§ 1. Introduction

In this paper we shall consider the family of meromorphic quadratic differentials with only simple poles on a compact Riemann surface. The quadratic differentials with closed trajectories are in a sense exceptional and have several extremal properties in above family (cf. [4], [8], [9]). Strebel ([10]) stated under a certain assumption that they are dense in the family. It is the first purpose of this paper to get rid of that assumption and show a new complete proof. The proof will be shown in §3. Relating to this fact we show in §4 that a holomorphic abelian differential whose square has closed trajectories is not always proportional to a holomorphic reproducing differential.

Strebel also considered the relation between the Teichmüller theory and the contractions of holomorphic quadratic differentials with closed trajectories. In the case of the unit disk with a finite set of preassigned points he gave successful results ([8]). As an application of our Theorem 1, the similar close relations between them are shown (§5) in the case of compact Riemann surfaces.

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§ 2. Preliminaries

(1) Notations and terminologies. Let $R$ be a compact Riemann surface of genus $g(>0)$. We write

$\mathcal{A} = \mathcal{A}_1(R) = \{\theta: \theta$ is a holomorphic abelian differential on $R\}.$

$\mathcal{A}_2 = \mathcal{A}_2(R) = \{\phi: \phi$ is a holomorphic quadratic differential on $R\}.$

$A'_2 D = A'_2 D(R) = \{\theta^2: \theta \in \mathcal{A}_1\}$

$A_2 D = A_2 D(R) = \{\phi: \phi$ is a meromorphic quadratic differential on $R$ such that $\|\phi\| = \int_R |\phi| < +\infty.\}$

We note that $\phi \in A_2 D$ may have only simple poles. Now let $\phi$ be such a meromorphic quadratic differential on $R$. The maximal regular curve $\alpha$ on $R$ along which $\phi > 0$ (or $< 0$) is called a trajectory (or orthogonal trajectory) of $\phi$. If $\alpha$ tends to a finite critical point (i.e., zero point or simple pole) of $\phi$ in at least one of its direction, then $\alpha$ is called a critical trajectory. Otherwise it is called a regular trajectory. If a regular trajectory is not a closed curve it is called to be divergent.

A quadratic differential $\phi$ is said to have the closed trajectories if all of its regular trajectories are closed. Here we define several subfamilies of $A_2 D$ and $\mathcal{A}_1$.

$CA_2 D = CA_2 D(R) = \{\phi \in A_2 D: \phi$ has closed trajectories.$\}$

$CA'_2 D = CA'_2 D(R) = A'_2 D \cap CA_2 D$

$C\mathcal{A}_1 = C\mathcal{A}_1(R) = \{\theta \in \mathcal{A}_1: \theta^2 \in CA_2 D\}$

$C\mathcal{A}_2 = CA_2 \cap \mathcal{A}_2$

A simply connected domain $D$ on $R$ is called a trajectory rectangle of $\phi$ if the following conditions are satisfied:

(i) $\phi$ is holomorphic and non-zero on $D$
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(ii) $\zeta = \sqrt[4]{\phi}$ is univalent on $D$ and maps $D$ onto a rectangle with horizontal and vertical sides in the $\zeta$-plane.

and a trajectory rectangle $D$ is called regular if $\phi$ is holomorphic and non-zero on $\bar{D}$ (the closure is taken in $\mathbb{R}$.)

Next the line element $|\phi|^\frac{1}{2}$ defines a metric on $\mathbb{R}$. In particular the length of a rectifiable curve $\gamma$ measured by this metric is called $\phi$-length of $\gamma$, and is denoted by $|\gamma|_{\phi}$ or simply $|\gamma|$.

Finally let $\Gamma_h$ be the space of real harmonic differentials on $\mathbb{R}$. It is well known that for a fixed 1-cycle $\gamma$, there exists a unique $\sigma_{\gamma} \in \Gamma_h$ such that

$$\int_{\gamma} \omega = (\omega, \sigma_{\gamma}) \quad \text{for every } \omega \in \Gamma_h.$$ 

This $\sigma_{\gamma}$ is called the reproducing differential for $\gamma$. And the holomorphic differential $\theta_{\gamma} = \sigma_{\gamma} + i*\sigma_{\gamma}$ is called the holomorphic reproducing differential for $\gamma$.

(II) Strebel's main results. The element of $CA_2D$ has several extremal properties, some of which were proved by Strebel ([8], [9]). To state them explicitly we need some more definitions.

**Definitions.** Let $\phi \in CA_2D$ and $\{P_j\}_{j=1}^n$ be a finite set of points on $\mathbb{R}$ including the poles of $\phi$. If we cut $\mathbb{R}$ along all critical trajectories of $\phi$ and closed regular trajectories of $\phi$ through $P_j$, then we get a system $\mathbb{R}(\phi)$ of ring domains (i.e. doubly connected domains) on $\mathbb{R}$, which is called the characteristic ring domains of $\phi$ on $\tilde{\mathbb{R}} = \mathbb{R} - \{P_j\}_{j=1}^n$.

A system of non intersecting Jordan closed curves $\{\gamma_i\}_{i=1}^n$ on $\tilde{\mathbb{R}} = \mathbb{R} - \{P_j\}_{j=1}^n$ is called admissible if no two of them are freely homotopic on $\tilde{\mathbb{R}}$ and none is freely homotopic to zero or to a Jordan closed curve surrounding only one point of $\{P_j\}_{j=1}^n$ on $\tilde{\mathbb{R}}$.

We say that a system of non overlapping ring domains $\{R_k\}$ on $\tilde{\mathbb{R}}$ belongs to an admissible curve system $\{\gamma_i\}_{i=1}^n$ if for every $R_k$ a Jordan closed curve $\gamma'_k \subset R_k$ separating its boundary components is freely homotopic to a $\gamma_i$ of $\{\gamma_i\}_{i=1}^n$ and $\gamma'_n \sim \gamma'_m$ for $n \sim m$, where $\sim$
means that $\alpha$ is freely homotopic to $\beta$ on $\hat{R}$.

**Theorem A.** Given an admissible curve system $\{\gamma_i\}_{i=1}^p$ on $\hat{R}=R-\{P_j\}_{j=1}^f$ and a set of numbers $\{m_i\}_{i=1}^p$ such that every $m_i$ is nonnegative and not all of them equal to zero, then there exists a $\phi_{\tilde{m}} \in CA_2D$ which is holomorphic on $\hat{R}$ and whose characteristic ring domains $\{R_i\}_{i=1}^p$ on $\hat{R}$ (for $R_i$ corresponds to $\gamma_i$, and may be an empty set.) belongs to the system $\{\gamma_i\}_{i=1}^p$, such that the moduli $M_i$ of $R_i$ satisfy the condition

$$M_1: M_2: M_3: \cdots: M_p = m_1: m_2: m_3: \cdots: m_p$$

(Here $M_i=0$ for the empty $R_i$). This solution $\phi_{\tilde{m}}$ is uniquely determined up to a positive constant factor.

Now give an admissible curve system $\{\gamma_i\}_{i=1}^p$ on $\hat{R}=R-\{P_j\}_{j=1}^f$ and let

$$E^p = \{\tilde{m} = (m_1, m_2, \ldots, m_p) = (m_i) : m_i \geq 0, \sum_{i=1}^p m_i^2 = 1\}.$$ 

Then for every $\tilde{m} \in E^p$ there exists a $\phi_{\tilde{m}} (\in CA_2D)$ as in Theorem A, and the vector $\tilde{M} = (M_i) = |M| \cdot \tilde{m}$ is uniquely determined, which is called the moduli vector of $\phi_{\tilde{m}}$. As $\tilde{m}$ varies, $\tilde{M}$ describes a surface in $R^p$, which is denoted by $\mathcal{A}\{\gamma_i\}$ and called the moduli surface of $\{\gamma_i\}$ (cf. [9], [11]).

**Theorem B.** The vector $\tilde{M}$ depends continuously on its direction $\tilde{m}$. And let $\tilde{M}$ be an interior point of $\mathcal{A} = \mathcal{A}\{\gamma_i\}$ and let

$$\Pi = \{\tilde{X} \in R^p, such that the scalar product (\tilde{X} - \tilde{M}, \tilde{a}) = 0\}$$

where $\tilde{a} = (a_i^2), a_i = |\gamma_i|$. Then $\Pi$ is the tangent plane and every point $\tilde{M}' \in \mathcal{A}$ lies on the same side of $\Pi$ as the origin. $\Pi$ has exactly the point $\tilde{M}$ in common with $\mathcal{A}$, and varies continuously with $\tilde{M}$.

§ 3. **On Strebel's conjecture**

We start with several lemmas.
Lemma 1. Let $\theta \in \overline{A}_1$. If there exists a real constant $C$ such that for every 1-cycle $\gamma$ on $R$, $\text{Im} \left( \frac{1}{C} \int_{\gamma} \theta \right)$ is an integer, then $\theta \in CA_1$.

Proof. If $\phi = \theta^2 \notin CA_2D$, then there exists a divergent trajectory $\alpha$, so we may assume that for a point $P \in \alpha$ the trajectory ray $\alpha^+(t)$ ($0 \leq t < +\infty$) starting from $P$ is divergent. Now the cluster set of $\alpha^+(t)$;

$$A^+ = \bigcap_{\alpha} \{ \alpha^+(t) : t \geq a \}$$

is a non empty closed set on $R$. Let $P_0 \in A^+$ be a regular point of $\phi$, $D$ be a regular trajectory rectangle such that $P_0$ is mapped to the center of the rectangle which is the image of $D$ on the $\zeta$-plane, and $\beta$ be the orthogonal trajectory in $D$ through $P_0$. Then there exists a sequence $\{P_n\} \subset \alpha^+ \cap \beta$ such that $P_n$ converges to $P_0$ and $P_n \approx P_m$ for $n \approx m$. (cf. [5], [10])

Now we define

$$C_{n,m} = [P_n \rightarrow P_m, \alpha^+] \cup [P_m \rightarrow P_n, \beta],$$

where $[P \rightarrow Q, \gamma]$ denote the curve from $P$ to $Q$ along $\gamma$, then $C_{n,m}$ is a closed curve on $R$. Hence

$$\int_{[P_n \rightarrow P_m, \beta]} |\theta| = |\text{Im} \int_{[P_n \rightarrow P_m, \beta]} \theta|$$

$$= |\text{Im} \int_{C_{n,m}} \theta| \geq C > 0.$$  

While, the integral gives the distance between $P_n$ and $P_m$ in the $\zeta$-plane, so must converges to zero, which is a contradiction. Thus $\phi \in CA_2D$. q.e.d.

Corollary 1. (cf. [1]) Let $\gamma$ be an arbitrary 1-cycle on $R$. Then the holomorphic reproducing differential $\theta_\gamma$ for $\gamma$ belongs to $CA_1$.

Proof. It suffices to recall that for an arbitrary 1-cycle $\delta$
\[ \text{Im} \int_\delta \theta' = \gamma \times \delta \]

is an integer. (cf. [7]) q.e.d.

**Remark.** Every \( \theta \in \tilde{A}_1 \) satisfying the condition in Lemma 1 is proportional to a holomorphic reproducing differential for a suitable \( \gamma \) (i.e. \( \theta = C \theta_\gamma, C: \text{real} \)). So Lemma 1 and Corollary 1 state the same thing.

**Example.** We consider the case of genus one. Given \( \theta_0 \in \tilde{A}_1 \), then a branch of the inverse of \( \zeta = \int \theta_0 \) can be considered a projection mapping from the \( \zeta \)-plane to \( R \). Let \( \{A_1, B_1\} \) be a canonical homology base, \( \omega_1 = \int_{A_1} \theta_0 \), and \( \omega_2 = \int_{B_1} \theta_0 \), then we can take the parallelogram with vertices \( 0, \omega_1, \omega_2, \) and \( \omega_1 + \omega_2 \) as a fundamental region of \( R \).

For an arbitrary \( \theta(\equiv 0) \in \tilde{A}_1 \), there exists a complex constant \( C \) such that \( \theta = C \theta_0 \), so any trajectory of \( \theta^2 \) must be lifted to a line which is parallel to the line

\[ \{ \zeta: \text{arg } \zeta \equiv -\text{arg } C \pmod{\pi} \} \]

on the \( \zeta \)-plane. Hence \( \theta \in C \tilde{A}_1 \) if and only if there exist integers \( n, m \) such that

\[ \text{arg } C \equiv -\text{arg } (m \omega_1 + n \omega_2) \pmod{\pi}. \]

And it can be shown that this is the case if and only if \( \theta = k \theta_{mA_1 + nB_1} \) for a suitable real constant \( k \). Thus we conclude that \( C \tilde{A}_1 \) consists of essentially only holomorphic reproducing differentials.

**Lemma 2.** The set \( C \tilde{A}_1 \) is dense in \( \tilde{A}_1 \) with respect to the Dirichlet norm.

**Proof.** Let \( \{A_i, B_i\}_{i=1}^q \) be a canonical homology base on \( R \), then \( \{\theta_{A_i}, \theta_{B_i}\}_{i=1}^q \) is a base of the real vector space \( \tilde{A}_1 \), for if \( \int_{A_i} \theta \) and \( \int_{B_i} \theta (\theta \in \tilde{A}_1) \) are real for every \( i \), then \( \theta \) is identically zero. Hence for every \( \theta \in \tilde{A}_1 \)
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\[ \theta = \sum_{i=1}^{g} \{ c_i \theta_A + d_i \theta_B \}, \]

where \( c_i = \text{Im} \int_{B_i} \theta, \quad d_i = -\text{Im} \int_{A_i} \theta. \) For every \( i, \) take two sequences of rational numbers \( \{ c_{i,n} \}_{n=1}^{\infty}, \{ d_{i,n} \}_{n=1}^{\infty} \) such that

\[ \lim_{n \to \infty} c_{i,n} = c_i, \quad \lim_{n \to \infty} d_{i,n} = d_i, \]

and define

\[ \theta_n = \sum_{i=1}^{g} \{ c_{i,n} \theta_A + d_{i,n} \theta_B \}, \]

then \( \theta_n \in \bar{A}_1 \) (by Lemma 1) and

\[ \| \theta - \theta_n \| \leq \sum_{i=1}^{g} \{ |c_{i,n} - c_i| \cdot \| \theta_A \| + |d_{i,n} - d_i| \cdot \| \theta_B \| \} \]

\[ \longrightarrow 0 \quad \text{as} \quad n \to \infty, \]

which implies the assertion.

**Lemma 3.** Let \( \{ \theta_n \} \subset \bar{A}_1 \) be a sequence converging to \( \theta \in \bar{A}_1 \) with respect to the Dirichlet norm. Then

\[ \lim_{n \to \infty} \| \theta_n^2 - \theta^2 \| = 0. \]

**Proof.**

\[ \| \theta_n^2 - \theta^2 \| = \int_R |\theta_n^2 - \theta^2| \]

\[ \leq \left\{ \int_R |\theta_n - \theta|^2 \cdot \int_R |\theta_n + \theta|^2 \right\}^{\frac{1}{2}} \]

\[ \leq 2\| \theta_n - \theta \| \cdot \{ \| \theta_n \|^2 + \| \theta \|^2 \}^{\frac{1}{2}}. \]

Here \( \| \theta_n \| \) is bounded, so \( \lim_{n \to \infty} \| \theta_n^2 - \theta^2 \| = 0. \)

**Corollary 2.** The set \( CA_2 \) is dense in \( A_2 D \) with respect to the \( \| \| \)-norm.
Lemma 4. Let $\phi_n, \phi$ belong to $\tilde{A}_2$. Then $\lim_{n \to \infty} \| \phi_n - \phi \| = 0$ if and only if $\phi_n$ converges locally uniformly to $\phi$; i.e. for every $P \in \mathbb{R}$ and every local parameter $z$ ($|z| < 1, z(P) = 0$), $\phi_n$ converges uniformly to $\phi$ on $\{|z| \leq r\}$ for every positive $r < 1$.

In particular zero points of $\phi_n$ converge to zero points of $\phi$ including multiplicity.

Proof. ([8], [11]) Let $f(z)$ be a regular function on $\{|z| \leq 1\}$ and $|\xi| \leq r$, then

$$|f(\xi)| \leq \frac{1}{\pi(1-r)^2} \int_{|z| \leq 1} |f| \, dx \, dy \quad (z = x + iy),$$

which implies the sufficiency.

To show the converse for every $P \in \mathbb{R}$ we choose a local parameter $z_p$ on $\{|z_p| \leq 1\}$ such that $z_p(P) = 0$, then there exists a finite set of points $\{P_i\}$ such that $\cup \{|z_p| < r\}$ already covers $\mathbb{R}$ for any given $r$ ($0 < r < 1$). Hence for any $\varepsilon > 0$ there exists an integer $n(\varepsilon)$ such that for all $n \geq n(\varepsilon)$ and all $i$

$$|\phi_n - \phi| < \varepsilon \quad \text{on} \quad \{|z_p| < r\}.$$

Thus we conclude that

$$\lim_{n \to \infty} \| \phi_n - \phi \| \geq \lim_{n \to \infty} \sum_i \int_{|z_p| \leq r} |\phi_n - \phi| = 0. \quad \text{q.e.d.}$$

Now we will prove the following

**Theorem 1.** The set $CA_2D$ is dense in $A_2D$ with respect to the $\| \cdot \|$-norm. Moreover if $\phi \in A_2D$ has poles at $\{P_i\}_{i=1}^r$ then $\phi$ can be approximated by the elements of $CA_2D$ with poles at $\{P_i\}_{i=1}^r$.

Proof. Let $\phi$ be any element of $A_2D$, then we will show that $\phi$ can be approximated by the elements of $CA_2D$. Now let $\left\{P_i\right\}_{i=1}^r$ be the set of poles and zeros of $\phi$ of odd order. This number $r$ is even since $\deg \phi = 4g - 4$ by Riemann-Roch's theorem.

1) Suppose $r > 0$. Then we can construct a two sheeted covering
surface $\bar{R}$ satisfying following conditions;

1. $\bar{R}$ has branch points over $\{P_i\}_{i=1}^g$.
2. There exist the sheet interchange $J: \bar{R} \rightarrow \bar{R}$ and the projection mapping $\Pi: \bar{R} \rightarrow R$ such that $\Pi \circ J = \Pi$.
3. There exists canonical homology bases $\{A_i, B_i\}_{i=1}^g$ and $\{\tilde{A}_i, \tilde{B}_i\}_{i=1}^{2g+\frac{g^2-g+1}{2}}$ on $R$ and $\bar{R}$ respectively such that

\[
\begin{align*}
\Pi(\tilde{A}_i) &= A_i, & \Pi(\tilde{B}_i) &= B_i \\
J(\tilde{A}_i) &= \tilde{A}_{g+i}, & J(\tilde{B}_i) &= \tilde{B}_{g+i} \\
J(\tilde{A}_{2g+i}) &= -\tilde{A}_{2g+j}, & J(\tilde{B}_{2g+j}) &= -\tilde{B}_{2g+j} & 1 \leq j \leq \frac{r}{2} - 1.
\end{align*}
\]

x) First let the square root of $\phi$ can be lifted to an abelian differential $\vartheta$ on $\bar{R}$. In this case $\vartheta \circ J = -\vartheta$ and we can take local parameters $z_i$ and $\tilde{z}_i$ at $P_i$ and $\tilde{P}_i = \Pi^{-1}(P_i)$ on $R$ and $\bar{R}$ respectively such that $z_i = \pi(\tilde{z}_i) = \tilde{z}_i^2$, then if $\phi$ has the expansion

\[(c_{2m+1}z_i^{2m+1} + c_{2m+2}z_i^{2m+2} + \cdots)dz_i^2 \quad \text{at} \quad P_i \]

then $\vartheta$ has the expansion

\[(\tilde{c}_{2m+2}z_i^{2m+2} + \tilde{c}_{2m+3}z_i^{2m+3} + \cdots) \tilde{d}z_i \quad \text{at} \quad \tilde{P}_i, \]

Here $2m+2 \geq 0$, for $\phi$ may have only simple poles (i.e. $m \geq -1$). Thus $\vartheta \in \tilde{A}_1(\bar{R})$.

It is well known that the real vector space $\tilde{A}_1(\bar{R})$ has the orthogonal decomposition with respect to the Dirichlet norm such that

\[\tilde{A}_1(\bar{R}) = \tilde{A}_0 + \tilde{A}_e\]

where $\tilde{A}_0 = \{\theta \in \tilde{A}_1(\bar{R}): \theta \circ J = -\theta\}$ and $\tilde{A}_e = \{\theta \in \tilde{A}_1(\bar{R}): \theta \circ J = \theta\}$.

In this case we can take

\[
\begin{align*}
\theta_{\tilde{A}_{g+i}}, & \quad \theta_{\tilde{B}_{g+i}} & 1 \leq i \leq g \\
\theta_{\tilde{A}_{2g+j}}, & \quad \theta_{\tilde{B}_{2g+j}} & 1 \leq j \leq \frac{r}{2} - 1
\end{align*}
\]
as a base of \( \tilde{A}_0 \), and \( \tilde{\theta} \) can be approximated by the elements of \( \tilde{A}_0 \cap C\tilde{A}_1(\tilde{R}) \) as in the proof of Lemma 2. Now we show that for any \( \tilde{\omega} \in \tilde{A}_0 \cap C\tilde{A}_1(\tilde{R}) \) \( \tilde{\omega}^2 \cdot \pi^{-1} \) belongs to \( CA_2D(R) \), which shows the assertion. First of all

\[
\| \tilde{\omega}^2 \cdot \pi^{-1} \|_R = \frac{1}{2} \| \tilde{\omega}^2 \|_R + \frac{1}{2} \| \tilde{\omega} \|_R^2 < +\infty
\]

Hence \( \tilde{\omega}^2 \cdot \pi^{-1} \in A_2D(R) \) and if \( \lim_{n \to \infty} \| \omega_n - \omega \|_R = 0 \), then

\[
\lim_{n \to \infty} \| \tilde{\omega}_n^2 \cdot \pi^{-1} - \tilde{\omega}^2 \cdot \pi^{-1} \|_R = 0.
\]

Next for any point \( P \in R \) such that the trajectory \( \gamma \) of \( \tilde{\omega}^2 \cdot \pi^{-1} \) through \( P \) is regular, the trajectory \( \tilde{\gamma} \) of \( \tilde{\omega}^2 \) through \( \tilde{P} \in \pi^{-1}(P) \) is also regular and hence closed. This implies that \( \gamma \) is closed, so \( \tilde{\omega}^2 \cdot \pi^{-1} \in CA_2D(R) \).

Now if \( \phi \) has poles \( \{ P_i \} \), then \( \tilde{\theta} \) is non-zero at \( \tilde{P}_i \) as shown above, so by Lemma 4 the approximating sequence \( \{ \tilde{\theta}_n \} \) can be taken to be also non-zero at \( \tilde{P}_i \) for every \( n \). This implies that every \( \tilde{\theta}_n^2 \cdot \pi^{-1} \) has poles \( \{ P_i \} \).

\( \beta) \) Next suppose that the square root of \( \phi \) can not be lifted to any abelian differential on \( \tilde{R} \). (Of course it is a Prym differential on \( \tilde{R} \).) In this case we take a two sheeted covering surface \( \tilde{R} \) over \( R \) on which the square root of \( \phi \) can be lifted to an abelian differential \( \tilde{\theta} \). Then there exist the sheet interchange \( J \) and the lift \( \tilde{J} \) of \( J \) such that \( \tilde{\theta} \cdot \tilde{J} = -\tilde{\theta} \).

(i.e. \( \tilde{J} : (P, \sqrt{\phi}) \longrightarrow (P, -\sqrt{\phi}) \) \( P \in \tilde{R} \)

\( \tilde{J} : (P, \sqrt{\phi}) \longrightarrow (J(P), \sqrt{\phi}) \) or \( (J(P), -\sqrt{\phi}) \) \( P \in \tilde{R} \)

so that \( \tilde{\theta} \cdot \tilde{J} = -\tilde{\theta} \).)

Now let \( \{ \tilde{A}_i, \tilde{B}_i \}_{i=1}^{g+r-3} \) be a homology base on \( \tilde{R} \), and

\( \tilde{A}_0 = \{ \tilde{\omega} \in \tilde{A}_1(\tilde{R}) : \tilde{\omega} \cdot \tilde{J} = -\tilde{\omega}, \tilde{\omega} \cdot \tilde{J} = -\tilde{\omega} \} \).

Then \( \tilde{\theta} \in \tilde{A}_0 \), and the system of holomorphic reproducing differentials \( \{ \theta_{\mu}, \theta_{\nu} \}_{i=1}^{g+r-3} \) with
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\[ \gamma_i = \{ \hat{A}_i - \hat{J}(\hat{A}_i) \} - \hat{J}(\hat{A}_i - \hat{J}(\hat{A}_i)) \]

\[ \delta_i = \{ \hat{B}_i - \hat{J}(\hat{B}_i) \} - \hat{J}(\hat{B}_i - \hat{J}(\hat{B}_i)) \]

spans the real vector space \( \hat{A}_0 \), so \( \hat{\theta} \) can be approximated by the elements of \( \hat{A}_0 \cap C\hat{A}_1(\mathcal{R}) \). By the similar argument as in I)–\( \alpha \), we can thus prove the assertion for this case.

II) Finally suppose \( r=0 \). In this case \( \phi \in \hat{A}_2(\mathcal{R}) \). If the square root of \( \phi \) is an abelian differential on \( \mathcal{R} \), then \( \phi \in A_2'(R) \), so \( \phi \) can be approximated by the elements of \( CA_2'(R) \). (cf. Corollary 2) Otherwise, taking the two sheeted covering surface of \( \mathcal{R} \) on which the square root of \( \phi \) can be lifted to an abelian differential, the assertion can be shown as in I).

Thus Theorem is completely proved. q.e.d.

§ 4. The structure of \( C\hat{A}_1 \)

In § 3 we have seen that \( C\hat{A}_1 \) consists of holomorphic reproducing differentials multiplied by a real constant in the case of genus one. But this is not true if \( g \geq 2 \). Theorem 2 in the sequel includes this assertion.

Lemma 5. Let \( \theta \in \hat{A}_1 \) and \( \phi=0^2 \). Suppose \( \phi \) has a closed trajectory \( \gamma \), then there exists an \( \varepsilon > 0 \) such that for every \( \tilde{\theta} \in \hat{A}_1 \) satisfying

\[ \| \theta - \tilde{\theta} \| < \varepsilon, \int_{\gamma} \tilde{\theta}; \text{real}, \]

\( \tilde{\theta}^2 \) has a closed trajectory \( \tilde{\gamma} \), which is freely homotopic to \( \gamma \) on \( \mathcal{R} \).

Proof. The set of all closed trajectories \( \gamma' \) of \( \phi \) such that \( \gamma' \sim \gamma \) makes a ring domain on \( \mathcal{R} \). Cutting it by an orthogonal trajectory \( \beta \), we have a trajectory rectangle \( D \). Let \( P_0 \) be an intersecting point of \( \gamma \) and \( \beta \). \( D \) is mapped by \( \zeta = \int_{P_0} \theta \) onto \( \{(x, y): 0 < x < a, b_2 < y < b_1\} \) \( (\zeta = x + iy) \). Let \( \delta \) be a positive constant such that

\[ 0 < 2\delta < \min \{ b_1, -b_2 \} \]
and $D'$ be the trajectory rectangle corresponding to the rectangle

$$\{(x, y): 0 < x < a, b_2 + \delta < y < b_1 - \delta\}$$

on the $\zeta$-plane. Then by Lemmas 3 and 4 there exists an $\varepsilon > 0$ such that for every $\bar{\theta}$ satisfying $\|\theta - \bar{\theta}\| < \varepsilon$, there holds

1. $\bar{\theta} \neq 0$ on $D'$, and
2. $|\zeta(P) - \tilde{\zeta}(P)| \leq \frac{1}{2} \min\{b_1 - 2\delta, -(b_2 + 2\delta)\}$ on $D'$, where $\tilde{\zeta}(P) = \frac{1}{\bar{\theta}}$.

Now let $F_1 = \{P \in D': \text{Im} \zeta(P) = b_1 - 2\delta\}$, then

$$\inf_{P \in F_1} \{\text{Im} \tilde{\zeta}(P)\} \geq (b_1 - 2\delta) - \inf_{P \in F_1} |\zeta(P) - \tilde{\zeta}(P)| \geq \frac{1}{2} (b_1 - 2\delta) > 0.$$  

Similarly let $F_2 = \{P \in D': \text{Im} \zeta(P) = b_2 + 2\delta\}$, then

$$\sup_{P \in F_2} \{\text{Im} \tilde{\zeta}(P)\} \leq -\frac{1}{2} (b_2 + 2\delta) < 0.$$ 

Moreover $\xi$ maps $D'$ locally conformally into the $\zeta$-plane and a pair of points on the boundary of $D'$ which are the same point on $R$ and mapped to a pair of points on the $\zeta$-plane which have the same imaginary part, for $\bar{\theta}$ is real. Images $\xi(F_1)$ and $\xi(F_2)$ do not intersect with the real axis of the $\zeta$-plane, so lifting the map $\xi = \int_{P_0}^{P} \bar{\theta}$ to the universal covering surface $U$ of $\text{Int.} D'$, the inverse of it can be continued analytically from $P_0$ along the real axis of the $\zeta$-plane. But here $\{P \in D': \text{Im} \tilde{\zeta}(P) = 0\}$ is a closed trajectory of $\bar{\theta}^2$, which is obviously freely homotopic to $\gamma$ on $R$. q.e.d.

**Lemma 6.** ([8], [11]) Given $\{\gamma_i\}$, let $\phi_n \in CA_2D$, the moduli vector of which belongs to the moduli surface $\mathcal{M}\{\gamma_i\}$ for every $n$, and $\|\phi_n\| = 1$. If $\lim_{n \to \infty} \phi_n$ exists and is equal to $\phi$, then $\phi \in CA_2D$, $\|\phi\| = 1$, and the moduli vector of $\phi$ belongs to $\mathcal{M}\{\gamma_i\}$. 

Here we recall that the maximal number \( N \) of elements in an admissible curve system on \( \hat{R} = R - \{ P_j \}_{j=1}^r \) is \( N = 3(g - 1) + r \) except for \( g = 1, r = 0 \) (then \( N = 1 \)) and \( g = 0, r \leq 3 \) (then \( N = 0 \)).

**Lemma 7.** Let \( \{ \gamma_i \} \) be an admissible curve system. If \( \{ \tilde{\gamma}_j \}_{j=1}^k \) is a mutually homologically independent subsystem of \( \{ \gamma_i \} \), then \( k \leq g \). Moreover, the set

\[
D\{ \gamma_i \} = \{ \theta \in C\bar{A}_1 : \text{the moduli vector of } \theta^2 \text{ belongs to } \mathcal{M}\{ \gamma_i \} \}
\]

is contained in a real \( g \)-dimensional manifold.

**Proof.** Suppose \( k \geq g \). We cut \( R \) along \( \tilde{\gamma}_j \) \((j = 1, \ldots, g)\), then we have a planer domain whose boundary components are \( \{ \tilde{\gamma}_j, -\tilde{\gamma}_j \}_{j=1}^g \). Hence every \( \gamma_i \) is homologous to \( \sum_{j=1}^g m_{i,j} \tilde{\gamma}_j \) with suitable integers \( m_{i,j} \). Thus we have proved that \( k \leq g \).

Next we note that for the moduli vector of the element of \( D\{ \gamma_i \} \), the normal vector \( (a_i) \) (cf. Theorem B) is uniquely determined by the periods associated with \( \tilde{\gamma}_j \) \((j = 1, \ldots, k)\). Thus the assertion follows from above fact. (cf. [11] Theorem 15.5.) q.e.d.

While, \( D\{ \gamma_i \} \) is not so small. Actually we have the following

**Theorem 2.** There exists an admissible curve system \( \{ \gamma_i \} \) such that \( D\{ \gamma_i \} \) contains a real \( g \)-dimensional manifold \( M \), which is contained in

\[
P\{ A_i \} = \{ \theta \in C\bar{A}_1 : \theta = \sum_{i=1}^g c_i A_i, \text{ for real } c_i \}
\]

for a suitable canonical homology base \( \{ A_i, B_i \}_{i=1}^g \) on \( R \).

**Proof.** Let \( \theta_0 \in C\bar{A}_1 \) be fixed, \( \phi = \theta_0^2 \), and suppose the moduli vector of \( \phi \) is an interior point of \( \mathcal{M}\{ \gamma_i \} \). We take such a subsystem \( \{ \tilde{\gamma}_j \}_{j=1}^k \) of \( \{ \gamma_i \}_{i=1}^r \) as in Lemma 7, and define

\[
M(\theta_0) = \{ \theta \in C\bar{A}_1 : \int_{\tilde{\gamma}_j} \theta \text{ is real for every } j (1 \leq j \leq k) \}
\]
Then $M(0_0)$ is a real $(2g - k)$-dimensional manifold.

By Lemma 5 there exists a neighbourhood $U(0_0)$ of $0_0$ in $M(0_0)$ such that for every $0 \in U(0_0)$ \{\theta'\}^2$ has closed trajectories $\{\gamma_i\}_{i=1}^n$ such that $\gamma'_i \sim \gamma_i$ for every $i$.

Case (1): $k = g$. We make a canonical homology base $\{A_i, B_i\}_{i=1}^g$ such that $A_i = \tilde{\gamma}_i$ for every $i$.

(α) Suppose $U(0_0) \cap C\tilde{A}_1$ is contained in $D\{\gamma_i\}$. In this case $M(0_0) = P\{A_i\}$ so by Lemma 2 and 6 we obtain $U(0_0) \subset C\tilde{A}_1$. Thus $M = U(0_0)$ has required properties.

(β) Suppose there exists a $\tilde{\gamma}_1 \in U(0_0) \cap C\tilde{A}_1$ such that $\tilde{\gamma}_1 \notin D\{\gamma_i\}$. Then $\tilde{\gamma}_1$ has a closed trajectory $\gamma$ such that $\gamma \sim \gamma_i$ for every $i$. In this case, $\int_\gamma \theta'$ is real for every $\theta' \in U(0_0)$. So by Lemma 5 there exists a neighbourhood $U(\tilde{\gamma}_1)$ of $\tilde{\gamma}_1$ in $M(0_0)$ (which is contained in $U(0_0)$) such that for every $\theta'' \in U(\tilde{\gamma}_1)$ \{\theta''\}^2$ has a closed trajectory $\gamma'$ such that $\gamma' \sim \gamma$.

If every $\theta' \in U(\tilde{\gamma}_1) \cap C\tilde{A}_1$ belongs to $D\{\gamma, \gamma_i\}$ then we can reduce the proof to Case (1)–(α), and if not, we start from Case (1)–(β) with $U(\tilde{\gamma}_1)$ instead of $U(0_0)$ and repeat this process. Such procedure ends in finite steps, because the number of elements of an admissible curve system on $R$ is bounded by $3(g-1)$ (or 1 if $g = 1$). And finally the proof is reduce to Case (1)–(α).

Case (2): $k < g$. First suppose that every trajectory $\gamma$ of the square of every element of $U(0_0) \cap C\tilde{A}_1$ is homologous to $\sum_{j=1}^\infty m_j \tilde{\gamma}_j$ with suitable integers $m_j$ (which of course depends on $\gamma$). Then by the same argument as in Case (1)–(β) we find a non-empty open set $U$ in $M(0_0)$ such that $U \subset C\tilde{A}_1$ and for a suitable curve system $\{\gamma_i\}$ $U$ is contained in $D\{\gamma_i\}$ which is contained in a real $g$-dimensional manifold by Lemma 7. But $M(0_0)$ is $(2g - k)$-dimensional, so is $U$, which is a contradiction, because $k < g$.

Hence there exists a $\theta'_0 \in U(0_0) \cap C\tilde{A}_1$ such that $\{\theta'_0\}^2$ has a closed trajectory $\gamma$ which is homologously independent of $\{\tilde{\gamma}_j\}_{j=1}^k$. Thus if we take this $\theta'_0$ instead of $\theta_0$ from the beginning, the number $k$ increases by at least 1. Repeating this argument until we have $k = g$, then we reduce the proof to Case (1). Hence Theorem is completely
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proved. q.e.d.

Remark. \( \theta = \sum_{i=1}^{g} c_i \theta A_i (\in P(A_i)) \) is proportional to a holomorphic reproducing differential if and only if there exist a real constant \( c \) and integers \( q_i (i=1, \ldots, g) \) such that \( c_i = c q_i \) for every \( i \). Hence if \( g \geq 2 \), \( C \bar{A}_1 \) contains a differential which is not proportional to any holomorphic reproducing differential, and the condition of Lemma 1 is not necessary.

§ 5. Theorems on contractions

Let \( T_g \) be a Teichmüller space of compact Riemann surfaces of genus \( g (>1) \), \( d(\cdot, \cdot) \) be the Teichmüller distance on \( T_g \). It is well-known that the metric space \( T_g \) is homeomorphic to the unit ball \( B = \{ (x_j): \sum_{j=1}^{g} x_j^2 < 1 \} \), where the metric on \( B \) is the usual Euclidean metric. (cf. [2], [3])

We recall about the homeomorphism from \( B \) to \( T_g \). Let \( R_0 \in T_g \) be fixed and \( \{ \varphi_j \}_{j=1}^{g} \) be a base of the real vector space \( \bar{A}_2(R_0) \). (We will use \( R_0, R \), etc. both as a Riemann surface and as a point of \( T_g \).) Then to each point \( (x_j) \in B \) we assign the point of \( T_g \) associated with the Beltrami coefficient \( \mu(z) = \left\{ \sum x_j^2 \right\}^{1/2} \frac{\sum x_j \bar{\varphi}_j}{|\sum x_j \varphi_j|} \) (with the base point \( R_0 \)).

Now we define

\[ U(R_0) = \{ \Sigma x_j \varphi_j; (x_j) \in B \}(\subset \bar{A}_2(R_0)) \]

and for \( \varphi = \Sigma x_j \varphi_j \in U(R_0) \), let \( R_\varphi \) be the point of \( T_g \) associated with the Beltrami coefficient

\[ \mu(z) = \left\{ \sum x_j^2 \right\}^{1/2} \frac{\sum x_j \bar{\varphi}_j}{|\sum x_j \varphi_j|}. \]

Here we remark that on account of the homeomorphism from \( B \) to \( T_g \), “\( \varphi_n \) converges to \( \varphi \) with respect to the \( \| \| -\text{norm} \)” is equivalent to “\( R_{\varphi_n} \) converges to \( R_\varphi \) in \( T_g \)”.

For an arbitrary \( R \in T_g \), the Teichmüller mapping from \( R_0 \) to \( R \)
is uniquely determined, hence we denote its Beltrami coefficient and the maximal dilatation by $B_{R_0}(R)$ and $D_{R_0}(R)$ respectively. Then the following theorem is well-known. (cf. [2], [3], [12])

**Theorem I.** There exists a $\varphi \in U(R_0)$ and $k$ $(1 > k \geq 0)$ such that

$$B_{R_0}(R) = k \frac{\overline{\varphi}}{|\varphi|}, \quad D_{R_0}(R) = K \frac{1+k}{1-k} \geq 1$$

and $d(R_0, R) = \log D_{R_0}(R)$.

Here $\varphi = \Sigma x_j \varphi_j$ is uniquely determined by the condition $\{\Sigma x_j^2\}^{1/2} = k$, and is denoted by $Q_{R_0}(R)$.

**Definition** (cf. [11]). Let $\varphi \in C\overline{A}_2(R_0), \{R_i\}$ be the characteristic ring domains $R(\varphi)$ of $\varphi, \{\gamma_i\}$ be the admissible curve system induced by $R(\varphi)$, and $\tilde{M}$ be the moduli vector of $\varphi$. If $\{\gamma'_i\}$ is an admissible curve system on $R$ corresponding to $\{\gamma_i\}$ by the homeomorphism $F: R_0 \rightarrow R$ in $T^{\varphi}$, there exists a $\varphi' \in C\overline{A}_2(R)$ whose characteristic ring domains $R(\varphi')$ belongs to $\{\gamma'_i\}$, and whose moduli vector $\tilde{M}'$ satisfies the equation $\tilde{M}' = C\tilde{M}$ for a suitable $C > 0$. (cf. Theorem A.)

We call this constant the *contraction of $\varphi$ from $R_0$ to $R$* and denote it by $C_R(\varphi)$. And $\varphi'$ is uniquely determined up to a positive constant factor, and is denoted by $I_R(\varphi)$.

**Theorem II.** ([8]) If $D_{R_0}(R) = K$, then for every $\varphi \in C\overline{A}_2(R_0)$

$$\frac{1}{K} \leq C_R(\varphi) \leq K, \quad \text{and}$$

$$C_R(\varphi) = \frac{1}{K} \quad \text{if and only if} \quad B_{R_0}(R) = k \frac{\overline{\varphi}}{|\varphi|}$$

$$C_R(\varphi) = K \quad \text{if and only if} \quad B_{R_0}(R) = k \frac{-\overline{\varphi}}{|\varphi|}$$

Now as an application of Theorem 1, we have the following

**Theorem 3.** If $D_{R_0}(R) = K$, then
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\[ K = \sup_{\varphi \in \mathcal{A}_2(R_0)} C_R(\varphi) \quad \frac{1}{K} = \inf_{\varphi \in \mathcal{A}_2(R_0)} C_R(\varphi) \]

**Proof.** Let \( Q_{R_0}(R) = \varphi_0 \in U(R_0) \) and \( k = \frac{K-1}{K+1} \). Note that for an arbitrary \( \varphi \in \mathcal{A}_2(R_0) \) there exists a suitable positive constant \( C \) such that \( C\varphi = \sum \varphi_j \varphi_j \in U(R_0) \) and \( \{\sum \varphi_j^2\}^{\frac{1}{2}} = k \). Hence in the sequel we consider only elements satisfying these conditions.

First let \( \varphi \in \mathcal{A}_2(R_0) \) and \( \tilde{\varphi} = I_R(\varphi) \), then

\[ C_{R,\varphi}(\varphi) = C_R(\varphi) \cdot C_{R,\varphi}(\tilde{\varphi}) \]

and by Theorem II, \( C_{R,\varphi}(\varphi) = \frac{1}{K} \). Now we take a sequence \( \{\varphi_n\} \subset \mathcal{A}_2(R_0) \) such that \( \varphi_n \) converges to \( \varphi_0 \) with respect to the \( \| \| \) norm. Then \( R_{\varphi_n} \) converges to \( R_{\varphi_0} = R \) in \( T_\varphi \). Let \( \tilde{\varphi}_n = I_R(\varphi_n) \), then

\[ C_{R,\varphi_n}(\varphi_n) = \frac{1}{K} = C_R(\varphi_n) \cdot C_{R,\varphi_n}(\tilde{\varphi}_n) \]

and \( \lim_{n \to \infty} C_{R,\varphi_n}(\tilde{\varphi}_n) = 1 \), because \( \lim_{n \to \infty} D_{R_0}(R_{\varphi_n}) = 1 \) (cf. Theorem II). Hence

\[ \lim_{n \to \infty} C_R(\varphi_n) = \frac{1}{K} \]. By Theorem II, we have

\[ \inf_{\varphi \in \mathcal{A}_2(R_0)} C_R(\varphi) = \frac{1}{K} \].

Similarly if we consider a sequence \( \{\psi_n\} \subset \mathcal{A}_2(R_0) \) which converges to \( -\varphi_0 \), then we can prove that

\[ \sup_{\varphi \in \mathcal{A}_2(R_0)} C_R(\varphi) = K \].

**Corollary 3.** If \( C_R(\varphi) \geq 1 \) for every \( \varphi \in \mathcal{A}_2(R_0) \), then \( R \) is conformally equivalent to \( R_0 \).

**Proof.** Since \( 1 \geq \inf C_R(\varphi) = \frac{1}{D_{R_0}(R)} \geq 1 \), we have

\[ D_{R_0}(R) = 1 \], i.e. \( R_0 = R \) in \( T_\varphi \). q.e.d.

**Remark.** From the proof of Theorem 3 for every \( \varepsilon > 0 \) there
exist neighbourhoods $U_\varepsilon$ and $V_\varepsilon$ of $\varphi_0$ and $-\varphi_0$ respectively in $\tilde{A}_2(R_0)$ such that
\[
C_R(\varphi') \leq \frac{1}{K} + \varepsilon \quad \text{for every } \varphi' \in U_\varepsilon \cap C\tilde{A}_2(R_0).
\]
\[
C_R(\psi') \geq K - \varepsilon \quad \text{for every } \psi' \in V_\varepsilon \cap C\tilde{A}_2(R_0).
\]

Lemma 8. (cf. [6]). Let $R_0$, $R$, $R'$ be points of $T_\theta$ and
\[
B_{R_0}(R) = k \frac{\bar{\varphi}}{\varphi}, \quad B_{R_0}(R') = k \frac{\bar{\psi}}{\psi}
\]
with $\varphi = Q_{R_0}(R)$, and $\psi = Q_{R_0}(R')$. If $d(R, R') = 2d(R_0, R)$, then
\[
\psi = -\varphi.
\]

Proof. Let $f, g$ be the Teichmüller mappings from $R_0$ to $R$ and to $R'$ respectively. Let $\mu = B_{R_0}(R)$, $\tau = B_{R_0}(R')$, then the quasiconformal mapping $h = g \circ f^{-1}$ from $R$ to $R'$ has the Beltrami coefficient
\[
\rho = \left\{ \frac{\tau - \mu}{1 - \tau \bar{\mu}} \cdot \frac{f_z}{(f_z)} \right\} \circ f^{-1},
\]
which can be shown by elementary calculation. And from $|\tau| = |\mu| = k$ we have $|\rho| = \rho |\tau| |f| = \left| \frac{\tau - \mu}{1 - \tau \bar{\mu}} \right| \leq \frac{2k}{1 + k^2}$, where the equality can hold only the points on $R_0$ where $\tau = -\mu$ holds. Hence
\[
d(R', R) \leq \sup \log \frac{1 + |\rho|}{1 - |\rho|} \leq 2 \log \frac{1 + k}{1 - k} = 2d(R_0, R).
\]
From our assumption we have therefore $|\rho| = \frac{2k}{1 + k^2}$, i.e. $\tau = -\mu$. Thus we can conclude that $\varphi = -\psi$. q.e.d.

Theorem 4. Let $B_{R_0}(R) = k \frac{\varphi_0}{|\varphi_0|}$ with $\varphi_0 = Q_{R_0}(R)$. If a sequence $\{\varphi_n\}$ in $C\tilde{A}_2(R_0)$ satisfies the condition that
\[
\lim_{n \to \infty} C_R(\varphi_n) = \frac{1}{K},
\]
then \( \varphi_n \) converges to \( \varphi_0 \) with respect to \( \| \| \)-norm. (Note that \( \varphi_n \) are normalized as in the proof of Theorem 3.)

**Proof.** By taking a subsequence, we may assume that \( \varphi_n \) converges. Let \( \varphi' = \lim_{n \to \infty} \varphi_n \). Note that for \( \varphi \in C\mathcal{A}_2(R_0) \) and \( \tilde{\varphi} = I_R(\varphi) \)

\[
C_R(\varphi) \cdot C_{R_0}(\tilde{\varphi}) = 1.
\]

Let \( \phi_n = I_{R_0}(\varphi_n) \), then

\[
C_R(\phi_n) = C_{R_0}(\phi_n) \cdot C_R(\varphi_n)
\]

By the previous remark for every \( \varepsilon > 0 \) there exists an \( n(\varepsilon) \) such that

\[
C_{R_0}(\phi_n) \geq K - \varepsilon \quad \text{and} \quad C_R(\varphi_n) \leq \frac{1}{K} + \varepsilon \quad \text{for every } n \geq n(\varepsilon).
\]

Thus we have \( \lim_{n \to \infty} C_R(\phi_n) \leq \frac{1}{K^2} \). But

\[
d(R, R \varphi) \leq d(R, R_0) + d(R_0, R \varphi') = 2 \log K.
\]

Hence by Theorem 3 we have \( d(R, R \varphi) = 2 \log K \). Then \( R = R \varphi \), by Lemma 8, thus \( \lim_{n \to \infty} \varphi_n = \varphi_0 \). q.e.d.

**Example.** For given \( \gamma \) the moduli vector of \( \theta_1^2 \) is not always an interior point of the moduli surface \( \mathcal{M}(\gamma_i) \) of the same admissible curve system \( \gamma_i \) when \( R \) varies in \( T_\gamma \). For example we consider a point of \( T_2 \), which is represented by \( w^2 = (z^2 - 4) \prod_{i=1}^{4} (z - z_i) \) where each \( z_i \) is a complex number. It can be considered as a two sheeted covering surface of the \( z \)-plane branched at \( \{ -2, 2, z_i \ (1 \leq i \leq 4) \} \). Cutting it by closed curve lying over non-intersecting homotopically fixed arcs \( \{ L_i \}_{i=1}^{4} \) on the \( z \)-plane such that \( L_1 = (-\infty, 2), L_3 = (2, +\infty) \) on the real axis and \( L_2, L_4 \) are the arcs joining \( z_1 \) and \( z_2, z_3 \) and \( z_4 \).
respectively, we have then two regions $D^+, D^-$ which are called the upper sheet and the lower sheet respectively. Now we choose a closed Jordan curve on $D^+$ separating $\{L_4^+, L_4^-\}$ from $\{L_4^u, L_4^\nu\}$ as $\gamma$.

First take $z_1 = -1$, $z_2 = -\frac{1}{2}$, $z_3 = \frac{1}{2}$, $z_4 = 1$, then

$$\theta^2 = \left( \frac{zd\z}{w} \right)^2 = \frac{z^2 dz^2}{(z^2 - \frac{1}{4})(z^2 - 1)(z^2 - 4)}$$

belongs to $CA_2^uD$, and it can be shown that $\theta$ is proportional to $\theta_\gamma$. In this case the characteristic ring domains $R(\theta^2)$ consists of three elements. See the figure 1, where the curves $\gamma'$ denotes the projection onto the $z$-plane of the critical trajectories of $\theta^2$.

Next take $z_1 = 1$, $z_2 = -1$, $z_3 = -i$, $z_4 = i$, then

$$\theta^2 = \left( \frac{zd\z}{w} \right)^2 = \frac{z^2 dz^2}{(z^4 - 1)(z^2 - 4)}$$

belongs to $CA_2^uD$ and $R(\theta^2)$ consists of a single element, so it is obvious that $\theta$ is proportional to $\theta_\gamma$. See the figure 2.
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References


Added in proof. (Nov., 1, 1976) The author found that A. Douady and J. Hubbard had proved that \( C \bar{A}^2 \) is dense in \( \bar{A}^2 \) (Invent. Math. 30
175–179, 1975). This is exactly the holomorphic case of our Theorem 1, so it is contained in our result. The proofs are quite different and our proof is rather constructive.