K(f)$ for any $f \in \mathfrak{A}$.

Proof. Since any $\varphi \in IM(\mathfrak{A})$ can be regarded as an invariant sublinear functional on $\mathfrak{A}$, with (6.1), (1) implies (2). Conversely it follows from Lemma 6.7 that (2) implies (1). Let $p$ be as in (2) and $f \in \mathfrak{A}$.

Then by Lemma 6.7 there exists a $\varphi \in IM(\mathfrak{A})$ such that $\varphi(f) = p(f)$. So it follows from Lemma 6.2 that $p(f) \in K(f)$. Hence (2) implies (3). Conversely let $p$ be as in (3). Then it follows from Lemma 6.1 that $p$ is invariant and satisfies (6.1). Hence (3) implies (2). \(\text{q.e.d.}\)

Remark 6.9. (1) Let $\mathfrak{A}$ be an amenable $X$-space. For any $f \in \mathfrak{A}$, we put:

\[ \tilde{K}(f) = \{ \varphi(f) : \varphi \in IM(\mathfrak{A}) \} , \]

\[ \tilde{p}(f) = \sup \{ c \in \mathbb{R} : c \in \tilde{K}(f) \} , \]

\[ \hat{p}(f) = \sup \{ c \in \mathbb{R} : c \in K(f) \} . \]

Then from the results in the above we can conclude that:

(i) $\tilde{p}$ is an invariant sublinear functional on $\mathfrak{A}$, with (6.1),
(ii) $\tilde{K}(f) = [ -\tilde{p}(-f), \tilde{p}(f) ] \subseteq K(f)$ for any $f \in \mathfrak{A}$,
(iii) $\hat{p}$ satisfies $\hat{p}(\lambda I_x) = \lambda$, $\hat{p}(f) \leq \| f \|$ and $\hat{p}(f-s) = 0$ for any $\lambda \in \mathbb{R}$,
(iv) $\hat{p}$ is sublinear if and only if $\tilde{K}(f) = K(f)$ for any $f \in \mathfrak{A}$.

In general $\hat{p}$ is not sublinear. For example let $(S, X)$ be the $\tau$-semigroup as in Remark 2.11 (1). Then $\delta(x_0)$ is a unique invariant mean on $B(X)$. So $\tilde{K}(f) = \{ f(x_0) \}$ for any $f \in B(X)$. On the other hand let us define an $f \in B(X)$ by $f(x_0) = 1$ and $f(x) = 0$ for otherwise $x \in X$. Then $K(f)$ becomes the interval $[0, 1]$, which contains properly $\tilde{K}(f) = \{ 1 \}$. In this case $\hat{p}$ is not sublinear.

(2) Let $S$ be a semigroup and $\mathfrak{A}$ a right stationary $S_r$-space.
We now assume that $\mathcal{A}$ is left-introverted, i.e., $\varphi \varphi \mathcal{A} \subseteq \mathcal{A}$ for any $\varphi \in M(\mathcal{A})$. Then as shown in [10, Lemma 2], $Z(f) \subseteq \mathcal{A}$ for any $f \in \mathcal{A}$. Moreover it is proved by the same way as in [17, Lemmas 4 and 5] that $\beta$ is sublinear. So for any semigroup $S$ it follows that a left-introverted $S_r$-space is left amenable if and only if it is right stationary.

6.3. Let $\mathcal{A}$ be an $X$-algebra. We now consider a sublinear functional $p$ on $\mathcal{A}$, with the following property:

(6.2) $p(fg) \leq p(f)p(g)$ for any $f \in \mathcal{A}^*$ and $g \in \mathcal{A}$, with $p(g) \geq 0$.

**Lemma 6.10.** The following conditions for a sublinear functional $p$ on $\mathcal{A}$, are equivalent:

1. $p$ has the properties (6.1) and (6.2).
2. For any $f_0 \in \mathcal{A}$, there exists a $\varphi \in \mathcal{M}(\mathcal{A})$ which is dominated by $p$ and satisfies $\varphi(f_0) = p(f_0)$.

**Proof.** At first assume that $p$ has (6.1) and (6.2). Let $f_0$ be any fixed element in $\mathcal{A}$, and put

$$M(f_0, p) = \{ \varphi \in \mathcal{A}^* : \varphi \text{ is dominated by } p \text{ and } \varphi(f_0) = p(f_0) \}.$$  

Then $M(f_0, p)$ is a nonempty, convex and $w^*$-compact subset of $M(\mathcal{A})$. Hence there exists at least one extreme point $\varphi_0$ of $M(f_0, p)$. We now show that $\varphi_0$ is multiplicative. To see this it suffices to prove that $\varphi_0(fg) = \varphi_0(f)\varphi_0(g)$ for any $g \in \mathcal{A}$, and $f \in \mathcal{A}^*$ with $\|f\| \leq 1$. For such $f$ we define $\psi \in \mathcal{A}^*$ by $\psi(g) = \varphi_0(fg) - \varphi_0(f)\varphi_0(g)$ ($g \in \mathcal{A}$). Using (6.2) and the fact that $p(g - \varphi_0(g)I_X) = p(g) - \varphi_0(g) \geq 0$, $0 \leq p(f) \leq 1$ and $0 \leq p(I_X - f) \leq 1$, we have for any $g \in \mathcal{A}$,

$$(\varphi_0 + \psi)(g) = \varphi_0(f(g - \varphi_0(g)I_X)) + \varphi_0(g) \leq p(f(g - \varphi_0(g)I_X)) + \varphi_0(g) \leq p(f) (p(g) - \varphi_0(g)I_X) + \varphi_0(g),$$

$$\leq p(f) (p(g) - \varphi(g)) + \varphi_0(g) \leq p(g).$$

$$(\varphi_0 - \psi)(g) = \varphi_0((I_X - f)(g - \varphi_0(g)I_X)) + \varphi_0(g) \leq p(I_X - f) (p(g) - \varphi_0(g)) + \varphi_0(g) \leq p(g),$$

$$\psi(f_0) = \varphi_0(f(f_0 - \varphi_0(f_0)I_X)) \leq p(f) (p(f_0) - \varphi_0(f_0)) = 0.$$
\[
\psi(-f_0) = \varphi_0(I_x - f)(f_0 - \varphi_0(f_0)I_x) \leq p(I_x - f)(p(f_0) - \varphi_0(f_0)) = 0.
\]

Thus \( \varphi_0 \pm \psi \) are dominated by \( p \) and \( \psi(f_0) = 0 \). i.e., \( \varphi_0 \pm \psi \in M(f_0, p) \). Since \( \varphi_0 \) is extreme, we get \( \psi = 0 \). Therefore (1) implies (2). Conversely assume that (2) holds. Then it is easily seen that \( p \) has (6.1). Let \( f \in \mathfrak{A}^+ \) and \( g \in \mathfrak{A} \) satisfy \( p(g) \geq 0 \). We take a \( \varphi \in \mathfrak{M}(\mathfrak{A}_r) \) which is dominated by \( p \) and satisfies \( \varphi(fg) = p(fg) \). Since \( 0 \leq \varphi(f) \leq p(f) \) and \( \varphi(g) \leq p(g) \), we have
\[
p(fg) = \varphi(fg) = \varphi(f)\varphi(g) \leq \varphi(f)p(g) \leq p(f)p(g).
\]
Hence \( p \) has (6.2) and (2) implies (1).

**Theorem 6.11.** The following conditions for an \( X \)-algebra \( \mathfrak{A} \) are mutually equivalent:

1. \( \mathfrak{A} \) is extremely amenable,
2. There exists an invariant sublinear functional \( p \) on \( \mathfrak{A} \), with (6.1) and (6.2),
3. \( \mathfrak{A} \) is extremely right stationary and there exists a sublinear functional \( p \) on \( \mathfrak{A} \), such that \( p(f) \in H(f) \) for any \( f \in \mathfrak{A}_r \).

**Proof.** Since any \( \varphi \in \mathfrak{M}(\mathfrak{A}) \) is regarded as an invariant sublinear functional on \( \mathfrak{A}_r \) with (6.1) and (6.2), (1) implies (2). Let \( p \) be as in (2) and \( f \in \mathfrak{A}_r \). Then by Lemmas 6.5 (3) and 6.10, there exists a \( \varphi \in \mathfrak{M}(\mathfrak{A}_r) \) such that \( \varphi(f) = p(f) \). So \( p(f) \in H(f) \) by Lemma 6.2. Hence (2) implies (3). Let \( p \) be as in (3). Then by Lemma 6.1 (5), we have \( p(h(f-sf)) = 0 \) for any \( f, s \in S \) and \( h \in \mathfrak{A}^+ \). Since \( p \) is sublinear, \( p(h) = 0 \) for any \( h \in H(\mathfrak{A}_r) \). Hence \( H(h) \) contains 0 for any \( h \in H(\mathfrak{A}_r) \). So it follows from Theorem 6.4 that \( \mathfrak{A} \) is extremely amenable. Thus (3) implies (1). q.e.d.

**Remark 6.12.** Let \( S \) be a semigroup and \( \mathfrak{A} \) an extremely right stationary \( S_r \)-algebra. We put \( \hat{p}(f) = \sup \{ c \in R ; c \in H(f) \} \) (\( f \in \mathfrak{A}_r \)). If \( \mathfrak{A} \) is left-\( m \)-introverted, i.e., \( \varphi \in \mathfrak{A} \) for any \( \varphi \in \mathfrak{M}(\mathfrak{A}) \), then we can see that \( \hat{p} \) is sublinear. So since \( H(f) \) is a closed subset of \( R \), it follows from Theorem 6.11 (3) that \( \mathfrak{A} \) is extremely left amenable.
Hence combining this fact and Theorem 6.3 we have: *Any left-m-introverted \( S^r \)-algebra is extremely left amenable if and only if it is extremely right stationary.*

**References**

Semigroups I

(1967), 63–77.


