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Formal fibers and openness of loci

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Introduction

Many loci are Zariski open for a large class of rings (algebro-geometric, analytic, complete, excellent) and such openness of loci is variously related to the good properties of formal fibers. To quote the well known examples, the geometric regularity of formal fibers implies, for a noetherian local ring A, the openness of regular locus for Spec (A'), where A' is any A-algebra of finite type, while the geometric reduceness of fibers carries the openness of normal locus.

The converse arrow is also true for some class of rings: for instance, if A is complete for some $\mathfrak{m}$-adic topology and excellent modulo $\mathfrak{m}$, then the openness of regular locus implies the geometric regularity of formal fibers (see [12], theorem 4).

In the present paper we investigate fibers and loci for a property $\boldsymbol{P}$ meaningful in any noetherian ring, submitted to the following conditions:

1—every field has $\boldsymbol{P}$;

2—$\boldsymbol{P}$ is local;

3—if A is a complete local ring, then the $\boldsymbol{P}$-locus of A is Zariski open;

4—if $(A, \mathfrak{m})\rightarrow(B, \mathfrak{n})$ is a faithfully flat local homomorphism, then $\boldsymbol{P}$ descends from B to A; if moreover $B/\mathfrak{m}B$ has $\boldsymbol{P}$, then $\boldsymbol{P}$ ascends;

5—if A is regular, then A has $\boldsymbol{P}$.

In n. 1. after a short recall on the main properties we need in the paper (Cohen-Macaulay, Gorenstein, complete intersection), we discuss the openness of $\boldsymbol{P}$-locus on a ring A and on finite A-algebras, giving a list of examples.

In n. 2 we discuss the so called "Nagata's criterion for the openness of loci", formally the same as the criterion for the openness of regular locus, but concerning a property $\boldsymbol{P}$ of the type considered above.

We discuss also the following condition, closely related to Nagata's criterion: if a ring A has $\boldsymbol{P}$, then every domain which is a quotient of A contains a non empty open set having $\boldsymbol{P}$.

Using Nagata's criterion and the condition on quotients we can prove the permanence of the openness of $\boldsymbol{P}$-loci under morphisms with good fibers, like completions or henselizations.

In n. 3 we prove the following lifting result, which generalizes [12], theorem 4: if A is separated and complete for some $\mathfrak{m}$-topology and the formal fibers of $A/\mathfrak{m}$ are geometrically $\boldsymbol{P}$ (where $\boldsymbol{P}$ satisfies just 1-5), then the openness of $\boldsymbol{P}$-loci for every A-algebra of finite type implies that the formal fibers of A are also geometrically $\boldsymbol{P}$.

When $\boldsymbol{P}$ satisfies Nagata's criterion and the quotient condition (*e.g.* when $\boldsymbol{P}$=Cohen-Macaulay, Gorenstein or complete intersection) the preceding theorem implies that the good properties of fibers and loci pass to $\mathfrak{m}$-adic completion.

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n. 1

All the rings are assumed to be commutative with 1 and noetherian; our terminology will freely follow [10].

We now shortly recall a few facts and definitions:

1—A local ring A is Cohen-Macaulay (CM) iff $\operatorname{depth}(A)=\dim(A)$; a ring A is CM iff $A_{\mathfrak{Q}}$ is CM for every $\mathfrak{Q}\in\operatorname{Spec}(A)$;

2—A local ring A is Gorenstein (Gor) iff A is CM and there is a system of parameters which generates an irreducible ideal; a ring A is Gor iff $A_{\mathfrak{Q}}$ is Gor for every $\mathfrak{Q}\in\operatorname{Spec}(A)$;

3—A local ring is called (absolute) complete intersection (CI) iff $\hat{A}=$ completion of A is a homomorphic image of a regular local ring modulo a regular sequence (we follow [6], (19. 3. 1); so we do not assume that A is a homomorphic image of a regular local ring); a ring A is CI iff $A_{\mathfrak{Q}}$ is CI for every $\mathfrak{Q}\in\operatorname{Spec}(A)$;

4—Let $(A, \mathfrak{m})\to(B, \mathfrak{n})$ be a faithfully flat local homomorphism; then, if B is regular (CI, Gor, CM) also A is regular (CI, Gor, CM); if A and $B/\mathfrak{m}B$ are regular (CI, Gor, CM) also B is (see: [10], theorem 51 and (21. *C*), corollary 1; [1], theorem 2; [7], proposition 9. 6);

5—The formal fibers of a local ring A are the rings $\hat{A}\otimes_A k(\mathfrak{p})$, where $\hat{A}=$completion of A, $k(\mathfrak{p})=$fraction field of $A/\mathfrak{p}$, $\mathfrak{p}\in\operatorname{Spec}(A)$; the formal fibers of A are the formal fibers of all localizations, if A is any ring;

6—If $\boldsymbol{P}$ is any property meaningful for a ring A (like regularity, CI, Gor, CM,...), we say that the formal fibers of A are geometrically $\boldsymbol{P}$ (shortly: A is a $\boldsymbol{P}$-ring) iff, for every $\mathfrak{Q}\in\mathrm{Spec}(A)$ and every $\mathfrak{p}\in\mathrm{Spec}(A_{\mathfrak{Q}})$, the ring $\hat{A}_{\mathfrak{Q}}\otimes_{A_{\mathfrak{Q}}}k(\mathfrak{p})\otimes_{k(\mathfrak{p})}L$ has $\boldsymbol{P}$, L being any finite extension of $k(\mathfrak{p})$;

7—A ring A is quasi excellent (q. excellent) iff the formal fibers of A are geometrically regular and the regular locus of $\mathrm{Spec}(A')$ is Zariski open, whenever A' is any A-algebra of finite type;

8—We say that a morphism $f: A\to B$ is a $\boldsymbol{P}$-morphism ($\boldsymbol{P}$ being as in 6–) iff it is flat and its fibers are geometrically $\boldsymbol{P}$;

9—Convention: if A is any ring, $\boldsymbol{P}(A)=\boldsymbol{P}$-locus of $A=\{\mathfrak{Q}\in\mathrm{Spec}(A)\,|\,A_{\mathfrak{Q}}$ has the property $\boldsymbol{P}\}$;

10—If A is a complete semilocal ring, then $\boldsymbol{P}(A)$ is Zariski open, whenever $\boldsymbol{P}=$ (i) regularity ([10], theorem 74) ; (ii) CI ([6], (19.3.3)) ; (iii) Gor ([11], theorem 3. 1 or [9], theorem 8) ; (iv) CM ([6], 6. 11. 2)).

We shall from now on consider a property $\boldsymbol{P}$ meaningful for a noetherian ring A and satisfying the following

AXIOMS of $\boldsymbol{P}$:

1—*every field has* $\boldsymbol{P}$;

2—$\boldsymbol{P}$ *is local*;

3—*if* A *is a complete local ring, then* $\boldsymbol{P}(A)$ *is Zariski open*;

4—*if* $(A, \mathfrak{m})\to(B, \mathfrak{n})$ *is a faithfully flat local homomorphism, then* $\boldsymbol{P}$ *descends from* B *to* A; *if both* A *and* $B/\mathfrak{m}B$ *have* $\boldsymbol{P}$, *then* $\boldsymbol{P}$ *ascends from* A *to* B;

5—*if* A *is regular, then* A *has* $\boldsymbol{P}$.

Remark 1: Axioms 1—5 are fulfilled whenever $\boldsymbol{P}$=any of the following properties:

1—regularity; 2– CI; 3– Gor; 4– CM.

On the other hand, properties like normality, reduceness, (R_k), (S_h) are forbidden because of axiom 4, since the property on the fiber over the closed point is not enough to make them ascend.

Remark 2: A property $\boldsymbol{P}$ of the type considerd above passes to polynomial rings ($A\to A[X]$ is a morphism with regular fibers).

Remark 3: Let $f: A\to B$ and $g: B\to C$ be two morphisms; if both f ang g are $\boldsymbol{P}$-morphisms, then also their product $g\circ f$ is a $\boldsymbol{P}$-morphism; if $g\circ f$ is a $\boldsymbol{P}$-morphism and g is faithfully flat, then f is a $\boldsymbol{P}$-morphism.

Remark 4: If A is any $\boldsymbol{P}$-ring then the morphism $A\to B$ is a $\boldsymbol{P}$-

morphism whenever B is any $\mathfrak{m}$-adic completion or henselization, with respect to $\mathfrak{m} \subseteq \mathrm{Rad}(A)$ ([6], (7. 4. 6) and [5], lemma 5. 1; really the morphism into the henselization is even regular, i. e., with geometrically regular fibers).

Remark 5: A is a $\boldsymbol{P}$-ring iff $A_{\mathfrak{M}}$ is a $\boldsymbol{P}$-ring for every maximal ideal $\mathfrak{M}$ ([6], (7. 4. 5)).

Now we introduce and discuss some conditions of openness of loci on a ring A and on finitely generated algebras.

Definition 1: *A ring A is $\boldsymbol{P}-0$ (where $\boldsymbol{P}$ satisfies $1-5$) iff $\boldsymbol{P}(A)$ contains a non empty open set.*

A is $\boldsymbol{P}-1$ iff $\boldsymbol{P}(A)$ is Zariski open (maybe empty).

A is $\boldsymbol{P}-2$ iff every A-algebra of finite type is $\boldsymbol{P}-1$.

Remarks and Examples:

1—Property $\boldsymbol{P}-2$ passes to homomorphic images and localizations, as well as to algebras of finite type;

2—If A is q. excellent and $\boldsymbol{P}$=regularity, then A is $\boldsymbol{P}-2$ ([10], (34. A));

3—If A is a homomorphic image of a regular ring, then A is $\boldsymbol{P}-2$, with $\boldsymbol{P}$=CI ([6], (19. 3. 3));

4—If A is a homomorphic image of a Gorenstein ring of finite Krull dimension, then A is $\boldsymbol{P}-2$, with $\boldsymbol{P}$=Gor ([9], theorem 8 or [11], theorem 3. 1);

5—In [6] (6. 11. 8) it is introduced the following condition: (CMU): Let A be a noetherian ring; for every $\mathfrak{P} \in \mathrm{Spec}(A)$, $\mathrm{Spec}(A/\mathfrak{P})$ contains a non empty open set being CM.

(CMU) implies: a) CM(A) is Zariski open ([6], (6. 11. 8); b) if (CMU) is true for A, it is automatically true also for any $A'=$ A-algebra of finite type; c) finally, if a ring A has (CMU), then it is $\boldsymbol{P}-2$, with $\boldsymbol{P}$=CM.

6—In [6] it is proved that (CMU) is true for a ring A such that $A=$ $=B/\mathfrak{J}$, where B is regular ([6], (6. 11. 2)); moreover there is the implicit conjecture that (CMU) be valid for any noetherian ring. On the contrary in [8] Hochster gives counterexamples to the openness of CM-locus, hence to (CMU) even in dimension 3 and for rings which are locally geometric; other counterxamples can be found in [3].

On the other hand in [9] it is shown that (CMU) is always true for homomorphic images of CM rings ([9], theorem 3).

n. 2

In the present section we want to discuss the consequences on openness of loci produced by Nagata's criterion, when it is assumed to be valid for a property $\boldsymbol{P}$ satisfying axioms $1-5$, $\boldsymbol{P}$ being eventually different from regularity.

Therefore we introduce the following

Definition 2: *A property* $\boldsymbol{P}$ *satisfying axioms* $1-5$ *has Nagata's criterion (shortly*: ***NC****) iff the following theorem is true for* $\boldsymbol{P}$:

Let $X=\operatorname{Spec}(A)$, *where* A *is any noetherian ring; then* $\boldsymbol{P}(A)$ *is open if, for every* $\mathfrak{Q}\in\operatorname{Spec}(A)$, *there is a non empty open set* $\mathfrak{U}$ *of* $\operatorname{Spec}(A/\mathfrak{Q})$ *contained in* $\boldsymbol{P}(A/\mathfrak{Q})$.

Remark 1: ***NC*** is valid and well known when $\boldsymbol{P}=$regularity (see for instance [10], (32. A)); it is a key result to construct the theory of excellent rings.

Recently proofs of ***NC*** have been given also for other properties like:
1—CM ([9], theorem 4);
2—Gor ([4]);
3—CI ([4]).

Remark 2: There are many properties with ***NC***, but not fulfilling axioms $1-5$, like (R_k) ([9], theorem 1) or (S_k) ([9], theorem 6); the results of the present section are generally not valid for them.

NC allows us to state equivalent conditions for $\boldsymbol{P}-2$ quite similar to the well known equivalences for regular loci (the so called property $J-2$ of [10], theorem 73). In fact we have:

Proposition 1: *Let* $\boldsymbol{P}$ *be any property satisfying axioms* $1-5$ *and* ***NC***. *Then the following conditions on a noetherian ring* A *are equivalent*:
1—*A has* $\boldsymbol{P}-2$;
2—*every finite algebra is* $\boldsymbol{P}-1$;
3—*for every* $\mathfrak{Q}\in\operatorname{Spec}(A)$ *and for every* $L=$*finite extension of the fraction field* $k(\mathfrak{Q})$ *of* $A/\mathfrak{Q}$ *there is a finite A-algebra B, containing* $A/\mathfrak{Q}$ *and having L as fraction field, such that B is* $\boldsymbol{P}-0$.

Proof: Enough to show that $3\Longrightarrow1$. Choose $\mathfrak{Q}\in\operatorname{Spec}(A)$, $L=$finite extension of $k(\mathfrak{Q})$ and B as in 3. Then B contains a linear basis of L over $k(\mathfrak{Q})$, say $b_1,\dots,b_n$, and there is an $f\neq0$ in $A/\mathfrak{Q}$ such that $B_f=\sum_{i=1}^{n}(A/\mathfrak{Q})_f b_i=$finite free module. Hence, by axiom 4 on $\boldsymbol{P}$, $A/\mathfrak{Q}$ is $\boldsymbol{P}-0$. Therefore, by ***NC***, A is $\boldsymbol{P}-1$, together with every quotient $A/\mathfrak{P}$, with $\mathfrak{P}\in\mathrm{Spec}(A)$.

Now we pass to consider finitely generated A-algebras; by ***NC*** it is enough to show that, if C is any domain finitely generated as an A-algebra, then C is $\boldsymbol{P}-0$. If $\mathfrak{Q}=\ker(A\to C)$, we can replace A by $A/\mathfrak{Q}$ and assume that A is contained in C. Passing to a suitable open set of $\mathrm{Spec}(A)$, we can assume also that A has $\boldsymbol{P}$. Let now K and L be the fraction fields of A and C respectively. There are two alternatives:

Case $1-L/K$ is separable, hence L has a separating transcendence base over K, say $(t_1,\dots,t_n)$, which can be chosen in C. Put: $A_1=A[t_1,\dots,t_n]$, $K_1=K(t_1,\dots,t_n)$. Then A_1 has $\boldsymbol{P}$ since it is a polynomial ring over a ring having $\boldsymbol{P}$. Replacing A by A_1, we can assume that L/K is separable algebraic; moreover we can choose a linear base of L over K, say $e_1,\dots,e_n$, contained in C and select $f\in A$ such that $C_f=\sum_{i=1}^{n}A_f e_i=$finite free A_f-module. Now replace A by A_f and C by C_f. Since L/K is separable algebraic we have: $d=\det(\mathrm{tr}_{L/K}(e_ie_j))\neq 0$. We claim that C_d has $\boldsymbol{P}$. In fact if $d\notin\mathfrak{p}'\in\mathrm{Spec}(C)$ and $\mathfrak{p}=\mathfrak{p}'\cap A$, then the canonical image of d in $C\otimes k(\mathfrak{p})$ is non zero in $k(\mathfrak{p})$, which means that $C\otimes k(\mathfrak{p})$ is a product of fields; hence $C_{\mathfrak{p}'}/\mathfrak{p}C_{\mathfrak{p}'}$ is a field. But $A_{\mathfrak{p}}\to C_{\mathfrak{p}'}$ is faithfully flat and both $A_{\mathfrak{p}}$ and $C_{\mathfrak{p}'}/\mathfrak{p}C_{\mathfrak{p}'}$ have $\boldsymbol{P}$; so that also $C_{\mathfrak{p}'}$ has $\boldsymbol{P}$, as we had to show.

Case $2-\mathrm{char}(L)=p>0$; then there exists a finite radical extension K_1 of K such that $L_1=L(K_1)$ is separable over K_1. We can choose $A_1\subseteq K_1$ as in 3, so that A_1 is $\boldsymbol{P}-0$ and also $A_1[C]$ is $\boldsymbol{P}-0$ by case 1.
Since $A_1[C]$ is finite over C, C itself is $\boldsymbol{P}-0$ (use axiom 4 on $\boldsymbol{P}$).

Remark 1: Our result is very close to property $J-2$ not only in the formulation, but also in the technique of proof (see [10], theorem 73).

Such a proof is based essentially on the following facts, valid both for regularity and for $\boldsymbol{P}$:

1—$\boldsymbol{P}$ ascends by faithful flatness if the fiber over the closed point has $\boldsymbol{P}$ (hence properties like normality, (S_k),... are excluded; on the other hand we remark that for (R_k) there is a proof of R_k-2 based on Nagata's criterion: [9], theorem 2);

2—$\boldsymbol{P}$ descends by faithful flatness;

3—$\boldsymbol{P}$ passes to polynomials;

4—$\boldsymbol{P}$ has ***NC***: this property allows us to restrict our investigation to *domains*.

Remark 2: In [11] Sharp introduces the class of acceptable rings, quite parallel to excellent rings, but with regularity replaced everywhere by Gorenstein; our proposition 1, using ***NC*** for Gor proved in [4], gives for acceptable rings the until now missing equivalent of property $J-2$.

Now we consider another condition on $\boldsymbol{P}$ concerning quotients and strictly related to Nagata's criterion:

Definition 3: *A property* $\boldsymbol{P}$ *satisfying axioms* $1-5$ *has the quotient condition (shortly*: $\boldsymbol{QC}$*) if the following theorem is true*:
Let A *be a noetherian ring having* $\boldsymbol{P}$ *and* $\mathfrak{P}\in\mathrm{Spec}(A)$; *then* $A/\mathfrak{P}$ *is* $\boldsymbol{P}-0$.

Examples:
1—$\boldsymbol{P}$=CM ([9], theorem 3) ;
2—$\boldsymbol{P}$=Gor ([4]) ;
3—$\boldsymbol{P}$=CI ([4]).

On the other hand we do not know whether regularity has $\boldsymbol{QC}$ or not, at least in char. 0.

Using $\boldsymbol{NC}$ and $\boldsymbol{QC}$ we can give the following permanence theorem for $\boldsymbol{P}-2$.

Theorem 2: *Let* A *be a noetherian ring,* $\boldsymbol{P}$ *a property satisfying axioms* $1-5$, $\boldsymbol{NC}$ *and* $\boldsymbol{QC}$, *and let* f: A→B *be a* $\boldsymbol{P}$*-morphism.*
Then, if A *is a ring with* $\boldsymbol{P}-2$, *also* B *has* $\boldsymbol{P}-2$.

Proof: By $\boldsymbol{NC}$ it is enough to show that, if C is any polynomial ring over B and $\mathfrak{Q}\in\mathrm{Spec}(C)$, then $C/\mathfrak{Q}$ is $\boldsymbol{P}-0$. Let $\mathfrak{q}=\mathfrak{Q}\cap B$ and $\mathfrak{p}=$ $=\mathfrak{Q}\cap A$. Replacing A by $A/\mathfrak{p}$, B by $B/\mathfrak{p}B$, C by $C/\mathfrak{p}C$, we may assume that $\mathfrak{p}=(0)$. By hypothesis there is an $f\neq 0$ in A such that A_f has $\boldsymbol{P}$; therefore also B_f and hence C_f has $\boldsymbol{P}$. But $f\notin\mathfrak{Q}$, since $\mathfrak{Q}\cap A=(0)$; therefore we can conclude, using $\boldsymbol{QC}$, that $C_f/\mathfrak{Q}$ is $\boldsymbol{P}-0$.

Remark: Theorem 2 can be applied when $\boldsymbol{P}$=CM, Gor or CI ; moreover B can be chosen to be any $\mathfrak{m}$-adic completion or henselization of A, with $\mathfrak{m}\subseteq\mathrm{Rad}(A)$ ([6], (7. 4. 6) ; [5], lemma 5. 1). When A is local, B can be chosen to be the strict henselization ${}^{hs}A$ ([6], (18. 8. 12), (ii)).

n. 3

In the present section we state a lifting result for the property of being a $\boldsymbol{P}$-ring, i. e. we lift it from $A/\mathfrak{m}$ to A when A is $\mathfrak{m}$-adically complete ; the result can be refined when $\boldsymbol{P}$ is supposed to have $\boldsymbol{NC}$ and $\boldsymbol{QC}$.

First we need a lemma :

Lemma: *A noetherian local ring A is a* $\boldsymbol{P}$*-ring iff, for every finite A-algebra B which is a domain, and for every* $\mathfrak{Q}\in \mathrm{Spec}(\hat{B})$ *(where* $\hat{B}=$ $=$*completion of B) with* $\mathfrak{Q}\cap B=(0)$*, the local ring* $\hat{B}_{\mathfrak{Q}}$ *has* $\boldsymbol{P}$*.*

Proof: Quite the same as the well known proof valid for $\boldsymbol{P}=$regularity (see, for instance, [10], (33. *E*), lemma 3).

Now we can prove the following

Theorem 3: *Let A be a noetherian ring,* $\mathfrak{m}$ *an ideal and* $\boldsymbol{P}$ *a property satisfying axioms* $1-5$*. Assume that*:
1—*A is* $\mathfrak{m}$*-adically separated and complete*;
2—$A/\mathfrak{m}$ *is a* $\boldsymbol{P}$*-ring*;
3—*A is* $\boldsymbol{P}-2$.
Then also A is a $\boldsymbol{P}$*-ring.*

Proof: By [6], (7. 4. 5), we can restrict our attention to the formal fibers of $A_{\mathfrak{M}}$, where $\mathfrak{M}$ is an arbitrary maximal ideal. By the lemma, it is enough to show that, if D is a domain finite as an $A_{\mathfrak{M}}$-module and $\mathfrak{Q}$ is a prime ideal of $\mathrm{Spec}(\hat{D})$ lying over $(0)\in\mathrm{Spec}(D)$, then $\hat{D}_{\mathfrak{Q}}$ has $\boldsymbol{P}$.

If $\mathfrak{m}D=(0)$, D is a finite $A_{\mathfrak{M}}/\mathfrak{m}A_{\mathfrak{M}}-$module; so, by hypothesis 2, $\hat{D}_{\mathfrak{Q}}$ has $\boldsymbol{P}$. Therefore we can assume that $\mathfrak{m}D\neq(0)$.

Let us now consider the canonical map f: $\mathrm{Spec}(\hat{D})\rightarrow\mathrm{Spec}(D)$ and the set $Y=f^{-1}(\boldsymbol{P}(D))\cap \mathrm{Non}\,\boldsymbol{P}(\hat{D})$, where $\mathrm{Non}\,\boldsymbol{P}(\hat{D})=\mathrm{Spec}(\hat{D})-\boldsymbol{P}(\hat{D})$. We want to show that $Y=\phi$, which will prove our claim.

Assume $Y\neq\phi$. Since A is $\boldsymbol{P}-2$, then $A_{\mathfrak{M}}$ is also $\boldsymbol{P}-2$ and D is $\boldsymbol{P}-1$; so, using axiom 3 on $\boldsymbol{P}$, we see that Y is locally closed and, by [10], (33. *F*), lemma 5, it contains a prime ideal $\mathfrak{P}'\in\mathrm{Spec}(\hat{D})$ such that $\dim(\hat{D}/\mathfrak{P}')\leqq 1$. If $\dim(\hat{D}/\mathfrak{P}')=0$, then $\mathfrak{P}'$ is maximal as well as $\mathfrak{P}=\mathfrak{P}'\cap D$. But $D_{\mathfrak{P}}\rightarrow\hat{D}_{\mathfrak{P}'}$ is faithfully flat and the fiber over the closed point is a field. Therefore, by axiom 4 on $\boldsymbol{P}$, $\hat{D}_{\mathfrak{P}'}$ should have $\boldsymbol{P}$, which contradicts the choice of $\mathfrak{P}'$. Hence $\dim(\hat{D}/\mathfrak{P}')=1$.

We now consider $\mathfrak{m}'=\mathfrak{m}E$, where $E=\hat{D}/\mathfrak{P}'$. There are two alternatives:

1—$\mathfrak{m}'=(0)$; then $\mathfrak{m}\hat{D}\subseteq\mathfrak{P}'$, so that $\mathfrak{m}D\subseteq\mathfrak{P}=\mathfrak{P}'\cap D$. Therefore $D/\mathfrak{P}$ is a finite module over $A_{\mathfrak{M}}/\mathfrak{m}A_{\mathfrak{M}}$ and has formal fibers with property $\boldsymbol{P}$, by hypothesis 2. In particular $\hat{D}_{\mathfrak{P}'}/\mathfrak{P}\hat{D}_{\mathfrak{P}'}$ must have $\boldsymbol{P}$, so that also $\hat{D}_{\mathfrak{P}'}$ has $\boldsymbol{P}$ by axiom 4 on $\boldsymbol{P}$; but this is absurd and, finally, $\mathfrak{m}'$ cannot be (0).

2—$\mathfrak{m}'\neq(0)$; since E is a local domain of dimension 1, $\mathfrak{m}'$ contains a suitable power of $\mathfrak{m}_E=\mathrm{Rad}(E)$, say $\mathfrak{m}_E^r$. Hence we have: $E/\mathfrak{m}'=$ homomorphic image of $\hat{D}/\mathfrak{m}_{\hat{D}}^r=D/\mathfrak{m}_D^r=$finite $A_{\mathfrak{M}}/\mathfrak{M}^r A_{\mathfrak{M}}$-module$=$finite

$A/\mathfrak{M}^r$-module. So $E/\mathfrak{m}E$ is finite over A, which is $\mathfrak{m}$-separated and complete; this means that E is a finite A-module ([10], (28. P), lemma). We have now the following finite inclusions:

$$A/(\mathfrak{P}\cap A)\hookrightarrow D/\mathfrak{P}\hookrightarrow E.$$

Since $A/(\mathfrak{P}\cap A)$ is $\mathfrak{m}$-complete and separated, the same is true for $D/\mathfrak{P}$; but $D/\mathfrak{P}$ is a local domain of dimension 1, so that it is complete as a local ring, i.e. $D/\mathfrak{P}=\hat{D}/\mathfrak{P}\hat{D}$; finally we see that $\mathfrak{P}\hat{D}=\mathfrak{P}'$ and $\hat{D}_{\mathfrak{P}'}/\mathfrak{P}\hat{D}_{\mathfrak{P}'}$ is a field. Therefore $\hat{D}_{\mathfrak{P}'}$ must have $\boldsymbol{P}$ by axiom 4 on $\boldsymbol{P}$.

We get again an absurd and the unique possibility is $Y=\phi$, which proves our claim.

Remark: The theorem is true when $\boldsymbol{P}$= regularity, CI, Gor, CM. In particular, in the case of regularity, we find exactly the result of [12], theorem 4.

Corollary 1: *Let A be a normal local ring of dimension 3, $\mathfrak{m}$-complete and separated for some ideal $\mathfrak{m}$. If $A/\mathfrak{m}$ is a $\boldsymbol{P}$-ring, with $\boldsymbol{P}=CM$, then the same is true for A.*

Proof: A has (CMU), hence it is $\boldsymbol{P}-2$, with $\boldsymbol{P}$=CM. In fact, if $f\neq 0$, $f\in \mathrm{Rad}(A)$, then A_f is a normal domain of dimension 2 and, if $\mathfrak{P}\in \mathrm{Spec}(A)$, $\mathfrak{P}\neq(0)$, then there is $f\in \mathrm{Rad}(A/\mathfrak{P})$ such that $(A/\mathfrak{P})_f$ is a domain of dimension not greater than 1; in any case we get a CM ring localizing at some suitable f.

When $\boldsymbol{P}$ has ***NC*** and ***QC*** we can deduce the following

Corollary 2: *Let A be a noetherian ring and let $\boldsymbol{P}$ satisfy axioms* 1 − 5, ***NC*** *and* ***QC***, *Assume moreover that*:

1—*A has $\boldsymbol{P}-2$*;

2—*A is $\boldsymbol{P}$-ring.*

Then if $\mathfrak{m}\subseteq \mathrm{Rad}(A)$ and $B=(A, \mathfrak{m})^\wedge$, also B satisfies 1 *and* 2.

Proof: By proposition 1, $\boldsymbol{P}-2$ passes to B, since $A\to B$ is a $\boldsymbol{P}$-morphism. Now apply theorem 3.

Remark: Corollary 2 states in particular the following facts:

1—If A is acceptable ([11]), then also $(A, \mathfrak{m})^\wedge$ is acceptable;

2—If A is q. excellent then $(A, \mathfrak{m})^\wedge$ has $\boldsymbol{P}-2$ with $\boldsymbol{P}$=CI, Gor, CM (the claim on fibers in this case is well known: it is easy to see that the fibers of $(A, \mathfrak{m})^\wedge$ are even CI without the machinery of the theorem).

Unfortunately the corollary cannot be employed when $\boldsymbol{P}$=regularity since $\boldsymbol{QC}$ is not known in this case (and $\boldsymbol{NC}$ is not enough).

We remark explicitly that, if $\boldsymbol{QC}$ is valid for some class of regular rings, like, for instance, regular local rings containing a field of characteristic 0, then it gives automatically the passage to completion of the excellent property, within the class considered, by corollary 2.

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