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Complete intersections

By

N. MOHAN KUMAR

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In [2], D. Ferrand has given some characterisation of a reduced scheme X which is a local complete intersection, in terms of $\Omega_X{}^1$ [the sheaf of 1-differentials]. In the global affine case, Murthy and Towber [3] have proved that a smooth affine curve over an algebraically closed field is a complete intersection in any embedding of it in an affine space if only and if the module of 1-differentials of the curve is trivial. It is not known whether there exist any intrinsic properties of an affine shceme, which will determine whether it is a complete intersection in any embedding of it in an affine space. Here we prove the following:

Let R be a finite type k-algebra which is a domain, where k is any field and the quotient field of R is separable over k. Then R is a complete intersection in some embedding of it in an affine space over k if and only if the module of 1-differentials, $\Omega^1_{R/k}$, has a free resolution of length $\leqslant 1$. We also prove that when R is smooth over k, for embeddings in large dimensional affine spaces it is a complete intersection, if it is so in some embedding. As a corollary we deduce that the conormal bundle of a local complete intersection in any embedding, is a complete intersection in some embedding. Finally we give examples of smooth affine varieties which have trivial canonical line bundles, but not a complete intersection in any embedding of it in affine space, thereby settling a question of M. P. Murthy [6].

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We will first prove an elementary lemma which is the key lemma.

Lemma. *Let R be a commutative ring with unity and I a finitely generated ideal of R. Let I/I^2 be generated by r elements as an R/I-module. Let F be any element of R. Then the ideal $(I, F)\subset R$ is generated by $r+1$ elements.*

Proof. Let $a_1, \cdots, a_r$ be elements of I such that their residues mod I^2 generate I/I^2. So in the ring $R/(a_1, \cdots, a_r)R$, the ideal $\bar{I}=I/(a_1, \cdots, a_r)R$ has the property that $\bar{I}/\bar{I}^2=0$. i.e. $\bar{I}=\bar{I}^2$. Since $\bar{I}$ is finitely generated, we see that $\bar{I}$ is generated by an idempotent. Let $h\in I$ be any lift of this idempotent in $\bar{I}$.

Thus we see that $I=(a_1, \cdots, a_r, h)R$ and $h(1-h)$ is in $(a_1, \cdots a_r)R$. So $(I, F) =(a_1, \cdots a_r, h, F)R$. We claim that the ideal, $J=(a_1, \cdots a_r, h+(1-h)F) \subset(I, F)$ is actually equal to (I, F).

By multiplying $h+(1-h)F$ by h, and since $h(1-h)$ belongs to J, we see that $h^2\in J$. Since $h=h^2+h(1-h)$, $h\in J$. Since $F=(h+(1-h)F)+h(F-1)$, $F\in J$. Thus $J=(I, F)$ which proves the claim.

Definition. Let R be a finite type algebra over a field k, which is a domain. (We call such an R, an affine domain over k.) We say that R is an *abstract complete intersection* (hereafter denoted by ACI) if there exists a polynomial ring $k[X_1\cdots X_N]$ over k such that R is a quotient of this polynomial ring with the kernel generated by codim $R=N-\dim R$ elements.

In this paper we would write 'embedding of R in an affine space' to mean 'embedding of Spec R in an affine space'.

Theorem. *Let R be an n-dimensional affine domain over a field k. Assume that the quotient field of R is separable over k.*

1) R is an ACI if and only if $\Omega^1_{R|k}$ has a free resolution of length less than or equal to 1. Furthermore any embedding of an R, which is an ACI, in an affine space such that codim $R\leqslant 2$, is a complete intersection.

2) Assume that R is smooth over k and k is an infinite field. If R is an ACI, then any embedding of R in any N-dimensional affine space is a complete intersection if $N\geqslant 2n+2$.

Proof. 1) If R is a complete intersection in some A^N, and P is the ideal of R in A^N, then P/P^2 is R-free. We have an exact sequence,

$$0\longrightarrow P/P^2\longrightarrow \Omega^1_{A^N|k}\otimes R\longrightarrow \Omega^1_{R|k}\longrightarrow 0.$$

Since $\Omega^1_{A^N|k}\otimes R$ is R-free, we see thet $\Omega^1_{R|k}$ has a free resolution of length$\leqslant 1$.

Conversely assume that $\Omega^1_{R|k}$ has a free resolution of length$\leqslant 1$.

Imbed R in some A^N and let P be its ideal. As before we have an exact sequence,

$$P/P^2\longrightarrow \Omega^1_{A^N|k}\otimes R\longrightarrow \Omega^1_{R|k}\longrightarrow 0.$$

But by [2, Theorem 2, p. 428] we see that if the quotient field of R is separable over k, the first map is an injection and P/P^2 is projective. So we see that, the sequence,

$$0\longrightarrow P/P^2\longrightarrow \Omega^1_{A^N|k}\otimes R\longrightarrow \Omega^1_{R|k}\longrightarrow 0$$

is exact and P/P^2 is projective. This is a projective resolution of $\Omega^1_{R|k}$, since $\Omega^1_{A^N|k}\otimes R$ is free. Since $\Omega^1_{R|k}$ has a free resolution of length$\leqslant 1$, we see that P/P^2 is stably free. Let $P/P^2\oplus R^m\cong R^{N-n+m}$. If we imbed R in A^{N+m}, by embedding A^N in A^{N+m} as a linear subspace, we see that, if I is the ideal of R

in A^{N+m}, $I/I^2 \cong P/P^2 \oplus R^m \cong R^{N-n+m}$. Hence by the lemma, we see that, if we imbed R in A^{N+m+1}, by embedding A^{N+m} as a hyperplane, the ideal of R is generated by $N-n+m+1$ elements, which is the codimension of R in A^{N+m+1}. Thus R is an ACI.

Now assume R is an ACI and assume that it is embedded in A^N where $N \leqslant n+2$. If $N=n$ or $n+1$, clearly R is a complete interesction. So let $N=n+2$. Since R is an ACI, it is a local complete intersection in A^N and $\mathrm{Ext}^2_{\mathcal{O}_{A^N}}(R, \mathcal{O}_{A^N}) \cong R$. So by [5], R is the section of a rank two vector bundle over $\mathcal{O}_{A^N}$. Using [4], we see that R is a complete intersection.

2) Now let R be smooth over an infinite field and let R be an ACI. Let R be a quotient of $k[X_1 \cdots X_N]$, where $N \geqslant 2n+2$. Then one can project R into A^{N-1} isomorphically. So after a change of co-ordinates, we have a map, $k[X_1 \cdots X_{N-1}] \longrightarrow k[X_1 \cdots X_N]$ and if P is the ideal of R in $k[X_1 \cdots X_N]$ and $Q = P \cap k[X_1 \cdots X_{N-1}]$, we see that, the corresponding morphism, $k[X_1 \cdots X_{N-1}]/Q \to k[X_1 \cdots X_N]/P$ is an isomorphism and isomorphic to R. So we get an induced map, $R \to R[X_N]$, and if $\bar{P}$ is the image of P in $R[X_N]$, then the composite, $R \to R[X_N] \to R[X_N]/\bar{P}$ is an isomorphism. Hence $\bar{P} = (X_N - t) \cdot R[X_N]$, where $t \in R$. So we see that,

$$P = (Q, X_N - t) \text{ where } t \in k[X_1 \cdots X_{N-1}].$$

Now since R is smooth, we have a split exact sequence,

$$0 \longrightarrow Q/Q^2 \longrightarrow \Omega^1_{A^{N-1}|k} \otimes R \longrightarrow \Omega^1_{R|k} \longrightarrow 0.$$

Since R is an ACI, $\Omega^1_{R|k}$ is stably free. So Q/Q^2 is stably free. But since rank of $Q/Q^2 = N-n-1 \geqslant n+1$, by [1, Theorem 9.3, p. 28] Q/Q^2 is free of rank $N-n-1$. Hence by the lemma, $P = (Q, X_N - t)$ is $N-n$ generated, i.e. R is a complete intersection in A^N.

Note. If we put $n=1$, the above bounds become, $N \leqslant 3$ or $N \geqslant 4$. Thus every embedding of a smooth ACI curve is a complete interesction. This was proved by Murthy and Towber in [3, Corollary p. 188].

Remark. 1) We see by the above result that a smooth affine variety X over an infinite field k is an ACI if and only if ω_X (the dualising module) is free and $\mu(\Omega^1_{X|k}) \leqslant \dim X + 1$, where $\mu(\Omega^1)$ denotes the minimal number of generators of $\Omega^1_{X|k}$.

2) We also see that any smooth affine variety X in A^n of codimension $\geqslant n/2+1$ is a complete intersection if and only if $\Omega^1_{X|k}$ is stably free (i.e. if P is the ideal of X in A^n, then P/P^2 is free of rank = codimension of X in A^n).

We deduce as a corollary the following result on conormal bundles:

Corollary. *Let R be an affine domain over k and let it be the quotient of a polynomial ring in N variables over k. Let R be a local complete intersection and the quotient field of R be separable over k. Then the conormal bundle of R under this embedding is an ACI.*

Proof. Let P be the ideal of R under this embedding. By our assumptions, we see that the conormal bundle S=symmetric algebra over P/P^2, is a domain and the quotient field of S is separable over k. So to prove that S is an ACI, we only have to show that $\Omega^1_{S|k}$ has a free resolution of length $\leqslant 1$.

We have an exact sequence,

$$0 \longrightarrow P/P^2 \longrightarrow R^N \longrightarrow \Omega^1_{R|k} \longrightarrow 0. \tag{i}$$

with P/P^2 projective.
Again we have an exact sequence

$$0 \longrightarrow \Omega^1_{R|k} \otimes_R S \longrightarrow \Omega^1_{S|k} \longrightarrow \Omega^1_{S|R} \longrightarrow 0.$$

We note that $\Omega^1_{S|R} = P/P^2 \otimes S$. So the above exact sequence becomes,

$$0 \longrightarrow \Omega^1_{R|k} \otimes_R S \longrightarrow \Omega^1_{S|k} \longrightarrow P/P^2 \otimes_R S \longrightarrow 0.$$

Since P/P^2 is R-projective, $P/P^2 \otimes_R S$ is S-projective and hence the above exact sequence splits. Thus we have

$$\Omega^1_{S|k} = (\Omega^1_{R|k} \otimes_R S) \oplus (P/P^2 \otimes_R S) \tag{ii}$$

Tensoring (i) by S which is an R- flat module, we have an exact sequence,

$$0 \longrightarrow P/P^2 \otimes_R S \longrightarrow S^N \longrightarrow \Omega^1_{R|k} \otimes_R S \longrightarrow 0.$$

From this we get an exact sequence,

$$\begin{aligned} 0 \longrightarrow P/P^2 \otimes_R S &\longrightarrow S^N \oplus (P/P^2 \otimes_R S) \\ &\longrightarrow (\Omega^1_{R|k} \otimes_R S) \oplus (P/P^2 \otimes_R S) \longrightarrow 0. \end{aligned}$$

From (ii) we get,

$$0 \longrightarrow P/P^2 \otimes_R S \longrightarrow S^N \oplus (P/P^2 \otimes_R S) \longrightarrow \Omega^1_{S|k} \longrightarrow 0 \text{ is exact.}$$

Let M be any module over R such that $P/P^2 \oplus M \simeq R^l$. ($M$ exists since P/P^2 is R- projective).
Then,

$$0 \longrightarrow (P/P^2 \oplus M) \otimes_R S \longrightarrow S^N \oplus (P/P^2 \oplus M) \otimes_R S \longrightarrow \Omega^1_{S|k} \longrightarrow 0 \text{ is exact}$$

i.e.

$$0 \longrightarrow S^l \longrightarrow S^{N+l} \longrightarrow \Omega^1_{S|k} \longrightarrow 0 \text{ is exact.}$$

Thus by the theorem, S is an ACI.

Remark. The question (a) of M. P. Murthy [6] reads as follows: If Y is a smooth affine sub-variety of $\boldsymbol{A}^N$ over a field k, of dimension d and $\bigwedge^d(\Omega_Y{}^1)$ (the canonical bundle of Y) is trivial, then is Y a complete intersection in $\boldsymbol{A}^N$? We answer this in the negative by the following example.

Let X be a smooth hypersurface in $\mathbf{P}_C^{n-1}$ of degree n. Then $Y=\mathbf{P}^{n-1}-X$ is affine and $\wedge^{n-1}(\Omega_Y^1)$ is isomorphic to $\mathcal{O}_{\mathbf{P}^{n-1}}(-n)|_Y$ and is therefore trivial. However, if n is composite and $n\neq 4$, Ω_Y^1 is not stably trivial, and hence Y is not an ACI.

To prove this, it suffices to show that some chern class of Ω_Y^1 is non-zero. In fact, we show that for any prime q dividing n such that $n\neq 2q$, $c_q(\Omega_Y^1)\in H^{2q}(Y, \mathbf{Z})$ is non-zero. The exact sequence:

$$0 \longrightarrow \Omega^1_{\mathbf{P}^{n-1}} \longrightarrow \mathcal{O}(-1)^n \longrightarrow \mathcal{O} \longrightarrow 0$$

yields $c(\Omega^1_{\mathbf{P}^{n-1}})\cdot c(\mathcal{O})=c(\mathcal{O}(-1)^n)=c(\mathcal{O}(-1)^n)$, where c is the total chern class. Let $t=c_1(\mathcal{O}(-1))\in H^2(\mathbf{P}^{n-1}, \mathbf{Z})$. Since $c(\mathcal{O})=1$, $c(\Omega^1_{\mathbf{P}^{n-1}})=(1+t)^n=\sum_{r=0}^{n}\binom{n}{r}t^r$. It follows that $c_q(\Omega_Y^1)=\binom{n}{q}i^*(t^q)$, where $i: Y\to\mathbf{P}^{n-1}$ is the inclusion. We shall show that $i^*(t)$ has order exactly equal to n in $H^{2q}(Y, \mathbf{Z})$, or equivalently that the image of f in the sequence:

$$\cdots \longrightarrow H^{2q}(\mathbf{P}^{n-1}, Y) \xrightarrow{f} H^{2q}(\mathbf{P}^{n-1}) \xrightarrow{i^*} H^{2q}(Y) \longrightarrow \cdots$$

is equal to $\mathbf{Z}nt^q$.

Let $g: H^{2q-2}(X)\to H^{2q}(\mathbf{P}^{n-1}, Y)$ be the Thom isomorphism and $h: H^{2q-2}(\mathbf{P}^{n-1})\to H^{2q-2}(X)$ be the map induced by the inclusion of X in $\mathbf{P}^{n-1}$. Then h is an isomorphism (because Lefschetz theorem on hyper plane sections states that $H^k(\mathbf{P}^{n-1})\cong H^k(X)$ for $k\leqslant n-3$, and $2q-2\leqslant n-3$; see for instance "Morse Theory" by Milnor, J. W.). Therefore, the image of f=the image of j: $H^{2q-2}(\mathbf{P}^{n-1})\to H^{2q}(\mathbf{P}^n)$, where $j=fgh$. Now, it is well-known that j is given by cupping with the cohomology class that gives the submanifold X of $\mathbf{P}^{n-1}$, which, in this case, is $-nt$, because X is a hypersurface of degree n. Which means that $j(t^{q-1})=-nt^q$.

Now, since n does not divide $\binom{n}{q}$, $\binom{n}{q}i^*(t^q)=c_q(\Omega_Y{}^1)$ is non-zero. Thus Y is not an ACI.

However, we prove that, if n is prime, Ω_Y^1 is stably trivial, and therefore Y is an ACI. This follows from an easy computation in $K^\cdot$: $K^\cdot(\mathbf{P}^{n-1})\approx\mathbf{Z}[t]/(t-1)^n$ where t denotes the class of $\mathcal{O}(-1)$. Let j: $K^\cdot(\mathbf{P}^{n-1})\to K^\cdot(Y)$ be the ring homomorphism induced by the inclusion of Y in $\mathbf{P}^{n-1}$. Put $s=j(t)$, and $s=v+1$. Then $v=j(t-1)$, and therefore v is nilpotent. Also, since $\mathcal{O}(-1)^n|Y$ is trivial, $1=s^n=(1+v)^n=1+nvf(v)$, where f is a polynomial with integer coefficients such that $f(0)=1$. Consequently $f(v)$ is a unit, because v is nilpotent, proving that $nv=0$. We have already seen that $[\Omega^1_{\mathbf{P}^{n-1}}]+[\mathcal{O}_{\mathbf{P}^{n-1}}]=n[\mathcal{O}_{\mathbf{P}^{n-1}}(-1)]$; by restriction to Y, $[\Omega_Y^1]+1=ns=n+nv=n$. But the assertion: $n-1=[\Omega_Y{}^1]$ in $K^\cdot(Y)$ is equivalent to the fact that Ω_Y^1 is stably trivial, since Y is affine. Thus by the theorem, Y is an ACI.

Finally, if $n=4$, Ω^1_Y is stably trivial if and only if there is a curve C of degree $4m+2$, for some m, lying on X. This follows again by appealing to $K^{\bullet}$.

SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
BOMBAY 400 005

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