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On the structure of minimal surfaces of general type with $2p_g=(K^2)+2$

By

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Introduction. Let S be a minimal nonsingular projective surface of general type defined over an algebraically closed field k of characteristic 0. We denote by p_g and K_S, respectively the geometric genus and the canonical divisor of S. In a series of papers [5], [6] and [7], Horikawa studied the structure (the number of moduli, the deformation type, etc.) of minimal nonsingular projective surfaces S of general type satisfying the equality: $2p_g=(K^2)+3$ or $2p_g=(K^2)+4$. The surfaces studied by Horikawa are, however, the extreme cases in the sense that if the value of p_g is given, $(K^2)=2p_g-3$ or $2p_g-4$ is the smallest possible value of (K^2) (cf. [3], Theorem 9). In the present article, by employing the methods introduced in [5] and used effectively in [6] and [7], we shall study the structures of minimal nonsingular projective surfaces of general type satisfying the equality $2p_g=(K^2)+2$, of which we shall give a description under several mild restrictions. In the first section of the present article, various results are collected, which we use below frequently and sometimes without specified references. In the second section we prove that the irregularity q vanishes for minimal surfaces of general type with $2p_g=(K^2)+2$. In the third and fourth sections we have to limit ourselves to the case where $|K|$ has no fixed component. This assumption implies that $|K|$ is not composed of a pencil. On the other hand, $|K|$ has at most two base points. In the third section we consider the case where $|K|$ has no base point and $n:=p_g-1\geqslant 3$. Then, morphism $\varphi:=\Phi_{|K|}: S\longrightarrow V\subset \boldsymbol{P}^n$(where $V:=\varphi(S)$) defined by $|K|$ is a morphism of degree 2 except when $n=3$ and deg $\varphi=3$. If one assumes that $n\geqslant 3$ and deg $\varphi=2$ then V is a Del Pezzo surface of degree n; thus $n\leqslant 9$. The construction of minimal surfaces S of this kind is given in Theorem 3. 7. Assuming that the Del Pezzo surface V is nonsingular and the branch locus B_φ is general (see 3. 8), we study the elliptic curves on the surface S (cf. Theorem 3. 9). In the fourth section we consider the case where $|K|$ has base points and $n:=p_g-1\geqslant 3$. Then $|K|$ has exactly two base points. Let $\pi: \tilde{S}\longrightarrow S$ be a composition of blowings-up with centers at the base points of $|K|$ such that the variable part $|L|$ of $|\pi^*K|$ has no base point, and let $\varphi:=\Phi_{|L|}: \tilde{S}\longrightarrow V\subset \boldsymbol{P}^n$ (where $V:=\varphi(\tilde{S})$) be the morphism defined by $|L|$. Then, deg $\varphi=2$ and V is an irreducible surface of degree $n-1$ in $\boldsymbol{P}^n$ studied by Nagata [10]. Now, using the structure theorem on V and employing the methods from [5], we can describe the structures and constructions of minimal surfaces S of this kind under an additional assumption that $\tilde{S}$ is the canonical resolution (cf. 1. 3) of the double covering of V with branch locus B_φ(cf. Theorem 4. 15).

The notations and the terminology which we use below are as follows: k is an algebraically closed field of characteristic 0, which we fix throughout the paper; every surface considered below are projective surfaces unless otherwise mentioned. Let S be a nonsingular projective surface and let D be a divisor on S. Then $|D|$ denotes the complete linear system defined by D. If $x_1, \ldots, x_r$ are points on S and if $m_1, \ldots, m_r$ are positive integers, $|D|-\sum m_ix_i$ is the linear subsystem of $|D|$ consisting of members of $|D|$ which pass through x_i's with multiplicity $\geqslant m_i$. If every member of $|D|-\sum m_ix_i$ passes through some points among x_i's with multiplicities greater than the assigned ones, or passes through new points other than the assigned base points, we say that $|D|-\sum m_ix_i$ has *accidental* base points. Let $f: S\longrightarrow V$ be a morphism of finite degree. Then, for an irreducible curve C on S we denote by $f(C)$ the set-theoretic image; for a divisor D on S we denote by $f_*(D)$ the direct image as a cycle; for an irreducible curve C' on V we denote by $f^{-1}(C')$ the set-theoretic inverse image; for a divisor D' on V we denote by $f^*(D')$ the inverse image as a cycle; if f is birational and if A is an irreducible curve on V, $f'(A)$ denotes the proper transform of A by f. The other notations are as follows:

p_g (or $p_g(S)$) : the geometric genus of S,
q (or q_S) : the irregularity of S,
K (or K_S, or $K(S)$) : the canonical divisor of S,
$\chi(\mathcal{O}_S)$ (or $\chi(S, \mathcal{O}_S)$) : the Euler-Poincaré characteristic of S,
e.g. $(D\cdot D')$ (or (D^2)) : the intersection number of D and D' (or D with itself),
$D\sim D'$: D is linearly equivalent to D',

$D \approx D'$: D is algebraically equivalent to D',
$\mathcal{O}(D)$: the invertible sheaf associated with D,
$p_a(D)$: the arithmetic genus of D,
[ ] : the Gauss symbol.

§ 1. Preliminaries

In this section we shall summarize various results which we frequently use below.

1. 1. Lemma (Bombieri [3]). *Let S be a minimal surface of general type. We have then the following:*

(1) *Assume that $|K|$ is not composed of a pencil. If $|K| = |C| + X$ with a fixed part X we have*

$$p_g \leqslant \frac{1}{2}(K^2) + 2 - \frac{1}{2}q - \frac{1}{2}(K \cdot X) - \frac{1}{4}(C \cdot X),$$

and $(C \cdot X) \geqslant 2$ if $X > 0$.

(2) *If $q \geqslant 2$ and if $|K|$ is composed of a pencil plus a fixed part we have*

$$p_g \leqslant \frac{1}{2}(K^2)$$

provided that (K^2) is even.

(3) *If $q \geqslant 1$ then $\chi(\mathcal{O}_S) \leqslant \frac{1}{2}(K^2)$; if $q = 1$ we have $p_g \leqslant \frac{1}{2}(K^2)$.*

(4) *If $q = 0$ and S has a torsion group of order m then we have*

$$p_g \leqslant \frac{1}{2}(K^2) + \frac{3}{m} - 1.$$

1. 2. Lemma (Horikawa [5]). *Let S be a minimal surface of general type with $p_g \geqslant 3$ such that $|K|$ is not composed of a pencil. Let $\pi: \tilde{S} \longrightarrow S$ be a composition of quadric transformations such that the variable part $|L|$ of $|\pi^* K|$ has no base point. Then we have $2p_g - 4 \leqslant (L^2) \leqslant (K^2)$. Moreover,*

(i) *if $(L^2) = (K^2)$ then $|K|$ has no base point,*

(ii) *if $(L^2) = 2p_g - 4$ then any general member of $|L|$ is a hyperelliptic curve.*

1. 3. The results of this paragraph are mainly due to Horikawa [5], [6]. Let $f: S \longrightarrow W$ be a surjective morphism of degree 2 between nonsingular algebraic surfaces. Assume that there is no exceptional curve of the first kind on S which is mapped to a point by f. Let R (or R_f) be the ramification divisor of f. Then $R \sim K_S - f^*(K_W)$; R is a sum (as cycles) of irreducible curves C on S such that either $f(C)$ is a point or $f^*(f(C)) = 2C + F$ with $F \geqslant 0$. Since $\deg f = 2$, any component C of R such that $f(C)$ is a curve has coefficient 1. Define the branch locus B (or B_f) by $B = f_*(R_f)$. Then B is a reduced divisor on W and $f^*B - 2R$ is a non-negative divisor. If there is a divisor F on W such that $B \sim 2F$ and if there is a non-negative divisor Z on S such that $f^*B - 2R = 2Z$ and $R + Z \in |f^*F|$ then f factors through the double covering of W with branch locus B, $f' : S' \longrightarrow W$ (see [5], p. 48 for the construction of $f' : S' \longrightarrow W$), which is the normalization of W in $k(S)$. Moreover, the condition that $Z = 0$ is equivalent to one of the following:

1) S' has at most rational double points as its singularities,

2) B has no singular point of multiplicity $\geqslant 4$; every triple point w of B (if any) decomposes into a singularity of multiplicity $\leqslant 2$ after a quadric transformation with center at w.

When the condition 2) is satisfied, we say that B has *no infinitely near triple point.*

Conversely, let B be a reduced (effective) divisor on W such that $B \sim 2F$ for some divisor F on W. Then we can construct explicitly the double covering $f' : S' \longrightarrow W$ with branch locus B (cf. [5], p. 48). If B is nonsingular then S' is nonsingular too, and the canonical divisor K_S is given as $f'^*(K_W + F)$. If B has a singular point w_1 of multiplicity m_1, let $q_1 : W_1 \longrightarrow W$ be a quadric transformation with center at w_1. Set $B_1 = q_1^*(B) - 2\left[\frac{m_1}{2}\right]E_1$ and $F_1 = q_1^*(F) - \left[\frac{m_1}{2}\right]E_1$, where $E_1 = q_1^{-1}(w_1)$ and $\left[\frac{m_1}{2}\right]$ is the greatest integer not more than $\frac{m_1}{2}$. Then $B_1 \sim 2F_1$, and we can construct the double covering $f_1' : S_1' \longrightarrow W_1$ with branch locus B_1. Moreover, there exists a birational morphism $p_1 : S_1' \longrightarrow S'$ such that $f' \cdot p_1 = q_1 \cdot f_1'$. If B_1 is not nonsingular we repeat the above process for S_1'. After a finite number of these processes we have the following commutative diagram,

$$\begin{array}{ccccccccccc}
S^* & = & S_n' & \xrightarrow{p_n} & S_{n-1}' & \xrightarrow{p_{n-1}} & \cdots\cdots & \xrightarrow{p_2} & S_1' & \xrightarrow{p_1} & S' \\
\downarrow f^* & & \downarrow f_n' & & \downarrow f_{n-1}' & & & & \downarrow f_1' & & \downarrow f' \\
W^* & = & W_n & \xrightarrow{q_n} & W_{n-1} & \xrightarrow{q_{n-1}} & \cdots\cdots & \xrightarrow{q_2} & W_1 & \xrightarrow{q_1} & W
\end{array}$$

where $q_i : W_i \longrightarrow W_{i-1}$ is a quadric transformation with center at a singular point w_i of B_{i-1} with multiplicity $m_i > 1$ for $i = 1, \ldots, n$, and

$B^*=B_n$ is nonsingular. We call S^* *the canonical resolution of* S'. The numerical characters of S^* are given as follows:

Lemma. *With the above notations we have*

$$\chi(S^*, \mathcal{O}_{S^*})=\frac{1}{2}(F\cdot K_W+F)+2\chi(W, \mathcal{O}_W)-\frac{1}{2}\sum_i\left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right]-1\right)$$

$$(K_{S^*}^2)=2((K_W+F)^2)-2\sum_i\left(\left[\frac{m_i}{2}\right]-1\right)^2.$$

Moreover, if $p_g(W)=q_W=0$ *and if* B *has no infinitely near triple point then we have*:

$$p_g(S^*)=\dim \mathrm{H}^0(W, \mathcal{O}(K_W+F)),$$
$$q_{S^*}=\dim \mathrm{H}^1(W, \mathcal{O}(K_W+F)),$$
$$(K_{S^*}^2)=2((K_W+F)^2).$$

1. 4. Lemma (Castelnuovo [4]). *Let* C *be an irreducible* (not necessarily nonsingular) *curve of degree* d *in the projective* n-*space* $\boldsymbol{P}^n$, *but not in any hyperplane. Let* χ *be the smallest integer not less than* $(d-n)/(n-1)$. *Then the* (geometric) *genus* $g(C)$ *of* C *is equal to or less than*

$$\chi\{d-n-\frac{1}{2}(n-1)(\chi-1)\}.$$

1. 5. Let Σ_d denote the Hirzebruch surface of degree d; Σ_d is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ which has a section M such that $(M^2)=-d$. We denote by l a fibre of the projection $\Sigma_d \longrightarrow \boldsymbol{P}^1$. We make use of the following three lemmas:

1. 5. 1. Lemma (Nagata [10]). *Let* V *be an irreducible surface of degree* $n-1$ *in* $\boldsymbol{P}^n$, *but not in any hyperplane. Then* V *is one of the followings*:

(i) $n=2$ *and* $V=\boldsymbol{P}^2$;

(ii) $n=5$ *and* $V=\boldsymbol{P}^2$ *embedded in* $\boldsymbol{P}^5$ *by* $|2H|$ *where* H *denotes a line on* $\boldsymbol{P}^2$;

(iii) $n=3, 4, \ldots$, $V=\Sigma_d$ *where* $n-d-3$ *is a nonnegative even integer*; V *is embedded into* $\boldsymbol{P}^n$ *by* $|M+\frac{(n-1+d)}{2}l|$;

(iv) $n=3, 4, \ldots$, *and* V *is a cone over a rational curve of degree* $n-1$ *in* $\boldsymbol{P}^{n-1}$.

1. 5. 2. Lemma (Nagata, ibid.) *Let* V *be an irreducible surface of degree* n *in* $\boldsymbol{P}^n$, *but not in any hyperplane. Then* V *is one of the followings*:

(i) *A projection of one in Lemma* 1. 5. 1 *with center outside of the* *surface* V;

(ii) *The system* L *of hyperplane sections of* V *is represented on* $\boldsymbol{P}^2$ *as a system of cubic curves with at most* 6 *base points and whose general members are nonsingular cubic curves*;

(iii) $n=8$ *and* $V=\Sigma_0$; *the Veronese transform of* Σ_0 *in* $\boldsymbol{P}^3$;

(iv) $n=8$ *and* V *is biregular to a cone in* $\boldsymbol{P}^3$ *with a nonsingular plane conic as a base curve*; *the Veronese transform of the cone*;

(v) V *is a cone with a nonsingular elliptic base curve.*

The surface V *is normal if* V *is not of type* (i).

1. 5. 3. Lemma (Nagata, ibid.). *Let* $P_1, \ldots, P_s (0 \leqslant s \leqslant 6)$ *be points such that* $\mathrm{dil}_{(P_1,\ldots,P_s)}$ (see [10] for the notation) *is well-defined on* $\boldsymbol{P}^2$. *Then the system* L^* *of cubic curves on* $\boldsymbol{P}^2$ *with pre-assigned base points* $P_1, \ldots, P_s$ *represents such an* L *as in Lemma* 1. 5. 2, (ii), *if and only if the* P_i *satisfies the following two conditions*:

(i) *Any four points among the* P_i *are not collinear.*

(ii) *For each* j, $\mathrm{dil}_{P_j} P_j$ *carries at most one of the* P_i.

1. 6. Lemma (Hurwitz's formula). *Let* $f: S \longrightarrow W$ *be as in* 1. 3. *Let* C *be an effective divisor, and let* $D=f^*(C)$. *Then we have*:

$$2(p_a(D)-1)=4(p_a(C)-1)+(B \cdot C).$$

1. 7. Lemma. *Let* r *and* s *be nonnegative integers. Then we have*:

$$\dim \mathrm{H}^0(\Sigma_d, \mathcal{O}(rM+sl))=\begin{cases}(r+1)(s+1)-\frac{1}{2}r(r+1)d & \text{if } s \geqslant rd \\ (a+1)(s+1)-\frac{1}{2}a(a+1)d & \text{if } s<rd,\end{cases}$$

$$\dim \mathrm{H}^1(\Sigma_d, \mathcal{O}(rM+sl))=\begin{cases}0 & \text{if } s \geqslant rd \\ (r-a)\{\frac{d}{2}(r+a+1)-(s+1)\} & \text{if } s<rd,\end{cases}$$

where $a=\left[\frac{s}{d}\right]$ (the Gauss symbol). *Moreover, we have the following*:

(1) $|M+nl|$ *is very ample if* $n>d$.

(2) *Let* $\rho:=\Phi_{|M+dl|}: \Sigma_d \longrightarrow V \subset \boldsymbol{P}^{d+1}$. *Then* ρ *is a morphism, and* $V=\rho(\Sigma_d)$ *is a cone over a nonsingular rational curve of degree* d *in* $\boldsymbol{P}^d$, *whose vertex is* $\rho(M)$.

The proof of 1. 6 is easy; the proof of the first assertion of 1. 7 is tedious but standard; the remaining assertions of 1. 7 are well-known and not hard to show. We omit the proofs of 1. 6 and 1. 7.

§ 2. Vanishing of the irregularity

2. 1. Let S be a minimal surface of general type defined over k such that $2p_g=(K^2)+2$. Set $p_g=n+1$ and $(K^2)=2n$, where $n\geqslant 1$. In this section we shall prove

Theorem. *The irregularity q of S is zero.*

The case where $n=1$ was proved by Bombieri ([3], Theorem 12). Hence we shall *assume* below that $n\geqslant 2$. The proof consists in showing in several steps that the assumption $q>0$ leads us to a contradiction. *We assume that* $q>0$.

2. 2. Lemma. *The following assertions hold:*

(1) $q=2$.

(2) $|K|$ *is not composed of a pencil.*

(3) $|K|$ *has no fixed component.*

(4) $|K|$ *has at most two base points, one of which is possibly an infinitely near point.*

Proof. If $q=1$ then $p_g\leqslant\frac{1}{2}(K^2)$, which contradicts our assumption (cf. 1. 1, (3)). Hence $q\geqslant 2$. If $|K|$ is composed of a pencil plus a fixed part then $p_g\leqslant\frac{1}{2}(K^2)$, which is again contradictory (cf. 1. 1, (2)). Hence $|K|$ is not composed of a pencil plus a fixed part. If $|K|=|C|+X$ with the fixed part $X>0$, then we have

$$n+1\leqslant n+2-1-\frac{1}{2}(K\cdot X)-\frac{1}{2}=n+\frac{1}{2}-\frac{1}{2}(K\cdot X)$$

which follows from 1. 1, (1). Since $(K\cdot X)\geqslant 0$ (cf. [3], Prop. 1), this contradicts our assumption. Thus, $|K|$ has no fixed component, and we have

$$p_g\leqslant\frac{1}{2}(K^2)+2-\frac{1}{2}q$$

from which follows that $q=2$. The last assertion follows from 1. 2.

Q. E. D.

2. 3. Let l be a prime number, and let $f\colon \hat{S}\longrightarrow S$ be a nontrivial cyclic covering with group Z_l (=the cyclic group of order l). Such a nontrivial covering exists because $q>0$ and $H^1_{et}(S, Z_l)\cong \mathrm{Pic}(S)_l$ (=the group of l-torsion elements). Then the surface $\hat{S}$ satisfies the condition of the following:

Lemma. *Let $\hat{S}$ be as above. Then $\hat{S}$ is a minimal surface of general type with $p_g(\hat{S})=ln+1$, $q_{\hat{S}}=2$ and $(K_{\hat{S}}^2)=2ln$.*

Proof. It is well-known that $\hat{S}$ is a minimal (nonsingular) surface satisfying $\chi(\hat{S}, \mathcal{O}_{\hat{S}})=l\chi(S, \mathcal{O}_S)=ln$ and $(K_{\hat{S}}^2)=l(K_S^2)=2ln$. Moreover, $f^*|K_S|\subset|f^*K_S|=|K_{\hat{S}}|$ and $q_{\hat{S}}\geqslant 2$. Since $p_g(\hat{S})\geqslant 3$ we know that $\hat{S}$ is a surface of general type (cf. [3], Theorem 1). Since $f^*|K_S|$ is a linear subsystem of $|K_{\hat{S}}|$ and since $|K_S|$ has no fixed component, $|K_{\hat{S}}|$ has no fixed component. Similarly, since $|K_S|$ is not composed of a pencil, $|K_{\hat{S}}|$ is not composed of a pencil. Then we have,

$$p_g(\hat{S})\leqslant\frac{1}{2}(K_{\hat{S}}^2)+2-\frac{1}{2}q_{\hat{S}}$$

whence follows that

$$ln+q_{\hat{S}}-1\leqslant ln+2-\frac{1}{2}q_{\hat{S}}.$$

Hence $q_S=2$ becaue $q_{\hat{S}}\geqslant 2$, and $p_g(\hat{S})=ln+1$. Q. E. D.

2. 4. In the paragraphs 2. 4∼2. 8 we assume that $|K|$ has no base point. We let $\varphi:=\Phi_{|K|}: S\longrightarrow V\subset \boldsymbol{P}^n$ denote the morphism defined by $|K|$ with $V=\varphi(S)$.

Lemma. *Assume that $|K|$ has no base point. Then φ is not birational.*

Proof. Assume that $\deg\varphi=1$. Then $\deg V=2n$. Let H be a general hyperplane of $\boldsymbol{P}^n$ and let $C=H\cdot V$, which is an irreducible curve of degree $2n$ in $\boldsymbol{P}^{n-1}$ but not in any hyperplane of $\boldsymbol{P}^{n-1}$. We apply 1. 4 to the curve C when p$\geqslant$5. Then

$$g(C)\leqslant 2\{2n-(n-1)-\frac{1}{2}(n-2)\}=n+4,$$

while $g(C)=p_a(K)=2n+1$ because C is birational to a general member of $|K|$, which is a nonsingular irreducible curve. This is a contradiction if $n\geqslant 5$. Assume that $n=4$. Then V is an irreducible surface of degree 8 in $\boldsymbol{P}^4$. Hence there exists at most one quadric hypersurface of $\boldsymbol{P}^4$ containing V. This implies that if $\{s_0, s_1, \ldots, s_4\}$ is a basis of $H^0(S, \mathcal{O}(K))$ then there is at most one linear (dependence) relation among s_is_j's $(i, j=0, \ldots, 4)$. Hence, the bigenus P_2 of S is

$$P_2\geqslant\binom{6}{2}-1=14,$$

while $P_2=(K^2)+\chi(\mathcal{O}_S)=8+(1-2+5)=12$ (cf. [3], Cor., p. 185). This is a contradiction. Assume that $n=3$. Then V is an irreducible surface of degree 6 in $\boldsymbol{P}^3$. Since a quadric hypersurface of $\boldsymbol{P}^3$ is rationnal, there is no quadric hypersurface of $\boldsymbol{P}^3$ containing V. Hence,

$$P_2 \geqslant \binom{5}{2}=10,$$

while $P_2=(K^2)+\chi(\mathcal{O}_S)=9$. This is a contradiction. If $n=2$, it is clear that φ is not birational. Q. E. D.

Lemma. *Assume that* $|K|$ *has no base point. Then we have one of the following cases*:

(i) $n \geqslant 4$, deg $\varphi=2$ *and* deg $V=n$; V *is either a normal rational surface or an elliptic cone.*

(ii) $n=3$, *either* deg $\varphi=2$ *and* deg $V=3$ *or* deg $\varphi=3$ *and* deg $V=2$; V *is either a normal rational surface or an elliptic cone.*

(iii) $n=2$, deg $\varphi=4$ *and* $V=\boldsymbol{P}^2$.

Proof. Note that deg $\varphi\cdot$deg $V=2n$ and deg $\varphi \geqslant 2$. On the other hand, deg $V \geqslant n-1$, for, if otherwise, V would be contained in a hyperplane of $\boldsymbol{P}^n$. Hence, if $n \geqslant 4$ we have deg $\varphi=2$ and deg $V=n$; if $n=3$, either deg $\varphi=2$ and deg $V=3$ or deg $\varphi=3$ and deg $V=2$; if $n=2$ we have deg $\varphi=4$ and $V=\boldsymbol{P}^2$. If deg $\varphi=2$, V is an irreducible surface of degree n in $\boldsymbol{P}^n$ but not in any hyperplane. Note that the case (i) of 1. 5. 2 does not occur in the present situation because $\varphi^* L$ is not a complete linear system, where L is the system of hyperplanes of $\boldsymbol{P}^n$. Hence, if deg $\varphi=2$ then V is either a normal rational surface or an elliptic cone by virtue of 1. 5. 2. If $n=3$ and deg $V=2$, V is isomorphic to either Σ_0 or a quadric cone in $\boldsymbol{P}^3$. Hence, V is normal. Q. E. D.

2. 5. In order to derive a contradiction from the assumption that $q_S>0$ we may assume by virtue of 2. 3 that $n \geqslant 4$. Then we have the following

Lemma. *Assume that* S *satisfies the conditions*:

(i) $q_S>0$ *and* $p_g=n+1 \geqslant 5$,

(ii) $|K|$ *has no base point,*

(iii) V *is a normal rational surface.*

Then we have a contradiction.

Proof. Our proof consists of six steps.

(I) Let l be a sufficiently large prime number, and let $f\colon \hat{S} \longrightarrow S$ be a nontrivial cyclic covering of S with group $\boldsymbol{Z}_l$. Let $\hat{\varphi}:=\Phi_{|K_{\hat{S}}|}\colon \hat{S} \longrightarrow \hat{V}$

$\subset \boldsymbol{P}^{ln}$ be the morphism defined by $|K_{\hat{S}}|$, where $\hat{V}=\hat{\varphi}(\hat{S})$ (cf. 2. 3). (Here note that $|K_{\hat{S}}|$ has no base point because $f^*|K_S| \subset |K_{\hat{S}}|$ and $|K_S|$ has no base point.) Then deg $\hat{\varphi}=2$, deg $\hat{V}=ln$ and $\hat{V}$ is a normal surface. On the other hand, the group $\boldsymbol{Z}_l$ acts on $\hat{V}$, and $\hat{\varphi}$ commutes with the actions of $\boldsymbol{Z}_l$ on $\hat{S}$ and $\hat{V}$, because the action of $\boldsymbol{Z}_l$ on $\hat{S}$ induces a linear representation on $H^0(\hat{S}, \mathcal{O}(K_{\hat{S}}))$; the action of $\boldsymbol{Z}_l$ on $\hat{V}$ is non-trivial because deg $\hat{\varphi}=2$. We have thus the following commutative diagram

$$\begin{array}{ccccc} \hat{S} & \xrightarrow{\hat{\varphi}} & \hat{V} & \hookrightarrow & \boldsymbol{P}^{ln} \\ \downarrow f & & \downarrow h & & \\ S & \xrightarrow{\varphi} & V & \hookrightarrow & \boldsymbol{P}^{n} \end{array}$$

where h is a projection corresponding to the inclusion $f^*|K_S| \subset |K_{\hat{S}}|$, and where both f and h kill the actions of $\boldsymbol{Z}_l$. Noting that $[k(\hat{V}) : k(V)]=l$, we know that $k(\hat{V})$ is a Galois extension of $k(V)$ with group $\boldsymbol{Z}_l$.

(II) We claim that h is a finite morphism. In fact, assume that an irreducible curve Z on $\hat{V}$ is mapped to a point on V by h. Let $\hat{Z}=\hat{\varphi}^{-1}(Z)$, and let $\hat{E}$ be an irreducible component of $\hat{Z}$. Then $\hat{E}$ is disjoint from a general member of $f^*|K_S|$, and hence, $(\hat{E}\cdot K_{\hat{S}})=0$. This implies that $\hat{\varphi}(\hat{E})$ (hence $\hat{\varphi}(\hat{Z})$) is a point on $\hat{V}$. This is a contradiction. Thus, h is a finite morphism.

(III) We claim that h is unramified at a point of V of codimension 1. In fact, let C be an irreducible curve on V, and let $\mathfrak{o}=\mathcal{O}_{C,V}$. Let $\tilde{\mathfrak{o}}$ be the normalization of $\mathfrak{o}$ in $k(\hat{V})$; then $\tilde{\mathfrak{o}}=\bigcap_{1\leqslant i\leqslant t}\mathcal{O}_{\hat{C}_i,\hat{V}}$ if $\mathrm{Supp}(h^{-1}(C))=\bigcup_{1\leqslant i\leqslant t}\hat{C}_i$, $\hat{C}_i$ being an irreducible component of $h^{-1}(C)$. Let $\hat{\mathfrak{o}}_i=\mathcal{O}_{\hat{C}_i,\hat{V}}$ for $1\leqslant i\leqslant t$, and let e_i and μ_i be respectively the ramification index and the residue field degree of $\hat{\mathfrak{o}}_i$ over $\mathfrak{o}$. Since $k(\hat{V})$ is a Galois extenison of $k(V)$, the Galois group $\boldsymbol{Z}_l$ acts transitively on $\{\hat{\mathfrak{o}}_1, \ldots, \hat{\mathfrak{o}}_t\}$, whence $e_1=\ldots=e_t(:=e)$, $\mu_1=\ldots=\mu_t(:=\mu)$ and $te\mu=l$. If either $t=l$ or $\mu=l$ then h is unramified at C. If $e=l$ (hence $t=\mu=1$), each place $\mathfrak{p}$ of $k(\hat{S})$ dominating C is easily seen to have the ramification index at least l over $\mathfrak{o}$ on the one hand and at most 2 over $\mathfrak{o}$ on the other hand. This is a contradiction because l is sufficiently large.

(IV) Let $U=V-\mathrm{Sing}(V)$. Then $\Gamma(U, \mathcal{O}_U^*)=k^*$ and $H^1_{et}(U, \boldsymbol{Z}_l)\cong \mathrm{Pic}(U)_l$. In fact, since V is a normal surface and hence, Sing (V) consists of finitely many points, we know that $\Gamma(U, \mathcal{O}_U^*)=k^*$. Then, the isomorphism $H^1_{et}(U, \boldsymbol{Z}_l)\cong \mathrm{Pic}(U)_l$ follows from the exact sequence :

$$0 \longrightarrow \boldsymbol{Z}_l \longrightarrow \Gamma(U, \mathcal{O}_U^*) \xrightarrow{\times l} \Gamma(U, \mathcal{O}_U^*) \longrightarrow H^1_{et}(U, \boldsymbol{Z}_l)$$
$$\longrightarrow \mathrm{Pic}(U) \xrightarrow{\times l} \mathrm{Pic}(U).$$

(V) $\mathrm{Pic}(U)$ is a finitely generated abelian group; hence $\mathrm{Pic}(U)_{tor}$ is a finite group. To show this assertion, note that U is embedded as a dense open set into a nonsingular projective rational surface Y, and that the restriction map, res: $\mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(U)$ is surjective. Since $\mathrm{Pic}(Y)$ is a finitely generated free abelian group, $\mathrm{Pic}(U)$ is a finitely generated abelian group.
(VI) We may assume that $l > |\mathrm{Pic}(U)_{tor}|$. Then $\mathrm{Pic}(U)_l = (0)$. Hence, there is no nontrivial cyclic covering of U with group $\boldsymbol{Z}_l$. However, if we set $\hat{U} = h^{-1}(U)$, $h|_{\hat{U}} : \hat{U} \longrightarrow U$ is a nontrivial étale finite covering with group $\boldsymbol{Z}_l$ as we observed in the steps (II) and (III). ($h|_{\hat{U}}$ is unramified everywhere on U by purity of branch loci.) This is a contradiction. Q. E. D.

2. 6. The following lemma together with Lemma 2. 5 shows that the assumption that $q_S > 0$ and $|K_S|$ has no base point leads us to a contradiction.

Lemma. *Assume that S satisfies the conditions:*
(i) $q_S > 0$, $p_g \geq 4$ *and* $|K_S|$ *has no base point,*
(ii) V *is an elliptic cone.*
Then we have a contradiction.

2. 7. In order to prove the above lemma we need the following auxiliary

Lemma. *Let $C \subset \boldsymbol{P}^{n-1}$ be the base curve of the elliptic cone V, which is a nonsingular elliptic curve, not in any hyperplane of $\boldsymbol{P}^{n-1}$, and let $\delta = C \cdot H'$ be a hyperplane section of C, and let $\Sigma := \mathrm{Proj}(\mathcal{O}_C \oplus \mathcal{O}_C(\delta))$. Then Σ is the minimal resolution of singularities of V; that is, there exists a birational morphism $q : \Sigma \longrightarrow V$ which is the contraction of the section Z to the vertex of V, where $(Z^2) = -n$ and $(Z \cdot l) = 1$ for any fibre l of the projection $p : \Sigma \longrightarrow C$. Moreover, there exists a morphism $\psi : S \longrightarrow \Sigma$ such that $\varphi = q \cdot \psi$.*

Proof. Taking a hyperplane H' of $\boldsymbol{P}^{n-1}$ to be general, we may assume that δ is a sum of distinct n points $P_1, \ldots, P_n$ on C. Then Σ is obtained from the direct product $\boldsymbol{P}^1 \times C$ by performing elementary transformations at the points $P_1, \ldots, P_n$ on the section $C_\infty = (\infty) \times C$, which is identified with C, (cf. Maruyama [9], Prop. 4. 1). The proper transform of C_∞ is the section Z. Hence, $(Z^2) = -n$, and $K_\Sigma \sim -2Z - p^{-1}(\delta) \approx$

$-2Z-nl$. Moreover, it is not hard to see that dim $|Z+p^{-1}(\delta)|=n$ and that $|Z+p^{-1}(\delta)|$ has no base point. Then, the morphism $\Phi_{|Z+p^{-1}(\delta)|}$: $\Sigma \longrightarrow \boldsymbol{P}^n$ gives us the morphism $q: \Sigma \longrightarrow V$ which contracts the section Z.

To prove the second assertion we may assume that the vertex Q of V is the point $(1, 0, \ldots, 0)$ and the base curve C is contained in the hyperplane $X_0=0$. Let L be the linear system of hyperplanes of $\boldsymbol{P}^n$ through Q, and let $\tilde{L}=\varphi^*(L)$. L is then a linear subsystem of $|K|$. Choose a basis $\{x_0, x_1, \ldots, x_n\}$ of $H^0(S, \mathcal{O}_S(K))$ such that $\{x_1, \ldots, x_n\}$ spans the module of $\tilde{L}$. Let G be the fixed part of $\tilde{L}$, and let $\bar{L}=\tilde{L}-G$. Then it is not hard to show:

(i) $\bar{L}$ is composed of a pencil Λ parametrized by C,

(ii) $(x_i)=\sum_{j=1}^n D_{ij}+G$ for $1 \leqslant i \leqslant n$, where $D_{ij} \in \Lambda$ for $1 \leqslant j \leqslant n$,

(iii) $\mathrm{Supp}((x_0)) \cap \mathrm{Supp}(G)=\phi$.

Let D be a general member of Λ. Then $K_S \approx nD+G$. We shall show that $(D^2)=0$, $(D \cdot G)=2$ and $(G^2)=-2n$. In fact, we hav $(G \cdot K)=0$ by virtue of the above condition (iii). If $G=0$, we have $((nD)^2)=2n$, whence $(D^2)=2/n$. This is a contradiction since $n \geqslant 3$. Thus $G>0$. Then $(G^2) \leqslant -2$ by the Hodge index theorem. On the other hand, $(G \cdot K)=0$ implies that $n(D \cdot G)=-(G^2) \geqslant 2$. Moreover, we have:

$$2n=((nD+G)^2)=n^2(D^2)+n(D \cdot G)$$
$$\text{or} \quad 2=n(D^2)+(D \cdot G).$$

Since $(D^2) \geqslant 0$, $(D \cdot G) \geqslant 1$ and $n \geqslant 3$, we must have: $(D^2)=0$ and $(D \cdot G)=2$. Then $(G^2)=-2n$, and Λ has no base point. Then $\rho:=\Phi_L: S \longrightarrow C \subset \boldsymbol{P}^{n-1}$ is everywhere defined, and $\rho^*(\mathcal{O}_C(\delta)) \cong \mathcal{O}_S(\sum_{j=1}^n D_{ij})$ for $1 \leqslant i \leqslant n$. Let τ be a section of $H^0(S, \mathcal{O}_S(G))$ corresponding to G. Then, $(\tau, x_0): \mathcal{O}_S \longrightarrow \mathcal{O}_S(G) \oplus \mathcal{O}_S(K)$ defines a section $\sigma: S \longrightarrow S \times_C \Sigma$ because $\mathrm{Supp}((x_0)) \cap \mathrm{Supp}(G)=\phi$. Let $\psi=p_2 \cdot \sigma: S \longrightarrow \Sigma$, where p_2 is the projection of $S \times_C \Sigma$ on the second factor. Then it is easy to see that $\varphi=q \cdot \psi$ up to an automorphism of $\boldsymbol{P}^n$. Q. E. D.

2. 8. *Proof of Lemma* 2. 6. Let us compute the branch locus B_ψ of the double covering $\psi: S \longrightarrow \Sigma$. For this purpose, note that $\psi^*(l) \approx D \in \Lambda$ and $\psi^*(Z)=G$. Then, the ramification locus R_ψ is given as $R_\psi \sim K_S-\psi^*(K_\Sigma) \approx nD+G+nD+2G=2nD+3G$, whence $B_\psi \approx 4nl+6Z$. On the other hand, since B_ψ is a reduced divisor, we must have: $(B_\psi \cdot Z) \geqslant (Z^2)=-n$. Hence, $4n-6n \geqslant -n$, *i.e.*, $n \leqslant 0$. This is a contradiction.

2. 9. Finally, we consider the case where $|K_S|$ has base points. We have then the following:

Lemma. *Assume that S satisfies the conditions:*

(i) $q_S>0$ *and* $p_g\geqslant 3$,
(ii) $|K_S|$ *has base points.*
Then we have a contradiction.

Proof. Let l be a sufficiently large prime number, and let $f: \hat{S}\longrightarrow S$ be a nontrivial cyclic covering of S with group $\boldsymbol{Z}_l$ (cf. 2. 3). If $|K_{\hat{S}}|$ has no base point we have a contradiction as we saw in the previous argument. If $|K_{\hat{S}}|$ has base points, then $\hat{S}$ should have more than l base points because $\boldsymbol{Z}_l$ acts freely on $\hat{S}$. This contradicts the assertion (4) of 2. 2. Q. E. D.

§ 3. Double Del Pezzo surfaces

3. 1. Let S be a minimal surface of general type such that $p_g=n+1$ and $(K^2)=2n$ with $n\geqslant 3$. In the following, we assume:

(i) $|K|$ *has no fixed component.*

Then, $|K|$ *is not composed of a pencil.* For the proof of this fact, see [3], the first three lines of the proof of Lemma 13. Then, $|K|$ has at most two base points (cf. 1. 2). In this section, we assume more strongly:

(i′) $|K|$ *has no base point.*

Let $\varphi: S\longrightarrow V\subset \boldsymbol{P}^n$ be the morphism defined by $|K|$, where $V=\varphi(S)$. Then, as in Lemma 2. 4, deg $\varphi=2$ and deg $V=n$ *if* $n\geqslant 4$; either deg $\varphi=2$ and deg $V=3$ or deg $\varphi=3$ and deg $V=2$, if $n=3$. We assume that

(ii) deg $\varphi=2$ *and* deg $V=n$.

Then V is an irreducible surface of type (ii), (iii) or (iv) of 1. 5. 2 because $q=0$. (The case (i) of 1. 5. 2 does not occur because φ^*L is not a complete linear system, where L is the system of hyperplanes of $\boldsymbol{P}^n$.) Moreover we have $3\leqslant n\leqslant 9$ by virtue of 1. 5. 2.

3. 2. Lemma. *Let V be an irreducible surface of degree n in $\boldsymbol{P}^n$ satisfying one of the conditions* (ii), (iii) *and* (iv) *of* 1. 5. 2 *Then we have*:

(1) *V is a normal surface having at most rational double points as singularities*

(2) *The canonical divisor K_V of V is linearly equivalent to $-H_V$, where H_V is a hyperplane section of V. Therefore, $-K_V$ is a very ample divisor of V.*

Proof. Since the assertions are clearly true for the surface V of type (iii) or (iv) we only consider the surface V of type (ii). Let

$W=\mathrm{dil}_{(P_1,\dots,P_s)}\boldsymbol{P}^2$, let p: $W\longrightarrow\boldsymbol{P}^2$ be the inverse of $\mathrm{dil}_{(P_1,\dots,P_s)}$ and let q: $W\longrightarrow V$ be the contraction map such that $q\cdot p^{-1}$: $\boldsymbol{P}^2\longrightarrow V$ is the representation of V given in 1. 5. 2. It is easy to see that if C is an irreducible curve on $\boldsymbol{P}^2$ such that the proper transform $p'(C)$ of C is contracted to a point on V by q then C is one of the following:

(1) $s=6$; C is a conic passing through all P_i's,

(2) $s\geqslant 3$; C is a line passing through three of P_i's.

Let Γ be the union of

1) irreducible components of $\mathrm{dil}_{(P_1,\dots,P_s)}(P_1,\dots,P_s)$ with irreducible exceptional curves of the first kind deleted off,

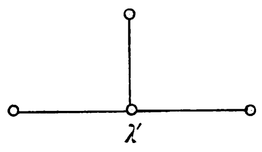
2) the proper transform $C'=p'(C)$ if C is an irreducible conic passing through all P_i's $(s=6)$,

3) the proper transforms $\lambda'=p'(\lambda)$ of lines λ which pass through three of P_i's $(s\geqslant 3)$.

The conditions (i) and (ii) of 1. 5. 3 imply that every irreducible component of Γ is a nonsingular rational curve with self-intersection multiplicity -2, and Lemma 1. 5. 2, (ii) implies that Γ is the union of all irreducible curves on W which are contracted to points by q. Moreover, by virtue of the condition (ii) of 1. 5. 3 and the fact that $s\leqslant 6$, we know that the weighted graph of every connected component of Γ is a linear chain except only when:

$s=6$; three ordinary points P_1, P_2, P_3 lie on a line λ and the other three points P_4, P_5, P_6 are infinitely near to P_1, P_2, P_3 respectively.

The weighted graph of this case is:



Therefore, the contraction q: $W\longrightarrow V$ produces as many rational double points on V as the connected components of Γ. This completes the proof of the first assertion. In order to show the second assertion note that:

a) K_V is a Cartier divisor; $q_*(K_W)\sim K_V$; $q^*(K_V)\sim K_W$; we may take K_W so that $\mathrm{Supp}(K_W)\cap\Gamma=\phi$, (cf. Artin [1], Theorem 2. 7),

b) $-K_W\sim$the proper transform by p of a nonsingular cubic curve on $\boldsymbol{P}^2$ passing through the pre-assigned base points.

Therefore, we have: $K_V\sim -H_V$. Q. E. D.

An irreducible surface V of degree n in $\boldsymbol{P}^n$, which is of type (ii), (iii) or (iv) of 1. 5. 2 is called a *Del Pezzo surface*. A nonsingular projective surface S is called a *double Del Pezzo surface* if there exists a surjective morphism $\varphi: S \longrightarrow V$ of degree 2 onto a Del Pezzo surface V.

3. 3. Let $\varphi: S \longrightarrow V$ be as in 3. 1. The authors do not know whether or not φ factors through $q: W \longrightarrow V$ (cf. 3. 2), that is, whether or not there exist a morphism $\psi: S \longrightarrow W$ such that $\varphi = q \cdot \psi$, except in the following case:

Lemma. *Suppose that V is of type* (iv) *of* 1. 5. 2. *Then there exists a morphism $\psi: S \longrightarrow W$* (=the Hirzebruch surface Σ_2 of degree 2) *such that $\varphi = q \cdot \psi$.*

Proof. The proof is essentially the same as the one in ([5], p. 46), except some minor modifications. Since V is biregular to a quadric cone in $\boldsymbol{P}^3$, there exists a linear system L on S such that:

a) $\dim L = 3$, and $2L \subset |K_S|$,

b) L is generated by 4 elements $(x_0) = 2D + G$, $(x_1) = 2D_1 + G$, $(x_2) = D + D_1 + G$ and (x_3), where D, D_1 and G have no common components, and $\mathrm{Supp}((x_3)) \cap \mathrm{Supp}(G) = \phi$, *esp.*, $(G \cdot K) = 0$.

Since $16 = (K^2) = (4D \cdot 4D + 2G)$, we have:

$$2(D^2) = (D \cdot G) = 2.$$

Since $(D^2) \geqslant 0$ and $(D \cdot G) \geqslant 0$, we have $(D^2) = 0$ or 1. Suppose that $(D^2) = 1$. Then $(D \cdot G) = 0$. By the Hodge index theorem, we have $(G^2) \leqslant 0$, while $2(G^2) = (G \cdot 4D + 2G) = (G \cdot K) = 0$. Thus $G = 0$. This case leads to a contradiction as follows: Since $K \sim 4D$, we have $p_a(D) = (D \cdot D + K)/2 + 1 = 7/2$, which is a contradiction because $p_a(D)$ is an integer. Therefore, $G \neq 0$; and we have $(D^2) = 0$, $(D \cdot G) = 2$ and $(G^2) = -4$. Now, by a similar argument as in ([5], p. 46), we have a morphism $\psi: S \longrightarrow \Sigma_2$ such that $\varphi = q \cdot \psi$. Q. E. D.

In the following paragraphs we assume that

(iii) *there exists a morphism $\psi: S \longrightarrow W$ such that $\varphi = q \cdot \psi$ when V is singular and of type* (ii) *of* 1. 5. 2, *where $q: W \longrightarrow V$ is the morphism given in* 3. 2.

When V is nonsingular we understand that $W = V$ and $\psi = \varphi$. We let H_W denote $q^* H_V$, where H_V is a hyperplane section of V.

3. 4. We denote by R_ψ and B_ψ respectively the ramification locus and the branch locus of $\psi: S \longrightarrow W$. Then we have

Lemma. *With the above notations and assumptions,* $R_\phi \sim 2K_S$ *and* $B_\phi \sim 4H_W$. *Moreover, a general member of* $|4H_W|$ *is a nonsingular irreducible curve of genus* $6n+1$.

Proof. We have $R_\phi \sim K_S - \phi^*(K_W) \sim K_S - \phi^*(q^*(K_V)) = K_S - \varphi^*(K_V) \sim K_S + \varphi^*(H_V) \sim 2K_S$ (cf. 3. 2). Hence $B_\phi = \phi_*(R_\phi) \sim 4H_W$. It is clear that a general member of $|4H_W|$ is a nonsingular irreducible curve. Then $p_a(4H_W) = (4H_W \cdot 3H_W)/2 + 1 = 6n+1$. Q. E. D.

3. 5. Lemma. *S is isomorphic to the canonical resolution* S^* (cf. 1. 3) *of the double covering of W with branch locus* B_ϕ.

Proof. With the notations of 1. 3, we have:

$$\chi(S^*, \mathcal{O}_{S^*}) = n+2-\frac{1}{2}\sum_i \left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right]-1\right),$$

$$(K_{S^*}^2) = 2n - 2\sum_i \left(\left[\frac{m_i}{2}\right]-1\right)^2.$$

Since $\chi(S^*, \mathcal{O}_{S^*}) = \chi(S, \mathcal{O}_S) = n+2$, we know that $\left[\frac{m_i}{2}\right] = 1$ for all indices i. Namely, B_ϕ has no infinitely near triple point. Then $(K_{S^*}^2) = 2n = (K_S^2)$. This implies that the natural birational morphism $\tilde{p}: S^* \longrightarrow S$, whose existence follows from the minimality of S, is an isomorphism. Q. E. D.

3. 6. As for the existence of surfaces S, we have the following:

Lemma. *Let B be a reduced divisor of* $|4H_W|$ *such that B has no infinitely near triple point and* $\mathrm{Supp}(B) \cap \Gamma = \phi$ (cf. 3. 2). *Let S be the canonical resolution of the double covering of W with branch locus B. Then S is a minimal surface of general type with* $p_g = n+1$ *and* $(K_S^2) = 2n$. *Moreover, S satisfies the assumptions* (i′) *and* (ii) *of* 3. 1 *and* (iii) *of* 3. 3.

Proof. Let $\phi: S \longrightarrow W$ be the natural morphism. Then, by virtue of Lemma 1. 3, we have $K_S \sim \phi^*(H_W)$, which implies that S is minimal. Moreover, $p_g(S) = \dim\ H^0(W, \mathcal{O}(H_W)) = \dim\ H^0(V, \mathcal{O}(H_V)) = n+1$; and $(K_S^2) = 2(H_W^2) = 2n$. Thus, S is a minimal surface of general type with $p_g = n+1$ and $(K_S^2) = 2n$. The remaining assertions are clear. Q. E. D.

3. 7. Summarizing the above results, we have

Theorem. *Let S be a minimal surface of general type such that* $p_g = n+1$ *and* $(K^2) = 2n$ *with* $n \geqslant 3$. *Assume that the conditions* (i′), (ii) *of* 3. 1 *and* (iii) *of* 3. 3 *hold. Then we have the following*:

(1) $3\leqslant n\leqslant 9$.

(2) *The surface* V *which is the image of* S *by* $\varphi:=\Phi_{|K|}: S\longrightarrow \boldsymbol{P}^n$, *is a Del Pezzo surface of degree* n.

(3) *According to the condition* (iii) *of* 3. 3, *split* φ *into* $S\xrightarrow{\phi} W\xrightarrow{q} V$, *where* $q: W\longrightarrow V$ *is the smallest blowings-up which resolve the singular points of* V. *Then the branch locus* B_ϕ *is linearly equivalent to* $4H_W$, *where* H_W *is the total transform by* q *of a hyperplane section of* V; B_ϕ *has no infinitely near triple point*; S *is the canonical resolution of the double covering of* W *with branch locus* B_ϕ.

(4) *Conversely, if* B *is a reduced divisor of* $|4H_W|$ *such that* B *has no infinitely near triple point and that* B *does not meet any curve contractible by* q, *the canonical resolution of the double covering of* W *with branch locus* B *is a minimal surface with* $p_g=n+1$ *and* $(K_S^2)=2n$ *satisfying the conditions* (i′), (ii) *of* 3. 1 *and* (iii) *of* 3. 3. *Such a surface exists for every Del Pezzo surface of degree* n *with* $3\leqslant n\leqslant 9$.

3. 8. In the following paragraphs of this section we shall study nonsingular elliptic curves lying on S. For the sake of simplicity we assume that V is nonsingular and that the branch locus $B:=B_\varphi$ is a general member of $|4H_V|$; hence, B is an irreducible nonsingular curve. Then $\varphi: S\longrightarrow V$ is a finite morphism of degree 2. Let C be a nonsingular elliptic curve on S. C is said to be *accidental* if $\varphi_*(C)=\varphi(C)$, and *non-accidental* if otherwise. Then we have the following

Lemma. *With the above notations and assumptions, we have the following*:

(1) *If* C *is a non-accidental elliptic curve on* S *then* $D:=\varphi(C)$ *is a line on* V. *Conversely, if* D *is a line on* V *then* $\varphi^{-1}(D)$ *is a non-accidental elliptic curve.*

(2) *If* C *is an accidental elliptic curve on* S *then* $D:=\varphi(C)$ *is an irredurible curve, whose singular points (if any) are cuspidal singular points centered at the points in* $B\cap D$; $B\cdot D$ *is a divisor on* B *(or* D*) of the form* $2(\sum b_iP_i)$ *with integers* $b_i>0$. *Conversely, if* D *is a nonsingular elliptic curve on* V *such that* $B\cdot D=2(\sum b_iP_i)$ *with* $b_i>0$ *and that* $\varphi^*(D)$ *is of the form* $C+C'$, *then* C *is an accidental elliptic curve.*

Proof. (1) Since B is a nonsingular curve of genus $6n+1$ $(\geqslant 19)$, $D\not\subset \mathrm{Supp}(B)$. Hence $\varphi_*(C)=2D$. Since $(C^2)=-(C\cdot K_S)$, $(C\cdot K_S)=(\varphi^*(D)\cdot\varphi^*(H_V))=2(D\cdot H_V)$ and $(C^2)=2(D^2)$, we have: $(D^2)=-(D\cdot H_V)$. Hence $p_a(D)=\{(D^2)-(D\cdot H_V)\}/2+1=1-(D\cdot H_V)\geqslant 0$, which implies that $(D\cdot H_V)=1$. Thus D is a line on V. Conversely, if D is a line, $(D\cdot B)=4$ because $B\sim 4H_V$, and D and B meet transversally each other because

B is a general member of $|4H_V|$. Hence $C:=\varphi^{-1}(D)$ is a nonsingular elliptic curve which is non-accidental.

(2) Let ι be a generator of $\mathrm{Gal}(k(S)/k(V))\cong \boldsymbol{Z}_2$, which acts on S. Since $\varphi_*(C)=\varphi(C)$ and $\varphi(C)\not\subset \mathrm{Supp}(B)$, we know that $\varphi^*(D)=C+C'$ $(C\neq C')$ with $C'=\iota(C)\cong C$, where $D=\varphi(C)$. It is clear that D is nonsingular outside of $B\cap D$, and that D has at most cuspidal singularity at a point P of $B\cap D$ because there is only one point $\tilde{P}$ above P. Noting that $\varphi^*(B)=2R$ with ramification locus R we have:

$$(D\cdot B)_P(:=i(D,\ B;\ P))=\frac{1}{2}(\varphi^*(D)\cdot\varphi^*(B))_{\tilde{P}}$$
$$=\frac{1}{2}(C+C'\cdot 2R)_{\tilde{P}}=2(C\cdot R)_{\tilde{P}}.$$

Therefore, $D\cdot B$ is a divisor on D of the form $2\sum b_iP_i$ with integers $b_i>0$. Conversely, let D be a nonsingular elliptic curve such that $D\cdot B=2\sum b_iP_i$ with $b_i>0$. Let $P\in D\cap B$, and let x, y be a system of local parameters at P such that

(i) $y=0$ is a local equation of B at P,
(ii) $y=x^{2b}$ is a local equation of D at P.

Let $\mathfrak{o}=k[[x,\ y]]=\hat{\mathcal{O}}_{P,V}$, and let $\tilde{\mathfrak{o}}=\hat{\mathcal{O}}_{\tilde{P},S}$, where $\tilde{P}$ is a unique point of S above P. Then $\tilde{\mathfrak{o}}=k[[t,\ x,\ y]]/(t^2-y)$. Hence we have

$$t^2=x^{2b}+(y-x^{2b})\quad\text{in } \tilde{\mathfrak{o}}.$$

This implies that $\varphi^{-1}(D)$ has two smooth analytic branches $t=x^b$ and $t=-x^b$ at $\tilde{P}$, which intersect each other with multiplicity b. Thus, if $\varphi^*(D)=C+C'$ $(C\neq C')$, both C and C' are nonsingular. On the other hand, we have:

$$(D^2)=(C^2)+(C\cdot C'),\ \ (C\cdot K_S)=(D\cdot H_V)=(D^2),$$
$$(C\cdot C')=\sum_i b_i \text{ and } \sum_i b_i=2(D^2),$$

where the last equality follows from $(D\cdot B)=2\sum_i b_i$, $B\sim 4H_V$ and $(D^2)=(D\cdot H_V)$. Then we have:

$$p_a(C)=(C\cdot C+K_S)/2+1=\{-(D^2)+(D^2)\}/2+1=1.$$

Therefore, C is an accidental elliptic curve on S. Q. E. D.

The authors do not know whether or not there exist accidental elliptic curves on S, under the assumptions that V is nonsingular and that B is a general member of $|4H_V|$.

3. 9. Theorem. *With the same assumptions on V and B as above, the* *number N of non-accidental elliptic curves on S is given as in the following table*:

n	3	4	5	6	7	8	8	9
N	27	16	10	6	3	$V\cong\Sigma_0$ none	$V\cong\Sigma_1$ 1	none

Proof. By virtue of 3. 8, (1), N is equal to the number of lines lying on V. Hence we have the table as above, a part of which is given in Manin [8], p. 136. Q. E. D.

§4. Double coverings of Hirzebruch surfaces

4. 1. Let S be a minimal surface of general type such that $p_g=n+1$ and $(K^2)=2n$ with $n\geqslant 3$. We assume that S satisfies the condition (i) of 3. 1 and that $|K|$ has base points. By virtue of 1. 2, $|K|$ has at most two base points. More precisely, we have the following

Lemma. *With the assumptions as above, $|K|$ has necessarily two base points.*

Proof. Assume that $|K|$ has only one base point P. Let $\pi: \tilde{S}\longrightarrow S$ be the blowing-up with center at P, let $E=\pi^{-1}(P)$ and let $|\pi^*K|=|L|+E$. Then $(L^2)=2n-1$. Let $\varphi:=\Phi_{|L|}: \tilde{S}\longrightarrow V\subset \boldsymbol{P}^n$ be the morphism defined by $|L|$, where $V=\varphi(\tilde{S})$. First of all, we shall show that φ is not birational. In fact, assume that φ is birational. Let C be a general hyperplane section of V. Then C is an irreducible curve of degree $2n-1$ in $\boldsymbol{P}^{n-1}$, but not in any hyperplane of $\boldsymbol{P}^{n-1}$. By virtue of 1. 4, the (geometric) genus $g(C)$ of C satisfies:

$$g(C)\leqslant\begin{cases} n+2 & \text{if } n\geqslant 4 \\ 6 & \text{if } n=3 \end{cases} .$$

However, since C is birational to a general member of $|L|$ which is an irreducible nonsingular curve of genus $p_a(K)=2n+1$, we have a contradiction. Therefore, $\deg\varphi\geqslant 3$ because $\deg\varphi\cdot\deg V=2n-1$. If $n\geqslant 5$ then $(2n-1)/\deg\varphi\leqslant n-2$. This implies that V is contained in a hyperplane of $\boldsymbol{P}^n$ if $n\geqslant 5$, which is a contradiction. If $n=4$ or 3, then $\deg V=1$ because $\deg\varphi\cdot\deg V=2n-1$ is a prime number and $\deg\varphi\geqslant 3$. Hence V is contained in a hyperplane of $\boldsymbol{P}^n$. Thus, we get a contradiction in both cases $n=4$ and $n=3$. Q. E. D.

4. 2. In the following paragraphs we assume that:

(iv) $|K|$ *has two base points.*

Let $\pi: \tilde{S} \longrightarrow S$ be a composition of blowings-up with centers at the base points of $|K|$ such that the variable part $|L|$ has no base points if we write $|\pi^* K| = |L| + X$, where X is the fixed part of $|\pi^* K|$. Then $(L^2) = 2n-2$. Let $\varphi := \Phi_{|L|}: \tilde{S} \longrightarrow V \subset \boldsymbol{P}^n$ be the morphism defined by $|L|$, where $V = \varphi(\tilde{S})$. Let P_1 and P_2 be base points of $|K|$, where P_2 is possibly infinitely near to P_1. Let $\pi_1: \tilde{S}_1 \longrightarrow S$ and $\pi_2: \tilde{S} \longrightarrow \tilde{S}_1$ be the blowings-up with centers at P_1 and P_2 respectively. Then $\pi = \pi_1 \cdot \pi_2$. Let $E_1 = \pi_2'(\pi_1^{-1}(P_1))$ and $E_2 = \pi_2^{-1}(P_2)$. Noting that a general member of $|K|$ is an irreducible nonsingular curve and that, if P_2 is infinitely near to P_1, two general members of $|K|$ meet each other at P_1 with intersection multiplicity 2, it is easy to show that

$$X = \begin{cases} E_1 + E_2 & \text{with } (E_1^2) = (E_2^2) = -1 \text{ and } (L \cdot E_1) = \\ & (L \cdot E_2) = 1 \text{ if } P_2 \text{ is not infinitely near to } P_1, \\ E_1 + 2E_2 & \text{with } (E_1^2) = -2,\ (E_2^2) = -1,\ (L \cdot E_1) = 0 \\ & \text{and } (L \cdot E_2) = 1 \text{ if } P_2 \text{ is infinitely near to } P_1. \end{cases}$$

4. 3. Lemma. *With the notations as above, we have*

$$\deg \varphi = 2 \text{ and } \deg V = n-1.$$

Proof. Assume that φ is birational. Let C be a general hyperplane section of V; then C is an irreducible curve of degree $2n-2$ in $\boldsymbol{P}^{n-1}$, but not in any hyperplane of $\boldsymbol{P}^{n-1}$. By virtue of 1. 4, the (geometric) genus $g(C)$ of C is not larger than n. However, since C is birational to a general member of $|L|$ which is an irreducible nonsingular curve of genus $p_a(K) = 2n+1$, we have a contradiction. Thus, $\deg \varphi \geqslant 2$. We shall show that $\deg \varphi = 2$. In fact, if $\deg \varphi \geqslant 3$ and $n \geqslant 4$ then $\deg V \leqslant n-2$, which is impossible. If $n=3$ then $\deg \varphi \cdot \deg V = 4$. Since $\deg V \neq 1$, we must have: $\deg \varphi = 2$. Q. E. D.

Therefore, V is an irreducible surface of type (ii), (iii) or (iv) of 1. 5. 1.

4. 4. Lemma. *V is not of type* (ii) *of* 1. 5. 1.

Proof. Let us compute the branch locus B_φ of φ. Let λ be a line on $\boldsymbol{P}^2$. Then the ramification locus R_φ of φ is given as follows:

$$R_\varphi \sim K_{\tilde{S}} - \varphi^*(K_V) \sim L + 2X + \varphi^*(3\lambda) \sim 5\varphi^*(\lambda) + 2X.$$

Hence $B_\varphi = \varphi_*(R_\varphi) \sim 10\lambda + 2\varphi_*(X)$. Write $B_\varphi \sim a\lambda$ with an integer a. Since $p_a(2\lambda) = 0$ and $p_a(L) = 2n+1 = 11$, we have by the Hurwitz's formula:

$$2(11-1) = -4 + 2a,$$

whence $a=12$. Therefore, $\varphi_*(X)\sim\lambda$, and hence, $\varphi_*(X)$ is a line on $\boldsymbol{P}^2$. Note that both $\varphi(E_1)$ and $\varphi(E_2)$ are irreducible curves if P_2 is not infinitely near to P_1, and that $\varphi(E_2)$ is an irreducible curve and $\varphi(E_1)$ is a point on $\varphi(E_2)$ if P_2 is infinitely near to P_1, (cf. 4. 2). Thus, we get a contradiction. Q. E. D.

4. 5. In the paragraphs 4. 5~4. 11 we assume that V is of type (iii) of 1. 5. 1. We use the notations of 1. 5. We shall list up all possible cases in the following:

Lemma. *If $V=\Sigma_d$ we have one of the followings:*

(1) $X=E_1+E_2$; $n\geqslant 2d-7$; $R_\varphi\sim 3\varphi^*(M)+\left(\frac{n+3+3d}{2}\right)\varphi^*(l)+2X$; $B_\varphi\sim 6M+(n+7+3d)l$; $\varphi_*(E_1)=l_1$ *and* $\varphi_*(E_2)=l_2$ *with* $l_1\sim l_2\sim l$; *moreover, if* $l_1\neq l_2$ *both* l_1 *and* l_2 *are contained in* $\mathrm{Supp}(B_\varphi)$, *and if* $l_1=l_2$ *we have* $l_1\not\subset\mathrm{Supp}(B_\varphi)$.

(2) $X=E_1+E_2$; $n=d+3$ *with* $0\leqslant d\leqslant 2$; $R_\varphi\sim 3\varphi^*(M)+(2d+3)\varphi^*(l)+2X$; $B_\varphi\sim 8M+(4d+8)l$; *either* $\varphi_*(E_1)=M$ *and* $\varphi_*(E_2)=l_0\sim l$, *or* $\varphi_*(E_1)=l_0\sim l$ *and* $\varphi_*(E_2)=M$; *moreover, both* M *and* l_0 *are contained in* $\mathrm{Supp}(B_\varphi)$.

(3) $X=E_1+E_2$; $d=1$ *and* $n=4$; $R_\varphi\sim 3\varphi^*(M)+5\varphi^*(l)+2X$; $B_\varphi\sim 10M+10l$; $\varphi_*(E_1)=\varphi_*(E_2)=M$ *and* $M\cap\mathrm{Supp}(B_\varphi)=\phi$.

(1′) $X=E_1+2E_2$; $n\geqslant 2d-7$; $R_\varphi\sim 3\varphi^*(M)+\left(\frac{n+3+3d}{2}\right)\varphi^*(l)+2X$; $B_\varphi\sim 6M+(n+7+3d)l$; $\varphi_*(E_2)=l_0\sim l$, *and* $\varphi(E_1)$ *is a point on* l_0; $l_0\subset\mathrm{Supp}(B_\varphi)$.

(3′) $X=E_1+2E_2$; $d=1$ *and* $n=4$; $R_\varphi\sim 3\varphi^*(M)+5\varphi^*(l)+2X$; $B_\varphi\sim 10M+10l$; $\varphi_*(E_2)=M$ *and* $\varphi(E_1)$ *is a point on* M; $M\subset\mathrm{Supp}(B_\varphi)$.

Proof. Since $L\sim\varphi^*\left(M+\frac{n-1+d}{2}l\right)$, $K_S\sim L+2X$ and $K_V\sim -2M-(d+2)l$, we have:

$$R_\varphi\sim 3\varphi^*(M)+\left(\frac{n+3+3d}{2}\right)\varphi^*(l)+2X,$$
$$B_\varphi\sim 6M+(n+3+3d)l+2\varphi_*(X).$$

Writing $B_\varphi\sim aM+bl$ with integers a and b, we shall determine a and b by virtue of the Hurwitz's formula. Since $p_a\left(M+\frac{n-1+d}{2}l\right)=0$ and $p_a(L)=2n+1$ we have:

$$b+\left(\frac{n-1-d}{2}\right)a=4n+4.$$

On the other hand, since $(L\cdot X)=2$ and $|M+dl|$ is a linear system with no base point, we have:

$$(\varphi^*(M)\cdot X)+\frac{n-1+d}{2}(\varphi^*(l)\cdot X)=2$$
$$(\varphi^*(M)\cdot X)+d(\psi^*(l)\cdot X)\geqslant 0,$$

where $(\varphi^*(l)\cdot X)\geqslant 0$. Therefore, we have:

$$\left(\frac{n-1-d}{2}\right)(\varphi^*(l)\cdot X)\leqslant 2,$$

where $2\leqslant(n-1-d)$ because $n-d-3$ is a nonnegative even integer. Here, we consider the cases $X=E_1+E_2$ and $X=E_1+2E_2$ separately.

Case: $X=E_1+E_2$. (1) Assume that $(\varphi^*(l)\cdot X)=0$. Then $p_a(\varphi^*(l))=2$. Hence, applying the Hurwitz's formula to l and $\varphi^*(l)$ we have $a=6$ and $b=n+7+3d$. Then $\varphi_*(X)\sim 2l$, whence follows that $\varphi_*(E_1)=l_1$ and $\varphi_*(E_2)=l_2$ with $l_1\sim l_2\sim l$ because $\varphi(E_1)$ and $\varphi(E_2)$ are irreducible curves. Here, note that if ι is a generator of $\mathrm{Gal}(k(S)/k(V))\cong \boldsymbol{Z}_2$ which acts on S by minimality of S, then either both P_1 and P_2 are fixed by ι, or we have $\iota(P_1)=P_2$. This remark implies that ι acts on $\tilde{S}$, and that either both E_1 and E_2 are fixed by ι, or $\iota(E_1)=E_2$. Hence we know that either $l_1\neq l_2$ and l_1, $l_2\subset \mathrm{Supp}(B_\varphi)$ or $l_1=l_2\not\subset \mathrm{Supp}(B_\varphi)$. Moreover, since B_φ is a reduced divisor, we have $(B_\varphi\cdot M)\geqslant -d$, whence $n\geqslant 2d-7$. (2) Assume that $(\varphi^*(l)\cdot X)=1$. Then $\left(\frac{n-1-d}{2}\right)=1$ or 2, *i. e.*, $n=d+3$ or $n=d+5$, and $p_a(\varphi^*(l))=3$. By the Hurwitz's formula applied to l and $\varphi^*(l)$ we have $a=8$. Assume that $n=d+3$, and hence $b=4d+8$. Then $\varphi_*(X)\sim M+l$, which implies that either $\varphi_*(E_1)=M$ and $\varphi_*(E_2)=l_0\sim l$ or $\varphi_*(E_1)=l_0\sim l$ and $\varphi_*(E_2)=M$, where M, $l_0\subset \mathrm{Supp}(B_\varphi)$ by a similar argument as in (1). Moreover, $(B_\varphi\cdot M)\geqslant -d$, *i. e.*, $0\leqslant d\leqslant 2$. Assume that $n=d+5$. Then, $b=4d+8$ and $\varphi_*(X)\sim M$. This is impossible because $\varphi(X)$ is a reducible curve. (3) Assume that $(\varphi^*(l)\cdot X)=2$. Then we have: $n=d+3$, $p_a(\varphi^*(l))=4$, and hence $a=10$ and $b=4d+6$. Moreover, $\varphi_*(X)\sim 2M$, whence $\varphi_*(E_1)\sim\varphi_*(E_2)\sim M$. Since $(B_\varphi\cdot M)\geqslant -d$, we have: $d=0$ or 1. If $d=1$, we have $\varphi_*(E_1)=\varphi_*(E_2)=M$; since $(B_\varphi\cdot M)=0$ and $M\not\subset \mathrm{Supp}(B_\varphi)$, we have $M\cap \mathrm{Supp}(B_\varphi)=\phi$. If $d=0$ (hence $n=3$), either $\varphi_*(E_1)=l_1'$ and $\varphi_*(E_2)=l_2'$ with $l_1'\neq l_2'$ or $\varphi_*(E_1)=\varphi_*(E_2)=l'$, where $l'(\sim l_1'\sim l_2')$ is a fibre of Σ_0 perpendicular to l; if $l_1'\neq l_2'$ then l_1', $l_2'\subset \mathrm{Supp}(B_\varphi)$; if $\varphi_*(E_1)=\varphi_*(E_2)$ then $l'\not\subset \mathrm{Supp}(B_\varphi)$. This is the case (1) above, where the roles of l and l' are interchanged with each other.

Case: $X=E_1+2E_2$. (1′) Assume that $(\varphi^*(l)\cdot X)=0$. Then we have

$a=6$, $b=n+7+3d$ $(n \geqslant 2d-7)$ and $\varphi_*(X) \sim 2l$ as in the case (1) above. Since $\varphi(E_1)$ is a point on $\varphi(E_2)$, we have $\varphi_*(E_2)=l_0 \sim l$ with $l_0 \subset \mathrm{Supp}(B_\varphi)$. (2′) Assume that $(\varphi^*(l) \cdot X)=1$. Then $n=d+3$ or $n=d+5$, $a=8$ and $b=4d+8$ with $0 \leqslant d \leqslant 2$. If $n=d+3$, we have $\varphi_*(X)=2\varphi_*(E_2) \sim M+l$, which is impossible because $2(\varphi_*(E_2) \cdot l)=(M+l \cdot l)=1$. If $n=d+5$, we have $\varphi_*(X)=2\varphi_*(E_2) \sim M$, which is impossible. (3′) Assume that $(\varphi^*(l) \cdot X)=2$. Then we have $a=10$, $b=4d+6$ ($d=0$ or 1) and $\varphi_*(X)=2\varphi_*(E_2) \sim 2M$. Hence $\varphi_*(E_2) \sim M$. If $d=1$ (hence $n=4$), $\varphi_*(E_2)=M$ and $M \subset \mathrm{Supp}(B_\varphi)$. If $d=0$ (hence $n=3$), $\varphi_*(E_2)=l'$ which is a fibre of Σ_0 perpendicular to l. This is the case (1′) above, where the roles of l and l' are interchanged with each other. Q. E. D.

4. 6. In this paragraph and the next, we shall study the surfaces of type (1) of 4. 5.

Lemma. *Assume that* $\varphi: \hat{S} \longrightarrow V$ *satisfies the conditions* (1) *of* 4. 5. *Then we have*:

(1) *If* $l_1 \neq l_2$ *and if* $\hat{S}$ *is the canonical resolution of the double covering of* V *with branch locus* B_φ, *the surface* S *can be constructed as follows*:

(i) *Let* $q_1: W_1 \longrightarrow V$ *be the blowing-up with centers at* x_1 *and* x_2 *on the fibres* l_1 *and* l_2 *respectively, and let* $q_2: \tilde{W} \longrightarrow W_1$ *be the blowings-up with centers at* y_1 *and* y_2 *on* $q_1'(l_1)$ *and* $q_1'(l_2)$ *respectively, which may be infinitely near to* x_1 *and* x_2. *Let* $q=q_1 \cdot q_2$, *and let* $E_{x_i}=q^{-1}(x_i)$ *and* $E_{y_i}=q_2^{-1}(y_i)$ *for* $i=1, 2$.

(ii) *Let* $\tilde{B}$ *be a reduced divisor of* $|6q^*(M)+(n+7+3d)q^*(l)-4E_{x_1}-4E_{y_1}-4E_{x_2}-4E_{y_2}|$, *which has no infinitely near triple point. Let* $\hat{S}$ *be the canonical resolution of the double covering of* $\tilde{W}$ *with branch locus* $\tilde{B}$. *The proper transforms of* l_1 *and* l_2 *by* q *are necessarily nonsingular components of* $\tilde{B}$, *which give rise to two exceptional curves* E_1 *and* E_2 *of the first kind on* $\tilde{S}$. *Contracting* E_1 *and* E_2 *we get a minimal surface* S *of general type with* $p_g=n+1$, $q=0$ *and* $(K_S^2)=2n$.

(2) *If* $l_1=l_2$, $\hat{S}$ *is not the canonical resolution of the double covering of* V *with branch locus* B_φ.

Proof. Let S^* be the canonical resolution of the double covering of V with branch locus B_φ. By virtue of Lemma 1. 3, we have:

$$n+2=\chi(S^*, \mathcal{O}_{S^*})=n+6-\frac{1}{2}\sum_i\left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right]-1\right),$$

$$(K_{S^*}^2)=2n+6-2\sum_i\left(\left[\frac{m_i}{2}\right]-1\right)^2.$$

Hence, we have $\sum_i\left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right]-1\right)=8$, and we know that one of the following two cases takes place;

1) there are four indices, say $i=1, 2, 3, 4$, such that $\left[\frac{m_i}{2}\right]=2$ for $i=1, 2, 3, 4$ and $\left[\frac{m_i}{2}\right]=1$ for $i\neq 1, 2, 3, 4$;

2) there are two indices, say $i=1, 2$, such that $\left[\frac{m_1}{2}\right]=3$, $\left[\frac{m_2}{2}\right]=2$ and $\left[\frac{m_i}{2}\right]=1$ for $i\neq 1, 2$.

Then $(K_{S^*}^2)=2n-2$ in the first case; $(K_{S^*}^2)=2n-4$ in the second case. On the other hand, it is easily seen that the natural morphism $p: S^*\longrightarrow S$, whose existence follows from the minimality of S, factors through $\tilde{S}$, *i. e.*, $p: S^*\xrightarrow{\tilde{p}}\tilde{S}\xrightarrow{\pi}S$. Thus, $\tilde{S}\cong S^*$ in the first case; $\tilde{p}$ is a composition of two quadric transformations in the second case.

We shall show that $l_1\neq l_2$ in the first case. Let $F:=3M+\left(\frac{n+7+3d}{2}\right)l$, and let $Z:=\varphi^*(l_1)+\varphi^*(l_2)-2X\geqslant 0$. Then it is easy to see that $\varphi^*(B)-2R\sim 2Z$. Since any irreducible component of $\varphi^*(B)-2R$ is a curve contractible to a point by φ and since $\varphi^*(B)-2R\geqslant 0$, we know that $\dim|\varphi^*(B)-2R|=0$. Hence $\varphi^*(B)-2R=2Z$. Since $\tilde{S}$ has no torsion by virtue of 1. 1, (4), we know that $R+Z\in|\varphi^*F|$. Moreover, we have $(E_1\cdot Z)=(E_2\cdot Z)=2$. Let $\tilde{x}_1$ be a point of $E_1\cap\mathrm{Supp}(Z)$, and let $x_1=\varphi(\tilde{x}_1)\in l_1$. Let $\sigma: V_1\longrightarrow V$ be the blowing-up with center at x_1, and let $D=\sigma^{-1}(x_1)$. Then there exists a morphism $\psi: \tilde{S}\longrightarrow V_1$ of degree 2 such that $\varphi=\sigma\cdot\psi$. Since $R_\psi\sim R_\varphi-\psi^*(D)$, and $\psi_*(E_1)$ and $\psi_*(E_2)$ are respectively the proper transforms of l_1 and l_2 by σ, we can easily show that:

$$B_\psi=\begin{cases}6\sigma^*(M)+(n+7+3d)\sigma^*(l)-4D & \text{if } l_1\neq l_2\\ 6\sigma^*(M)+(n+7+3d)\sigma^*(l)-6D & \text{if } l_1=l_2,\end{cases}$$

and hence

$$(B_\psi\cdot D)=\begin{cases}4 & \text{if } l_1\neq l_2\\ 6 & \text{if } l_1=l_2.\end{cases}$$

Let $B_0:=B_\varphi-(l_1+l_2)$ if $l_1\neq l_2$ and $B_0:=B_\varphi$ if $l_1=l_2$. Let μ be the multiplicity of a reduced divisor B_0 at x_1. Then B_ψ is written as:

$$B_\psi=\begin{cases}\sigma'(B_0)+\sigma'(l_1)+\sigma^*(l_2)+\left(\mu+1-2\left[\frac{\mu+1}{2}\right]\right)D & \text{if } l_1\neq l_2\\ \sigma'(B_0)+\left(\mu-2\left[\frac{\mu}{2}\right]\right)D & \text{if } l_1=l_2.\end{cases}$$

Moreover, $\mu\leqslant(l_1\cdot B_0)=6$. Thence, we conclude that $\mu=3$ or 4 if $l_1\neq l_2$ and $\mu=6$ if $l_1=l_2$. However, $\mu=6$ is impossible because $\left[\frac{m_i}{2}\right]\leqslant 2$ for all

i. Therefore, $l_1 \neq l_2$ in the first case. If $l_1 \neq l_2$ and $\tilde{S} \cong S^*$, the same arguments as in ([5], p. 51) leads us to the construction stated as above. Q. E. D.

4. 7. We shall consider when there exists a reduced divisor $\tilde{B}$ having no infinitely near triple point (cf. the case (1) of Lemma 4. 6). Write $\tilde{B} = \tilde{B}_0 + \tilde{l}_1 + \tilde{l}_2$, where $\tilde{l}_1$ and $\tilde{l}_2$ are respectively the proper transforms of l_1 and l_2 by q, and $\tilde{B}_0 \in |6q^*(M) + (n+5+3d)q^*(l) - 3E_{x_1} - 3E_{y_1} - 3E_{x_2} - 3E_{y_2}|$. In a similar fashion as in [7] we can show the following

Lemma. (1) *Assume that* $n \geqslant 3d-5$, *and when* $d > 0$ *assume also that* x_1, y_1, x_2 *and* y_2 *do not lie on* M. *Then the linear system* $|\tilde{B}_0|$ *has no base point. Hence its general members are nonsingular.*

(2) *Assume that* $3d-5 \geqslant n \geqslant 2d-1$ *and that both* x_1 *and* x_2 *are on* M *but neither* y_1 *nor* y_2 *is on* M. *Then any general member of* $|\tilde{B}_0|$ *has no infinitely near triple point and is disjoint from* $\tilde{l}_1$ *and* $\tilde{l}_2$.

(3) *Assume that* $3d-5 > n$ *and that not both of* x_1 *and* x_2 *are on* M *and neither* y_1 *nor* y_2 *is on* M. *Then any divisor of* $|\tilde{B}_0|$ *has multiple components.*

(4) *Assume that* $2d-1 > n$. *Then any divisor of* $|\tilde{B}|$ *has a multiple component.*

Proof. (1) Assume that $d > 0$. We shall show that any general member of the linear system $\Lambda := |2M + 2dl| - (x_1 + y_1 + x_2 + y_2)$ is an irreducible nonsingular curve and Λ has no accidental base point. In fact, since $|M + 2(d-1)l| + M + l_1 + l_2$ is a linear subsystem of Λ, the fixed components of Λ (if any) are possibly M, l_1 and l_2. If M is a fixed component of Λ, then $\dim \Lambda = \dim |M + 2dl| - (x_1 + y_1 + x_2 + y_2) = 3d-3$. [Since $(M + 2dl \cdot l) = 1$, l_1 and l_2 are then fixed components of $|M + 2dl| - (x_1 + y_1 + x_2 + y_2)$. Hence $\dim |M + 2dl| - (x_1 + y_1 + x_2 + y_2) = \dim |M + (2d-2)l| = 3d-3$ (cf. 1. 7)] Hence M is not a fixed component of Λ because $\dim \Lambda \geqslant 3d-2$. Neither l_1 nor l_2 is a fixed component of Λ, for, if otherwise, M is also a fixed component of Λ, which is impossible as shown in the above argument. Therefore Λ has no fixed component. Λ is not composed of a pencil because $|M + (2d-2)l| + M + l_1 + l_2$ is a linear subsystem of Λ. On the other hand, since $|M + (2d-2)l| + M + l_1 + l_2 \subset \Lambda$, accidental base points of Λ (if any) lie on M, l_1 or l_2. However, there is no accidental base point on l_1 or l_2, for, if otherwise, l_1 (or l_2) is a fixed component of Λ. Similarly, there is no accidental base point on M, because $(M \cdot 2M + 2dl) = 0$ and a general member of Λ is irreducible by Bertini's Theorem. Therefore, Λ has no accidental base points. Thus, we obtain our assertions by Bertini's Theorem.

Let Δ be a general member of $|2M+2dl|-(x_1+y_1+x_2+y_2)$. The linear system $|\tilde{B}_0|$ contains a linear subsystem $|(n+5-3d)q^*(l)|+3\tilde{\Delta}$, where $\tilde{\Delta}$ is the proper transform of Δ by q. On the other hand, the same linear system contains a linear subsystem $|4q^*(M)+(n-1+3d)q^*(l)|+2q^*(M)+3\tilde{l}_1+3\tilde{l}_2$. Two linear systems $|(n+5-3d)q^*(l)|$ and $|4q^*(M)+(n-1+3d)q^*(l)|$ have no base point, while, on the other hand, the support of $\tilde{\Delta}$ does not meet that of $2q^*(M)+3\tilde{l}_1+3\tilde{l}_2$. Hence $|\tilde{B}_0|$ has no base point. If $d=0$, we can make a similar argument by replacing $|2M+2dl|$ by $|2M+2l|$, where M is now a fibre of Σ_0 perpendicular to l.

(2) Let $\tilde{M}$ be the proper transform of M by q. Then $\tilde{M}$ is a fixed component of $|\tilde{B}_0|$. Hence we write $\tilde{B}_0=\tilde{M}+C$ with $C\in|5q^*(M)+(n+5+3d)q^*(l)-2E_{x_1}-2E_{x_2}-3E_{y_1}-3E_{y_2}|$. We shall show that $|C|$ has no base point. Let Δ be a general irreducible member of $|M+dl|-y_1-y_2$. [It is not hard to show that a general member of $|M+dl|-y_1-y_2$ is irreducible and nonsingular.] Then $|C|$ contains a linear subsystem $|(n+5)q^*(l)|+2\tilde{M}+3\tilde{\Delta}$, where $\tilde{\Delta}$ is the proper transform of Δ by q. On the other hand, $|C|$ contains a linear subsystem $|5q^*(M)+(n+1+3d)q^*(l)-E_{y_1}-E_{y_2}|+2\tilde{l}_1+2\tilde{l}_2$. Since $n\geqslant 2d-1$, $|5M+(n+1+3d)l|-y_1-y_2$ contains a linear subsystem $|(n+1-2d)l|+(|5M+5dl|-y_1-y_2)$. Thence, it follows that $|5M+(n+1+3d)l|-y_1-y_2$ has no base point other than y_1 and y_2, and that any general member of $|5M+(n+1+3d)l|-y_1-y_2$ passes simply through y_1 and y_2. Hence $|5q^*(M)+(n+1+3d)q^*(l)-E_{y_1}-E_{y_2}|$ has no base point. These facts imply that $|C|$ has no base point. In view of the equality $(C\cdot\tilde{M})=n+1-2d$, C intersects $\tilde{M}$ unless the equality $n=2d-1$ holds. By the same argument as in ([7], p. 125) we can show that if C is a general member then C intersects $\tilde{M}$ transversally. Thus, $\tilde{B}_0=\tilde{M}+C$ has no infinitely near triple point.

(3) It is easy to see that the proper transform $\tilde{M}$ of M is a fixed component of $|\tilde{B}_0|$ and that either $\tilde{l}_1$ or $\tilde{l}_2$ is a fixed component of $|\tilde{B}_0-\tilde{M}|$.

(4) The assumptions that $n-3-d$ is a nonnegative even integer and that $n<2d-1$ imply that $d>4$. But, for later use, we only assume that $d\geqslant 4$. Then, in view of (3) above, we have only to consider the case where both x_1 and x_2 lie on M. Then, neither y_1 nor y_2 is on the proper transform $\tilde{M}$ of M by q, whence $(\tilde{M}\cdot E_{x_1})=(\tilde{M}\cdot E_{x_2})=1$. Note that $\tilde{M}$ is a fixed component of $|\tilde{B}_0|$. Since $(\tilde{M}\cdot\tilde{B}_0-\tilde{M})=-5d+(n+5+3d)-2-2=n+1-2d<0$, $\tilde{M}$ is a multiple component of $|\tilde{B}_0|$.

Q. E. D.

4. 8. In this paragraph and the next we shall study the surfaces of type (2) of 4. 5.

Lemma. (1) *If $d=2$ there do not exist the surfaces of type* (2) *of* 4. 5.

(2) *If $d=0$ or 1, $\tilde{S}$ is the canonical resolution of the double covering of $V=\Sigma_d$ with branch locus B_φ.*

(3) *If $d=1$, the surface S can be constructed as follows:*

(i) *Let $x_1=M\cap l_0$, let $q_1: W_1 \longrightarrow V$ be the blowing-up with center at x_1, and let $q_2: \tilde{W} \longrightarrow W_1$ be the blowing-up with center at a point x_2 on $q_1'(l_0)$, which may be infinitely near to x_1. Let $q=q_1\cdot q_2$, and let $E_{x_1}=q^{-1}(x_1)$ and $E_{x_2}=q_2^{-1}(x_2)$.*

(ii) *Let $\tilde{B}$ be a reduced divisor of $|8q^*(M)+12q^*(l)-6E_{x_1}-4E_{x_2}|$, which has no infinitely near triple point. Let $\tilde{S}$ be the canonical resolution of the double covering of $\tilde{W}$ with branch locus $\tilde{B}$. The proper transforms $\tilde{M}$ and $\tilde{l}_0$ of M and l_0 by q respectively are nonsingular components of $\tilde{B}$, which give rise to exceptional curves E_1 and E_2 of the first kind on $\tilde{S}$. Contracting E_1 and E_2 we get a minimal surface S of general type with $p_g=5$, $q=0$ and $(K_S^2)=8$.*

(4) *If $d=0$, the surface S can be constructed as follows:*

(i) *Let $x_1=M^{*)}\cap l_0$, let $q_1: W_1 \longrightarrow V$ be the blowing-up with center at x_1, and let $q_2: \tilde{W} \longrightarrow W_1$ be the blowings-up with centers at x_2 and y_2 on $q_1'(l_0)$ and $q_1'(M)$ respectively, which may be infinitely near to x_1. Let $q=q_1\cdot q_2$, and let $E_{x_1}=q^{-1}(x_1)$, $E_{x_2}=q_2^{-1}(x_2)$ and $E_{y_2}=q_2^{-1}(y_2)$.*

(ii) *Let $\tilde{B}$ be a reduced divisor of $|8q^*(M)+8q^*(l)-6E_{x_1}-4E_{x_2}-4E_{y_2}|$, which has no infinitely near triple point. Let $\tilde{S}$ be the canonical resolution of the double covering of $\tilde{W}$ with branch locus $\tilde{B}$. The proper transforms $\tilde{M}$ and $\tilde{l}_0$ of M and l_0 by q respectively are nonsingular components of $\tilde{B}$, which give rise to exceptional curves E_1 and E_2 of the first kind on $\tilde{S}$. Contracting F_1 and E_2 we get a minimal surface S of general type with $p_g=4$, $q=0$ and $(K_S^2)=6$.*

Proof. Let S^* be the canonical resolution of the double covering of V with branch locus B_φ. By virtue of Lemma 1. 3 we have:

$$d+5=\chi(S^*, \mathcal{O}_{S^*})=10-\frac{1}{2}\sum_i\left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right]-1\right),$$

$$(K_{S^*}^2)=16-2\sum_i\left(\left[\frac{m_i}{2}\right]-1\right)^2.$$

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*) Note that M is a fixed fibre of Σ_0 perpendicular to l.

On the other hand, it is easy to see that the natural morphism $p: S^* \longrightarrow S$, whose existence follows from the minimality of S, factors through $\tilde{S}$, *i.e.*, $p: S^* \xrightarrow{\tilde{p}} \tilde{S} \xrightarrow{\pi} S$. Hence we have $(K_{S^*}^2) \leqslant (K_{\tilde{S}}^2) = 2d+4$. Taking this inequality into account, we can easily show that;

1) if $d=2$, there is one index, say $i=1$, such that $\left[\frac{m_1}{2}\right]=3$ and $\left[\frac{m_i}{2}\right]=1$ for all other indices,
2) if $d=1$, there are two indices, say $i=1, 2$, such that $\left[\frac{m_1}{2}\right]=3$, $\left[\frac{m_2}{2}\right]=2$ and $\left[\frac{m_i}{2}\right]=1$ for $i\neq 1, 2$,
3) if $d=0$, there are three indices, say $i=1, 2, 3$, such that $\left[\frac{m_1}{2}\right]=3$, $\left[\frac{m_2}{2}\right]=\left[\frac{m_3}{2}\right]=2$ and $\left[\frac{m_i}{2}\right]=1$ for $i\neq 1, 2, 3$.

In each case, $(K_{S^*}^2)=(K_{\tilde{S}}^2)$, whence $\tilde{p}: S^* \longrightarrow \tilde{S}$ is an isomorphism.

Now, note that M and l_0 are irreducible components of B_φ, and that the point $x_1 = M \cap l_0$ should be blown up in the process $\tilde{q}: V^* \longrightarrow V$, which is the shortest composition of blowings-up such that the (new) branch locus B^* is nonsingular (cf. 1. 3). Let M^* be the proper transform of M by $\tilde{q}$. Then $(M^{*2}) \leqslant -d-1$. On the other hand, since we may assume that M^* gives rise to E_1, we have $(M^{*2})=-2$. Then, since the proper transform l_0^* of l_0 by $\tilde{q}$ gives rise to E_2, we have $(l_0^{*2})=-2$. This shows that;

1′) if $d=2$, there do not exist surfaces of type (2) of 4. 5,
2′) if $d=1$, x_1 is the only point on M which is blown up in the process $\tilde{q}$; there is exactly one more point x_2 on l_0 which is blown up in the process $\tilde{q}$; x_2 may be infinitely near to x_1,
3′) if $d=0$, there are points x_2 and y_2 on l_0 and M respectively, which may be infinitely near to x_1; the points x_1, x_2 and y_2 are the points on $l_0 \cup M$, which are blown up in the process $\tilde{q}$.

Now, by the same arguments as in ([5], pp. 51—52) and the above observations taken into account, we have the constructions of surfaces S given in the above statements. Q. E. D.

Let $\bar{B} := B_\varphi - l_0 - M$, and let μ_1, μ_2 and ν_2 be respectively the multiplicities of $\bar{B}$ at x_1, x_2 and y_2, when $d=0$. Assume that both x_2 and y_2 are infinitely near to x_1. Then, the argument as in the proof of Lemma 4. 6 shows that either $\mu_1=4$ and $\mu_2=\nu_2=3$, or $\mu_1=5$ and $\mu_2=\nu_2=2$. However, the first case is apparently impossible.

4. 9. We shall consider the existence of a reduced divisor $\tilde{B}$ having no infinitely near triple point, whose existence was assumed in Lemma 4. 8. Write $\tilde{B}=\tilde{B}_0+\tilde{M}+\tilde{l}_0$, where $\tilde{B}_0 \in |7q^*(M)+11q^*(l)-4E_{x_1}-3E_{x_2}|$ if

$d=1$, and $\tilde{B}_0 \in |7q^*(M)+7q^*(l)-4E_{x_1}-3E_{x_2}-3E_{y_2}|$ if $d=0$. Then we have the following:

Lemma. (1) *Case* $d=1$. *The linear system* $|7q^*(M)+11q^*(l)-4E_{x_1}-3E_{x_2}|$ *has no base point. Hence its general members are nonsingular and irreducible; moreover they are disjoint from* $\tilde{M}$ *and* $\tilde{l}_0$.

(2) *Case* $d=0$. *Assume that not both of* x_2 *and* y_2 *are infinitely near to* x_1. *Then the linear system* $|7q^*(M)+7q^*(l)-4E_{x_1}-3E_{x_2}-3E_{y_2}|$ *has no base point. Hence its general members are nonsingular and irreducible; moreover they are disjoint from* $\tilde{M}$ *and* $\tilde{l}_0$.

Proof. (1) We shall show that any general member of $|2M+3l|-x_1-x_2$ is a nonsingular irreducible curve and that $|2M+3l|-x_1-x_2$ has no accidental base point. In fact, since $|2M+2l|+l_0$ is a linear subsystem of $|2M+3l|-x_1-x_2$ and $|2M+2l|$ has no base point, the fixed component (if any) of $|2M+3l|-x_1-x_2$ is possibly l_0. However, since $\dim|2M+3l|-x_1-x_2 \geqslant 6$ and $\dim|2M+2l|=5$, $|2M+3l|-x_1-x_2$ has no fixed component. Moreover, since $|2M+3l|-x_1-x_2$ has a linear subsystem $|M+2l|+M+l_0$ and $|M+2l|$ is very ample, $|2M+3l|-x_1-x_2$ is not composed of a pencil. Since $((2M+3l)^2)=8$ and $\dim|2M+3l|-x_1-x_2 \geqslant 6$, $|2M+3l|-x_1-x_2$ does not have accidental base points. Hence we get our assertions by Bertini's Theorem. Let Δ be a general member of $|2M+3l|-x_1-x_2$, and let Γ be a general member of $|M+2l|-x_1$, which meets l_0 transversally. Let $\tilde{\Delta}$ and $\tilde{\Gamma}$ be the proper transforms of Δ and Γ by q respectively. Then $3\tilde{\Delta}+\tilde{\Gamma}$ is a member of $|\tilde{B}_0|$. On the other hand, $|7q^*(M)+8q^*(l)-E_{x_1}|+3\tilde{l}_0$ is a linear subsystem of $|\tilde{B}_0|$, and $|7q^*(M)+8q^*(l)-E_{x_1}|$ has no base point because $|7M+8l|$ is very ample. Then, since $\tilde{l}_0 \cap (\tilde{\Delta} \cup \tilde{\Gamma})=\phi$, we know that $|\tilde{B}_0|$ has no base point.

(2) Since one of x_2 and y_2 is not infinitely near to x_1, we may assume that y_2 is not. Let l_2 be a fibre passing through y_2 and linearly equivalent to l. Now, we shall show that any general member of $|2M+2l|-x_1-x_2-y_2$ is a nonsingular irreducible curve and that $|2M+2l|-x_1-x_2-y_2$ has no accidental base point. In fact, since $|2M|+l_0+l_2$ is a linear subsystem of $|2M+2l|-x_1-x_2-y_2$ and $|2M|$ has no base point, a fixed component of $|2M+2l|-x_1-x_2-y_2$ is possibly l_0 or l_2. However, since $\dim|2M+2l|-x_1-x_2-y_2 \geqslant 5$, $\dim|2M+l|+l_0-y_2=4$ and $\dim|2M+l|+l_2-x_1-x_2 \leqslant 4$, we know that neither l_0 nor l_2 is a fixed component of $|2M+2l|-x_1-x_2-y_2$. Since $|M+l|+M+l_0 \subset |2M+2l|-x_1-x_2-y_2$, $|2M+2l|-x_1-x_2-y_2$ is not composed of a pencil. Moreover, since $\dim|2M+2l|-x_1-x_2-y_2 \geqslant 5$ and $((2M+2l)^2)=8$, we know that $|2M+2l|-x_1-x_2-y_2$ has no accidental base point. Thus we

get our assertions by Bertini's Theorem. Therefore, $|2q^*(M)+2q^*(l)-E_{x_1}-E_{x_2}-E_{y_2}|$ has no base point. Note also that $|q^*(M)+q^*(l)-E_{x_1}|$ has no base point. Then, since $3|2q^*(M)+2q^*(l)-E_{x_1}-E_{x_2}-E_{y_2}|+|q^*(M)+q^*(l)-E_{x_1}|\subset|7q^*(M)+7q^*(l)-4E_{x_1}-3E_{x_2}-3E_{y_2}|$, we know that $|\tilde{B}_0|$ has no base point. Q. E. D.

4. 10. Lemma. *There is no surface of type* (3) *of* 4. 5.

Proof. The conditions imply that $X=\varphi^*(M)$. Then, $K_S\sim\varphi^*(M+2l)+2X\sim\varphi^*(3M+2l)$. Hence $p_g(S)=p_g(\tilde{S})\geqslant\dim \mathrm{H}^0(\Sigma_1, \mathcal{O}(3M+2l))=\dim \mathrm{H}^0(\Sigma_1, \mathcal{O}(2M+2l))=6$, which contradicts the assumption that $p_g(S)=5$.
Q. E. D.

4. 11. Lemma. *Let S (or $\tilde{S}$) be a surface of type* (1′) *of* 4. 5. *Then $\tilde{S}$ is not the canonical resolution of the double covering of $V=\Sigma_d$ with branch locus B_φ.*

Proof. Let S^* be the canonical resolution of the double covering of V with branch locus B_φ. Then there exists a birational morphism $\tilde{p}$: $S^*\longrightarrow\tilde{S}$ such that $\pi\cdot\tilde{p}$: $S^*\longrightarrow S$ is the natural morphism, whose existence follows from the minimality of S. By virtue of 1. 3, we have:

$$n+2=\chi(S^*, \mathcal{O}_{S^*})=n+6-\frac{1}{2}\sum_i\left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right]-1\right),$$

$$(K_{S^*}^2)=2n+6-2\sum_i\left(\left[\frac{m_i}{2}\right]-1\right)^2.$$

Thence, we have one of the following two cases;

(i) there exist four indices, say $i=1, 2, 3, 4$, such that $\left[\frac{m_i}{2}\right]=2$ for $i=1, 2, 3, 4$ and $\left[\frac{m_i}{2}\right]=1$ for $i\neq1, 2, 3, 4$; $(K_{S^*}^2)=2n-2$; $\tilde{p}$: $S^*\longrightarrow\tilde{S}$ is an isomorphism,

(ii) there exist two indices, say $i=1, 2$, such that $\left[\frac{m_1}{2}\right]=3$, $\left[\frac{m_2}{2}\right]=2$ and $\left[\frac{m_i}{2}\right]=1$ for $i\neq1, 2$; $(K_{S^*}^2)=2n-4$; $\tilde{p}$ is a composition of two quadric transformations.

Assume now that the first case takes place. By 1. 3 (cf. the proof of Lemma 4. 6), we have an effective divisor Z on S such that $Z=2\varphi^*(l_0)-2X$, $\varphi^*(B_\varphi)-2R_\varphi=2Z$ and $R_\varphi+Z\in|\varphi^*F|$, where $F:=3M+\left(\frac{n+7+3d}{2}\right)l$. Then, $(Z\cdot E_1)=0$ and $(Z\cdot E_2)=2$. Let $\tilde{x}\in E_2\cap\mathrm{Supp}(Z)$ and let $x=\varphi(\tilde{x})$. Let σ: $V_1\longrightarrow V$ be the blowing-up with center at x. Then, there exists a morphism ψ: $\tilde{S}\longrightarrow V_1$ of degree 2 such that $\varphi=\sigma\cdot\psi$; $R_\psi=R_\varphi-\psi^*(D)$, where $D=\sigma^{-1}(x)$. Then $B_\psi\sim\sigma^*(B_\varphi)+2\psi_*(E_1)-6D$. Hence we have:

1) $(B_\phi \cdot D)=6$ if $x \neq \varphi(E_1)$,
2) $(B_\phi \cdot D)=4$ if $\psi(E_1)=D$ and $\psi_*(E_1)=D$,
3) $(B_\phi \cdot D)=2$ if $\psi(E_1)=D$ and $\psi_*(E_1)=2D$.

We shall show that none of these cases takes place. *Case* 1). Let μ be the multiplicity of B_φ at x. Then, $\left[\frac{\mu}{2}\right]=3$ (cf. the proof of Lemma 4. 6), which contradicts the condition (i). *Case* 2). Let ι be a generator of $\mathrm{Gal}(k(S)/k(V)) \cong \boldsymbol{Z}_2$, which acts on S (and hence on $\tilde{S}$). Since E_1 is ι-stable, the condition 2) says that D is branched, *i. e.*, $D \subset \mathrm{Supp}(B_\phi)$. Then the point $\sigma'(l_0) \cap D$ should be blown up in the process $\tilde{q}: V^* \longrightarrow V$, which is the shortest composition of blowings-up such that the (new) branch locus B^* is nonsingular. This implies that $E_1 \cap E_2 = \phi$, which is a contradiction. *Case* 3). Let μ be as above. Then, $\left[\frac{\mu}{2}\right]=1$, whence $\mu=2$ or 3. Assume that $\mu=2$. Write $B_\varphi = l_0 + B_1$. Then l_0 and B_1 intersect each other transversally at x. Since $(B_1 \cdot l_0)=6$, $B_1 \cap l_0$ contains another point y distinct from x. If $B_1 \cap l_0$ contains the third point $z(\neq x, y)$, then, $4(E_2^2) \leqslant -6$, which is a contradiction. Thus, $B_1 \cap l_0 = \{x, y\}$ and $i(B_1, l_0; y)=5$. The multiplicty ν of B_1 at y must be 5, for $4(E_2^2) \leqslant -6$ otherwise. Then the multiplicity of B_φ at y is 6, which contradicts the condition (i). Assume that $\mu=3$. Then, $D \subset \mathrm{Supp}(B_\phi)$, which is a contradiction again by the same reason as in the case 2). Therefore, the case (i) does not take place. Q. E. D.

4. 12. Lemma. *There is no surface of type* (3′) *of* 4. 5.

Proof. We shall show that $\varphi^*(M)=X$. Then, we get a contradiction by the same reason as in 4. 10. Let S^* be the canonical resolution of the double covering of $V=\Sigma_1$ with branch locus B_φ. If $\tilde{q}: V^* \longrightarrow V$ is the shortest composition of blowings-up such that the (new) branch locus B^* is nonsingular (cf. 1. 3), we have the following commutative diagram;

$$\begin{array}{ccc} S^* & \xrightarrow{\psi} & V^* \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle \tilde{q}} \\ \tilde{S} & \xrightarrow{\varphi} & V \end{array}$$

where ψ is a finite morphism of degree 2; the existence of $\tilde{p}$ was mentioned repeatedly (cf. 4. 6, 4. 11). By virtue of 1. 3, we have;

$$6=\chi(S^*, \mathcal{O}_{S^*})=7-\frac{1}{2}\sum_i\left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right]-1\right),$$

$$(K_{S^*}^2)=6-2\sum_i(\left[\frac{m_i}{2}\right]-1)^2.$$

Hence there exists one index, say $i=1$, such that $\left[\frac{m_1}{2}\right]=2$ and $\left[\frac{m_i}{2}\right]=1$ for $i\neq 1$. Then $(K_{S^*}^2)=4<(K_{\tilde{S}}^2)=6$, which implies that $\tilde{p}$ is a composition of two quadric transformations. On the other hand, M is an irreducible component of B_φ. Write $B_\varphi=M+B_1$ with $B_1\sim 9M+10l$ and $(B_1\cdot M)=1$. Then, M and B_1 meet each other only in one point x, at which B_1 is nonsingular. The point is nothing but $\varphi(E_1)$, because $\varphi(E_1)$ must be blown up in the process $\tilde{q}$. Noting that $\tilde{q}$ is the shortest process to get the nonsingular branch locus B^*, we know that no points on $\sigma'(M)$ and D are blown up in the process $\tilde{q}$, where $\sigma: V_1\longrightarrow V$ is the blowing-up of V with center at x, and $D=\sigma^{-1}(x)$. Then, it is easy to see that $\psi^*(D+\sigma'(M))$ is a divisor on S^* having the same property as X. This implies that the support of $\psi^*(D+\sigma'(M))$ does not meet any fundamental curve of $\tilde{p}$. Thus $\varphi^*(D)=X$. Q. E. D.

4. 13. In the remaining paragraphs of this section we shall assume that V is of type (iv) of 1. 5. 1, *i.e.*, V is a cone over a nonsingular rational curve of degree $n-1$ in $\boldsymbol{P}^{n-1}$.

Lemma. *Let* $q: W:=\Sigma_{n-1}\longrightarrow V$ *be the minimal resolution of singularities of* V. *Then, there exists a morphism* $\psi: \tilde{S}\longrightarrow W$ *of degree* 2 *such that* $\varphi=q\cdot\psi$.

Proof. Our proof is almost parallel to the proof of Lemma 1. 5 of [6]. There exists a basis $\{x_0, x_1, \ldots, x_n\}$ of $H^0(\tilde{S}, \mathcal{O}(L))$ such that

$$\frac{x_1}{x_2}=\frac{x_2}{x_3}=\ldots=\frac{x_{n-1}}{x_n},$$

where x_1/x_2 defines a rational function g on $\tilde{S}$. Write $(g)=D-D_1$, where D and D_1 are effective divisors without common components. Then we can write

$$(x_i)=(n-i)D+(i-1)D_1+G \qquad \text{for } 1\leqslant i\leqslant n.$$

Since $|L|$ has no base point, we know that $\mathrm{Supp}((x_0))\cap G=\phi$, *esp.*, $(L\cdot G)=0$, and, noting that $(L\cdot X)=2$, we know that

(i) if $X=E_1+E_2$, neither E_1 nor E_2 is a component of G,

(ii) if $X=E_1+2E_2$, E_2 is not a component of G.

In any case, $(G\cdot E_2)\geqslant 0$. Since $(L\cdot E_2)=1$, $(D\cdot E_2)\geqslant 0$ and $n\geqslant 3$, the equality

$$1=(n-1)(D\cdot E_2)+(G\cdot E_2)$$

implies that $(D\cdot E_2)=0$ and $(G\cdot E_2)=1$. (If $X=E_1+E_2$, we have $(D\cdot E_1)=0$ and $(G\cdot E_1)=1$.) On the other hand, $(L\cdot D)=2$ because $(L^2)=(n-1)(L\cdot D)+(L\cdot G)=2n-2$. Hence, we have:

$$(n-1)(D^2)+(D\cdot G)=2.$$

If $n\geqslant 4$, this implies that $(D^2)=0$ and $(D\cdot G)=2$. If $n=3$, $(D^2)=1$ and $(D\cdot G)=0$ then we have $(G^2)\leqslant 0$ by the Hodge index theorem. Then $(G^2)=0$ because $(L\cdot G)=(n-1)(D\cdot G)+(G^2)=0$. Hence $G=0$, which is contrary to $(G\cdot E_2)=1$. Hence $(D^2)=0$ and $(D\cdot G)=2$ if $n=3$. Then, the same argument as in [6] shows the existence of ψ. Q. E. D.

As a consequence of the above lemma we know that $\psi^*(M)=G$ and $\psi^*(l)\sim D$, where M is the section of Σ_{n-1} with $(M^2)=-(n-1)$.

4. 14. Let R_ψ and B_ψ be respectively the ramification locus and the branch locus of $\psi: \tilde{S}\longrightarrow W=\Sigma_{n-1}$. Since $K_W\sim -2M-(n+1)l$ and $L\sim \psi^*(M+(n-1)l)$ we have:

$$R_\psi\sim 3\psi^*(M)+2n\psi^*(l)+2X,$$
$$B_\psi\sim 6M+4nl+2\psi_*(X).$$

Write $B_\psi\sim aM+bl$ with integers a and $b\geqslant 0$. Since $p_a(L)=2n+1$, the Hurwitz's formula tells us:

$$2(2n)=-4+(aM+bl\cdot M+(n-1)l),$$

whence $b=4n+4$. On the other hand, since $\psi_*(l)\sim D$ and $p_a(D)=2$, we have $a=6$. Therefore, $B_\psi\sim 6M+(4n+4)l$, and $\psi_*(X)\sim 2l$. If $X=E_1+E_2$, then $\psi_*(E_1)=l_1$ and $\psi_*(E_2)=l_2$ with $l_1\sim l_2\sim l$; if $l_1\neq l_2$ both l_1 and l_2 are contained in $\mathrm{Supp}(B_\psi)$; if $l_1=l_2$ then $l_1\not\subset \mathrm{Supp}(B_\psi)$. If $X=E_1+2E_2$, $\psi_*(E_2)=l_0\sim l$ and $\psi(E_1)$ is a point on l_0; $l_0\subset \mathrm{Supp}(B_\psi)$. Since $(B_\psi\cdot M)\geqslant -(n-1)$ we have $n\leqslant 9$. Now, the same observations in Lemmas 4. 6, 4. 7 and 4. 10 lead us to the following

Lemma. (1) *Case* $X=E_1+E_2$. If $l_1\neq l_2$ *and if* $\tilde{S}$ *is the canonical resolution of the double covering of* W *with branch locus* B_ψ, *the surface* S *can be constructed as in Lemma* 4. 6, (1) *with* d *replaced by* $n-1$; n *is necessarily* 3 *or* 4. *If* $l_1=l_2$, $\tilde{S}$ *is not the canonical resolution of the double covering of* W *with branch locus* B_ψ.

(2) *Case* $X=E_1+2E_2$. $\tilde{S}$ *is not the canonical resolution of the double covering of* W *with branch locus* B_ψ.

Proof. In applying Lemma 4. 7, (4), note that $n-1\geqslant 4$ (cf. its proof). The same lemma implies that there exists a nonsingular curve $\tilde{B}_0$ if $n\leqslant 4$. Q. E. D.

4. 15. Summarizing the above results 4. 1$\sim$4. 14 we have the following

Theorem. *Let S be a minimal surface of general type such that $p_g = n+1$ and $(K^2) = 2n$ with $n \geqslant 3$. Assume that $|K|$ has no fixed component and that $|K|$ has base point. Then we have the following*:

(1) *$|K|$ has exactly two base points, and $|K|$ is not composed of a pencil.*

(2) Let P_1 and P_2 be base points of $|K|$ with P_2 possibly infinitely near to P_1, let $\pi_1: \tilde{S}_1 \longrightarrow S$ and $\pi_2: \tilde{S} \longrightarrow \tilde{S}_1$ be the blowings-up with centers at P_1 and P_2 respectively. Let $\pi = \pi_1 \cdot \pi_2$, let $E_1 = \pi_2'(\pi_1^{-1}(P_1))$ and let $E_2 = \pi_2^{-1}(P_2)$. Let $|\pi^* K| = |L| + X$ with the fixed part X, and let $\varphi := \Phi_{|L|}: \tilde{S} \longrightarrow V \subset \boldsymbol{P}^n$ with $V = \varphi(\tilde{S})$. *Then either $X = E_1 + E_2$ or $X = E_1 + 2E_2$, respectively P_2 is not, or is infinitely near to P_1*; $\deg \varphi = 2$ *and* $\deg V = n-1$; *V is a surface of type* (iii) *or* (iv) *of* 1. 5. 1.

(3) *If $X = E_1 + 2E_2$, $\tilde{S}$ is not the canonical resolution of the double covering of V (or, the minimal resolution W of V if V is singular) with branch locus B_φ (or B_ϕ)*, (cf. Lemmas 4. 6 and 4. 13).

(4) *If $X = E_1 + E_2$ and if $\tilde{S}$ is the canonical resolution of the double covering of V (or the minimal resolution W of V) with branch locus B_φ (or B_ϕ) we have the following three cases*:

(i) *$V = \Sigma_d$; $n-3-d$ is a nonnegative even integer, and $n \geqslant 2d-1$; $B_\varphi \sim 6M + (n+7+3d)l$; $\varphi_*(E_1) = l_1$ and $\varphi_*(E_2) = l_2$ such that $l_1 \sim l_2 \sim l$; $l_1 \neq l_2$ and l_1, $l_2 \subset$* Supp(B_φ), *(the construction of such surfaces is given in Lemma* 4. 6, (1)).

(ii) *$V = \Sigma_d$; $d = 0$ or 1, and $n = d+3$; $B_\varphi \sim 8M + (4d+8)l$; either $\varphi_*(E_1) = M$ and $\varphi_*(E_2) = l_0$, or $\varphi_*(E_1) = l_0$ and $\varphi_*(E_2) = M$, where $l_0 \sim l$ and M, $l_0 \subset$* Supp(B_φ), *(the construction of such surfaces is given in Lemma* 4. 8).

(iii) *V is a cone* (cf. 1. 5. 1, (iv)) ; *$n = 3$ or 4*; let *$q: W := \Sigma_{n-1} \longrightarrow V$ be the minimal resolution of singularities of V, and let $\phi: \tilde{S} \longrightarrow W$ be a morphism such that $\varphi = q \cdot \phi$; $B_\phi \sim 6M + (4n+4)l$; $\phi_*(E_1) = l_1$ and $\phi_*(E_2) = l_2$ such that $l_1 \sim l_2 \sim l$, $l_1 \neq l_2$ and l_1, $l_2 \subset$* Supp(B_ϕ), *(the construction of such surfaces is given in Lemma* 4. 6, (1)).

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