J. Math. Kyoto Univ. (JMKYAZ)
18-1 (1978) 1-70

Rigidity for isometric imbeddings

By

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(Received, March 11, 1977)

Introduction.

Let $\boldsymbol{f}$ be an immersion of a manifold M into the m-dimensional Euclidean space $\boldsymbol{R}^m$. Assume that $\boldsymbol{f}$ is non-degenerate (see § 1). In his paper [15] one of the authors has shown that there is associated to $\boldsymbol{f}$ a linear differential operator L in a natural manner, and that it is equivalent, in a sense, to the differential operator $d\Phi_f$ of infinitesimal isometric deformations of $\boldsymbol{f}$ ([15], Theorem 1.2; Theorem 1.1 in the present paper). Especially the infinitesimal isometric deformations $\boldsymbol{u}$ of $\boldsymbol{f}$ are in a one-to-one correspondence with the solutions φ of the equation $L\varphi=0$. It should be here pointed out that the symbol of the operator $d\Phi_f$ necessarily degenerates, while the symbol of the operator L does not necessarily degenerate. Thus we have the notion of ellipticity for the operator L. These facts indicate that the equation $L\varphi=0$ plays an important role in the study of the rigidity problem for the immersion $\boldsymbol{f}$.

Owing the operator L, he has indeed established a global rigidity theorem ([15], Theorem 2.4) which may be stated as follows: Let $\boldsymbol{f}_0$ be an immersion $M\rightarrow\boldsymbol{R}^m$ which satisfies the following conditions: 1) $\boldsymbol{f}_0$ is elliptic, i. e., $\boldsymbol{f}_0$ is non-degenerate and the associated equation $L\varphi=0$ is elliptic; 2) $\boldsymbol{f}_0$ is globally infinitesimally rigid, i. e., every global solution of the equation $L\varphi=0$ is derived from an infinitesimal Euclidean transformation of $\boldsymbol{R}^m$; 3) M is compact. Then the theorem states that if two imbeddings $\boldsymbol{f}$ and $\boldsymbol{f}'$: $M\rightarrow\boldsymbol{R}^m$ lie both near to $\boldsymbol{f}_0$ with respect to the C^3-topology, and if they induce the same Riemannian metric g, then there is a unique Euclidean transformation a of $\boldsymbol{R}^m$ such that $\boldsymbol{f}'=a\boldsymbol{f}$. He has also applied this theorem to the canonical isometric imbedding $\boldsymbol{f}_0$ of a compact hermitian symmetric space, $M=K/K_0$, into the Euclidean space $\mathfrak{k}=\boldsymbol{R}^m$, $\mathfrak{k}$ being the Lie algebra of K, and has obtained a rigidity theorem ([15], Theorem 3. 5) for the imbeddings arround $\boldsymbol{f}_0$, which partially generalizes the famous theorem of Cohn-Vossen.

The main purpose of the present paper is first to make a general and systematic study on the rigidity problem for isometric imbeddings, based on the operators L, and second to establish actual rigidity theorems (analogous to the second theorem cited above) for some of the canonical isometric imbeddings of the so called symmetric R spaces into Euclidean spaces.

Corresponding to the global rigidity theorem mentioned above, we prove, in the present paper, a local rigidity theorem (Theorem 2. 8), which we shall explain from now on. Let us first introduce the notion for an immersion of being finite (see § 2) : An immersion $\boldsymbol{f}: M \rightarrow \boldsymbol{R}^m$ is of completely finite type if $\boldsymbol{f}$ is non-degenerate and if the second prolongation $\mathfrak{h}^{(2)}$ of the symbol $\mathfrak{h}$ of the equation $L\varphi=0$ is minimal in a sense. Now let $\boldsymbol{f}_0$ be an immersion $M \rightarrow \boldsymbol{R}^m$. Assume that $\boldsymbol{f}_0$ is of completely finite type and that M is connected. Then the local rigidity theorem asserts that if two immersions $\boldsymbol{f}$ and $\boldsymbol{f}'$: $M \rightarrow \boldsymbol{R}^m$ lie both near to $\boldsymbol{f}_0$ with respect to the C^2-topology and if they induce the same Riemannian metric g, then there is a unique Euclidean transformation a of $\boldsymbol{R}^m$ such that $\boldsymbol{f}'=a\boldsymbol{f}$.

Now the symmetric R spaces form a class of compact Riemannian symmetric spaces, which are associated with the simple graded Lie algebras of the first kind, $\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ (see [12] and § 3). It is known that every symmetric R space M can be canonically isometrically imbedded into a Euclidean space. It is also known that the irreducible hermitian symmetric spaces of compact type as well as the real Grassmann manifolds belong to this class. First of all we show that the canonical isometric imbedding $\boldsymbol{f}$ of an irreducible hermitian symmetric space of compact type, M, is of completely finite type except that M is a complex projective space (Theorem 3. 7). This is a refinement of [15], Theorem 3. 5 cited above. As for the canonical isometric imbedding $\boldsymbol{f}$ of the real Grassmann manifold $G^{p,q}(\boldsymbol{R})$ $(p \leqq q)$, we prove the following facts: (1) If $p \geqq 3$, then $\boldsymbol{f}$ is of completely finite type (Theorem 4. 1), and (2) if $p=2$ and $q \geqq 3$, then $\boldsymbol{f}$ is elliptic, of infinite type, and globally infinitesimally rigid (Theorems 3. 8 and 7. 12). Note that if $p=1$, then $\boldsymbol{f}$ is globally deformable. Accordingly the rigidity problem for the canonical imbedding of a symmetric R space considerably depends on the (group-theoretic) structure of the individual space.

In § 1 we first recall the definition of the operator L, and then study the symbol $\mathfrak{h}$ of the equation $L\varphi=0$ together with its prolongations

$\mathfrak{h}^{(l)}$. In §2 we prove the local rigidity theorem. In §3 we prove Theorems 3.7 and 3.8 after some general considerations on a symmetric R space. §4 is devoted to the proof of Theorem 4.1, which makes use of the root system associated with the simple Lie algebra $\mathfrak{sl}(p+q; \boldsymbol{R})$.

§5~§7 are devoted to the study of the global solutions of the equation $L\varphi=0$, associated with the canonical isometric imbedding $\boldsymbol{f}$ of a symmetric R space M. In §5 we give a sufficient condition for $\boldsymbol{f}$ to be globally infinitesimally rigid, which is described in terms of the eigenvalues of the Laplacian Δ on M (Proposition 5.10). §6 is preliminalry to the subsequent section, §7, and in this last section we prove Theorem 7.12 by the application of Proposition 5.10.

Finally in Appendix we study the non-linear differential equation of isometric imbeddings, and give another proof of the theorem of Janet-Cartan on the possibility for locally isometric imbeddings of Riemannian metrics. (A study in the same line can be found in the recent paper of Gasqui [5].) Through the discussions here we further find the close relation between the linear equation $L\varphi=0$ and the non-linear equation of isometric imbeddings.

§1. The linear differential equation $L\varphi=0$.

1.1. The differential operator L. Let M be an n-dimensional differentiable manifold. $T(M)$ or simply T denotes the tangent bundle of M, and T^* its dual. S^kT^* (resp. $\wedge^kT^*$) denotes the k-th symmetric product (resp. the k-th exterior product) of T^*.

Let $\Gamma(M, m)$ denote the set of all differentiable maps of M to the m-dimensional Euclidean space $\boldsymbol{R}^m$, and $\mathcal{I}(M, m)$ the subset of $\Gamma(M, m)$ composed of all immersions of M into $\boldsymbol{R}^m$. For $\boldsymbol{u}\in\Gamma(M, m)$ we define a cross section $\Phi(\boldsymbol{u})$ of S^2T^* by

$$\Phi(\boldsymbol{u})=\langle d\boldsymbol{u}, d\boldsymbol{u}\rangle,$$

where $\langle , \rangle$ stands for the inner product as a Euclidean vector space. If $\boldsymbol{f}\in\mathcal{I}(M, m)$, then $g=\Phi(\boldsymbol{f})$ is a Riemannian metric on M, which is called *induced* from the immersion $\boldsymbol{f}$. Conversely given a Rimannian metric g on M, an immersion $\boldsymbol{f}\in\mathcal{I}(M, m)$ is called an *isometric immersion* of the Riemannian manifold (M, g) into the Euclidean space $\boldsymbol{R}^m$ if it is a solution of the equation $\Phi(\boldsymbol{u})=g$.

The assignment $\boldsymbol{u}\rightarrow\Phi(\boldsymbol{u})$ gives a non-linear differential operator Φ of $\Gamma(M, m)$ to $\Gamma(S^2T^*)$. Let $\boldsymbol{f}\in\mathcal{I}(M, m)$. As usual $d\Phi_{\boldsymbol{f}}$ denotes the linearization of Φ at $\boldsymbol{f}$, which is the linear differential operator of

$\Gamma(M, m)$ to $\Gamma(S^2T^*)$ given by

$$d\Phi_f(\boldsymbol{u}) = 2\langle d\boldsymbol{f}, d\boldsymbol{u}\rangle \qquad \text{for all } \boldsymbol{u}\in\Gamma(M, m).$$

A solution $\boldsymbol{u}$ of the equation $d\Phi_f(\boldsymbol{u})=0$ is called an *infinitesimal isometric deformation* of $\boldsymbol{f}$.

We denote by N the *normal bundle* of the immersion $\boldsymbol{f}$, which may be regarded as a subbundle of the trivial bundle $M\times\boldsymbol{R}^m$. Let ∇ be the covariant differentiation associated with the induced Riemannian metric $g=\Phi(\boldsymbol{f})$. It is well known that, for any $x, y\in T_p$, the second covariant derivative $\nabla_x\nabla_y\boldsymbol{f}$ is in the fibre N_p of N at p.

Definition (cf. [15]). We say that the immersion $\boldsymbol{f}$ is *non-degenerate* if the fibre N_p of N at each $p\in M$ is spanned by the vectors of the form $\nabla_x\nabla_y\boldsymbol{f}$, $x, y\in T_p$.

For $\boldsymbol{v}\in N_p$ we define a symmetric bilinear form $\Theta(\boldsymbol{v})$ on T_p by

$$\Theta(\boldsymbol{v})(x, y) = \langle\boldsymbol{v}, \nabla_x\nabla_y\boldsymbol{f}\rangle \qquad \text{for all } x, y\in T_p,$$

which is usually called the *second fundamental form* of $\boldsymbol{f}$ corresponding to the normal vector $\boldsymbol{v}$. It is clear that $\boldsymbol{f}$ is non-degenerate if and only if the bundle homomorphism Θ: $N\ni\boldsymbol{v}\to\Theta(\boldsymbol{v})\in S^2T^*$ is injective.

Now assume that $\boldsymbol{f}$ is non-degenerate. Then the image $\mathfrak{n}=\Theta(N)$ of N by Θ forms a subbundle of S^2T^*, which will be called the *bundle of second fundamental forms* of $\boldsymbol{f}$.

We define a differential operator D of $\Gamma(T^*)$ to $\Gamma(S^2T^*)$ by

$$(D\varphi)(x, y) = (\nabla_x\varphi)(y) + (\nabla_y\varphi)(x) \qquad \text{for all } \varphi\in\Gamma(T^*) \text{ and } x, y\in T_p.$$

Let Π denote the projection of S^2T^* onto the quotient bundle $S^2T^*/\mathfrak{n}$. Then the composition $L=\Pi\circ D$ of Π and D gives a differential operator of $\Gamma(T^*)$ to $\Gamma(S^2T^*/\mathfrak{n})$.

The next theorem which is fundamental in our arguments indicates that the two operators $d\Phi_f$ and L are equivalent in a sense.

Theorem 1.1 ([15], Theorem 1.2). *Let $\alpha\in\Gamma(S^2T^*)$. Then there is a natural one-to-one correspondence between the set of solutions $\boldsymbol{u}$ of the equation $d\Phi_f(\boldsymbol{u})=\alpha$ and the set of solutions φ of the equation $L\varphi=\Pi\alpha$. Furthermore the two solutions $\boldsymbol{u}$ and φ are related as follows:*

$$\varphi = \langle\boldsymbol{u}, d\boldsymbol{f}\rangle$$

$$\boldsymbol{u} = V\boldsymbol{f} + \frac{1}{2}\Theta^{-1}(D\varphi-\alpha),$$

where V is the vector field dual to φ with respect to the Riemannian metric $g=\Phi(\boldsymbol{f})$.

1.2. The differential equation R and its prolongations. Let $\boldsymbol{f}\in\mathscr{I}(M, m)$. We denote by $J^k(T^*)$ the vector bundle of all k-jets of local cross sections of T^*, and denote by ρ^k_{k-1} the projection of $J^k(T^*)$ onto $J^{k-1}(T^*)$. As is well known, the vector bundle $S^kT^*\otimes T^*$ may be regarded as a subbundle of $J^k(T^*)$ and we have the exact sequence:

$$0\rightarrow S^kT^*\otimes T^*\xrightarrow{i}J^k(T^*)\rightarrow J^{k-1}(T^*)\rightarrow 0.$$

Now assume that $\boldsymbol{f}$ is non-degenerate and consider the associated operator $L: \Gamma(T^*)\rightarrow\Gamma(S^2T^*/\mathfrak{n})$. The equation $L\varphi=0$ may be represented as usual by the subvariety $R=\bigcup_{p\in M}R_p$ of $J^1(T^*)$, where

$$R_p=\{j^1_p\varphi \mid \varphi\in\Gamma(T^*),\ (L\varphi)_p=0\}.$$

The intersection $\mathfrak{h}=R\cap(T^*\otimes T^*)$ or the kernel of the map $\rho^1_0:R\rightarrow T^*$ is usually called the *symbol* of the equation R.

The assignment $j^1_p\varphi\rightarrow\varphi_p+(\nabla\varphi)_p$ gives an isomorphism of $J^1(T^*)$ onto $T^*+T^*\otimes T^*$, by which we identify these two vector bundles. Then $R_p=\{\varphi_p+(\nabla\varphi)_p | \varphi\in\Gamma(T^*),\ (D\varphi)_p\in\mathfrak{n}_p\}$. Therefore R and $\mathfrak{h}$ may be described respectively as follows:

$$R=T^*+(\mathfrak{n}+\wedge^2T^*),$$

$$\mathfrak{h}=\mathfrak{n}+\wedge^2T^*.$$

We now define the *l-th prolongation* $R^{(l)}$ of the equation R by

$$R^{(l)}=J^l(R)\cap J^{l+1}(T^*)$$

and define the *l-th prolongation* $\mathfrak{h}^{(l)}$ of the symbol $\mathfrak{h}$ to be the kernel of the map ρ^{l+1}_l: $R^{(l)}\rightarrow R^{(l-1)}$: $\mathfrak{h}^{(l)}=(S^{l+1}T^*\otimes T^*)\cap R^{(l)}$. Note that $\mathfrak{h}^{(l)}$ depends only on $\mathfrak{h}$, or more precisely,

$$\mathfrak{h}^{(l)}=S^lT^*\otimes\mathfrak{h}\cap S^{l+1}T^*\otimes T^*.$$

Our task from now on is to study the prolongations $R^{(l)}$ and $\mathfrak{h}^{(l)}$. For this purpose we first make a consideration on the special solutions φ^A of the equation R coming from the infinitesimal Euclidean transformations A of $\boldsymbol{R}^m$.

Let $E(m)$ be the Euclidean transformation group of $\boldsymbol{R}^m$ and $\mathfrak{e}(m)$ its Lie algebra. Let $\mathfrak{o}(m)$ be the Lie algebra of skew symmetric matrices of degree m. Every element A of $\mathfrak{e}(m)$ may be represented by a matrix A of degree $m+1$ of the form: $\begin{pmatrix} 0 & 0 \\ c & B \end{pmatrix}$ or by a map $\boldsymbol{R}^m \ni \boldsymbol{x} \to B\boldsymbol{x}+\boldsymbol{c} \in \boldsymbol{R}^m$, where $\boldsymbol{c} \in \boldsymbol{R}^m$, $B \in \mathfrak{o}(m)$.

For $A \in \mathfrak{e}(m)$ we define a 1-form φ^A on M by

$$\varphi^A = \langle A\boldsymbol{f}, d\boldsymbol{f} \rangle,$$

giving a solution of the equation $L\varphi=0$ or of R. It is known that the map $A \to \varphi^A$ is injective (see [15]). For $p \in M$ we define a subspace $\bar{R}^l_p$ of $J^{l+1}(T^*)_p$ by

$$\bar{R}^l_p = \{ j^{l+1}_p \varphi^A \mid A \in \mathfrak{e}(m) \}, \quad l \geqq -1,$$

and denote by $\bar{\mathfrak{h}}^l_p$ the kernel of the (surjective) map $\rho^{l+1}_l: \bar{R}^l_p \to \bar{R}^{l-1}_p$. φ^A being solutions of R, we have $\bar{R}^l_p \subset R^{(l)}_p$ and $\bar{\mathfrak{h}}^l_p \subset \mathfrak{h}^{(l)}_p$.

These being prepared, we shall now prove the following

Proposition 1.2. *$\dim \bar{R}^{-1}_p = n$, $\dim \bar{\mathfrak{h}}^0_p = r + \frac{1}{2} n(n-1)$, $\dim \bar{\mathfrak{h}}^1_p = nr$, $\dim \bar{\mathfrak{h}}^2_p = \frac{1}{2} r(r-1)$ and $\bar{\mathfrak{h}}^l_p = 0$ for $l \geqq 3$, where $r = m-n$.*

If $a \in E(m)$, the two immersions $\boldsymbol{f}$ and $a\boldsymbol{f}$ induce the same Riemannian metric g, the same bundle $\mathfrak{n}$ of second fundamental forms and hence the same operator L. Consequently we may assume that $\boldsymbol{f}(p) = \boldsymbol{0}$, $\boldsymbol{f}_* T_p = \sum_{i=1}^{n} \boldsymbol{R} e_i$ and $N_p = \sum_{i=n+1}^{m} \boldsymbol{R} e_i$, where $\{e_1, \dots, e_m\}$ denotes the canonical basis of the Euclidean vector space $\boldsymbol{R}^m$.

This being said, every $A \in \mathfrak{e}(m)$ may be represented by a matrix of degree $m+1$ of the form:

$$\begin{pmatrix} 0 & 0 & 0 \\ \boldsymbol{c}_{-1} & B_0 & -{}^tB_1 \\ \boldsymbol{c}_0 & B_1 & B_2 \end{pmatrix}$$

where $\boldsymbol{c}_{-1} \in \boldsymbol{R}^n$, $\boldsymbol{c}_0 \in \boldsymbol{R}^r$, $B_0 \in \mathfrak{o}(n)$, $B_2 \in \mathfrak{o}(r)$ and B_1 is an $r \times n$ matrix.

A simple calculation proves the following

Lemma 1.3. *Let $A \in \mathfrak{e}(m)$ and $x, y, z, w \in T_p$.*

(1) $\varphi^A(x) = \langle \boldsymbol{c}_{-1}, \nabla_x \boldsymbol{f} \rangle$.

(2) $(\nabla_y \varphi^A)(x) = \langle B_0(\nabla_y \boldsymbol{f}), \nabla_x \boldsymbol{f} \rangle + \langle \boldsymbol{c}_0, \nabla_y \nabla_x \boldsymbol{f} \rangle$.

(3) $(\nabla_z \nabla_y \varphi^A)(x) = -\langle \nabla_z \nabla_y \boldsymbol{f}, B_1(\nabla_x \boldsymbol{f}) \rangle + \langle B_1(\nabla_z \boldsymbol{f}), \nabla_y \nabla_x \boldsymbol{f} \rangle$

$$+<B_1(\nabla_y f),\ \nabla_z\nabla_x f>+<c_{-1}+c_0,\ \nabla_z\nabla_y\nabla_x f>.$$

(4) *If* $c_{-1}=c_0=B_0=B_1=0$, *then*

$$(\nabla_w\nabla_z\nabla_y\varphi^A)(x)=<B_2(\nabla_z\nabla_y f),\ \nabla_w\nabla_x f>+<B_2(\nabla_w\nabla_z f),\ \nabla_y\nabla_x f>$$
$$+<B_2(\nabla_w\nabla_y f),\ \nabla_z\nabla_x f>.$$

Now Proposition 1.2 follows immediately from the following

Lemma 1. 4. *Let* $A\in\mathfrak{e}(m)$.

(1) $\varphi_p^A=0$ *if and only if* $c_{-1}=\mathbf{0}$.

(2) $j_p^1\varphi^A=0$ *if and only if* $c_{-1}=c_0=B_0=0$.

(3) $j_p^2\varphi^A=0$ *if and only if* $c_{-1}=c_0=B_0=B_1=0$.

(4) $j_p^3\varphi^A=0$ *if and only if* $A=0$.

This fact follows easily from Lemma 1. 3. We only remark that $j_p^l\varphi^A=0$ if and only if $(\overbrace{\nabla\dots\nabla}^{k}\varphi^A)_p=0$ for all $0\leqq k\leqq l$.

We put $\bar{R}^l=\bigcup_{p\in M}\bar{R}_p^l$ and $\bar{\mathfrak{h}}^l=\bigcup_{p\in M}\bar{\mathfrak{h}}_p^l$, which are vector bundles by Proposition 1. 1.

Proposition 1. 5. $R=\bar{R}^0$ *and* $R^{(1)}=\bar{R}^1$.

Proof. By Proposition 1. 2 we have $\dim\bar{R}^0=n+r+\frac{1}{2}n(n-1)=\dim R$, whence $R=\bar{R}^0$. Since $R=\bar{R}^0$, the sequence $0\to\mathfrak{h}^{(1)}\to R^{(1)}\to R\to 0$ is exact. By Proposition 1. 2, we have $\dim\bar{\mathfrak{h}}^1=nr$, and we shall prove in the next paragraph that $\dim\mathfrak{h}^{(1)}=nr$. Therefore $\mathfrak{h}^{(1)}=\bar{\mathfrak{h}}^1$ and hence $R^{(1)}=\bar{R}^1$. Q. E. D.

Corollary. *Both the equations* R *and* $R^{(1)}$ *are vector bundles, and the following two sequences are exact*:

$$0\longrightarrow\mathfrak{h}^{(1)}\longrightarrow R^{(1)}\longrightarrow R\longrightarrow 0$$

$$0\longrightarrow\mathfrak{h}^{(2)}\longrightarrow R^{(2)}\longrightarrow R^{(1)}\longrightarrow 0$$

(Note that $\mathfrak{h}^{(2)}$ and hence $R^{(2)}$ are not necessarily vector bundles.)

1. 3. The algebraic prolongations. Let V be an n-dimensional vector space over a field K (of characteristic zero). In this paragraph we shall make a general consideration on the prolongations $\mathfrak{h}^{(l)}$ of the subspace $\mathfrak{h}=\mathfrak{n}+\wedge^2V^*$ of $\otimes^2V^*$, $\mathfrak{n}$ being a subspace of S^2V^*. This leads to the study of the prolongations $\mathfrak{h}^{(l)}$ of the symbol $\mathfrak{h}$ of the equation R, associated with a non-degenerate immersionn $\boldsymbol{f}$.

For $u \in V$ and $X \in \otimes^l V^*$, we denote by $u \lrcorner X$ the elements of $\otimes^{l-1} V^*$ defined by $(u \lrcorner X)(u_1, \ldots, u_{l-1}) = X(u, u_1, \ldots, u_{l-1})$ for all $u_1, \ldots, u_{l-1} \in V$. We have $\otimes^2 V^* = S^2 V^* + \wedge^2 V^*$ (direct sum). We denote by π the projection of $\otimes^2 V^*$ onto $S^2 V^*$, and define a linear map $\partial : \otimes^3 V^* \longrightarrow V^* \otimes \wedge^2 V^*$ by

$$(\partial X)(u, v, w) = X(v, u, w) - X(w, u, v)$$

$$\text{for all } X \in \otimes^3 V^* \text{ and } u, v, w \in V.$$

For any integer $l \geqq 1$, we define a linear map

$$\pi_l : \otimes^{l+2} V^* \longrightarrow \otimes^l V^* \otimes S^2 V^*$$

by $\pi_l = 1_l \otimes \pi$, and a linear map

$$\partial_l : \otimes^{l+2} V^* \longrightarrow \otimes^l V^* \otimes \wedge^2 V^*$$

by $\partial_l = 1_{l-1} \otimes \partial$, where 1_k denotes the identity of $\otimes^k V^*$ onto itself. Clearly we have $\pi_l \circ \partial_l = 0$ and

$$u \lrcorner \pi_{l+1} X = \pi_l (u \lrcorner X),$$

$$u \lrcorner \partial_{l+1} X = \partial_l (u \lrcorner X), \quad X \in \otimes^{l+2} V^*, \; u \in V.$$

Now let $\mathfrak{n}$ be an r-dimensional subspace of $S^2 V^*$. We define a subspace $\mathfrak{h}$ of $\otimes^2 V^*$ by

$$\mathfrak{h} = \mathfrak{n} + \wedge^2 V^*$$

and denote by $\mathfrak{h}^{(l)}$ its l-th prolongation:

$$\mathfrak{h}^{(l)} = S^l V^* \otimes \mathfrak{h} \cap S^{l+1} V^* \otimes V^*, \quad l \geqq 1.$$

Note that $\mathfrak{h}^{(l)} = V^* \otimes \mathfrak{h}^{(l-1)} \cap S^2 V^* \otimes \mathfrak{h}^{(l-2)}$, where $\mathfrak{h}^{(-1)} = V^*$ and $\mathfrak{h}^{(0)} = \mathfrak{h}$. Let us now define vector spaces $p^l(\mathfrak{n})$ $(l \geqq 1)$ as follows:

$$p^1(\mathfrak{n}) = V^* \otimes \mathfrak{n},$$

$$p^2(\mathfrak{n}) = S^2 V^* \otimes \mathfrak{n} \cap \partial_2^{-1}(S^2 V^* \otimes \wedge^2 V^*),$$

$$p^l(\mathfrak{n}) = S^l V^* \otimes \mathfrak{n} \cap S^{l-2} V^* \otimes p^2(\mathfrak{n}), \quad l \geqq 3.$$

Note that $p^l(\mathfrak{n}) = V^* \otimes p^{l-1}(\mathfrak{n}) \cap S^2 V^* \otimes p^{l-2}(\mathfrak{n})$, $l \geqq 3$.

Theorem 1.6. *For any* $l \geqq 1$, *the map*

$$\pi_l : \otimes^{l+2} V^* \longrightarrow \otimes^l V^* \otimes S^2 V^*$$

induces an isomorphism of $\mathfrak{h}^{(l)}$ *onto* $p^l(\mathfrak{n})$, *and the inverse of this isomorphism is given by the map*

$$\otimes^{l+2} V^* \ni Y \longrightarrow Y + \partial_l Y \in \otimes^{l+2} V^*,$$

restricted to $p^l(\mathfrak{n})$.

In particular we know from this theorem that dim $\mathfrak{h}^{(1)} = nr$.

Proof of Theorem 1. 6. First of all we assert that the map π_1 induces an isomorphism of $S^2V^* \otimes V^*$ onto $V^* \otimes S^2 V^*$ and that $X = \pi_1 X + \partial \pi_1 X$ for all $X \in S^2 V^* \otimes V^*$. Indeed let $Y \in V^* \otimes S^2 V^*$. Putting $X = Y + \partial Y$, we have $X(u, v, w) = Y(u, v, w) + Y(v, u, w) - Y(w, u, v)$ for all $u, v, w \in V$. Hence we easily find that 1°. $X \in S^2 V^* \otimes V^*$, 2°. $Y = 0$ if $X = 0$ and 3°. $Y = \pi_1 X$. These facts prove our assertion.

(1) The case $l = 1$. Let $X \in S^2 V^* \otimes V^*$. Then for all $u \in V$ we have $u \lrcorner X = u \lrcorner \pi_1 X + u \lrcorner \partial \pi_1 X$. We have $u \lrcorner \pi_1 X \in S^2 V^*$ and $u \lrcorner \partial \pi_1 X \in \wedge^2 V^*$. Hence $X \in \mathfrak{h}^{(1)}$ if and only if $\pi_1 X \in V^* \otimes \mathfrak{n} = p^1(\mathfrak{n})$. Therefore the map π_1 induces an isomorphism of $\mathfrak{h}^{(1)}$ onto $p^1(\mathfrak{n})$ and $X = \pi_1 X + \partial \pi_1 X$ for all $X \in \mathfrak{h}^{(1)}$.

(2) The case $l = 2$. By (1) we see that the map π_2 induces an isomorphism of $V^* \otimes \mathfrak{h}^{(1)}$ onto $V^* \otimes p^1(\mathfrak{n})$ and that $X = \pi_2 X + \partial_2 \pi_2 X$ for all $X \in V^* \otimes \mathfrak{h}^{(1)}$. Let $X \in V^* \otimes \mathfrak{h}^{(1)}$. Then for all $u, v \in V$, we have $v \lrcorner u \lrcorner X = v \lrcorner u \lrcorner \pi_2 X + v \lrcorner u \lrcorner \partial_2 \pi_2 X$. We have $v \lrcorner u \lrcorner \pi_2 X \in \mathfrak{n}$ and $v \lrcorner u \lrcorner \partial_2 \pi_2 X \in \wedge^2 V^*$. Hence $X \in \mathfrak{h}^{(2)}$ if and only if $\pi_2 X \in S^2 V^* \otimes \mathfrak{n}$ and $\partial_2 \pi_2 X \in S^2 V^* \otimes \wedge^2 V^*$, i. e., $\pi_2 X \in p^2(\mathfrak{n})$. Therefore the map π_2 induces an isomorphism of $\mathfrak{h}^{(2)}$ onto $p^2(\mathfrak{n})$ and $X = \pi_2 X + \partial_2 \pi_2 X$ for all $X \in \mathfrak{h}^{(2)}$.

(3) The case $l \geqq 3$. By (2) we see that the map π_3 induces an isomorphism of $V^* \otimes \mathfrak{h}^{(2)}$ onto $V^* \otimes p^2(\mathfrak{n})$ and that $X = \pi_3 X + \partial_3 \pi_3 X$ for all $X \in V^* \otimes \mathfrak{h}^{(2)}$. Let $X \in V^* \otimes \mathfrak{h}^{(2)}$. Then for all $u, v \in V$, we have $v \lrcorner u \lrcorner \pi_3 X \in p^1(\mathfrak{n}) = V^* \otimes \mathfrak{n}$ and $v \lrcorner u \lrcorner \partial_3 \pi_3 X = \partial(v \lrcorner u \lrcorner \pi_3 X) \in V^* \otimes \wedge^2 V^*$. Hence $X \in \mathfrak{h}^{(3)}$ if and only if $\pi_3 X \in S^2 V^* \otimes p^1(\mathfrak{n})$, i. e., $\pi_3 X \in p^3(\mathfrak{n})$. Therefore the map π_3 induces an isomorphism of $\mathfrak{h}^{(3)}$ onto $p^3(\mathfrak{n})$ and $X = \pi_3 X + \partial_3 \pi_3 X$ for all $X \in \mathfrak{h}^{(3)}$. In the same way or precisely by induction on the integer l, we can prove the theorem for $l \geqq 4$.

Q. E. D.

1. 4. Involutive immersions. We use the same notations as in the previous paragraph. Let $\{e_1, \ldots, e_n\}$ be a basis of V. For each $1 \leqq i \leqq n$ we define a subspace $\mathfrak{h}_i$ of $\mathfrak{h}$ by

$$\mathfrak{h}_i = \{X \in \mathfrak{h} \mid e_1 \lrcorner X = \ldots = e_i \lrcorner X = 0\}.$$

Then the basis $\{e_1, \ldots, e_n\}$ is called *regular* for $\mathfrak{h}$ if the following equality is satisfied:

$$\dim \mathfrak{h}^{(1)} = \sum_{i=0}^{n-1} \dim \mathfrak{h}_i.$$

By definition the subspace $\mathfrak{h}$ of $\otimes^2 V^*$ is *involutive* if V admits a regular basis for $\mathfrak{h}$.

Proposition 1.7. *A basis $\{e_1, \ldots, e_n\}$ of V is regular for $\mathfrak{h}$ if and only if*

$$\dim (\mathfrak{h}_{i-1}/\mathfrak{h}_i) = n, \quad 1 \leqq i \leqq n-1.$$

Proof. First assume that $\{e_1, \ldots, e_n\}$ is regular. Then if we put $r_i = \dim(\mathfrak{h}_{i-1}/\mathfrak{h}_i)$, we have $r_i \leqq n$ and $\dim \mathfrak{h}_i = \dim \mathfrak{h}_i - \sum_{j=1}^{i} r_j$. Furthermore we have $\dim \mathfrak{h} = r + \frac{1}{2}n(n-1)$ and $\dim \mathfrak{h}^{(1)} = nr$. Therefore the equality $\dim \mathfrak{h}^{(1)} = \sum_{i=1}^{n-1} \dim \mathfrak{h}_i$ gives

$$nr = \frac{1}{2}n^2(n-1) + nr - \sum_{i=1}^{n-1}(n-i)\, r_i.$$

Hence we obtain

$$\frac{1}{2}n^2(n-1) = \sum_{i=1}^{n-1}(n-i) r_i \leqq n \sum_{i=1}^{n-1}(n-i) = \frac{1}{2}n^2(n-1),$$

indicating that $r_i = n$, $1 \leqq i \leqq n-1$. The converse is clear.

Q. E. D.

Corollary 1. *A basis $\{e_1, \ldots, e_n\}$ of V is regular for $\mathfrak{h}$ if and only if $\mathfrak{n}$ contains $\frac{1}{2}n(n-1)$ independent symmetric forms Θ_{ij} $(1 \leqq i, j \leqq n-1)$ such that $\Theta_{ij} \equiv \omega_i \circ \omega_j$ (mod $\{\omega_n\} \circ V^*$), where $\{\omega_1, \ldots, \omega_n\}$ is the dual basis of $\{e_1, \ldots, e_n\}$.*

Corollary 2. *If the subspace $\mathfrak{h}$ of $\otimes^2 V^*$ is involutive, then*

$$\frac{1}{2}n(n-1) \leqq r \leqq \frac{1}{2}n(n+1).$$

Definition. An immersion $f \in \mathscr{I}(M, m)$ is said to be *involutive* if it is non-degenerate and if the symbol $\mathfrak{h} = \mathfrak{n} + \wedge^2 T^*$ of the equation R is involutive, i. e., each fibre $\mathfrak{h}_p$ of $\mathfrak{h}$ is involutive.

By definition, the equation R is *involutive* if it satisfies the following condition: 1°. Both R and $R^{(1)}$ are vector bundles and the map

$\rho_1^2: R^{(1)} \to R$ is surjective, and 2°. $\mathfrak{h}$ is involutive. Therefore we know from Corollary to Proposition 1.5 that an immersion $\boldsymbol{f}$ is involutive if and only if 1°. $\boldsymbol{f}$ is non-degenerate and 2°. the equation R is involutive.

1.5. Elliptic immersions. Let V be an n-dimensional vector space over the field $\boldsymbol{R}$ of real numbers, and let $\mathfrak{n}$ be an r-dimensional subspace of S^2V^*. Then the following three statements are mutually equivalent (cf. [15]) :

(1) The subspace $\mathfrak{h}=\mathfrak{n}+\wedge^2V^*$ of $\otimes^2V^*$ is elliptic. Namely $\mathfrak{h}$ contains no *non-zero decomposable elements* (w.r.t. the tensor product), i.e., contains no elements of the form $\alpha\otimes\beta$, where $\alpha, \beta\in V^*$ and $\alpha, \beta\neq 0$.

(2) The subspace $\mathfrak{n}$ of S^2V^* contains no *non-zero decomposable* elements (w. r. t. the symmetric product), i.e., contains no elements of the form $\alpha\circ\beta$, where $\alpha, \beta\in V^*$ and $\alpha, \beta\neq 0$.

(3) Every non-zero form $H\in\mathfrak{n}$ has at least two eigenvalues of the same sign $(\neq 0)$, where H should be identified with a symmetric endomorphism of V with respect to an inner product $<,>$ of V.

Proposition 1.8 ([15]). *If the subspace $\mathfrak{h}$ of $\otimes^2V^*$ is elliptic, then either $r<\frac{1}{2}n(n-1)$ or $n=2$ and $r=1$.*

Proof. We take an inner product $<,>$ in V and identify S^2V^* with the space $\mathfrak{S}(V)$ of symmetric endomorphisms of the Euclidean vector space V. The inner product $<,>$ in V induces an inner product $<,>$ in $\mathfrak{S}(V)$. This being said, we denote by $\mathfrak{n}^\perp$ the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{S}(V)$. We have $<Xu, v>=<X, u\circ v>$ for all $X\in\mathfrak{S}(V)$ and $u, v\in V$. Accordingly the assumption for $\mathfrak{n}$ means that, for every non-zero $u\in V$, the map $\mathfrak{n}^\perp\ni X\to Xu\in V$ is surjective. It follows that $\dim \mathfrak{n}^\perp=\frac{1}{2}n(n+1)-r\geqq n$, i.e., $r\leqq\frac{1}{2}n(n-1)$. Let us now consider the case where $r=\frac{1}{2}n(n-1)$, i.e., $\dim \mathfrak{n}=n$. Then, for every non-zero $u\in V$, the map $\mathfrak{n}^\perp\ni X\to Xu\in V$ is an isomorphism, meaning that $\det X\neq 0$ for every non-zero $X\in\mathfrak{n}$. Therefore we have $n=2$ and $r=1$ by a theorem of Adams-Lax-Phillips [1]. Q. E. D.

For example, Let V be a $2n$-dimensional vector space over $\boldsymbol{R}$ and I a complex structure on V. Then the space of (real) hermitain forms on V with respect to I contains no non-zero decomposable elements.

Definition. An immersion $f\in\mathscr{I}(M, m)$ is said to be *elliptic* if it is non-degenerate and if the symbol $\mathfrak{h}$ of the equation R is elliptic, i.e., each fibre $\mathfrak{h}_p$ of $\mathfrak{h}$ is elliptic.

It is easy to see that an immersion $f\in\mathscr{I}(M, m)$ is elliptic if and only if 1°. f is non-degenerate and 2°. the operator L is elliptic (see [15]).

§ 2. Immersions of completely finite type.

2. 1. Some facts on the prolongations $\mathfrak{h}^{(l)}$, $l\geqq 2$. The notations being as in 1. 3, we shall study, in this paragraph, the prolongations $\mathfrak{h}^{(l)}$, $l\geqq 2$, of the subspace $\mathfrak{h}=\mathfrak{n}+\wedge^2V^*$ of $\otimes^2V^*$. By virtue of Theorem 1. 6, the study is reduced to that of the spaces $p^l(\mathfrak{n})$, $l\geqq 2$. We fix a basis $\{e_1, \dots, e_n\}$ of V, and express covariant tensors in terms of this basis. This "index" expression of covariant tensors will be of particular use for the calculations of the space $p^2(\mathfrak{n})$ (see 2. 3, 2. 5 and § 4).

First of all the space $p^2(\mathfrak{n})$ consists of all $X\in S^2V^*\otimes\mathfrak{n}$ which satisfy the following relation:

$$X_{\alpha\beta\gamma\delta}-X_{\alpha\delta\gamma\beta}-X_{\gamma\beta\alpha\delta}+X_{\gamma\delta\alpha\beta}=0,$$

where $X_{\alpha\beta\gamma\delta}=X(e_\alpha, e_\beta, e_\gamma, e_\delta)$, the coefficients of X, and the indices α, β, ... run over the range 1, ..., n. Another expression of $p^2(\mathfrak{n})$ based on a basis of $\mathfrak{n}$, is given as follows: Let $\{H^1, \dots, H^r\}$ be a basis of $\mathfrak{n}$. Then every $X\in S^2V^*\otimes\mathfrak{n}$ may be written uniquelly as follows:

$$X=\sum_\lambda A^\lambda\otimes H^\lambda,$$

where $A^\lambda\in S^2V^*$. Thus $S^2V^*\otimes\mathfrak{n}$ may be identified with the space $(S^2V^*)^r=\overbrace{S^2V^*\times\dots\times S^2V^*}^{r}$. This being said, $p^2(\mathfrak{n})$ consists of all $(A^1,\dots, A^r)\in(S^2V^*)^r$ which satisfy the following relation:

$$\sum_\lambda(u\lrcorner A^\lambda)\wedge(v\lrcorner H^\lambda)-\sum_\lambda(v\lrcorner A^\lambda)\wedge(u\lrcorner H^\lambda)=0$$

for all u, $v\in V$.

Proposition 2. 1. *The dimension $p^2(\mathfrak{n})$ is larger than or equal to $\frac{1}{2}r(r-1)$. And the equality holds if and only if every $X\in p^2(\mathfrak{n})$ satisfies the following relation:*

$$X_{\alpha\beta\gamma\delta}=-X_{\gamma\delta\alpha\beta}.$$

Proof. Let $\mathfrak{a}$ denote the subspace of $S^2V^*\otimes S^2V^*$ consisting of all $X\in S^2V^*\otimes S^2V^*$ which satisfy $X_{\alpha\beta\gamma\delta}=-X_{\gamma\delta\alpha\beta}$. Then we easily see that $p^2(\mathfrak{n})\cap\mathfrak{a}=\mathfrak{n}\otimes\mathfrak{n}\cap\mathfrak{a}$. Therefore to prove the propostition, it sufficies to show that $\dim(\mathfrak{n}\otimes\mathfrak{n}\cap\mathfrak{a})=\frac{1}{2}r(r-1)$. Let $X\in\mathfrak{n}\otimes\mathfrak{n}$, which may be expressed uniquely as follows: $X=\sum_{\lambda,\mu}a_{\lambda\mu}H^{\lambda}\otimes H^{\mu}$. As is easily observed, $X\in\mathfrak{a}$ if and only if $a_{\lambda\mu}+a_{\mu\lambda}=0$. This fact implies that $\dim(\mathfrak{n}\otimes\mathfrak{n}\cap\mathfrak{a})=\frac{1}{2}r(r-1)$. Q. E. D.

Proposition 2. 2. *If* $\dim p^2(\mathfrak{n})=\frac{1}{2}r(r-1)$, *then the space* $p^3(\mathfrak{n})$ *vanishes.*

Proof. Let $X\in p^3(\mathfrak{n})$. Then $X_{\varepsilon\alpha\beta\gamma\delta}$ is symmetric with respect to the following pairs of indices: (ε,α), (α,β) and (γ,δ). Furthermore $X_{\varepsilon\alpha\beta\gamma\delta}=-X_{\varepsilon\gamma\delta\alpha\beta}$ by Proposition 2. 1. It follows that $X_{\varepsilon\alpha\beta\gamma\delta}=X_{\alpha\varepsilon\beta\gamma\delta}=-X_{\alpha\gamma\delta\varepsilon\beta}=-X_{\gamma\alpha\delta\varepsilon\beta}=X_{\gamma\varepsilon\beta\alpha\delta}=X_{\varepsilon\gamma\beta\alpha\delta}$, showing that $X_{\varepsilon\alpha\beta\gamma\delta}$ is symmetric with respect to the pair (α,γ). Moreover, using this fact, we have $X_{\varepsilon\alpha\beta\gamma\delta}=X_{\varepsilon\gamma\beta\alpha\delta}=X_{\varepsilon\beta\gamma\delta\alpha}=X_{\varepsilon\delta\gamma\beta\alpha}=X_{\varepsilon\gamma\delta\alpha\beta}$. Thus $X_{\varepsilon\alpha\beta\gamma\delta}=X_{\varepsilon\gamma\delta\alpha\beta}$. Since $X_{\varepsilon\alpha\beta\gamma\delta}=-X_{\varepsilon\gamma\delta\alpha\beta}$, we get $X_{\varepsilon\alpha\beta\gamma\delta}=0$, i. e., $X=0$. Hence the space $p^3(\mathfrak{n})$ vanishes. Q. E. D.

Let $\mathfrak{n}$ be an r-dimensional subspace of S^2V^*. By definition the subspace $\mathfrak{h}=\mathfrak{n}+\wedge^2V^*$ of $\otimes^2V^*$ is of finite type if the k-th prolongation $\mathfrak{h}^{(k)}$ of $\mathfrak{h}$ vanishes for some $k>0$. Note that if $\mathfrak{h}$ is of finite type, then it is elliptic.

Definition. The subspace $\mathfrak{h}$ is said to be *of completely finite type* (for simply *of c. finite type*) if the dimension of the second prolongation $\mathfrak{h}^{(2)}(\cong p^2(\mathfrak{n}))$ is just equal to $\frac{1}{2}r(r-1)$ (and hence the third prolongation $\mathfrak{h}^{(3)}(\cong p^3(\mathfrak{n}))$ vanishes).

The next proposition will be proved in Appendix.

Proposition 2. 3. *Let* $\mathfrak{n}$ *be an* r*-dimensional subspace of* S^2V^*. *We put* $N=\frac{1}{2}n(n-1)$.

(1) *The case* $r\geqq N$. *The subspace* $\mathfrak{h}=\mathfrak{n}+\wedge^2V^*$ *of* $\otimes^2V^*$ *is of infinite type (i. e., not of finite type) except for* $r=n=0$ *or* $r=0$, $n=1$.

(2) *The case* $r<N$. *If* $0\leqq l\leqq\frac{N+r}{N-r}$, *then the* l*-th prolongation* $\mathfrak{h}^{(l)}$ *of* $\mathfrak{h}$ *does not vanish.*

Corollary. *If the subspace* $\mathfrak{h}$ *is of c. finite type, then* $r\leqq\frac{1}{4}n(n-1)$.

2. 2. Rigidity theorems. At the outset we recall the (global) rigidity theorem for elliptic immersions which was proved by one of the authors in [15]. In general let $\boldsymbol{f}\in\mathscr{I}(M, m)$ and assume that it is non-degenerate. Then we denote by $\mathscr{A}(M, \boldsymbol{f})$ the space of global solutions of the equation $L\varphi=0$, and denote by $\mathscr{A}_E(M, \boldsymbol{f})$ the subspace of $\mathscr{A}(M, \boldsymbol{f})$ consisting of all the special solutions φ^A, $A\in\mathfrak{e}(m)$.

Theorem 2.4 ([15], Theorem 2. 4). *Let $\boldsymbol{f}_0\in\mathscr{I}(M, m)$. Assume the following*: 1°. *$\boldsymbol{f}_0$ is elliptic*: 2°. *$\mathscr{A}(M, \boldsymbol{f}_0)=\mathscr{A}_E(M, \boldsymbol{f}_0)$* : 3°. *$M$ is compact. Then there is a neighborhood U of $\boldsymbol{f}_0$ (in $\mathscr{I}(M, m)$) with respect to the C^3-topology having the following property : If $\boldsymbol{f}_1$, $\boldsymbol{f}_2\in U$ and if they induce the same Riemannian metric, i. e., $\Phi(\boldsymbol{f}_1)=\Phi(\boldsymbol{f}_2)$, then there is a unique Euclidean transformations a of $\boldsymbol{R}^m$ such that $a\boldsymbol{f}_1=\boldsymbol{f}_2$.*

We shall now introduce the notion of an *immersion of completely finite type* and prove a (local) rigidity theorem for immersions of c. finite type.

Definition. An immersion $\boldsymbol{f}\in\mathscr{I}(M, m)$ is said to be *of finite type* (resp. *of c. finite type*) if it is non-degenerate and if the symbol $\mathfrak{h}$ of the equation R is of finite type (resp. of c. finite type), i. e., each fibre $\mathfrak{h}_p$ of $\mathfrak{h}$ is of finite type (resp. of c. finite type).

Proposition 2.5. *Let $\boldsymbol{f}\in\mathscr{I}(M, m)$. If $\boldsymbol{f}$ is of c. finite type and if M is connected, then the two spaces $\mathscr{A}(M, \boldsymbol{f})$ and $\mathscr{A}_E(M, \boldsymbol{f})$ coincide.*

Proof. We have $\mathfrak{h}^{(2)}=\bar{\mathfrak{h}}^2$ and $\mathfrak{h}^{(3)}=\bar{\mathfrak{h}}^3=0$. Therefore it follows from Proposition 1. 5 that $R=\bar{R}^0$ and $R^{(l)}=\bar{R}^l$ for all $l\geqq 1$. We have thus seen that the equation R is formally integrable and the equation $R^{(3)}$ is involutive (with vanishing symbol). Hence $\dim\mathscr{A}(M, \boldsymbol{f})\leqq\dim\bar{R}^2=\dim\mathfrak{e}(m)$. Thus we get $\mathscr{A}(M, \boldsymbol{f})=\mathscr{A}_E(M, \boldsymbol{f})$. Q. E. D.

Proposition 2.6. *If an immersion $\boldsymbol{f}_0\in\mathscr{I}(M, m)$ is of c. finite type, then there is a neighborhood U of $\boldsymbol{f}_0$ (in $\mathscr{I}(M, m)$) with respect to the C^2-topology such that every $\boldsymbol{f}\in U$ is of c. finite type.*

Proof. Let $J^2(M, m)$ be the vector bundle of all 2-jets of (local) differentiable maps of M to $\boldsymbol{R}^m$ and ρ the projection of $J^2(M, m)$ onto M. Let $\bar{M}$ be the image of M by the map $M\ni p\to j_p^2\boldsymbol{f}_0\in J^2(M, m)$. Since the immersion $\boldsymbol{f}_0$ is non-degenerate, there is a neighborhood O of $\bar{M}$ such that every $z\in O$ may be expressed as $z=j_p^2\boldsymbol{f}$, where $p=\rho(z)$ and $\boldsymbol{f}$ is a non-degenerate immersion of a neighborhood of p into $\boldsymbol{R}^m$. Let $z=j_p^2\boldsymbol{f}\in O$. We denote by $\mathfrak{n}(z)$ the space of second fundamental forms of $\boldsymbol{f}$ defined at p, being a subspace of $S^2T_p^*$. (Note that $\mathfrak{n}(z)$ depends only on z.) Furthermore we define a subspace $\mathfrak{h}(z)$ of $\otimes^2T_p^*$ by $\mathfrak{h}(z)=\mathfrak{n}(z)+\wedge^2T_p^*$. Since $\mathfrak{h}(j_p^2\boldsymbol{f}_0)$ is of c. finite type for all $p\in M$, we see from Lemma 1. 6 below that there is a neighborhood O' $(\subset O)$

of $\tilde{M}$ such that $\mathfrak{h}(z)$ is of *c.* finite type for all $z\in O'$. Now let U be the subset of $\mathscr{I}(M, m)$ consisting of all $\boldsymbol{f}\in\mathscr{I}(M, m)$ such that $j_p^2\,\boldsymbol{f}\in O'$ for all $p\in M$, which is a neighborhood of $\boldsymbol{f}_0$ with respect to the C^2-topology. Then it is clear that every $\boldsymbol{f}\in U$ is of *c.* finite type.

Q. E. D.

Lemma 2.7. *Let V be an n-dimensional vector space over $\boldsymbol{R}$ and G^r the Grassmann manifold of all r-dimensional subspaces $\mathfrak{n}$ of S^2V^*. Let G_0^r be the subset of G^r consisting of all $\mathfrak{n}\in G^r$ such that the subspaces $\mathfrak{h}=\mathfrak{n}+\wedge^2V^*$ of $\otimes^2V^*$ are of c. finite type. If $G_0^r\neq\phi$, then G_0^r is an open (dense) subset of G^r.*

By using Propositions 2.5 and 2.6 we can prove the following theorem just in the same way as in the proof of Theorem 2.4.

Theorem 2.8. *Let $\boldsymbol{f}_0\in\mathscr{I}(M, m)$. Assume that $\boldsymbol{f}_0$ is of c. finite type and that M is connected. Then there is a neighborhood U of $\boldsymbol{f}_0$ (in $\mathscr{I}(M, m)$) with respect to the C^2-topology having the following property: If $\boldsymbol{f}_1$, $\boldsymbol{f}_2\in U$ and if they induce the same Riemannian metric, then there is a unique Euclidean transformation a of $\boldsymbol{R}^m$ such that $\boldsymbol{f}_2=a\boldsymbol{f}_1$.*

2.3. Direct sums of spaces of symmetric forms. Let V_i, $1\leqq i\leqq k$, be k vector spaces over a field K and, for each i, let $\mathfrak{n}_i$ be a subspace of $S^2V_i^*$. If we put $V=\sum_i V_i$, we have $V^*=\sum_i V_i^*$. Thus the direct sum $\mathfrak{n}=\sum_i \mathfrak{n}_i$ is a subspace of S^2V^*. $\mathfrak{n}_i$ being a subspace of $V_i^*\otimes V_i^*$, we may speak of the first prolongation $\mathfrak{n}_i^{(1)}$ of $\mathfrak{n}_i$:

$$\mathfrak{n}_i^{(1)}=V_i^*\otimes\mathfrak{n}_i\cap S^2V_i^*\otimes V_i^*.$$

We shall prove the following

Proposition 2.9. *Assume that $\mathfrak{n}_i^{(1)}=0$ for each i. Then the subspace $\mathfrak{h}=\mathfrak{n}+\wedge^2V^*$ of $\otimes^2V^*$ is of c. finite type if and only if the subspace $\mathfrak{h}_i=\mathfrak{n}_i+\wedge^2V_i^*$ of $\otimes^2V_i^*$ is of c. finite type for each i.*

Proof. Let $\mathfrak{n}'=\sum'\mathfrak{n}_i$ be a sum of $\mathfrak{n}_i$'s. Then the first prolongation $\mathfrak{n}'^{(1)}$ of $\mathfrak{n}'$ being $\sum'\mathfrak{n}_i^{(1)}$, vanishes. Accordingly to prove the proposition, it suffices to deal with the case where $k=2$.

The proof is based on Proposition 2.1. Thus we consider the spaces $p^2(\mathfrak{n})$ $(\cong\mathfrak{h}^{(2)})$ and $p^2(\mathfrak{n}_i)$ $(\cong\mathfrak{h}_i^{(2)})$ associated with $\mathfrak{n}$ and $\mathfrak{n}_i$ respectively. Note that $p^2(\mathfrak{n}_i)$ are subspaces of $p^2(\mathfrak{n})$. We put $n_i=\dim V_i(i=1, 2)$. We fix a basis $\{e_1, \dots, e_{n_1+n_2}\}$ of V such that $e_1,\dots, e_{n_1}$(resp. $e_{n_1+1}, \dots, e_{n_1+n_2}$) form a basis of V_1(resp. of V_2), and express covariant tensors (on V or V_i) in terms of this basis. We further promise that the indices λ, μ, ν, σ run over the range $1, \dots, n_1+n_2$, the indices α_1, β_1, γ_1, δ_1 over the range $1, \ldots, n_1$ and indces α_2, β_2, γ_2, δ_2 over the range $n_1+1, \ldots, n_1+n_2$.

First assume that $\mathfrak{h}$ is of c. finite type. Then have $X_{\lambda\mu\nu\sigma}=-X_{\nu\sigma\lambda\mu}$ for all $X\in p^2(\mathfrak{n})$ (Proposition 2.1). Since $p^2(\mathfrak{n}_i)\subset p^2(\mathfrak{n})$ it follows that $X_{\alpha_i\beta_i\gamma_i\delta_i}=-X_{\gamma_i\delta_i\alpha_i\beta_i}$ for all $X\in p^2(\mathfrak{n}_i)$ $(i=1, 2)$. This means that both $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are of c. finite type (Proposition 2.1).

Conversely assume that both $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are of c. finite type. By Lemma 2.10 below we have $X_{\lambda\mu\nu\sigma}=-X_{\nu\sigma\lambda\mu}$ for all $X\in p^2(\mathfrak{n})$ meaning that $\mathfrak{h}$ is of c. finite type.

Lemma 2.10. *Let* $X\in p^2(\mathfrak{n})$.

(1) $X_{\alpha_i\beta_i\gamma_i\delta_i}=-X_{\gamma_i\delta_i\alpha_i\beta_i} (i=1, 2)$.

(2) $X_{\lambda\mu\gamma_1\delta_2}=0$.

(3) $X_{\alpha_1\beta_1\gamma_2\delta_2}=-X_{\gamma_2\delta_2\alpha_1\beta_1}$.

(4) $X_{\alpha_1\beta_2\gamma_2\delta_2}=X_{\alpha_1\beta_2\gamma_1\delta_1}=0$.

Proof. (1) Fixing i, we define an element X^i of $S^2V_i^*\otimes\mathfrak{n}_i$ by $(X^i)_{\alpha_i\beta_i\gamma_i\delta_i}=X_{\alpha_i\beta_i\gamma_i\delta_i}$, which clearly belongs to $p^2(\mathfrak{n}_i)$. Since $\mathfrak{h}_i$ is of c. finite type, we have $(X^i)_{\alpha_i\beta_i\gamma_i\delta_i}=-(X^i)_{\gamma_i\delta_i\alpha_i\beta_i}$. Hence $X_{\alpha_i\beta_i\gamma_i\delta_i}=-X_{\gamma_i\delta_i\alpha_i\beta_i}$, proving (1).

(2) is clear.

(3) Since

$$X_{\alpha_1\beta_1\gamma_2\delta_2}-X_{\alpha_1\delta_2\gamma_2\beta_1}=X_{\gamma_2\beta_1\alpha_1\delta_2}-X_{\gamma_2\delta_2\alpha_1\beta_1},$$

it follows from (2) that $X_{\alpha_1\beta_1\gamma_2\delta_2}=-X_{\gamma_2\delta_2\alpha_1\beta_1}$, proving (3).

(4) Since

$$X_{\alpha_1\beta_2\gamma_2\delta_2}-X_{\alpha_1\delta_2\gamma_2\beta_2}=X_{\gamma_2\beta_2\alpha_1\delta_2}-X_{\gamma_2\delta_2\alpha_1\beta_2},$$

it follows from (2) that $X_{\alpha_1\beta_2\gamma_2\delta_2}=X_{\alpha_1\delta_2\gamma_2\beta_2}$. Fixing α_1, we define an element Y^{α_1} of $V_2^*\otimes\mathfrak{n}_2$ by $(Y^{\alpha_1})_{\beta_2\gamma_2\delta_2}=X_{\alpha_1\beta_2\gamma_2\delta_2}$. Then the equality just above means that Y^{α_1} belongs to $\mathfrak{n}_2^{(1)}$. Since $\mathfrak{n}_2^{(1)}=0$, we have $Y^{\alpha_1}=0$, i.e., $X_{\alpha_1\beta_2\gamma_2\delta_2}=0$. Similarly we get $X_{\alpha_1\beta_2\gamma_1\delta_1}=0$, proving (4).

Q. E. D.

For each i, $1\leqq i\leqq k$, let $\boldsymbol{f}_i$ be a non-degenerate immersion of a manifold M_i into the m_i-dimensional Euclidean space $\boldsymbol{R}^{m_i}$, and let $\mathfrak{n}_i$ be the bundle of second fundamental forms of $\boldsymbol{f}_i$. Let $\mathfrak{n}_i^{(1)}$ be the first prolongation of $\mathfrak{n}_i$:

$$\mathfrak{n}_i^{(1)}=T_i^*\otimes\mathfrak{n}_i\cap S^2T_i^*\otimes T_i^*,$$

T_i being the tangent bundle of M_i.

Corollary. *Assume that each* $\mathfrak{n}_i^{(1)}$ *vanishes. Then the product immersion*

$$f_1\times\ldots\times f_k:\ M_1\times\ldots\times M_k\to \boldsymbol{R}^m\ (m=\sum_i m_i)$$

is of c. finite type if and only if each f_i *is of c. finite type.*

2. 4. The type number. Let V be an n-dimensional vector space over a field K, and $\mathfrak{n}$ an r-dimensional subspace of S^2V^*. Let $\{H^1, \ldots, H^r\}$ be a basis of $\mathfrak{n}$. By definition the *type number* of $\mathfrak{n}$ is the largest integer k for which there are k vectors $u_1, \ldots, u_k$ in V such that rk covectors $u_i \lrcorner H^\lambda$, $1\leqq\lambda\leqq r$, $1\leqq i\leqq k$, are linearly independent.

Proposition 2. 11. (1) *If the type number is larger than or equal to* 2, *then the first prolongation* $\mathfrak{n}^{(1)}$ *of* $\mathfrak{n}$ *vanishes.*

(2) (cf. [2] and [4]) *If the type number is larger than or equal to* 3, *then the subspace* $\mathfrak{h}=\mathfrak{n}+\wedge^2V^*$ *of* $\otimes^2V^*$ *is of c. finite type.*

Proof. (1) By the assumption there are two vectors u_1 and u_2 in V such that $2r$ covectors $u_1\lrcorner H^\lambda$, $u_2\lrcorner H^\mu$, $1\leqq\lambda$, $\mu\leqq r$, are linearly independent. Let $X\in\mathfrak{n}^{(1)}$. Then we have $u_1\lrcorner X$, $u_2\lrcorner X\in\mathfrak{n}$ and $u_1\lrcorner u_2\lrcorner X=u_2\lrcorner u_1\lrcorner X$. Hence $u_i\lrcorner X$ may be expressed as $u_i\lrcorner X=\sum_\lambda a_\lambda^i H^\lambda$, and $\sum_\lambda a_\lambda^2 u_1\lrcorner H^\lambda=\sum_\lambda a_\lambda^1 u_2\lrcorner H^\lambda$. It follows that $a_\lambda^i=0$, i. e., $u_i\lrcorner X=0$. Now for any $u\lrcorner X$ we have $u_1\lrcorner(u\lrcorner X)=u\lrcorner(u_1\lrcorner X)=0$. Then it follows as above that $u\lrcorner X=0$. Hence $X=0$, proving (1).

(2) The space $p^2(\mathfrak{n})$ may be identified with the subspace of $(S^2V^*)^r$ consisting of all $(A^1, \ldots, A^r)\in(S^2V^*)^r$ such that

$$\sum_\lambda (u\lrcorner A^\lambda)\wedge(v\lrcorner H^\lambda)=\sum_\lambda (v\lrcorner A^\lambda)\wedge(u\lrcorner H^\lambda)$$

(See 2. 1.) Now by the assumption there are three vectors u_1, u_2 and u_3 in V such that $3r$ covectors $u_i\lrcorner H^\lambda$, $1\leqq i\leqq 3$, $1\leqq\lambda\leqq r$, are linearly independent. Let $(A^1, \ldots, A^r)\in p^2(\mathfrak{n})$. Then we have

$$\sum_\lambda (u_i\lrcorner A^\lambda)\wedge(u_j\lrcorner H^\lambda)=\sum_\lambda (u_j\lrcorner A^\lambda)\wedge(u_i\lrcorner H^\lambda).$$

Applying Cartan's lemma (cf. [11]) to these equalities, we see that $u_i\lrcorner A^\lambda$ may be expressed as follows:

$$u_i\lrcorner A^\lambda=\sum_\mu {}^i a_\mu^\lambda(u_i\lrcorner H^\mu),$$

where ${}^ia^\lambda_\mu = -{}^ja^\mu_\lambda$ if $i \neq j$. If we put $a^\lambda_\mu = {}^1a^\lambda_\mu$, we have ${}^ia^\lambda_\mu = a^\lambda_\mu$ and $a^\lambda_\mu = -a^\mu_\lambda$. We now assert that $A^\lambda = \sum_\mu a^\lambda_\mu H^\mu$. Indeed let u be any vector in V. Then we have

$$\sum_\lambda (u \rfloor A^\lambda) \wedge (u_i \rfloor H^\lambda) = \sum_\lambda (u_i \rfloor A^\lambda) \wedge (u \rfloor H^\lambda),$$

whence

$$\sum_\lambda (u \rfloor A^\lambda - \sum_\mu a^\lambda_\mu (u \rfloor H^\mu)) \wedge (u_i \rfloor H^\lambda) = 0.$$

It follows that $u \rfloor A^\lambda - \sum_\mu a^\lambda_\mu (u \rfloor H^\mu) = 0$, proving our assertion. We have thus seen that $\dim p^2(\mathfrak{n}) \leqq \frac{1}{2} r(r-1)$. Since $\dim p^2(\mathfrak{n}) \geqq \frac{1}{2} r(r-1)$, this proves (2).

2.5. Spaces of hermitian forms. Let V be a $2n$-dimensional vector space over $\boldsymbol{R}$ and I a complex structure on V. Let V^c be the complexification of V. (In general, W^c denotes the complexification of a vector space W over $\boldsymbol{R}$.) As usual we define subspaces V^+ and V^- of V^c respectively as follows:

$$V^+ = \{X \in V^c \mid IX = \sqrt{-1}\ X\}$$

$$V^- = \{X \in V^c \mid IX = -\sqrt{-1}\ X\}.$$

Then we have $V^- = \bar{V}^+$ and $V^c = V^+ + V^-$ (direct sum).

By definition a symmetric form H on V is *hermitian* if $H(Iu, Iv) = H(u, v)$ for all $u, v \in V$. This condition is equivalent to the condition that $H(u, v) = 0$ (or $H(\bar{u}, \bar{v}) = 0$) for all $u, v \in V^+$, where H should be considered as an element of $(S^2 V^*)^c = S^2(V^c)^*$.

Now let $\mathfrak{n}$ be a space of hermitian forms on V. For any $H \in \mathfrak{n}^c$ we define an element $\tilde{H}$ of $(V^+)^* \otimes (V^-)^*$ by $\tilde{H}(u, \bar{v}) = H(u, \bar{v})$ for all $u, v \in V^+$. We denote by $\tilde{\mathfrak{n}}$ the subspace of $(V^+)^* \otimes (V^-)^*$ consisting of all $\tilde{H}$, and denote by $\tilde{\mathfrak{n}}^{(2)}$ the second prolongation $\tilde{\mathfrak{n}}$:

$$\tilde{\mathfrak{n}}^{(2)} = S^2(V^+)^* \otimes \tilde{\mathfrak{n}} \cap S^3(V^+)^* \otimes (V^-)^*.$$

With these preparations we have the following

Proposition 2.12. (1) *The first prolongation $\mathfrak{n}^{(1)}$ of $\mathfrak{n}$ vanishes.*

(2) *The subspace $\mathfrak{h} = \mathfrak{n} + \wedge^2 V^*$ of $\otimes^2 V^*$ is of c. finite type if and only if the second prolongation $\tilde{\mathfrak{n}}^{(2)}$ of $\tilde{\mathfrak{n}}$ vanishes.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of V^+. If we put $e_{\bar{i}} = \bar{e}_i$, then the $2n$ vectors $e_1, \ldots, e_n, e_{\bar{1}}, \ldots, e_{\bar{n}}$ form a basis of V^c. We express covariant tensors on V^c in terms of this basis. We further promise that the indices λ, μ, ν, σ run over the range $1, \dots, n, \bar{1}, \dots, \bar{n}$ and the indices α, β, γ, δ over the range $1, \dots,$ n and that $\bar{\bar{\alpha}}=\alpha$.

(1) Let $X\in\mathfrak{n}^{(1)}\subset\otimes^3(V^c)^*$. Then X satisfies the following: $\bar{X}_{\lambda\mu\nu}=X_{\bar{\lambda}\bar{\mu}\bar{\nu}}$, $X_{\lambda\alpha\beta}=0$ and $X_{\lambda\mu\nu}=X_{\lambda\nu\mu}=X_{\mu\lambda\nu}$. It follows immediately that $X_{\lambda\mu\nu}=0$, i.e., $X=0$. Hence $\mathfrak{n}^{(1)}=0$, proving (1).

(2) Let $X\in p^2(\mathfrak{n})\subset\otimes^4(V^c)^*$. Then X satisfies the following: $\bar{X}_{\lambda\mu\nu\sigma}=X_{\bar{\lambda}\bar{\mu}\bar{\nu}\bar{\sigma}}$, $X_{\lambda\mu\alpha\beta}=0$, $X_{\lambda\mu\nu\sigma}=X_{\mu\lambda\nu\sigma}=X_{\lambda\mu\sigma\nu}$, and

$$X_{\lambda\mu\nu\sigma}-X_{\lambda\sigma\nu\mu}=X_{\nu\mu\lambda\sigma}-X_{\nu\sigma\lambda\mu}.$$

We consider this last equality in the following two cases: 1°. $\lambda=\alpha$, $\mu=\beta$, $\nu=\gamma$, and $\sigma=\bar{\delta}$; 2°. $\lambda=\bar{\beta}$, $\mu=\alpha$, $\nu=\gamma$ and $\sigma=\bar{\delta}$. Then we obtain $X_{\alpha\beta\gamma\bar{\delta}}=X_{\gamma\beta\alpha\bar{\delta}}$ and $X_{\alpha\bar{\beta}\gamma\bar{\delta}}=-X_{\gamma\bar{\delta}\alpha\bar{\beta}}$. Let us now define an element $\tilde{X}$ of $\otimes^3(V^+)^*\otimes(V^-)^*$ by $(\tilde{X})_{\alpha\beta\gamma\delta}=X_{\alpha\beta\gamma\bar{\delta}}$. Then we can easily verify the following: 1°. $\tilde{X}\in\tilde{\mathfrak{n}}^{(2)}$; 2°. The linear map (over $\boldsymbol{R}$) $p^2(\mathfrak{n})\ni X\rightarrow\tilde{X}\in\tilde{\mathfrak{n}}^{(2)}$ is suvjective; 3°. The kernel of this map consists of all $X\in p^2(\mathfrak{n})$ with $X_{\lambda\mu\nu\sigma}=-X_{\nu\sigma\lambda\mu}$. Therefore we see from Proposition 2.1 that $\mathfrak{h}$ is of c. finite type if and only if $\tilde{\mathfrak{n}}^{(2)}$ vanishes. Q. E. D.

§ 3. The canonical isometric imbeddings of symmetric $\boldsymbol{R}$ spaces.

3.1. Symmetric $\boldsymbol{R}$ spaces. Let $\mathfrak{g}=\sum_p\mathfrak{g}_p$ be a graded Lie algebra over $\boldsymbol{R}$ which satisfies the following conditions: 1°. $\mathfrak{g}$ is simple; 2°. $\mathfrak{g}_p=0$ for $|p|>1$; 3°. $\mathfrak{g}_{-1}\neq0$. (Such a graded Lie algebra will be called a simple graded Lie algebra of the first kind.) As is well known, there is a unique element Z_0 in the centre of $\mathfrak{g}_0$ such that $\mathfrak{g}_p=\{X\in\mathfrak{g}\,|\,[Z_0, X]=pX\}$ for all p (cf. [8]).

Remark (cf. [8]). Accordingly a simple graded Lie algebra of the first kind may be represented by the pair $(\mathfrak{g}, Z_0)$ of a simple Lie algebra $\mathfrak{g}$ and an element Z_0 of $\mathfrak{g}$ such that ad Z_0 is a semi-simple endomorphism of $\mathfrak{g}$ with eigenvalues -1, 0 and 1.

Let B be the Killing form of $\mathfrak{g}$. Then the following two lemmas are also well known.

Lemma 3.1 (cf. [8]). *$B(\mathfrak{g}_p, \mathfrak{g}_q)=0$ if $p+q\neq0$, and, for each p, the bilinear map $\mathfrak{g}_p\times\mathfrak{g}_{-p}\ni(X, Y)\rightarrow B(X, Y)\in\boldsymbol{R}$ is non-degenerate.*

Lemma 3.2 (cf. [8]). *There is an involutive automorphism σ of $\mathfrak{g}$ such that $\sigma Z_0=-Z_0$ or $\sigma(\mathfrak{g}_p)=\mathfrak{g}_{-p}$ for all p and such that the quadratic form $B(X, \sigma X)$, $X\in\mathfrak{g}$, is negative definite.*

By using the Killing form B and the involutive automorphism σ, we define an inner product $<,>$ in $\mathfrak{g}$ by

$$<X, Y>=-B(X, \sigma Y) \qquad \text{for all } X, Y\in\mathfrak{g}.$$

Then it is clear that the decomposition $\mathfrak{g}=\sum_p \mathfrak{g}_p$ is orthogonal with respect to the inner product $<,>$.

We now define subspaces $\mathfrak{k}$ and $\mathfrak{m}$ of $\mathfrak{g}$ respectively as follows:

$$\mathfrak{k}=\{X\in\mathfrak{g} \mid \sigma X=X\},$$

$$\mathfrak{m}=\{X\in\mathfrak{g} \mid \sigma X=-X\}.$$

Then we have $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ (orthogonal), and $[\mathfrak{k}, \mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}]\subset\mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}]\subset\mathfrak{k}$. Furthermore we define subspaces $\mathfrak{k}_0$, $\mathfrak{k}_1$, $\mathfrak{m}_0$ and $\mathfrak{m}_1$ of $\mathfrak{g}$ respectively as follows:

$$\mathfrak{k}_0=\mathfrak{k}\cap\mathfrak{g}_0, \quad \mathfrak{k}_1=\mathfrak{k}\cap(\mathfrak{g}_{-1}+\mathfrak{g}_1),$$

$$\mathfrak{m}_0=\mathfrak{m}\cap\mathfrak{g}_0, \quad \mathfrak{m}_1=\mathfrak{m}\cap(\mathfrak{g}_{-1}+\mathfrak{g}_1).$$

Then we have $\mathfrak{k}=\mathfrak{k}_0+\mathfrak{k}_1$ (orthogonal) and $\mathfrak{m}=\mathfrak{m}_0+\mathfrak{m}_1$ (orthogonal), and $[\mathfrak{k}_0, \mathfrak{k}_0]\subset\mathfrak{k}_0$, $[\mathfrak{k}_0, \mathfrak{k}_1]\subset\mathfrak{k}_1$, $[\mathfrak{k}_1, \mathfrak{k}_1]\subset\mathfrak{k}_0$, etc.

Now consider the adjoint group $Ad(\mathfrak{g})$ of the simple Lie algebra $\mathfrak{g}$. As usual the Lie algebra of $Ad(\mathfrak{g})$ may be identified with $\mathfrak{g}$. We denote by K the Lie subgroup of $Ad(\mathfrak{g})$ generated by the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, which is a maximal compact subgroup of $Ad(\mathfrak{g})$. We also denote by K_0 the subgroup of K consisting of all $a\in K$ which satisfy the equality $Ad(a)Z_0=Z_0$ or $Ad(a)\mathfrak{g}_p=\mathfrak{g}_p$ for all p. Here we note that the Lie algebra of K (resp. of K_0) is $\mathfrak{k}$ (resp. $\mathfrak{k}_0$).

We have thus obtained the homogeneous space

$$M=K/K_0.$$

(Note that the action of K on M is effective.) Since $\mathfrak{k}/\mathfrak{k}_0\cong\mathfrak{k}_1$, the tangent space $T_0=T(M)_0$ to M at the origin o may be identified with $\mathfrak{k}_1$. This being said, there is a unique K-invariant Riemannian metric g on M such that $g(X, Y)=<X, Y>$ for all $X, Y\in T_0=\mathfrak{k}_1$. We are now in a position to assert that $M=K/K_0$ is a homogeneous symmetric space with respect to the Riemannian metric g (cf. [12]). Following [12], the symmetric space M thus obtained will be called a symmetric R space.

3.2. The canonical isometric imbeddings. We have $Z_0\in\mathfrak{m}$, $Ad(K)\mathfrak{m}\subset\mathfrak{m}$ and $K_0=\{a\in K \mid Ad(a)\,Z_0=Z_0\}$. Hence the map $K\ni a\rightarrow$

$Ad(a)Z_0 \in \mathfrak{m}$ induces an imbedding of M into $\mathfrak{m}$, which we denote by $\boldsymbol{f}$. Clearly $\boldsymbol{f}$ is K-equivariant, i.e., $Ad(a)\boldsymbol{f}(p) = \boldsymbol{f}(ap)$ for all $a \in K$ and $p \in M$. Let ∇ be the covariant differentiation associated with the Riemannian metric g.

Lemma 3.3 (cf. [15], Lemma 3.2). *Let* $X, Y \in T_0 = \mathfrak{k}_1$.

(1) $\nabla_X \boldsymbol{f} = [X, Z_0]$ $(\in \mathfrak{m}_1)$.

(2) $\nabla_X \nabla_Y \boldsymbol{f} = [X, [Y, Z_0]]$ $(\in \mathfrak{m}_0)$.

Now if we put $m = \dim \mathfrak{m}$, $\mathfrak{m}$ may be regarded as the Euclidean space $\boldsymbol{R}^m$ with respect to a fixed orthonormal basis $X_1, \ldots, X_m$ of $\mathfrak{m}$. Using (1) of Lemma 3.3, we have

$$\begin{aligned} g(X, Y) &= \langle X, Y \rangle = \langle [X, Z_0], [Y, Z_0] \rangle \\ &= \langle \nabla_X \boldsymbol{f}, \nabla_Y \boldsymbol{f} \rangle. \end{aligned}$$

Since $\boldsymbol{f}$ is K-equivariant, this means that $\boldsymbol{f}$ is an isometric imbedding of the R space M to the Euclidean space $\boldsymbol{R}^m = \mathfrak{m}$ (cf. [12]).

Let N be the normal bundle of the imbedding $\boldsymbol{f}$. We have $\boldsymbol{f}_* T_0 = [\mathfrak{k}_1, Z_0] = \mathfrak{m}_1$, and hence the fibre N_0 of N at the origin coincides with the orthogonal complement $\mathfrak{m}_0$ of $\mathfrak{m}_1$ in $\mathfrak{m}$.

Proposition 3.4. *The canonical imbedding* $\boldsymbol{f}$ *is non-degenerate.*

Proof. We have $\mathfrak{k}_1 = \{X + \sigma X \mid X \in \mathfrak{g}_{-1}\}$ and $\mathfrak{m}_0 = \{X - \sigma X \mid X \in \mathfrak{g}_0\}$. And for any $X, Y \in \mathfrak{g}_{-1}$, we have $[X + \sigma X, [Y + \sigma Y, Z_0]] = [\sigma X, Y] - \sigma[\sigma X, Y]$. Since $\mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_{-1}]^{(*)}$, it follows that $\mathfrak{m}_0 = [\mathfrak{k}_1, [\mathfrak{k}_1, Z_0]]$. Therefore we see from (2) of Lemma 3.3 that N_0 is spanned by the vectors of the form $\nabla_X \nabla_Y \boldsymbol{f}$, $X, Y \in T_0$. Since $\boldsymbol{f}$ is K-equivariant, this means that $\boldsymbol{f}$ is non-degenerate. Q. E. D.

As is easily observed, the bundle $\mathfrak{n} = \Theta(N)$ of second fundamental forms is K-invariant. It follows that the operator L and hence the equation R, associated with $\boldsymbol{f}$, are K-invariant.

Let us consider the symbol $\mathfrak{h} = \mathfrak{n} + \wedge^2 T^*$ of the equation R, being K-invariant. If we identify T_0 and $\mathfrak{k}_1$ as before, then the fibre $\mathfrak{n}_0$ of $\mathfrak{n}$ at the origin may be identified with a subspace of $S^2\mathfrak{k}_1^*$, the fibre $\mathfrak{h}_0$ of $\mathfrak{h}$ at the origin with a subspace of $\otimes^2\mathfrak{k}_1^*$, and

$$\mathfrak{h}_0 = \mathfrak{n}_0 + \wedge^2 \mathfrak{k}_1^*.$$

Hereafter the fibres $\mathfrak{n}_0$ and $\mathfrak{h}_0$ will be simply written as $\mathfrak{n}$ and $\mathfrak{h}$

(*) This follows from the fact that $\mathfrak{a} = \mathfrak{g}_{-1} + [\mathfrak{g}_1, \mathfrak{g}_{-1}] + \mathfrak{g}_1$ is an ideal of $\mathfrak{g}$.

respectively.

We shall now describe the spaces $\mathfrak{n}$ and $\mathfrak{h}$ in terms of the space $\mathfrak{m}_0$. For this purpose we first define a linear isomorphism η of $\mathfrak{g}_{-1}$ onto $\mathfrak{k}_1$ by

$$\eta(X)=\frac{1}{\sqrt{2}}(X+\sigma X) \quad \text{for all } X\in\mathfrak{g}_{-1}.$$

It is easy to see that η is isometric, i. e.,

$$<\eta(X),\ \eta(Y)>=<X,\ Y> \quad \text{for all } X,\ Y\in\mathfrak{g}_{-1}.$$

Lemma 3. 5. *Let $A\in N_0=\mathfrak{m}_0$. Then the second fundamental form $\Theta(A)$ corresponding to the normal vector A may be expressed as follows:*

$$\Theta(A)(\eta(X),\ \eta(Y))=<[A,\ X],\ Y>=<A,\ [X,\ \sigma Y]> \quad \textit{for all } X,\ Y\in\mathfrak{g}_{-1}.$$

Proof. By (2) of Lemma 3. 3 we have

$$\Theta(A)(\eta(X),\ \eta(Y))=<A,\ \nabla_{\eta(X)}\nabla_{\eta(Y)}\boldsymbol{f}>$$
$$=<A,\ [\eta(X),\ [\eta(Y),\ Z_0]]>,$$

which can be shown to be equal to $<A,\ [X,\ \sigma Y]>=<[A,\ X],\ Y>$.

Q. E. D.

We define a representation ρ of the Lie algebra $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ by

$$\rho(A)X=[A,\ X] \quad \text{for all } A\in\mathfrak{g}_0 \text{ and } X\in\mathfrak{g}_{-1},$$

which is faithful (cf. [14]). We have $\rho(\mathfrak{m}_0)=\rho(\mathfrak{g}_0)\cap\mathfrak{s}(\mathfrak{g}_{-1})$ and $\rho(\mathfrak{k}_0)=\rho(\mathfrak{g}_0)\cap\mathfrak{o}(\mathfrak{g}_{-1})$, where $\mathfrak{s}(\mathfrak{g}_{-1})$ (resp. $\mathfrak{o}(\mathfrak{g}_{-1})$) denotes the space of all symmetric (resp. skew-symmetric) endomorphisms of $\mathfrak{g}_{-1}$ with respect to the inner product $<,>$. Let us now define an (injective) linear map $\check{\rho}$ of $\mathfrak{g}_0$ to $\otimes^2\mathfrak{g}_{-1}^*$ by

$$\check{\rho}(A)(X,\ Y)=<\rho(A)X,\ Y> \quad \text{for all } A\in\mathfrak{g}_0 \text{ and } X,\ Y\in\mathfrak{g}_{-1}.$$

Then it follows that $\check{\rho}(\mathfrak{m}_0)=\check{\rho}(\mathfrak{g}_0)\cap S^2\mathfrak{g}_{-1}^*$ and $\check{\rho}(\mathfrak{k}_0)=\check{\rho}(\mathfrak{g}_0)\cap\wedge^2\mathfrak{g}_{-1}^*$. Now the isomorphism η of $\mathfrak{g}_{-1}$ onto $\mathfrak{k}_1$ induces the isomorphism η^* of $\otimes^2\mathfrak{k}_1^*$ onto $\otimes^2\mathfrak{g}_{-1}^*$: $(\eta^*\alpha)(X,\ Y)=\alpha(\eta(X),\ \eta(Y))$ for all $\alpha\in\otimes^2\mathfrak{k}_1^*$ and $X,\ Y\in\mathfrak{g}_{-1}$.

Proposition 3. 6. (1) $\eta^*\Theta(A)=\check{\rho}(A),\ A\in\mathfrak{m}_0$.

(2) $\eta^*\mathfrak{n}=\check{\rho}(\mathfrak{m}_0)$.

(3) $\eta^*\mathfrak{h}=\not{p}(\mathfrak{m}_0)+\wedge^2\mathfrak{g}_{-1}^*$.

Proof. (1) is clear from Lemma 3.5. We have $\mathfrak{n}=\Theta(\mathfrak{m}_0)$ and $\mathfrak{h}=\mathfrak{n}+\wedge^2\mathfrak{k}_1^*$. Therefore (2) and (3) immediately follow from (1).

Q. E. D.

Corollary. *The canonical imbedding* $\boldsymbol{f}$ *of* $M=K/K_0$ *into* $\mathfrak{m}$ *is involutive (resp. elliptic ; resp. of finite type ; resp. of c. finite type) if and only if the same holds for the subspace* $\not{p}(\mathfrak{m}_0)+\wedge^2\mathfrak{g}_{-1}^*$ *of* $\otimes^2\mathfrak{g}_{-1}^*$.

The rest of the section will be devoted to the study of the canonical imbeddings of the following R spaces:

(I) Irreducible hermitian symmetric spaces of compact type.

(II) The real Grassmann manifolds $G^{p,q}(\boldsymbol{R})$.

3. 3. Irreducible hermitian symmetric spaces of compact type.

Let $M=K/K_0$ be an irreducible hermitian symmetric space of compact type, where K is the largest group of automorphisms (holomorphic isometries) of M, and K_0 is the isotropy group of K at a fixed point o of M. It is well known that $M=K/K_0$ may be represented by a symmetric R space (cf. [14]), which we first explain.

Let $\mathfrak{k}$ (resp. $\mathfrak{k}_0$) be the Lie algebra of K (resp. of K_0). Let $B_{\mathfrak{k}}$ be the Killing form of $\mathfrak{k}$, and $\mathfrak{k}_1$ the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{k}$ with respect to $B_{\mathfrak{k}}$: $\mathfrak{k}=\mathfrak{k}_0+\mathfrak{k}_1$ (direct sum). Furthermore let I be the tensor field of type $\binom{1}{1}$ on M representing the complex structure of M. Then there is a unique element I_0 in the centre of $\mathfrak{k}_0$ such that $IX=[I_0, X]$ for all $X\in T_0=\mathfrak{k}_1$, where $T_0=T(M)_0$ is identified with $\mathfrak{k}_1$ as before. Let us now consider the complexification $\mathfrak{g}=\mathfrak{k}^c$ of the Lie algebra $\mathfrak{k}$ as well as its subspaces $\mathfrak{g}_{-1}=\mathfrak{k}_1^+$, $\mathfrak{g}_0=\mathfrak{k}_0^c$ and $\mathfrak{g}_1=\mathfrak{k}_1^-$, where $\mathfrak{k}_1^{\pm}=\{X\in\mathfrak{k}_1^c \mid [I_0, X]=\pm\sqrt{-1}\ X\}$. Then we see that the subspaces $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ together give the structure of a simple graded Lie algebra of the first kind, in the Lie algebra $\mathfrak{g}$.

Let σ be the conjugation of $\mathfrak{g}=\mathfrak{k}^c$, which is an involutive automorphism of the Lie algebra $\mathfrak{g}$ and which has the properties in Lemma 3.2. Clearly we have $\mathfrak{k}=\{X\in\mathfrak{g} \mid \sigma X=X\}$ and $\mathfrak{k}_0=\mathfrak{g}_0\cap\mathfrak{k}$. Moreover we know that the centre of the group K is reduced to the identity element. We have thus seen that the given hermitian symmetric space $M=K/K_0$ is represented by the symmetric R space which is associated with the simple graded Lie algebra of the first kind, $\mathfrak{g}=\sum_p \mathfrak{g}_p$, (together with the involutive automorphism σ). Then notations being as in 3.1, we furhter see that $\mathfrak{m}=\sqrt{-1}\ \mathfrak{k}$, $\mathfrak{m}_0=\sqrt{-1}\ \mathfrak{k}_0$, $\mathfrak{m}_1=\sqrt{-1}\ \mathfrak{k}_1$ and

$$<X, Y>=-2\ \mathrm{Re}\ B_{\mathfrak{k}}(X, Y) \quad \text{for all } X, Y\in\mathfrak{g}.$$

Theorem 3. 7. *The canonical imbedding* $\boldsymbol{f}$ *of the irreducible hermitian symmetric space* $M=K/K_0$ *into the Euclidean space* $\mathfrak{m}$ *is of c. finite type if and only if* M *is not a complex projective space.*

Proof. Consider the subspace $\mathfrak{h}=\mathfrak{n}+\wedge^2\mathfrak{k}_1^*$ of $\otimes^2\mathfrak{k}_1^*$. We first remark that $\mathfrak{n}=\Theta(\mathfrak{m}_0)$ is a space of real hermitian forms of $\mathfrak{k}_1=T_0$ ([15]). (Indeed we have $\Theta(A)(X, Y)=\langle A, [X, [Y, \sqrt{-1}\ I_0]]\rangle$ for all $A\in\mathfrak{m}_0$ and $X, Y\in\mathfrak{k}_1$. We have $[[I_0, X], [[I_0, Y], \sqrt{-1}\ I_0]]=[X, [Y, \sqrt{-1}\ I_0]]$. Hence $\Theta(A)([I_0, X], [I_0, Y])=\Theta(A)(X, Y)$.) Thus we may apply the arguments in 2. 5 to the space $\mathfrak{n}$. Let us consider the subspace $\tilde{\mathfrak{n}}$ of $(\mathfrak{k}_1^+)\otimes(\mathfrak{k}_1^-)^*$. Then we see from Proposition 2. 12 that $\mathfrak{h}$ is of c. finite type if and only if the second prolongation $\tilde{\mathfrak{n}}^{(2)}$ of $\tilde{\mathfrak{n}}$ vanishes. Since $\mathfrak{m}_0^c=\mathfrak{g}_0$, the linear map Θ of $\mathfrak{m}_0$ to $S^2\mathfrak{k}_1^*$ is extended to a complex linear map $\tilde{\Theta}$ of $\mathfrak{g}_0$ to $(\mathfrak{k}_1^+)^*\otimes(\mathfrak{k}_1^-)^*$ by $\tilde{\Theta}(A)(X, \bar{Y})=\Theta(A)(X, \bar{Y})$ for all $A\in\mathfrak{g}_0$ and $X, Y\in\mathfrak{k}_1^+$. Then $\tilde{\mathfrak{n}}=\tilde{\Theta}(\mathfrak{g}_0)$, and we can easily verify the equality

$$\tilde{\Theta}(A)(X, \bar{Y})=-2B_t([A, X], \bar{Y}),$$
$$A\in\mathfrak{g}_0,\ X, Y\in\mathfrak{k}_1^+.$$

Let us now identify $(\mathfrak{k}_1^-)^*$ with $\mathfrak{k}_1^+$ by the non-degenerate bilinear map $\mathfrak{k}_1^+\times\mathfrak{k}_1^-\ni(X, Y)\rightarrow -B_t(X, Y)\in\boldsymbol{C}$. Then it follows that the two complex subspaces $\tilde{\mathfrak{n}}$ and $\rho(\mathfrak{g}_0)$ of $\mathrm{Hom}_c(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$ $(=(\mathfrak{k}_1^+)^*\otimes(\mathfrak{k}_1^-)^*)$ coincide. However we know from [8] that the second prolongation $\rho(\mathfrak{g}_0)^{(2)}$ of $\rho(\mathfrak{g}_0)$ vanishes if and only if M is not a complex projective space. Therefore we have seen that $\boldsymbol{f}$ is of c. finite type if and only if M is not a complex projective space. Q. E. D.

We shall now make some remarks in connection with Theorem 3. 7. Let $M_i=K_i/(K_i)_0$, $1\leqq i\leqq k$, be k irreducible hermitian symmetric spaces of compact type, and, for each i, let $\boldsymbol{f}_i$ be the canonical imbedding of M_i into $\mathfrak{m}_i=\sqrt{-1}\ \mathfrak{k}_i$. We consider the product imbedding $\boldsymbol{f}=\boldsymbol{f}_1\times\dots\times\boldsymbol{f}_k$ of the (reducible) hermitain symmetric space $M=M_1\times\dots\times M_k$ into the Euclidean space $\mathfrak{m}=\mathfrak{m}_1+\dots+\mathfrak{m}_k$. Then we remark that $\boldsymbol{f}$ is elliptic ([15]) and that $\boldsymbol{f}$ is of c. finite type if and only if none of M_i is a complex projective space. This last fact follows immediately from Corollary to Proposition 2. 9, Proposition 2. 12 and Theorem 3. 7. Now assume that none of M_i is a complex projective space. Then it follows from Proposition 2. 5 that $\mathscr{A}(U, \boldsymbol{f}\mid U)=\mathscr{A}_E(U, \boldsymbol{f}\mid U)$ for any connected open subset $U(\neq\phi)$ of M, where $\boldsymbol{f}\mid U$ denotes the restriction of $\boldsymbol{f}$ to U. In [15], one of the authors has shown that the two spaces $\mathscr{A}(M, \boldsymbol{f})$ and $\mathscr{A}_E(M, \boldsymbol{f})$ necessarily coincide (without assumptions on the spaces M_i). See also § 5.

3. 4. The real Grassmann manifolds $G^{p,q}(\boldsymbol{R})$. Let p and q be any positive integers with $p \leqq q$. Let $G^{p,q}(\boldsymbol{R})$ be the Grassmann manifold of all p-dimensional subspaces in the $(p+q)$-dimensional vector space $\boldsymbol{R}^{p+q}$. As usual $G^{p,q}(\boldsymbol{R})$ may be expressed as the symmetric homogeneous space K/K_0, where

$$K = SO(p+q)/C_0,$$

$$K_0 = (SO(p+q) \cap O(p) \times O(q))/C_0,$$

C_0 being the centre of $SO(p+q)$. Moreover as is well known, $G^{p,q}(\boldsymbol{R}) = K/K_0$ may be represented by a symmetric R space. This is associated with the simple graded Lie algebra of the first kind, $\mathfrak{g} = \sum_p \mathfrak{g}_p$ or $(\mathfrak{g}, Z_0)$, (together with the involutive automorphism σ of $\mathfrak{g}$) which is defined as follows:

$$\mathfrak{g} = \mathfrak{sl}(p+q\,;\, \boldsymbol{R}),$$

$$Z_0 = \begin{pmatrix} aI_p & 0 \\ 0 & bI_q \end{pmatrix} \quad \left(a = \frac{q}{p+q} \text{ and } b = \frac{-p}{p+q}\right),$$

$$\sigma(X) = -{}^tX, \quad X \in \mathfrak{g},$$

where I_r denotes the unit matrix of degree r.

We shall now give the explicit expressions for the various objects associated with the R space. In the following we shall use the following notations: $M_{r,s}(K)$ denotes the space of $r \times s$ matrices over a field K, and $S_r(K)$ the space of all symmetric matrices of degree r over K.

Then spaces $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are expressed respectively as follows:

$$\mathfrak{g}_{-1} = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \qquad X \in M_{q,p}(\boldsymbol{R}),$$

$$\mathfrak{g}_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad A \in \mathfrak{gl}(p;\, \boldsymbol{R}),\ B \in \mathfrak{gl}(q;\, \boldsymbol{R}) \text{ and } Tr\ A + Tr\ B = 0,$$

$$\mathfrak{g}_1 = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \qquad Y \in M_{p,q}(\boldsymbol{R}).$$

And the representation ρ of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is expressed as follows:

$$\rho(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}) X = BX - XA, \qquad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{g}_0, \quad X \in \mathfrak{g}_{-1},$$

where we have identified $\mathfrak{g}_{-1}$ and $M_{q,p}(\boldsymbol{R})$ in a natural manner. Analogously we have the expressions for $\mathfrak{k}$, $\mathfrak{k}_0$ and $\mathfrak{k}_1$:

$$\mathfrak{k} = \mathfrak{o}(p+q),$$

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in \mathfrak{o}(p), \ B \in \mathfrak{o}(q) \right\},$$

$$\mathfrak{k}_1 = \left\{ \begin{pmatrix} 0 & -{}^tX \\ X & 0 \end{pmatrix} \mid X \in M_{q,p}(\boldsymbol{R}) \right\},$$

and the expressions for $\mathfrak{m}$, $\mathfrak{m}_0$ and $\mathfrak{m}_1$:

$$\mathfrak{m} = \{X \in S_{p+q}(\boldsymbol{R}) \mid Tr X = 0\},$$

$$\mathfrak{m}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in S_p(\boldsymbol{R}), \ B \in S_q(\boldsymbol{R}) \text{ and } Tr\,A + Tr\,B = 0 \right\},$$

$$\mathfrak{m}_1 = \left\{ \begin{pmatrix} 0 & {}^tX \\ X & 0 \end{pmatrix} \mid X \in M_{q,p}(\boldsymbol{R}) \right\}.$$

Finally the inner product $<,>$ in $\mathfrak{g}$ is expressed as follows:

$$<X, Y> = 2(p+q)\,Tr(X\,{}^tY), \qquad X, Y \in \mathfrak{g}.$$

These being prepared, we shall prove the following

Theorem 3.8. *Consider the canonical isometric imbedding* $\boldsymbol{f}$ *of the Grassmann manifold* $G^{p,q}(\boldsymbol{R}) = K/K_0$ *into the Euclidean space* $\mathfrak{m}$.

(1) *The case where* $p=1$. *The imbedding* $\boldsymbol{f}$ *is involutive and not elliptic.*

(2) *The case where* $p=q=2$. *The imbedding* $\boldsymbol{f}$ *is not elliptic.*

(3) *The case where* $p=2$ *and* $q \geqq 3$. *The imbedding* $\boldsymbol{f}$ *is of infinite type and elliptic.*

(4) *The case where* $p \geqq 3$. *The imbedding* $\boldsymbol{f}$ *is of finite type.*

We shall show in §4 that $\boldsymbol{f}$ is of *c.* finite type in case (4), and in §7 that $\mathscr{A}(M, \boldsymbol{f}) = \mathscr{A}_E(M, \boldsymbol{f})$ in case (3).

Let us consider the (injective) linear map $\check{\rho}$ of $\mathfrak{m}_0$ to $S^2\mathfrak{g}_{-1}{}^*$ which is expressed as follows:

$$\phi\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right)(X,\ Y)=2(p+q)\ Tr((BX-XA)^{t}Y),$$

where $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\in\mathfrak{m}_0$ and $X,\ Y\in\mathfrak{g}_{-1}=M_{q,p}(\boldsymbol{R})$. The map ϕ is extended to a complex linear map of $\mathfrak{m}_0^c$ to $S^2(\mathfrak{g}_{-1}^c)^*$. Note that $\mathfrak{g}_{-1}^c=M_{q,p}(\boldsymbol{C})$ and $\mathfrak{m}_0^c$ consists of all $(A,\ B)\in S_p(\boldsymbol{C})\times S_q(\boldsymbol{C})$ with $TrA+TrB=0$. Also note that the extended map ϕ is defined by the same formula above.

We first prove the following

Lemma 3.9. *Assume that $p\geqq 2$ and $q\geqq 3$. Then the subspace $\phi(\mathfrak{m}_0^c)$ of $S^2(\mathfrak{g}_{-1}^c)^*$ contains non-zero decomposable elements (w. r. t. the symmetric product) if and only if $p=2$.*

Proof. Assume that $\phi(\mathfrak{m}_0^c)$ contains a non-zero decomposable element, say $\phi\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right)$. Then there are non-zero matrices $(X_{\lambda i})$ and $(Y_{\lambda i})$ in $M_{q,p}(\boldsymbol{C})$ such that

$$\delta_{ij}B_{\lambda\mu}-A_{ij}\delta_{\lambda\mu}=X_{\lambda i}Y_{\mu j}+Y_{\lambda i}X_{\mu j}\quad(1\leqq i,\ j\leqq p,\ 1\leqq\lambda,\ \mu\leqq q),$$

where $A=(A_{ij})$ and $B=(B_{\lambda\mu})$. For simplicity we put $Z_{\lambda\mu}^{ij}=X_{\lambda i}Y_{\mu j}+Y_{\lambda i}X_{\mu j}$. Now we have $-A_{ij}\delta_{\lambda\mu}=Z_{\lambda\mu}^{ij}(i\neq j)$. Since $q\geqq 3$ and since the rank of the matrix $(Z_{\lambda\mu}^{ij})_{1\leqq\lambda,\mu\leqq q}$ is at most 2, it follows that

$$A_{ij}=0(i\neq j)\ \text{ and }\ Z_{\lambda\mu}^{ij}=0(i\neq j).$$

For i, let $X^{(i)}$(resp. $Y^{(i)}$) denote the vector $(X_{\lambda i})_{1\leqq\lambda\leqq q}$(resp. $(Y_{\lambda i})_{1\leqq\lambda\leqq q}$). We assert that $X^{(i)}\neq 0$ and $Y^{(i)}\neq 0$ for every i. For example suppose that $X^{(1)}=0$ or $Y^{(1)}=0$. Then we have $B_{\lambda\mu}-A_{11}\delta_{\lambda\mu}=Z_{\lambda\mu}^{11}=0$. Hence $(A_{11}-A_{ii})\delta_{\lambda\mu}=B_{\lambda\mu}-A_{ii}\delta_{\lambda\mu}=Z_{\lambda\mu}^{ii}$. Therefore it follows as above that $A_{11}-A_{ii}=0$. Since $B_{\lambda\mu}=A_{11}\delta_{\lambda\mu}$, $A_{ij}=A_{11}\delta_{ij}$ and $TrA+TrB=0$, we obtain $A=B=0$. This is a contradiction, proving our assertion. Since $Z_{\lambda\mu}^{ij}=0$ $(i\neq j)$, we therefore see that the two vectors $X^{(i)}$ and $Y^{(i)}$ are linearly dependent for every i.

Accordingly we have

$$Y^{(1)}=aX^{(1)}\quad\text{with some } a\neq 0.$$

Since $Z_{\lambda\mu}^{i1}=0$ $(i\geqq 2)$, it follows that

$$Y^{(i)}=-aX^{(i)}\quad(i\geqq 2).$$

Furthermore we have $B_{\lambda\mu}-A_{11}\delta_{\lambda\mu}=2aX_{\lambda1}X_{\mu1}$ and $B_{\lambda\mu}-A_{22}\delta_{\lambda\mu}=-2aX_{\lambda2}X_{\mu2}$, whence

$$(A_{22}-A_{11})\delta_{\lambda\mu}=2a(X_{\lambda1}X_{\mu1}+X_{\lambda2}X_{\mu2}).$$

Hence we obtain

$$A_{22}=A_{11} \text{ and } X_{\lambda1}X_{\mu1}+X_{\lambda2}X_{\mu2}=0.$$

This second equality means that

$$X^{(2)}=\sqrt{-1}\ X^{(1)} \text{ or } -\sqrt{-1}\ X^{(1)}.$$

The case where $p\geqq 3$. We have $0=Z_{\lambda\mu}^{23}=-2aX_{\lambda2}X_{\mu3}$. This is a contradiction, because $a\neq 0$, $X^{(2)}\neq 0$ and $X^{(3)}\neq 0$.

The case where $p=2$. We have $A_{11}=A_{22}$, $B_{\lambda\mu}=A_{11}\delta_{\lambda\mu}+2a\ X_{\lambda1}X_{\mu1}$ and $TrA+TrB=0$. Hence $A_{11}=-\frac{2a}{q+2}\sum_{\lambda}(X_{\lambda1})^2$. Therefore putting $X_\lambda=2aX_{\lambda1}$, we obtain

$$A_{12}=0 \quad \text{and} \quad A_{11}=A_{22}=-\frac{1}{q+2}\sum_{\lambda}X_\lambda^2,$$

$$B_{\lambda\mu}=-\frac{1}{q+2}(\sum_{\kappa}X_\kappa^2)\delta_{\lambda\mu}+X_\lambda X_\mu.$$

Conversely take any non-zero vector $(X_\lambda)\in \boldsymbol{C}^q$, and define matrices $A\in S_2(\boldsymbol{C})$ and $B\in S_q(\boldsymbol{C})$ by the formula above. Then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\in \mathfrak{m}_0^c$, and $\check{\rho}(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix})$ is a non-zero decomposable element.

We have thereby proved Lemma 3. 9.

We are now in a position to prove Theorem 3. 8. First recall Corollary to Proposition 3. 6.

The case where $p\geqq 2$ and $q\geqq 3$. We know that the subspace $\check{\rho}(\mathfrak{m}_0)+\wedge^2\mathfrak{g}_{-1}^*$ of $\otimes^2\mathfrak{g}_{-1}^*$ is of infinite type if and only if the subspace $\check{\rho}(\mathfrak{m}_0^c)+\wedge^2(\mathfrak{g}_{-1}^c)^*$ of $\otimes^2(\mathfrak{g}_{-1}^c)^*$ contains non-zero decomposable element (w. r. t. the tensor product) ([7]). This last condition is equivalent to the condition that the subspace $\check{\rho}(\mathfrak{m}_0^c)$ of $S^2(\mathfrak{g}_{-1}^c)^*$ contains non-zero decomposable elements (w. r. t. the symmetric product). Therefore we see from Lemma 3. 9 that $\boldsymbol{f}$ is of infinite type if and only if $p=2$. Moreover we deduce from the proof of Lemma 3. 9 that the subspace $\check{\rho}(m_0)$ of $S^2\mathfrak{g}_{-1}^*$ contains no non-zero decomposable elements (w. r. t. the symmetric product), meaning that $\boldsymbol{f}$ is elliptic.

The case where $p=1$. We have dim $\mathfrak{g}_{-1}=q$ and dim $\mathfrak{m}_0=\frac{1}{2}q(q+1)$, whence $\check{\rho}(\mathfrak{m}_0)=S^2\mathfrak{g}_{-1}^*$. This clearly implies that $\boldsymbol{f}$ is involutive and not elliptic.

The case where $p=q=2$. For example, define matrices A and B in $S_2(\boldsymbol{R})$ by $A=B=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Then $\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}\in\mathfrak{m}_0$, and $\check{\rho}(\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix})$ is a non-zero decomposable element. Hence $\boldsymbol{f}$ is not elliptic.

We have thus completed the proof of Theorem 3. 8.

3. 5. Remark (the real quadrics $Q_r^n(\boldsymbol{R})$). Let (n, r) be any pair of integres n and r with $n\geqq 1$ and $0\leqq r\leqq[\frac{n}{2}]$. Let $x_0, \dots, x_{n+1}$ be the homogeneous coordinates of the $(n+1)$-dimensional real projective space $P^{n+1}(\boldsymbol{R})$, and let $Q_r^n(\boldsymbol{R})$ be the quadric of $P^{n+1}(\boldsymbol{R})$ defined by the equation

$$-2x_0x_{n+1}+\sum_{i=1}^{n}\varepsilon_i x_i^2=0,$$

where $\varepsilon_i=-1$ if $1\leqq i\leqq r$ and $\varepsilon_i=1$ otherwise. Let G be the group of all projective transformations which leave $Q_r^n(\boldsymbol{R})$ invariant. The group G acts effectively and transitively on $Q_r^n(\boldsymbol{R})$, and hence the quadric $Q_r^n(\boldsymbol{R})$ may be represented by the homogeneous space G/G', where G' denotes the isotropy group of G at the point $o=(1, 0, \dots, 0)\in Q_r^n(\boldsymbol{R})$. This homogeneous space is usually known as the n-dimensional Möbius space of index r (cf. [13]).

Let $\mathfrak{g}$ be the Lie algebra of G, which is isomorphic with the subalgebra of $\mathfrak{sl}(n+2; \boldsymbol{R})$ consisting of all matrices $S\in\mathfrak{sl}(n+2; \boldsymbol{R})$ with ${}^tX\tilde{J}+\tilde{J}X=0$, where

$$\tilde{J}=\begin{pmatrix}0 & 0 & -1\\ 0 & J & 0\\ -1 & 0 & 0\end{pmatrix},\quad J=\begin{pmatrix}\varepsilon_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \varepsilon_n\end{pmatrix}.$$

Let Z_0 be the element of $\mathfrak{g}$ defined by

$$Z_0=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Hereafter we assume that $(n, r)\neq(2, 1)$. Then the pair $(\mathfrak{g}, Z_0)$ defines a simple graded Lie algebra of the first kind, say $\mathfrak{g}=\sum_p\mathfrak{g}_p$, and the homogeneous space G/G' is naturally obtained from this graded Lie algebra (cf. 5.6 and [14]). Now the assignment $X \to -{}^tX$ gives an involutive automorphism of $\mathfrak{g}$, say σ, and it has the properties stated in Lemma 3.2. Hence the quadric $Q_r^n(\boldsymbol{R})$ may be further represented by the symmetric R space K/K_0 which is associated with the graded Lie algebra $\mathfrak{g}=\sum_p \mathfrak{g}_p$ together with the involutive automorphism σ. Here we note that the quadric $Q_2^4(\boldsymbol{R})$ is isomorphic with the Grassmann manifold $G^{2,2}(\boldsymbol{R})$ as symmetric R spaces.

These being prepared, we state the following proposition without proof.

Proposition 3.10. *Assume that* $(n, r) \neq (2, 1)$. *Consider the canonical isometric imbedding* $\boldsymbol{f}$ *of the symmetric R space* $Q_r^n(\boldsymbol{R})=K/K_0$ *into the Euclidean space* $\mathfrak{m}$.

(1) $\boldsymbol{f}$ *is involutive if and only if* $(n, r)=(1, 0)$ *or* $(2, 0)$ *or* $(3, 1)$.

(2) $\boldsymbol{f}$ *is elliptic if and only if* $n \geqq 2$ *and* $r=0$.

(3) $\boldsymbol{f}$ *is of c. finite type if and only if* $n \geqq 3$ *and* $r=0$.

(4) *The first prolongation* $\mathfrak{n}^{(1)}$ *of the bundle of second fundamental forms of* $\boldsymbol{f}$ *vanishes if and only if either* $n \geqq 2$ *and* $r=0$ *or* $r \geqq 2$.

Note that $\dim \mathfrak{m}=(r+1)(n-r+1)$.

§4. Rigidity for the canonical isometric imbeddings of the real Grassmann manifolds $G^{p,q}(R)$, $q \geqq p \geqq 3$.

Let $\boldsymbol{f}$ be the canonical isometric imbedding of the real Grassmann manifold $M=G^{p,q}(\boldsymbol{R})$, $q \geqq p$, into the Euclidean space $\mathfrak{m}$ (see 3.4). The main aim of this section is to prove the following

Theorem 4.1. (1) *If* $p \geqq 2$, *then the first prolongation*

$$\mathfrak{n}^{(1)}=T(M)^* \otimes \mathfrak{n} \cap S^2T(M)^* \otimes T(M)^*$$

of the bundle $\mathfrak{n}$ *of second fundamental forms vanishes.*

(2) *If* $p \geqq 3$, *then the imbedding* $\boldsymbol{f}$ *is of c. finite type.*

4.1. Preliminaries. Let us consider the graded Lie algebra $\mathfrak{g}=\sum_i \mathfrak{g}_i$ together with the involutive automorphism σ which is associated with $G^{p,q}(\boldsymbol{R})$ (see 3.4). We denote by $\mathfrak{a}$ the Cartan subalgebra of $\mathfrak{g}=\mathfrak{sl}(p+q;\boldsymbol{R})$ which consists of all diagonal matrices.

We first recall the root system with respect to $\mathfrak{a}$. Δ denotes the set of all non-zero roots, and for every $\alpha \in \Delta$, $\mathfrak{g}^\alpha$ denotes the root subspace of $\mathfrak{g}$ corresponding to $\alpha \in \Delta$. Then we have $\mathfrak{g}=\mathfrak{a}+\sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$ (direct sum). Let us now give the exact expressions of Δ and $\mathfrak{g}^\alpha$. For $1 \leqq i \leqq p+q$, let λ_i denote the linear function $\mathfrak{a}\ni H\rightarrow\lambda_i(H)\in\boldsymbol{R}$, where $\lambda_i(H)$ is the i-th diagonal component of H. Then Δ consists of all the linear functions $\lambda_i-\lambda_j(i\neq j)$. For every $\alpha=\lambda_i-\lambda_j$, let E_α denote the matrix $(\delta_{ai}\delta_{bj})_{1\leqq a,b\leqq p+q}$ in $\mathfrak{g}$. Then $\mathfrak{g}^\alpha$ is the 1-dimensional subspace of $\mathfrak{g}$ spanned by E_α.

As is well known, we have the following:

(4. 1) $\sigma E_\alpha=-E_{-\alpha}$, $\alpha\in\Delta$.

(4. 2) Let $\alpha\in\Delta$. If we put $H_\alpha=[E_\alpha, E_{-\alpha}]$, then $H_\alpha\in\mathfrak{a}$ and $\langle H_\alpha, H\rangle=2(p+q)\alpha(H)$, $H\in\mathfrak{a}$.

Let α, $\beta\in\Delta$.

(4. 3) If $\alpha+\beta$ is not a root, i. e., $\alpha+\beta\neq0$ and $\alpha+\beta\notin\Delta$, then $[E_\alpha, E_\beta]=0$.

(4. 4) If $\alpha+\beta\in\Delta$, then $[E_\alpha, E_\beta]=N_{\alpha,\beta}E_{\alpha+\beta}$, and $N_{\alpha,\beta}=1$ or -1. More precisely $N_{\alpha,\beta}=1$ if $\alpha=\lambda_i-\lambda_j$ and $\beta=\lambda_j-\lambda_k$ $(i\neq k)$, and $N_{\alpha,\beta}=-1$ if $\alpha=\lambda_i-\lambda_j$ and $\beta=\lambda_k-\lambda_i(j\neq k)$.

Let us now describe the graded structure of $\mathfrak{g}$ in terms of the root system. Consider the element $Z_0=\begin{pmatrix} aI_p & 0\\ 0 & bI_q\end{pmatrix}$ $\left(a=\frac{q}{p+q}, b=\frac{-p}{p+q}\right)$ in $\mathfrak{g}$ which gives the grading of $\mathfrak{g}$. Then we have $\alpha(Z_0)=-1$, 0 or 1 for every $\alpha\in\Delta$, and hence $\Delta=\Delta_{-1}\cup\Delta_0\cup\Delta_1$(disjoint), where we set $\Delta_i=\{\alpha\in\Delta\mid\alpha(Z_0)=i\}$. The subsets Δ_i are explicitly given as follows:

$$\Delta_{-1}=\{\lambda_r-\lambda_i\mid 1\leqq i\leqq p \text{ and } p+1\leqq r\leqq p+q\},$$

$$\Delta_0=\{\pm(\lambda_i-\lambda_j)\mid 1\leqq i<j\leqq p+q \text{ or } p+1\leqq i<j\leqq p+q\},$$

$$\Delta_1=\{\lambda_i-\lambda_r\mid 1\leqq i\leqq p \text{ and } p+1\leqq r\leqq p+q\}.$$

And the subspaces $\mathfrak{g}_i$ $(i=-1, 0, 1)$ of $\mathfrak{g}$ are discribed as follows:

$$\mathfrak{g}_{-1}=\sum_{\alpha\in\Delta_{-1}}\mathfrak{g}^\alpha,\ \mathfrak{g}_0=\mathfrak{a}+\sum_{\alpha\in\Delta_0}\mathfrak{g}^\alpha \text{ and } \mathfrak{g}_1=\sum_{\alpha\in\Delta_1}\mathfrak{g}^\alpha.$$

Finally we prove a useful lemma on the subspace $\rho(\mathfrak{m}_0)$ of $S^2\mathfrak{g}^*_{-1}$. First note that $E_\alpha(\alpha\in\Delta_{-1})$ forms a basis of $\mathfrak{g}_{-1}$. We express covariant tensors on $\mathfrak{g}_{-1}$ in terms of this basis, and promise that the indices i, j, k, ... run over the range 1, 2, ..., p and the indices r, s, t, ... over the range $p+1$, $p+2$, ..., $p+q$. We also promise that the Greek letters α, β, γ, ... mean roots in Δ_{-1}.

Lemma 4. 2. *Let* $X \in \rho(\mathfrak{m}_0)$.

(1) $X_{\alpha\beta}=0$ *if* $\alpha-\beta$ *is not a root.*

(2) $X_{\alpha\beta}=X_{\alpha_1\beta_1}$ *if* $\alpha=\lambda_r-\lambda_i$, $\beta=\lambda_s-\lambda_i$, $\alpha_1=\lambda_r-\lambda_j$, $\beta_1=\lambda_s-\lambda_j (r\neq s)$ *or if* $\alpha=\lambda_r-\lambda_i$, $\beta=\lambda_r-\lambda_j$, $\alpha_1=\lambda_s-\lambda_i$, $\beta_1=\lambda_s-\lambda_j (i\neq j)$.

(3) $X_{\alpha\alpha}=\sum_a X_{\beta_a\beta_a}-\sum_b X_{\gamma_b\gamma_b}$ *if* $\alpha=\sum_a \beta_a-\sum_b \gamma_b$.

Proof. By the very definition of the map ρ, there is a unique $A \in \mathfrak{m}_0$ such that

$$X_{\alpha\beta}=\langle A, [E_\alpha, \sigma E_\beta]\rangle=-\langle A, [E_\alpha, E_{-\beta}]\rangle.$$

Now (1) is clear. (2) follows from the equality: $[E_\alpha, E_{-\beta}]=[E_{\alpha_1}, E_{-\beta_1}]$, and (3) from the equality: $H_\alpha=\sum_a H_{\beta_a}-\sum_b H_{\gamma_b}$. Q. E. D.

4. 2. Proof of Theorem 4. 1, (1). By Proposition 3. 6 it suffices to show if $p\geqq 2$, then the first prolongation $\rho(\mathfrak{m}_0)^{(1)}$ of $\rho(\mathfrak{m}_0)$ (as a subspace of $\otimes^2\mathfrak{g}^*_{-1}$) vanishes.

Take any element X of $\rho(\mathfrak{m}_0)^{(1)}$. Then $X_{\alpha\beta\gamma}$ may be expressed as follows: $X_{\alpha\beta\gamma}=\langle A_\alpha, [E_\beta, \sigma E_\gamma]\rangle$, where $A_\alpha \in \mathfrak{m}_0$. Furthermore we have $X_{\alpha\beta\gamma}=X_{\beta\gamma\alpha}=X_{\gamma\alpha\beta}$. By Lemma 4. 2 we see that $X_{\alpha\beta\gamma}=0$ if one of $\alpha-\beta$, $\beta-\gamma$ and $\gamma-\alpha$ is not a root.

Lemma 4. 3. *If* $\alpha-\beta \in \Delta$, $\beta-\gamma \in \Delta$ *and* $\gamma-\alpha \in \Delta$, *then* $X_{\alpha\beta\gamma}=0$.

Proof. The following two cases are possible:

(i) $\alpha=\lambda_r-\lambda_i$, $\beta=\lambda_s-\lambda_i$ and $\gamma=\lambda_t-\lambda_i$, where $r\neq s$, $r\neq t$ and $t\neq s$,

(ii) $\alpha=\lambda_r-\lambda_i$, $\beta=\lambda_r-\lambda_j$ and $\gamma=\lambda_r-\lambda_k$, where $i\neq j$, $i\neq k$ and $j\neq k$.

We shall prove our assertion only for case (i). (Case (ii) can be similarly dealt with.) Since $p\geqq 2$, we can find j such that $j\neq i$. Putting $\beta_1=\lambda_s-\lambda_j$ and $\gamma_1=\lambda_t-\lambda_j$, we see that $\alpha-\gamma_1$ is not a root. Therefore using Lemma 4. 2 we have $X_{\alpha\beta\gamma}=X_{\alpha\beta_1\gamma_1}=0$. Q. E. D.

In the same way we can prove the following

Lemma 4. 4. *If* $\alpha-\gamma \in \Delta$, *then* $X_{\alpha\alpha\gamma}=0$.

Lemma 4. 5. $X_{\alpha\alpha\alpha}=0$.

Proof. Let $\alpha=\lambda_r-\lambda_i$. Choose j and s such that $j\neq i$ and $s\neq r$, and set $\beta_1=\lambda_s-\lambda_i$, $\gamma_1=\lambda_r-\lambda_j$ and $\delta_1=\lambda_s-\lambda_j$. Then $\alpha=\beta_1+\gamma_1-\delta_1$, both $\alpha-\beta_1$ and $\alpha-\gamma_1$ are in Δ, and $\alpha-\delta_1$ is not a root. Therefore it follows from

Lemmas 4. 2 and 4. 4 that $X_{\alpha\alpha\alpha}=X_{\alpha\beta_1\beta_1}+X_{\alpha\gamma_1\gamma_1}-X_{\alpha\delta_1\delta_1}=0$.

Q. E. D.

By Lemmas 4. 3～4. 5, we have shown that every $X\in\rho(\mathfrak{m}_0)^{(1)}$ satisfies $X_{\alpha\beta\gamma}=0$ for all α, β, γ, thus completing the proof of Theorem 4. 1, (1).

4. 3. Proof of Theorem 4. 1, (2). By Proposition 3. 6, it suffices to show that if $p\geqq 3$, then the subspace $\not{p}(\mathfrak{m}_0)+\wedge^2\mathfrak{g}_{-1}^*$ of $\otimes^2\mathfrak{g}_{-1}^*$ is of c. finite type or equivalently every $Y\in p^2(\not{p}(\mathfrak{m}_0))$ satisfies the relation: $Y_{\alpha\beta\gamma\delta}=-Y_{\gamma\delta\alpha\beta}$ for all α, β, γ, δ (cf. Proposition 2. 1).

Take any element Y of $p^2(\not{p}(\mathfrak{m}_0))$. Then $Y_{\alpha\beta\gamma\delta}$ can be expressed as follows: $Y_{\alpha\beta\gamma\delta}=\langle A_{\alpha\beta}, [E_\gamma, \sigma E_\delta]\rangle$, where $A_{\alpha\beta}\in\mathfrak{m}_0$. Furthermore we have $Y_{\alpha\beta\gamma\delta}=Y_{\beta\alpha\gamma\delta}$ (or $A_{\alpha\beta}=A_{\beta\alpha}$), and $Y_{\alpha\beta\gamma\delta}-Y_{\alpha\delta\gamma\beta}-Y_{\gamma\beta\alpha\delta}+Y_{\gamma\delta\alpha\beta}=0$. By Lemma 4. 2 we see that $Y_{\alpha\beta\gamma\delta}=0$ if $\gamma-\delta$ is not a root.

Lemma 4. 6. *If $\alpha-\beta$ is not a root and $\gamma-\delta\in\Delta$, then $Y_{\alpha\beta\gamma\delta}=0$.*

Proof. The following two cases are possible:

(i) $\alpha=\lambda_r-\lambda_i$, $\beta=\lambda_s-\lambda_j$, $\gamma=\lambda_t-\lambda_k$ and $\delta=\lambda_u-\lambda_k$,

where $i\neq j$, $r\neq s$ and $t\neq u$,

(ii) $\alpha=\lambda_r-\lambda_i$, $\beta=\lambda_s-\lambda_j$, $\gamma=\lambda_t-\lambda_k$ and $\delta=\lambda_t-\lambda_l$,

where $i\neq j$, $k\neq l$ and $r\neq s$.

We shall prove our assertion only for case (i). (Case (ii) can be similarly dealt with.) Since $Y_{\alpha\beta\gamma\delta}$ is symmetric with respect to the pairs (α, β) and (γ, δ), we may assume that $r\neq u$ and $s\neq t$. Since $p\geqq 3$, we can find l such that $l\neq i, j$. Putting $\gamma_1=\lambda_t-\lambda_l$ and $\delta_1=\lambda_u-\lambda_l$, we see that neither $\gamma_1-\beta$ nor $\alpha-\delta_1$ are roots. It follows from Lemma 4. 2 that $Y_{\alpha\beta\gamma\delta}=Y_{\alpha\beta\gamma_1\delta_1}=Y_{\alpha\delta_1\gamma_1\beta}+Y_{\gamma_1\beta\alpha\delta_1}-Y_{\gamma_1\delta_1\alpha\beta}=0$. Q. E. D.

Lemma 4. 7. *If $\beta\neq\gamma$ and $\alpha-\delta$ is not a root, then $Y_{\alpha\beta\gamma\delta}=-Y_{\gamma\delta\alpha\beta}$.*

Proof. Using Lemma 4. 6 we have $Y_{\alpha\delta\gamma\beta}=Y_{\gamma\beta\alpha\delta}=0$. Since $Y_{\alpha\beta\gamma\delta}-Y_{\alpha\delta\gamma\beta}=Y_{\gamma\beta\alpha\delta}-Y_{\gamma\delta\alpha\beta}$, it follows that $Y_{\alpha\beta\gamma\delta}=-Y_{\gamma\delta\alpha\beta}$. Q. E. D.

Lemma 4. 8. *If $\alpha-\beta\in\Delta$ and $\gamma-\delta\in\Delta$, then $Y_{\alpha\beta\gamma\delta}=-Y_{\gamma\delta\alpha\beta}$.*

Proof. The following four cases are possible:

(i) $\alpha=\lambda_r-\lambda_i$, $\beta=\lambda_s-\lambda_i$, $\gamma=\lambda_t-\lambda_j$ and $\delta=\lambda_u-\lambda_j$,

where $r\neq s$ and $t\neq u$.

(ii) $\alpha=\lambda_r-\lambda_i$, $\beta=\lambda_s-\lambda_i$, $\gamma=\lambda_t-\lambda_j$ and $\delta=\lambda_t-\lambda_k$, where $j \neq k$ and $r \neq s$,

(iii) $\alpha=\lambda_r-\lambda_i$, $\beta=\lambda_r-\lambda_j$, $\gamma=\lambda_s-\lambda_k$ and $\delta=\lambda_s-\lambda_l$,

where $i \neq j$ an $k \neq l$,

(iv) $\alpha=\lambda_r-\lambda_i$, $\beta=\lambda_r-\lambda_j$, $\gamma=\lambda_s-\lambda_k$ and $\delta=\lambda_t-\lambda_k$,

where $i \neq j$ and $s \neq t$.

We shall prove our assertion only for cases (i) and (ii). (Cases (iii) and (iv) can be similarly dealt with.)

Case (i). We first consider the case where $i \neq j$. Then $\alpha \neq \gamma$, δ and $\beta \neq \gamma$, δ and either $\alpha-\gamma$ or $\alpha-\delta$ is not a root. It follows from Lemma 4.7 that $Y_{\alpha\beta\gamma\delta}=-Y_{\gamma\delta\alpha\beta}$. We next consider the case where $i=j$. Chosse k and l such that $k \neq i$, $l \neq i$ and $k \neq l$, and set $\alpha_1=\lambda_r-\lambda_k$, $\beta_1=\lambda_s-\lambda_k$, $\gamma_1=\lambda_t-\lambda_l$ and $\delta_1=\lambda_u-\lambda_l$. Then both $\alpha_1-\beta_1$ and $\gamma_1-\delta_1$ are in Δ. Therefore using Lemma 4.2 and the result obtained above, we have $Y_{\alpha\beta\gamma\delta}=Y_{\alpha\beta\gamma_1\delta_1}$ $=-Y_{\gamma_1\delta_1\alpha\beta}=-Y_{\gamma_1\delta_1\alpha_1\beta_1}=Y_{\alpha_1\beta_1\gamma_1\delta_1}=Y_{\alpha_1\beta_1\gamma\delta}=-Y_{\gamma\delta\alpha_1\beta_1}=-Y_{\gamma\delta\alpha\beta}$.

Case (ii). Let us consider the case where $i \neq j$, k. Then $\alpha \neq \gamma$, δ and $\beta \neq \gamma$, δ and either $\alpha-\gamma$ or $\beta-\gamma$ is not a root. It follows from Lemma 4.7 that $Y_{\alpha\beta\gamma\delta}=-Y_{\gamma\delta\alpha\beta}$. Analogously we have $Y_{\alpha\beta\gamma\delta}=-Y_{\gamma\delta\alpha\beta}$ in the case where $r \neq t$, s. Thus to finish the proof in case (ii), we have only to discuss the following case: $\alpha=\lambda_r-\lambda_i$, $\beta=\lambda_s-\lambda_i$, $\gamma=\lambda_r-\lambda_i$ and $\delta=\lambda_r-\lambda_j$, where $i \neq j$ and $r \neq s$. Choose l and u such that $l \neq i$, j and $u \neq r$, s, and set $\alpha_1=\lambda_r-\lambda_l$, $\beta_1=\lambda_s-\lambda_l$, $\gamma_1=\lambda_u-\lambda_i$ and $\delta_1=\lambda_u-\lambda_j$. Then in the same manner as in case (i) we obtain $Y_{\alpha\beta\gamma\delta}=Y_{\alpha\beta\gamma_1\delta_1}=\ldots=-Y_{\gamma\delta\alpha\beta}$.

Q. E. D.

Lemma 4.9. *If $\alpha-\beta$ is not a root, then $Y_{\alpha\beta\gamma\gamma}=0$.*

Proof. We first consider the case where $\gamma \neq \alpha$, β. If either $\gamma-\alpha$ or $\gamma-\beta$ is not a root, then we see from Lemma 4.7 that $Y_{\alpha\gamma\beta\gamma}=-Y_{\beta\gamma\alpha\gamma}$, and hence that $Y_{\alpha\beta\gamma\gamma}=Y_{\alpha\gamma\gamma\beta}+Y_{\gamma\beta\alpha\gamma}-Y_{\gamma\gamma\alpha\beta}=0$. If both $\gamma-\alpha$ and $\gamma-\beta \in \Delta$, then we see from Lemma 4.8 that $Y_{\alpha\gamma\beta\gamma}=-Y_{\beta\gamma\alpha\gamma}$ and hence, as above, that $Y_{\alpha\beta\gamma\gamma}=0$. We next consider the case where $\gamma=\alpha$ or $\gamma=\beta$. Since $Y_{\alpha\beta\gamma\gamma}=Y_{\beta\alpha\gamma\gamma}$, we may assume that $\gamma=\beta \neq \alpha$. Let $\alpha=\lambda_r-\lambda_i$ and $\beta=\gamma=\lambda_s-\lambda_j$, where $i \neq j$ and $r \neq s$. We choose k and t such that $k \neq i$, j and $t \neq r$, s, and set $\gamma_1=\lambda_t-\lambda_j$, $\delta_1=\lambda_s-\lambda_k$ and $\varepsilon_1=\lambda_t-\lambda_k$. Then we have: $\alpha \neq \gamma_1$, δ_1, ε_1; $\beta \neq \gamma_1$, δ_1, ε_1; $\beta=\gamma_1+\delta_1-\varepsilon_1$. Therefore using Lemma 4.2 and the result obtained above, we have $Y_{\alpha\beta\beta\beta}=Y_{\alpha\beta\gamma_1\gamma_1}+Y_{\alpha\beta\delta_1\delta_1}-Y_{\alpha\beta\varepsilon_1\varepsilon_1}=0$.

Q. E. D.

Lemma 4.10. *If $\alpha-\beta \in \Delta$, then $Y_{\alpha\beta\gamma\gamma}=-Y_{\gamma\gamma\alpha\beta}$.*

Proof. We first consider the case where $\gamma \neq \alpha, \beta$. This case can be similarly treated to the first half of the proof of Lemma 4.9. If either $\gamma-\alpha$ or $\gamma-\beta$ is not a root, then we see from Lemma 4.7 that $Y_{\alpha\beta\gamma\gamma}=-Y_{\gamma\gamma\alpha\beta}$. If both $\gamma-\alpha$ and $\gamma-\beta$ are in Δ, we see from Lemma 4.8 that $Y_{\alpha\beta\gamma\gamma}=-Y_{\gamma\gamma\alpha\beta}$. We next consider the case where $\gamma=\alpha$ or $\gamma=\beta$. As before we may assume that $\alpha\neq\beta=\gamma$. The following two cases are possible:

(i) $\alpha=\lambda_r-\lambda_i$ and $\beta=\gamma=\lambda_s-\lambda_i$, where $r\neq s$,

(ii) $\alpha=\lambda_r-\lambda_i$ and $\beta=\gamma=\lambda_r-\lambda_j$, where $i\neq j$.

We shall prove our assertion only for case (i). (Case (ii) can be similarly dealt with.) Choose j, k and t such that $j\neq i$, $k\neq i$, $i\neq k$ and $t\neq r$, and set $\alpha_1=\lambda_r-\lambda_k$, $\beta_1=\lambda_s-\lambda_k$, $\gamma_1=\lambda_t-\lambda_i$, $\delta_1=\lambda_s-\lambda_j$ and $\varepsilon_1=\lambda_t-\lambda_j$. Then we have: $\alpha\neq\gamma_1, \delta_1, \varepsilon_1$; $\beta\neq\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1$; $\alpha_1\neq\gamma_1, \delta_1, \varepsilon_1$; $\beta_1\neq\gamma_1, \delta_1, \varepsilon_1$; $\alpha_1-\beta_1\in\Delta$; $\beta=\gamma_1+\delta_1-\varepsilon_1$. Therefore using Lemma 4.2 and the result obtained above, we have

$$\begin{aligned} Y_{\alpha\beta\beta\beta} &= Y_{\alpha\beta\gamma_1\gamma_1}+Y_{\alpha\beta\delta_1\delta_1}-Y_{\alpha\beta\varepsilon_1\varepsilon_1} \\ &= -Y_{\gamma_1\gamma_1\alpha\beta}-Y_{\delta_1\delta_1\alpha\beta}+Y_{\varepsilon_1\varepsilon_1\alpha\beta} \\ &= -Y_{\gamma_1\gamma_1\alpha_1\beta_1}-Y_{\delta_1\delta_1\alpha_1\beta_1}+Y_{\varepsilon_1\varepsilon_1\alpha_1\beta_1} \\ &= Y_{\alpha_1\beta_1\gamma_1\gamma_1}+Y_{\alpha_1\beta_1\delta_1\delta_1}-Y_{\alpha_1\beta_1\varepsilon_1\varepsilon_1} \\ &= Y_{\alpha_1\beta_1\beta\beta}=-Y_{\beta\beta\alpha_1\beta_1}=-Y_{\beta\beta\alpha\beta}. \end{aligned}$$

Q. E. D.

Lemma 4.11. $Y_{\alpha\alpha\beta\beta}=-Y_{\beta\beta\alpha\alpha}$.

Proof. We first consider the case where $\alpha\neq\beta$. If $\alpha-\beta\notin\Delta$, it follows from Lemma 4.7 that $Y_{\alpha\alpha\beta\beta}=-Y_{\beta\beta\alpha\alpha}$. If $\alpha-\beta\in\Delta$, it follows from Lemma 4.8 that $Y_{\alpha\beta\alpha\beta}=-Y_{\alpha\beta\alpha\beta}$, i. e., $Y_{\alpha\beta\alpha\beta}=0$. Hence $Y_{\alpha\alpha\beta\beta}=Y_{\alpha\beta\beta\alpha}+Y_{\beta\alpha\alpha\beta}-Y_{\beta\beta\alpha\alpha}=-Y_{\beta\beta\alpha\alpha}$. We next consider the case where $\alpha=\beta$. Let $\alpha=\beta=\lambda_r-\lambda_i$. We choose j, k, s and t such that $j\neq i$, $k\neq i$, $j\neq k$, $s\neq r$, $t\neq r$ and $s\neq t$, and set $\gamma=\lambda_s-\lambda_j$, $\delta=\lambda_t-\lambda_k$, $\varepsilon_1=\lambda_s-\lambda_i$, $\sigma_1=\lambda_t-\lambda_i$, $\varepsilon_2=\lambda_r-\lambda_j$ and $\sigma_2=\lambda_s-\lambda_k$. Then we have: $\gamma\neq\varepsilon_2, \sigma_2, \alpha, \delta$; $\delta\neq\varepsilon_1, \sigma_1, \alpha$; $\sigma_2\neq\varepsilon_1, \sigma_1, \alpha$; $\alpha\neq\varepsilon_1, \sigma_1$; $\gamma=\varepsilon_1+\sigma_1-\alpha$; $\delta=\varepsilon_2+\sigma_2-\alpha$. Therefore using Lemma 4.1 and the result obtained above, we have

$$\begin{aligned} Y_{\gamma\gamma\delta\delta} &= Y_{\gamma\gamma\varepsilon_2\varepsilon_2}+Y_{\gamma\gamma\sigma_2\sigma_2}-Y_{\gamma\gamma\alpha\alpha} \\ &= -Y_{\varepsilon_2\varepsilon_2\gamma\gamma}-Y_{\sigma_2\sigma_2\gamma\gamma}+Y_{\alpha\alpha\gamma\gamma} \\ &= -(Y_{\varepsilon_2\varepsilon_2\varepsilon_1\varepsilon_1}+Y_{\varepsilon_2\varepsilon_2\sigma_1\sigma_1}-Y_{\varepsilon_2\varepsilon_2\alpha\alpha}) \\ &\quad -(Y_{\sigma_2\sigma_2\varepsilon_1\varepsilon_1}+Y_{\sigma_2\sigma_2\sigma_1\sigma_1}-Y_{\sigma_2\sigma_2\alpha\alpha}) \\ &\quad +(Y_{\alpha\alpha\varepsilon_1\varepsilon_1}+Y_{\alpha\alpha\sigma_1\sigma_1}-Y_{\alpha\alpha\alpha\alpha}). \end{aligned}$$

In the same manner we have

$$\begin{aligned} Y_{\delta\delta\gamma\gamma} = & -(Y_{\varepsilon_1\varepsilon_1\varepsilon_2\varepsilon_2} + Y_{\varepsilon_1\varepsilon_1\sigma_2\sigma_2} - Y_{\varepsilon_1\varepsilon_1\alpha\alpha}) \\ & -(Y_{\sigma_1\sigma_1\varepsilon_2\varepsilon_2} + Y_{\sigma_1\sigma_1\sigma_2\sigma_2} - Y_{\sigma_1\sigma_1\alpha\alpha}) \\ & +(Y_{\alpha\alpha\varepsilon_2\varepsilon_2} + Y_{\alpha\alpha\sigma_2\sigma_2} - Y_{\alpha\alpha\alpha\alpha}). \end{aligned}$$

Consequently it follows that $0 = Y_{\gamma\gamma\delta\delta} + Y_{\delta\delta\gamma\gamma} = -2Y_{\alpha\alpha\alpha\alpha}$. Q. E. D.

By Lemmas 4. 6 and 4. 8~4. 11 we have shown that $Y_{\alpha\beta\gamma\delta} = -Y_{\gamma\delta\alpha\beta}$ for all α, β, γ, δ, thus completing the proof of Theorem 4. 1, (2).

Remark. Let p and q be any positive integers with $p \leqq q$. Let $\boldsymbol{Q}^{p+q}$ denote the space of quaternionic $(p+q)$-vectors, and $G^{p,q}(\boldsymbol{Q})$ the Grassmann manifold of p-dimensional quaternionic subspaces of $\boldsymbol{Q}^{p+q}$. As is well known, $G^{p,q}(\boldsymbol{Q})$ may be represented by a symmetric R space (cf. [12]). This is associated with the simple graded Lie algebra of the first kind, $(\mathfrak{g}, Z_0)$, (together with the involutive automorphism σ) which is defined as follows:

$$\begin{aligned} \mathfrak{g} &= \mathfrak{su}^*(2(p+q)) \\ &= \left\{ \begin{pmatrix} U & V \\ -\bar{V} & \bar{U} \end{pmatrix} \middle| \, U, \ V \in \mathfrak{gl}(p+q;\ \boldsymbol{C}), \ Tr(U+\bar{U}) = 0 \right\}, \end{aligned}$$

$$Z_0 = \begin{pmatrix} aI_p & & & 0 \\ & bI_q & & \\ & & aI_p & \\ 0 & & & bI_q \end{pmatrix} \quad \left(a = \frac{q}{p+q} \ \text{ and } \ b = \frac{-p}{p+q}\right),$$

$$\sigma X = -{}^t\bar{X}, \quad X \in \mathfrak{g}.$$

Now consider the canonical isometric imbedding $\boldsymbol{f}$ of the quaternionic Grassmann manifold $G^{p,q}(\boldsymbol{Q})$ into the Euclidean space $\mathfrak{m}$. Then it can be shown that the imbedding $\boldsymbol{f}$ is of finite type for any (p, q) (cf. Theorem 3. 8). Moreover we can prove the following theorem by a similar method to the proof of Theorem 4. 1.

Theorem 4. 12. (1) *For any (p, q) the first prolongation $\mathfrak{n}^{(1)}$ of the bundle $\mathfrak{n}$ of second fundamental forms vanishes.*

(2) *If $p \geqq 3$ or $p = q = 1$, then the imbedding $\boldsymbol{f}$ is of c. finite type.*

§ 5. Global solutions of the equation $L\varphi = 0$.

In this section, $M = K/K_0$ is the symmetric R space associated with a simple graded Lie algebra of the first kind, $\mathfrak{g} = \sum_p \mathfrak{g}_p$, (together with an involutive automorphism σ of $\mathfrak{g}$ having the properties in Lemma 3. 2). We use the notations as explained in 3. 1 and 3. 2. Above all $\boldsymbol{f}$ means the canonical isometric imbedding of M into the Euclidean space $\boldsymbol{R}^m=\mathfrak{m}$. We put dim $M=n$.

5. 1. The curvature $\boldsymbol{R}$ and the tensor field $\boldsymbol{S}$. As is well known, the curvature R of the symmetric space $M=K/K_0$, evaluated at the origin o, may be expressed as follows:

$$R(X, Y)Z=-[[X, Y], Z] \qquad \text{for all } X, Y, Z\in\mathfrak{k}_1=T_0.$$

The assignment $X\to[X, Z_0]$ gives an isomorphism of $\mathfrak{k}_1$ onto $\mathfrak{m}_1$, and $[\mathfrak{k}_1, [\mathfrak{k}_1, [\mathfrak{k}_1, Z_0]]]\subset\mathfrak{m}_1$. These being said, we define a tensor S of type $\binom{1}{3}$ on $\mathfrak{k}_1$ by

$$[S(X, Y)Z, Z_0]=[Z, [X, [Y, Z_0]]] \qquad \text{for all } X, Y, Z\in\mathfrak{k}_1.$$

For any $X, Y\in\mathfrak{g}_{-1}$ we denote by $-R'(X, Y)$ (resp. by $S'(X, Y)$) the $\mathfrak{k}_0$-component (resp. the $\mathfrak{m}_0$-component) of $[X, \sigma Y]$ $(\in\mathfrak{g}_0)$ with respect to the decomposition: $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{m}_0$. Clearly we have

$$R'(X, Y)=-\frac{1}{2}([X, \sigma Y]+[\sigma X, Y]),$$

$$S'(X, Y)=\frac{1}{2}([X, \sigma Y]-[\sigma X, Y]).$$

For simplicity let us idenify $\mathfrak{g}_0$ with a subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1})$ by the natural representation ρ of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$: $\mathfrak{g}_0=\rho(\mathfrak{g}_0)$. We also consider the isometric isomorphism η of $\mathfrak{g}_{-1}$ onto $\mathfrak{k}_1$. Then a simple calculation proves the following

Lemma 5. 1. *Let* $X, Y, Z\in\mathfrak{g}_{-1}$.

(1) $R(\eta(X), \eta(Y))\eta(Z)=\eta(R'(X, Y)Z)$.

(2) $S(\eta(X), \eta(Y))\eta(Z)=\eta(S'(X, Y)Z)$.

Lemma 5. 2. *Let* $X, Y, Z, W\in\mathfrak{k}_1$.

(1) $S(X, Y)=S(Y, X)$ *and* $\langle S(X, Y)Z, W\rangle=\langle Z, S(X, Y)W\rangle$.

(2) $\langle S(X, Y)Z, W\rangle=\langle S(Z, W)X, Y\rangle$

(3) $S(X, Y)Z=S(Z, Y)X+R(X, Z)Y$.

These equalities follow immediately from the corresponding equalities for S' (which can be easily derived) and Lemma 5. 1.

It is easy to see that S is invariant by the linear isotropy group of the homogeneous space K/K_0 at the origin. Hence S gives rise to a K-invariant tensor field of type $\binom{1}{3}$ on M, which we denote by the same letter S. Note that the tensor field S is parallel with respect to the Riemannian connection on M.

5.2. The bundle $\mathfrak{n}$ of second fundamental forms. By Lemma 3.1 the bilinear map $\mathfrak{g}_{-1}\times\mathfrak{g}_1\ni(X, Z)\to B(X, Z)\in\boldsymbol{R}$ is non-degenerate. Thus there corresponds to every $A\in\mathfrak{gl}(\mathfrak{g}_{-1})$ *its transpose* ${}^tA\in\mathfrak{gl}(\mathfrak{g}_1)$ as follows: $B(AX, Z)=B(X, {}^tAZ)$ for all $X\in\mathfrak{g}_{-1}$ and $Z\in\mathfrak{g}_1$. This being said, we denote by $\hat{\mathfrak{g}}_0$ the subspace of $\mathfrak{gl}(\mathfrak{g}_{-1})$ consisting of all $A\in\mathfrak{gl}(\mathfrak{g}_{-1})$ which satisfy the equation

$$A[X, Z]-[X, Z]A-[AX, Z]+[X, {}^tAZ]=0 \qquad \text{for all } X\in\mathfrak{g}_{-1} \text{ and } Z\in\mathfrak{g}_1.$$

Lemma 5.3. $\hat{\mathfrak{g}}_0=\mathfrak{g}_0$.

Proof. Let $A\in\mathfrak{g}_0$, $X\in\mathfrak{g}_{-1}$ and $Z\in\mathfrak{g}_1$. Then we have

$$[A, [X, Z]]=[[A, X], Z]+[X, [A, Z]] \quad\text{and}\quad [A, Z]=-{}^tAZ.$$

These facts show that $\mathfrak{g}_0\subset\hat{\mathfrak{g}}_0$. Furthermore $\hat{\mathfrak{g}}_0$ is a subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1})$, and the direct sum $\hat{\mathfrak{g}}=\mathfrak{g}_{-1}+\hat{\mathfrak{g}}_0+\mathfrak{g}_1$ is endowed with the structure of a graded Lie algebra, so that the given $\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ is a graded subalgebra of $\hat{\mathfrak{g}}=\mathfrak{g}_{-1}+\hat{\mathfrak{g}}_0+\mathfrak{g}_1$. Since the natural representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is irreducible (cf. [8] and [14]), so is the natural representation of $\hat{\mathfrak{g}}_0$ on $\mathfrak{g}_{-1}$. It follows that $\hat{\mathfrak{g}}$ is simple (cf. ibid.). Hence we obtain $\hat{\mathfrak{g}}_0=[\mathfrak{g}_{-1}, \mathfrak{g}_1]=\mathfrak{g}_0$ (cf. ibid.). Q. E. D.

Let us consider the space $\mathfrak{s}(\mathfrak{g}_{-1})$ of all symmetric endomorphisms of $\mathfrak{g}_{-1}$. By Lemma 5.3 we have $\mathfrak{m}_0=\mathfrak{g}_0\cap\mathfrak{s}(\mathfrak{g}_{-1})=\hat{\mathfrak{g}}_0\cap\mathfrak{s}(\mathfrak{g}_{-1})$.

Lemma 5.4. *The subspace $\mathfrak{m}_0$ of $\mathfrak{s}(\mathfrak{g}_{-1})$ consists of all $A\in\mathfrak{s}(\mathfrak{g}_{-1})$ which satisfy the equation*

$$AS'(X, Y)-S'(X, Y)A+R'(AX, Y)-R'(X, AY)=0 \qquad \textit{for all } X, Y\in\mathfrak{g}_{-1}.$$

Proof. Since $\mathfrak{m}_0=\hat{\mathfrak{g}}_0\cap\mathfrak{s}(\mathfrak{g}_{-1})$, $\mathfrak{m}_0$ consists of all $A\in\mathfrak{s}(\mathfrak{g}_{-1})$ which satisfy the equation

$$A[X, \sigma Y]-[X, \sigma Y]A-[AX, \sigma Y]+[X, {}^tA\sigma Y]=0 \qquad \text{for all } X, Y\in\mathfrak{g}_{-1}.$$

We have $[X, \sigma Y]=-R'(X, Y)+S'(X, Y)$ and ${}^tA\sigma Y=\sigma AY$. We also note that $R'(X, Y)\in\mathfrak{k}_0\subset\mathfrak{o}(\mathfrak{g}_{-1})$ and $S'(X, Y)\in\mathfrak{m}_0\subset\mathfrak{s}(\mathfrak{g}_{-1})$. Hence we see that the equation above is equivalent to the following two equations:

$$AR'(X, Y)-R'(X, Y)A+S'(AX, Y)-S'(X, AY)=0,$$

$$AS'(X, Y)-S'(X, Y)A+R'(AX, Y)-R'(X, AY)=0.$$

However these two equations (for A) are mutually equivalent, which can be easily verified by the use of the equalities:

$$\langle R'(X, Y)Z, W\rangle=\langle R'(Z, W)X, Y\rangle,$$
$$\langle S'(X, Y)Z, W\rangle=\langle S'(Z, W)X, Y\rangle \quad \text{(cf. Lemma 5. 2).}$$

Q. E. D.

We now define a bundle homomorphism Ψ of S^2T^* to $S^2T^*\otimes\wedge^2T^*$ by

$$(\Psi\alpha)(X, Y, Z, W)=\alpha(S(X, Y)Z, W)-\alpha(Z, S(X, Y)W)$$
$$-\alpha(X, R(Z, W)Y)-\alpha(R(Z, W)X, Y),$$

where $\alpha\in S^2T_p^*$ and $X, Y, Z, W\in T_p$.

Proposition 5. 5. *The bundle* $\mathfrak{n}$ *of second fundamental forms of the canonical imbedding* $\boldsymbol{f}$ *is the kernel of the bundle homomorphism* Ψ.

Proof. From Lemma 5. 4 we see that $\mathfrak{m}_0$ consists of all $A\in\mathfrak{s}(\mathfrak{g}_{-1})$ which satisfy the equation

$$\langle AS'(X, Y)Z, W\rangle-\langle AZ, S'(X, Y)W\rangle-\langle AX, R'(Z, W)Y\rangle$$
$$-\langle AR'(Z, W)X, Y\rangle=0 \qquad \text{for all } X, Y, Z, W\in\mathfrak{g}_{-1}.$$

(Note that $\langle R'(AX, Y)Z, W\rangle=-\langle AX, R'(Z, W)Y\rangle$, etc.) Furthermore we have $\eta^*\mathfrak{n}_0=\rho(\mathfrak{m}_0)$ by Proposition 3. 6. Therefore it follows from Lemma 5. 1 that $\mathfrak{n}_0=\{\alpha\in S^2\mathfrak{k}_1^* \mid \Psi\alpha=0\}$. Since both $\mathfrak{n}$ and Ψ are K-invariant, the proposition follows. Q. E. D.

5. 3. The covariant derivatives of the canonical imbedding *f*. As before let ∇ be the covariant differentiation associated with the canonical Riemannian metric $g(=\Phi(\boldsymbol{f}))$.

Proposition 5. 6 (cf. [15], Proposition 3. 1). Let $X, Y, Z\in T_p$.

(1) $\nabla_X\nabla_Y\boldsymbol{f}=[[\nabla_X\boldsymbol{f}, \boldsymbol{f}], \nabla_Y\boldsymbol{f}]$.

(2) $\nabla_Z\nabla_X\nabla_Y\boldsymbol{f}=(S(X, Y)Z)\boldsymbol{f}$.

Proof. (1) Let $X, Y\in\mathfrak{k}_1=T_0$. By Lemma 3. 3 we have $\nabla_X\boldsymbol{f}=[X, Z_0]$

and $\nabla_X\nabla_Y \boldsymbol{f}=[X, [Y, Z_0]]$. Furthermore we have $\boldsymbol{f}(o)=Z_0$ and $[[X, Z_0], Z_0]=X$. Hence it follows that $\nabla_X\nabla_Y\boldsymbol{f}=[[\nabla_X\boldsymbol{f}, \boldsymbol{f}], \nabla_Y\boldsymbol{f}]$. Since $\boldsymbol{f}$ is K-equivariant, (1) follows.

(2) Let $X, Y, Z\in T_p$. By using (1), we have

$$\nabla_Z\nabla_X\nabla_Y\boldsymbol{f}= [[\nabla_Z\nabla_X\boldsymbol{f}, \boldsymbol{f}], \nabla_Y\boldsymbol{f}]+[[\nabla_X\boldsymbol{f}, \nabla_Z\boldsymbol{f}], \nabla_Y\boldsymbol{f}] + [[\nabla_X\boldsymbol{f}, \boldsymbol{f}], \nabla_Z\nabla_Y\boldsymbol{f}].$$

Let us consider this equality at the origin, i. e., $p=o$. Then we have

$$\nabla_Z\nabla_X\nabla_Y\boldsymbol{f}= [[[Z, [X, Z_0]], Z_0], [Y, Z_0]] + [[[X, Z_0], [Z, Z_0]], [Y, Z_0]]+[X, [Z, [Y, Z_0]]].$$

An easy calculation shows that

$$[[[Z, [X, Z_0]], Z_0], [Y, Z_0]]=[[[Z, Z_0], [X, Z_0]], [Y, Z_0]] + [[Z, X], [Y, Z_0]].$$

Hence it follows that $\nabla_Z\nabla_X\nabla_Y\boldsymbol{f}=[Z, [X, [Y, Z_0]]]=(S(X, Y)Z)\boldsymbol{f}$. Since $\boldsymbol{f}$ is K-equivariant, (2) follows. Q. E. D.

We denote by $\mathscr{C}^k(M)$ the space of all (anti-symmetric) k-forms on M: $\mathscr{C}^k(M)=\Gamma(\wedge^k T^*)$. In terms of ∇, the *exterior differentiation* d: $\mathscr{C}^k(M)\to\mathscr{C}^{k+1}(M)$ may be described as follows:

$$(d\varphi)(X_1, \dots, X_{k+1})=\sum_i(-1)^{i+1}(\nabla_{X_i}\varphi)(X_1, \dots, \hat{X}_i, \dots, X_{k+1}),$$

where $\varphi\in\mathscr{C}^k(M)$ and $X_1, \dots, X_{k+1}\in T_p$. By denition the *co-differentiation* is the differential operator δ: $\mathscr{C}^k(M)\to\mathscr{C}^{k-1}(M)$ defined as follows: Let $\{e_1, \dots, e_n\}$ be any orthonormal basis of T_p. Then

$$(\delta\varphi)(X_1, \dots, X_{k-1})=-\sum_i(\nabla_{e_i}\varphi)(e_i, X_1, \dots, X_{k-1}),$$

where $\varphi\in\mathscr{C}^k(M)$ and $X_1, \dots, X_{k-1}\in T_p$. The differential operator $\Delta=\delta d+d\delta$: $\mathscr{C}^k(M)\to\mathscr{C}^k(M)$ is usually called the *Laplacian.* Note that Δ commutes with d and δ. In pacrticular we have

$$\Delta f=-\sum_i\nabla_{e_i}\nabla_{e_i}f \qquad \text{for any function } f.$$

Proposition 5. 7. $\Delta\boldsymbol{f}=\frac{1}{2}\boldsymbol{f}$.

Although this fact is known ([12]), we shall give a simple proof of it which uses (1) of Proposition 5. 6 and some elementary facts on the graded Lie algebra $\mathfrak{g}=\sum_p\mathfrak{g}_p$.

Proof of Proposition 5. 7. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of $\mathfrak{g}_{-1}$. Putting $e_i=\eta(u_i)$, we have $(\Delta f)(o)=-\sum_i \nabla_{e_i}\nabla_{e_i} f=-\sum_i [e_i, [e_i, Z_0]] =\sum_i [u_i, \sigma u_i]$. Furthermore putting $Z'_0=\sum_i [u_i, \sigma u_i]$, we see that Z'_0 is in the centre of $\mathfrak{g}_0$ and $\sigma Z'_0=-Z'_0$. It follows that $Z'_0=aZ_0$ with some a (cf. [14]). We have $B(Z_0, Z'_0)=\sum_i B(Z_0, [u_i, \sigma u_i])=-\sum_i B(u_i, \sigma u_i)=n$. And we can easily verify that $B(Z_0, Z_0)=2n$. Therefore we have $a=\frac{1}{2}$, and hence $(\Delta f)(o)=\frac{1}{2}Z_0=\frac{1}{2}f(o)$. Since f is K-equivariant, the proposition follows. Q. E. D.

5. 4. Relevant calculations on the solutions of the equations $d\Phi_f(u)=0$ ***and*** $L\varphi=0$. In this and the subsequent paragraphs we assume that $M=K/K_0$ is an Einstein space, that is, there is a constant κ such that

$$\sum_i R(X, e_i)e_i=\kappa X \qquad \text{for all } X\in T_p.$$

Under this assumption we have

$$\Delta\varphi=-\sum_i \nabla_{e_i}\nabla_{e_i}\varphi+\kappa\varphi \qquad \text{for any 1-form } \varphi.$$

Now recall the differential operators $d\Phi_f$ of $\Gamma(M, m)$ to $\Gamma(S^2T^*)$, D of $\Gamma(T^*)$ to $\Gamma(S^2T^*)$ and $L=\Pi\circ D$ of $\Gamma(T^*)$ to $\Gamma(S^2T^*/\mathfrak{n})$, which are all defined in 1. 1.

Proposition 5. 8 (cf. [15], Lemma 3. 11). *Let* u *be a solution of the equation* $d\Phi_f(u)=0$, *and* φ *the corresponding solution of the equation* $L\varphi=0$. *(The correspondence is given by Thorem* 1. 1.)

(1) $\langle \Delta u-\frac{1}{2}u, \nabla f\rangle=-d\delta\varphi.$

(2) $\langle \Delta u-\frac{1}{2}u, \nabla\nabla f\rangle=\frac{1}{2}D(\Delta\varphi-\varphi).$

Proof. We have $\langle u, \nabla_Y f\rangle=\varphi(Y)$ and hence

$$\langle \nabla_X u, \nabla_Y f\rangle+\langle u, \nabla_X\nabla_Y f\rangle=(\nabla_X\varphi)(Y).$$

Since $\nabla_X\nabla_Y f=\nabla_Y\nabla_X f$ and $\langle \nabla_X f, \nabla_Y u\rangle+\langle \nabla_Y f, \nabla_X u\rangle=0$, it follows that

(5. 1) $\langle u, \nabla_X\nabla_Y f\rangle=\frac{1}{2}(D\varphi)(X, Y).$

(5. 2) $\langle \nabla_X u, \nabla_Y f\rangle=\frac{1}{2}(d\varphi)(X, Y).$

From these two equalities we obtain

$$\sum_i <\nabla_{e_i}\boldsymbol{u},\ \nabla_{e_i}\nabla_X\boldsymbol{f}> + \sum_i <\boldsymbol{u},\ \nabla_{e_i}\nabla_{e_i}\nabla_X\boldsymbol{f}> = \frac{1}{2}\sum_i (\nabla_{e_i}D\varphi)(e_i,\ X),$$

$$\sum_i <\nabla_{e_i}\nabla_{e_i}\boldsymbol{u},\ \nabla_X\boldsymbol{f}> + \sum_i <\nabla_{e_i}\boldsymbol{u},\ \nabla_{e_i}\nabla_X\boldsymbol{f}> = \frac{1}{2}\sum_i (\nabla_{e_i}d\varphi)(e_i,\ X).$$

Hence

$$-\sum_i <\nabla_{e_i}\nabla_{e_i}\boldsymbol{u},\ \nabla_X\boldsymbol{f}> + \sum_i <\boldsymbol{u},\ \nabla_{e_i}\nabla_{e_i}\nabla_X\boldsymbol{f}>$$
$$= \frac{1}{2}\sum_i (\nabla_{e_i}D\varphi)(e_i,\ X) - \frac{1}{2}\sum_i (\nabla_{e_i}d\varphi)(e_i,\ X)$$

Now by using the fact that $\Delta\boldsymbol{f} = \frac{1}{2}\boldsymbol{f}$ (Proposition 5. 7), we have

$$\sum_i \nabla_{e_i}\nabla_{e_i}\nabla_X\boldsymbol{f} = \sum_i \nabla_X\nabla_{e_i}\nabla_{e_i}\boldsymbol{f} + \sum_i (R(X,\ e_i)e_i)\boldsymbol{f}$$
$$= (\kappa - \frac{1}{2})\nabla_X\boldsymbol{f},$$

whence

$$\sum_i <\boldsymbol{u},\ \nabla_{e_i}\nabla_{e_i}\nabla_X\boldsymbol{f}> = (\kappa - \frac{1}{2}) <\boldsymbol{u},\ \nabla_X\boldsymbol{f}> = (\kappa - \frac{1}{2})\varphi(X).$$

Furthermore we have $\sum_i \nabla_{e_i}\nabla_{e_i}\boldsymbol{u} = -\Delta\boldsymbol{u}$, $\sum_i (\nabla_{e_i}d\varphi)(e_i, X) = -(\delta d\varphi)(X)$, and

$$\sum_i (\nabla_{e_i}D\varphi)(e_i,\ X) = (-2d\delta\varphi + 2\kappa\varphi - \delta d\varphi)(X).$$

Therefore (1) follows.

Let us now prove (2). From (5. 1) we obtain

$$\sum_i <\nabla_{e_i}\nabla_{e_i}\boldsymbol{u},\ \nabla_X\nabla_Y\boldsymbol{f}> + 2\sum_i <\nabla_{e_i}\boldsymbol{u},\ \nabla_{e_i}\nabla_X\nabla_Y\boldsymbol{f}>$$
$$+ \sum_i <\boldsymbol{u},\ \nabla_{e_i}\nabla_{e_i}\nabla_X\nabla_Y\boldsymbol{f}> = \frac{1}{2}\sum_i (\nabla_{e_i}\nabla_{e_i}D\varphi)(X,\ Y)$$

First by (2) of Proposition 5. 6 combined with (5. 2), we have

$$\sum_i <\nabla_{e_i}\boldsymbol{u},\ \nabla_{e_i}\nabla_X\nabla_Y\boldsymbol{f}> = \frac{1}{2}\sum_i (d\varphi)(e_i,\ S(X,\ Y)e_i).$$

Since $S(X, Y)$ is symmetric with respect to g (cf. Lemma 5. 2), we obtain $\sum_i (d\varphi)(e_i,\ S(X,\ Y)e_i) = 0$ Hence

$$\sum_i <\nabla_{e_i}\boldsymbol{u},\ \nabla_{e_i}\nabla_X\nabla_Y\boldsymbol{f}> = 0.$$

Next by using the fact that $\Delta\boldsymbol{f} = \frac{1}{2}\boldsymbol{f}$, we have

$$\sum_i \nabla_{e_i}\nabla_{e_i}\nabla_X\nabla_Y f = (2\kappa - \frac{1}{2})\nabla_X\nabla_Y f + 2\sum_i \nabla_{e_i}\nabla_{R(X,e_i)Y} f.$$

From this equality together with (5. 1) follows that

$$\sum_i <u, \nabla_{e_i}\nabla_{e_i}\nabla_X\nabla_Y f> = \frac{1}{2}(2\kappa - \frac{1}{2})(D\varphi)(X, Y) + \sum_i (D\varphi)(e_i, R(X, e_i)Y).$$

Furthermore we have

$$\sum_i (\nabla_{e_i}\nabla_{e_i} D\varphi)(X, Y) = -(D\Delta\varphi)(X, Y) + 2\kappa(D\varphi)(X, Y) + 2\sum_i (D\varphi)(R(X, e_i)Y, e_i).$$

Therefore (2) follows. Q. E. D.

Remark. Incidentally we remark that the constant κ of the Einstein space M is smaller than $\frac{1}{2}$. Indeed we have $\sum_i \nabla_{e_i}\nabla_{e_i}\nabla_X f = (\kappa - \frac{1}{2})\nabla_X f$. Hence it follows that

$$|\nabla\nabla f|^2 + n(\kappa - \frac{1}{2}) = \delta\alpha,$$

where $|\nabla\nabla f|^2 = \sum_{i,j} <\nabla_{e_i}\nabla_{e_j} f, \nabla_{e_i}\nabla_{e_j} f>$ and α is the 1-form defined by $\alpha(X) = -\sum_j <\nabla_{e_j} f, \nabla_X\nabla_{e_j} f>$ for all $X \in T_p$. Since $\nabla\nabla f \neq 0$, we see from Green's theorem that $\kappa < \frac{1}{2}$, proving our remark.

5. 5. The space $\mathscr{A}_E(M)$ and $\mathscr{A}(M)$. In this paragraph we study the space $\mathscr{A}(M, f)$ of all global solutions φ of the equation $L\varphi = 0$ as well as the subspace $\mathscr{A}_E(M, f)$ of $\mathscr{A}(M, f)$ consisting of all the special solutions $\varphi^A (A \in \mathfrak{e}(m))$. For simplicity we put $\mathscr{A}(M) = \mathscr{A}(M, f)$ and $\mathscr{A}_E(M) = \mathscr{A}_E(M, f)$. For any real number λ and any integer p, we define a subspace $\mathscr{C}^p(M)_{(\lambda)}$ of $\mathscr{C}^p(M)$ by

$$\mathscr{C}^p(M)_{(\lambda)} = \{\varphi \in \mathscr{C}^p(M) \mid \Delta\varphi = \lambda\varphi\}.$$

By an *infinitesimal K-transformation* we mean the vector field V on $M = K/K_0$ which is induced from a one-parameter group $\{a_t\}$ of K. We denote by $\mathscr{A}_K(M)$ the subspace of $\mathscr{C}^1(M)$ consisting of all the 1-forms φ which are dual to infinitesimal K-transformations V (w. r. t. g). Here we notice that an infinitesimal K-transformation means an infinitesimal isometry (cf. Proposition 5. 12 in the next paragraph). Consequently the space $\mathscr{A}_K(M)$ coincides with the space of all the

Killing forms on M, i. e., $\mathscr{A}_K(M)=\{\varphi\in\mathscr{C}^1(M)\mid D\varphi=0\}$. Moreover it is known that this space is equal to the space $\{\varphi\in\mathscr{C}^1(M)_{(2\kappa)}\mid\delta\varphi=0\}=\mathscr{C}^1(M)_{(2\kappa)}\cap\delta^{-1}(0)$ ([16]). (We are assuming that M is an Einstein space.) Therefore we have

$$\mathscr{A}_K(M)=\mathscr{C}^1(M)_{(2\kappa)}\cap\delta^{-1}(0).$$

We denote by $\mathscr{P}(M)$ the subspace of $\mathscr{C}^0(M)$ consisting of all the functions of the form $\langle \boldsymbol{c}, \boldsymbol{f}\rangle(\boldsymbol{c}\in\mathfrak{m})$. By Proposition 5. 7 we have

$$\mathscr{P}(M)\subset\mathscr{C}^0(M)_{(\frac{1}{2})}.$$

We also denote by $\mathscr{A}_G(M)$ the subspace of $\mathscr{C}^1(M)$ defined by

$$\mathscr{A}_G(M)=\mathscr{A}_K(M)+d\mathscr{P}(M).$$

In the next paragraph we shall justify this notation.

From Proposition 5. 9 just below we shall see that $\Delta\mathscr{A}_E(M)\subset\mathscr{A}_E(M)$. We here assert that

$$\Delta\mathscr{A}(M)\subset\mathscr{A}(M).$$

Indeed let $\varphi\in\mathscr{A}(M)$. Then it follows from (2) of Proposition 5. 8 that $D(\Delta\varphi-\varphi)$ is a cross section of $\mathfrak{n}$. Hence $L\Delta\varphi=\Pi\circ D(\Delta\varphi-\varphi)=0$, i. e., $\Delta\varphi\in\mathscr{A}(M)$. This proves our assertion. For any real number λ we put $\mathscr{A}_E(M)_{(\lambda)}=\mathscr{A}_E(M)\cap\mathscr{C}^1(M)_{(\lambda)}$ and $\mathscr{A}(M)_{(\lambda)}=\mathscr{A}(M)\cap\mathscr{C}^1(M)_{(\lambda)}$.

Proposition 5. 9. (1) *$\Delta\mathscr{A}_E(M)\subset\mathscr{A}_E(M)$, and the eigenvalues of the operator $\Delta\mid\mathscr{A}_E(M)$, the restriction of Δ to $\mathscr{A}_E(M)$, are 2κ, $\frac{1}{2}$ and* 1.

(2) *$\mathscr{A}_G(M)=\mathscr{A}_E(M)_{(2\kappa)}+\mathscr{A}_E(M)_{(\frac{1}{2})}$ and $\delta\mathscr{A}_E(M)_{(1)}=0$.*

Note that $0\leqq 2\kappa<1$ (Remark at the end of the previous paragraph). Also note that the second assertion of (2) of the proposition may be restated as follows: 1°. If $2\kappa\neq\frac{1}{2}$, then $\mathscr{A}_E(M)_{(2\kappa)}=\mathscr{A}_K(M)$ and $\mathscr{A}_E(M)_{(\frac{1}{2})}=d\mathscr{P}(M)$; 2°. If $2\kappa=\frac{1}{2}$, then $\mathscr{A}_E(M)_{(\frac{1}{2})}=\mathscr{A}_G(M)$.

Proof of Proposition 5. 9. We denote by $\mathscr{A}_0(M)$ the subspace of $\mathscr{A}_E(M)$ consiting of all 1-forms of the form φ^A $(A\in\mathfrak{o}(m))$. Then we have $\mathscr{A}_E(M)=\mathscr{A}_0(M)+d\mathscr{P}(M)$. Furthermore we have $\mathscr{A}_K(M)\subset\mathscr{A}_0(M)$, because the canonical imbedding $\boldsymbol{f}$ is K-equivariant. Therefore to prove the proposition, it suffices to show that $\delta\mathscr{A}_0(M)=0$ and $\mathscr{A}_0(M)\subset\mathscr{A}_K(M)+\mathscr{C}^1(M)_{(1)}$. Let $\varphi=\varphi^A\in\mathscr{A}_0(M)$. Then $\boldsymbol{u}=A\boldsymbol{f}$ is the solution of the equation $d\Phi_f(\boldsymbol{u})=0$ which corresponds to the solution

φ of the equation $L\varphi=0$. (The correspondence is given by Theorem 1.1.) By Proposition 5.7 we have $\Delta \boldsymbol{u}=\frac{1}{2}\boldsymbol{u}$. Therefore we see from Proposition 5.8 that $d\delta\varphi=0$ and $D(\Delta\varphi-\varphi)=0$.
This last equality means that $\delta(\Delta\varphi-\varphi)=0$ and $\Delta(\Delta\varphi-\varphi)=2\kappa(\Delta\varphi-\varphi)$. It follows that $\delta\varphi=\delta\Delta\varphi=\Delta\delta\varphi=0$. Putting $\varphi_1=\frac{1}{2\kappa-1}(\Delta\varphi-\varphi)$ and $\varphi_2=\frac{1}{2\kappa-1}(2\kappa\varphi-\Delta\varphi)$, we further find that $\varphi_1\in\mathscr{A}_K(M)$, $\varphi_2\in\mathscr{C}^1(M)_{(1)}$ and $\varphi=\varphi_1+\varphi_2$. We have thereby proved that $\delta\mathscr{A}_O(M)=0$ and $\mathscr{A}_O(M)\subset\mathscr{A}_K(M)+\mathscr{C}^1(M)_{(1)}$. Q. E. D.

Let us now consider the following four conditions for the symmetric R space M:

(C_1) $\mathscr{C}^0(M)_{(\frac{1}{2})}=\mathscr{P}(M)$.

(C_2) The canonical imbedding $\boldsymbol{f}$ is elliptic.

(C_3) The eigenvalues of the operator $\Delta\mid\mathscr{A}(M)$ are 2κ, $\frac{1}{2}$ and 1.

(C_4) $\mathscr{A}_G(M)=\mathscr{A}(M)_{(2\kappa)}+\mathscr{A}(M)_{(\frac{1}{2})}$ and $\delta\mathscr{A}(M)_{(1)}=0$.

Proposition 5.10. *Assume conditions* (C_1)$\sim$(C_4). *Then the two spaces* $\mathscr{A}(M)$ *and* $\mathscr{A}_E(M)$ *coincide.*

Proof. (C_2) means that the equation $L\varphi=0$ is elliptic, whence $\mathscr{A}(M)$ is of finite dimension. From this fact combined with (C_3) and (C_4) we know that $\mathscr{A}(M)=\mathscr{A}_G(M)+\mathscr{A}(M)_{(1)}$. Therefore to prove the proposition, it suffices to show that $\mathscr{A}(M)_{(1)}\subset\mathscr{A}_E(M)$.
Let $\varphi\in\mathscr{A}(M)_{(1)}$, and $\boldsymbol{u}$ the corresponding solution of the equation $d\Phi_f(\boldsymbol{u})=0$. Since $\Delta\varphi-\varphi=0$ and $\delta\varphi=0$ (by (C_4)), it follows from Proposition 5.8 that

$$\langle\Delta\boldsymbol{u}-\frac{1}{2}\boldsymbol{u},\ \nabla\boldsymbol{f}\ \rangle=0,$$

$$\langle\Delta\boldsymbol{u}-\frac{1}{2}\boldsymbol{u},\ \nabla\nabla\boldsymbol{f}\rangle=0.$$

Since $\boldsymbol{f}$ is non-degenerate, these equalities mean that $\Delta\boldsymbol{u}=\frac{1}{2}\boldsymbol{u}$. Consequently we see from (C_1) that $\boldsymbol{u}$ can be written in the form: $\boldsymbol{u}=A\boldsymbol{f}$, where A is a matrix of degree m. We can further verify that $A\in\mathfrak{o}(m)$ (cf. [15], Lemma 3.14). Hence $\varphi=\langle A\boldsymbol{f},\ d\boldsymbol{f}\rangle=\varphi^A\in\mathscr{A}_E(M)$. We have thus shown that $\mathscr{A}(M)_{(1)}\subset\mathscr{A}_E(M)$.

It is known that (C_1) is satisfied for various symmetric R spaces including irreducible hermitian symmetric spaces of compact type, the real projective spaces, the real quadrics (or the Möbius spaces) of index 0, etc. In § 6 we shall directly verify (C_1) for the real Grassmann manifolds $G^{2,n}(\boldsymbol{R})$ with $n \geqq 2$.

Now consider the case where M is an irreducible hermitian symmetric space of compact type, being an Einstein space. In [15] one of the authors has shown that M satisfies $(C_1) \sim (C_4)$, and hence $\mathscr{A}(M) = \mathscr{A}_E(M)$. Note that the constant 2κ is just equal to $\frac{1}{2}$. Also note that the canonical imbedding is of infinite type in the special case where M is a complex projective space. Next consider the case where M is the real Grassmann manifold $G^{2,n}(\boldsymbol{R})$ with $n \geqq 3$, being an Einstein space. In 3. 5 we have seen that the canonical imbedding is elliptic and of infinite type (see Theorem 3. 8). In §§ 6 and 7 we shall see that M satisfies (C_1), (C_3) and (C_4), and hence $\mathscr{A}(M) = \mathscr{A}_E(M)$. Note that the constant 2κ is smaller than $\frac{1}{2}$.

In connection with the results above, recall the rigidity theorem (Theorem 2. 4) for elliptic imbeddings.

5. 6. The space $\mathscr{A}_G(M)$. Let G denote the adjoint group $Ad(\mathfrak{g})$ of $\mathfrak{g}$. We define a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ by $\mathfrak{g}' = \mathfrak{g}_0 + \mathfrak{g}_1$, and a closed subgroup G' of G by

$$G' = \{a \in G \mid Ad(a)\mathfrak{g}' = \mathfrak{g}'\}.$$

We also define a closed subgroup G_0 of G' by

$$G_0 = \{a \in G \mid Ad(a)\mathfrak{g}_p = \mathfrak{g}_p \text{ for all } p\}.$$

Then we know the following ([14]) : 1°. The Lie algebra of G' (resp. of G_0) is $\mathfrak{g}'$ (resp. $\mathfrak{g}_0$) ; 2°. $G' = G_0 \cdot \exp \mathfrak{g}_1$; 3°. The homogeneous space G/G' is effective ; 4 . $K \cap G' = K_0$ and

$$G/G' = K/K_0.$$

For example consider the case the symmetric R space $M = K/K_0$ is the real projective space $P^n(\boldsymbol{R})$ $(= G^{1,n}(\boldsymbol{R}))$. Then G may be characterized as the largest connected group of projective transformations.

By an *infinitesimal G-transformation* we mean the vector field V on $M = G/G'$ which is induced from a one-parameter group $\{a_t\}$ of G. By using the curvature R and the tensor field S, we define a tensor field F of type $\binom{1}{3}$ on M by

$$F = -R + S.$$

Note that $F_0(\eta(X), \eta(Y))\eta(Z)=\eta([X, \sigma Y]Z)$ for all $X, Y, Z\in\mathfrak{g}_{-1}$.

The next lemma whose proof is omitted, can be verified by the use of the results of [14].

Lemma 5.11. *Assume that M is neither the 1-dimensional real projective space $P^1(\boldsymbol{R})$ nor the 1-dimensional complex projective space $P^1(\boldsymbol{C})$. Let V be a vector field defined on an open set U of M. Then V is an infinitesimal G-transformation if and only if there is a vector field $\tilde{V}$ defined on the open set U such that*

$$(L_V\nabla)(X, Y)=F(X, \tilde{V})Y \qquad \textit{for all } X, Y\in T_p.$$

Here $L_V\nabla$ denotes the Lie derivative of the covariant differentiation ∇ w. r. t. V, which may be expressed as follows:

$$(L_V\nabla)(X, Y)=\nabla_X\nabla_Y V+R(V, X)Y.$$

Let us newly denote by $\mathscr{A}_G(M)$ the subspace of $\mathscr{C}^1(M)$ consisting of all the 1-forms which are dual to infinitesimal G-transformations.

Proposition 5.12. *The space $\mathscr{A}_K(M)$ coincides with the space of all Killing forms, and the space $\mathscr{A}_G(M)$ may be described as follows:*

$$\mathscr{A}_G(M)=\mathscr{A}_K(M)+d\mathscr{P}(M) \quad (\textit{direct sum}).$$

Proof. We first consider the case when M is neither $P^1(\boldsymbol{R})$ nor $P^1(\boldsymbol{C})$. Let $I(M)^0$ denote the largest connected group of isometries of M, which is compact. Since an isometry leaves ∇ invariant, we see from Lemma 5.11 that $I(M)^0\subset G$. Furthermore $K\subset I(M)^0$, and K is a maximal compact subgroup of G. From these facts it follows that $I(M)^0=K$. Now let $f\in\mathscr{P}(M)$, and let V be the vector field on M which is dual to df w. r. t. g. Then we can easily verify that $(L_V\nabla)(X, Y)=F(X, V)Y$. It follows from Lemma 5.11 that V is an infinitesimal G-transformation, whence $df\in\mathscr{A}_G(M)$. We have $\dim\mathscr{A}_K(M)=\dim\mathfrak{k}$, $\dim d\mathscr{P}(M)=\dim\mathfrak{m}$, $\dim\mathscr{A}_G(M)=\dim\mathfrak{g}=\dim\mathfrak{k}+\dim\mathfrak{m}$, and $\mathscr{A}_K(M)\cap d\mathscr{P}(M)=0$. Therefore we obtain $\mathscr{A}_G(M)=\mathscr{A}_K(M)+d\mathscr{P}(M)$. Finally the proposition can be directly verified in the case when $M=P^1(\boldsymbol{R})$ or $P^1(\boldsymbol{C})$. Q. E. D.

§ 6. The real Grassmann manifolds $G^{2,n}(\boldsymbol{R})$.

In this and the subsequent sections we observe the real Grassmann manifold $M=G^{2,n}(\boldsymbol{R})$ $(n\geqq 2)$, which is a symmetric R space (see 3.4). As for this space we preserve the notations as explained so far.

6.1. The curvature R and the tensor field S. We define an element I_0 in the centre of $\mathfrak{k}_0$ by

$$I_0=\begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$$

Then we have $[I_0, [I_0, X]]=-X$ for all $X\in\mathfrak{k}_1$.

Remark. Let K_0^0 be the connected component of the identity of K_0. Then $\tilde{M}=K/K_0^0$ is the universal covering space of $M=K/K_0$, and I_0 naturally induces a complex structure on $\tilde{M}$ so that $\tilde{M}$ becomes a hermitian symmetric space.

Now consider the complexification $\mathfrak{k}^c=\mathfrak{k}_0^c+\mathfrak{k}_1^c$ of $\mathfrak{k}$, and define subspaces $\mathfrak{k}_1^+$ and $\mathfrak{k}_1^-$ of $\mathfrak{k}_1^c$ in the same manner as in 3.3. We have $\mathfrak{k}_1^c=\mathfrak{k}_1^+ + \mathfrak{k}_1^-$ (direct sum), $\mathfrak{k}_1^-=\bar{\mathfrak{k}}_1^+$, $<\mathfrak{k}_1^+, \mathfrak{k}_1^+>=0$, and $[\mathfrak{k}_1^+, \mathfrak{k}_1^+]=0$. In particular it follows that the curvature R is completely determined by the values $R(X, \bar{Y})Z=-[[X, \bar{Y}], Z]$ for $X, Y, Z\in\mathfrak{k}_1^+$.

Lemma 6.1. *There is a basis* $\{e_1, \dots, e_n\}$ *of* $\mathfrak{k}_1^+$ *having the following properties:*

1) $<e_i, \bar{e}_j>=\delta_{ij}$.

2) $R(e_i, \bar{e}_j)e_k=\dfrac{1}{4(n+2)}(\delta_{jk}e_i-\delta_{ij}e_k+\delta_{ji}e_k)$.

3) $S(e_i, \bar{e}_j)e_k=-\dfrac{1}{4(n+2)}(\delta_{jk}e_i+\delta_{ik}e_j+\delta_{ji}e_k)$,

$S(e_i, e_j)\bar{e}_k=-\dfrac{1}{2(n+2)}\delta_{ij}e_k$,

$S(e_i, e_j)e_k=0$.

Proof. For each $1\leqq i\leqq n$, we define an element of $\mathfrak{g}_{-1}^c=M_{n,2}(\boldsymbol{C})$ by

$$u_i=c\begin{pmatrix} 0 & 0 \\ 1 & -\sqrt{-1} \\ 0 & 0 \end{pmatrix} \ i\text{-th line} \quad \left(c=\frac{1+\sqrt{-1}}{\sqrt{8(n+2)}}\right).$$

Then an easy calculation proves the following:

1′) $<u_i, \bar{u}_j>=\delta_{ij}$.

2') $R'(u_i, \bar{u}_j)u_k = \frac{1}{4(n+2)}(\delta_{jk}u_i - \delta_{ik}u_j + \delta_{ji}u_k)$.

3') $S'(u_i, \bar{u}_j)u_k = -\frac{1}{4(n+2)}(\delta_{jk}u_i + \delta_{ik}u_j + \delta_{ji}u_k)$,

$S'(u_i, u_j)\bar{u}_k = -\frac{1}{2(n+2)}\delta_{ij}u_k$,

$S'(u_i, u_j)u_k = 0$.

If we put $e_i = \eta(u_i)$, we easily see that $u_1, \ldots, u_n$ form a basis of $\mathfrak{k}_1^+$. Furthermore we see from 1'), 2') and 3'), and Lemma 5. 1 that the basis $\{e_1, \ldots, e_n\}$ satisfies 1), 2) and 3) in Lemma 6. 1.

Q. E. D.

Let $\{e_1, \ldots, e_n\}$ be as in Lemma 6. 1. Then we have

$$\sum_i R(e_j, \bar{e}_i)e_i = \sum_i R(e_i, \bar{e}_i)e_j = \frac{n}{4(n+2)}e_j,$$

showing that M is an Einstein space with constant $\kappa = \frac{n}{4(n+2)}$.

6. 2. Various spaces of forms and of functions. As is well known, the differential forms on $\tilde{M} = K/K_0^0$ may be represented by the suitable functions on K. The same holds for the differential forms on $M = K/K_0$. In this paragraph we formulate these facts together with some related facts. $\tilde{\pi}$ (resp. π) denotes the projection of K onto $\tilde{M}$ (resp. onto M), and $\tilde{\omega}$ the projection of $\tilde{M}$ onto M.

The spaces $\mathscr{T}^p(M')$ **and** $\mathscr{D}^p(M')$ ($M' = \tilde{M}$ or M). $\mathscr{T}^p(M')$ denotes the space of all complex valued covariant tensor fields of degree p on M'. Since $\tilde{M}$ is a covering space of M, we may identify $\mathscr{T}^p(M)$ with a subspace of $\mathscr{T}^p(\tilde{M})$ in a natural manner: $\tilde{\omega}_*\varphi = \varphi$ for all $\varphi \in \mathscr{T}^p(M)$. Clearly we have $\tilde{\nabla}\varphi = \nabla\varphi$ for all $\varphi \in \mathscr{T}^p(M)$, where $\tilde{\nabla}$ denotes the covariant differentiation in the Riemannian manifold $\tilde{M}$. $\mathscr{D}^p(M')$ denotes the space of all forms in $\mathscr{T}^p(M')$ which are anti-symmetric. $\tilde{M}$ (resp. M) being a Riemannian manifold, we have the operators $\tilde{d}$, $\tilde{\delta}$ and $\tilde{\Delta}$ (resp. d, δ and Δ) acting on the spaces $\mathscr{D}^p(\tilde{M})$ (resp. on $\mathscr{D}^p(M)$) (cf. the operators d, δ and Δ defined in 5. 3). We have $\mathscr{D}^p(M) \subset \mathscr{D}^p(\tilde{M})$, and $\tilde{d}\varphi = d\varphi$, $\tilde{\delta}\varphi = \delta\varphi$ and $\tilde{\Delta}\varphi = \Delta\varphi$ for all $\varphi \in \mathscr{D}^p(M)$. Hereafter $\tilde{\nabla}$, $\tilde{d}$, $\tilde{\delta}$ and $\tilde{\Delta}$ will be simply written as ∇, d, δ and Δ respectively.

The spaces $\mathscr{T}^p(K, K_0^0)$. We denote by $\mathscr{T}^p(K, K_0^0)$ the space of all functions $\varphi: K \to \otimes^p(\mathfrak{k}_1^c)^*$ which satisfy the equality

$$\varphi(za)(X_1, \dots, X_p) = \varphi(z)(Ad(a)X_1, \dots, Ad(a)X_p)$$
$$\text{for all } z \in K,\ a \in K_0^0 \text{ and } X_1, \dots, X_p \in \mathfrak{k}_1^c.$$

Let φ be a function $K \to \otimes^p(\mathfrak{k}_1^c)^*$. Then φ is in $\mathscr{T}^p(K, K_0^0)$ if and only if it satisfies the equality

$$X\varphi(X_1, \dots, X_p) = \sum_i \varphi(X_1, \dots, [X, X_i], \dots, X_p)$$
$$\text{for all } X \in \mathfrak{k}_0^c \text{ and } X_1, \dots, X_p \in \mathfrak{k}_1^c.$$

($\varphi(X_1, \dots, X_p)$ stands for the function on K defined by $\varphi(X_1, \dots, X_p)(z) = \varphi(z)(X_1, \dots, X_p)$ for all $z \in K$.) For $\varphi \in \mathscr{T}^p(\tilde{M})$ we define a function $\iota_K\varphi$: $K \to \otimes^p(\mathfrak{k}_1^c)^*$ by

$$(\iota_K\varphi)(z)(X_1, \dots, X_p) = \varphi(\tilde{\pi}_*(X_1)_z, \dots, \tilde{\pi}_*(X_p)_z)$$
$$\text{for all } z \in K \text{ and } X_1, \dots, X_p \in \mathfrak{k}_1^c.$$

Then we have $\iota_K\varphi \in \mathscr{T}^p(K, K_0^0)$, and the assignment $\varphi \to \iota_K\varphi$ gives an isomorphism ι_K of $\mathscr{T}^p(\tilde{M})$ onto $\mathscr{T}^p(K, K_0^0)$.

Lemma 6.2. *Let $\varphi \in \mathscr{T}^p(\tilde{M})$ and $X_1, \dots, X_{p+1} \in \mathfrak{k}_1^c$.*

$$(\iota_K \nabla\varphi)(X_1, \dots, X_{p+1}) = X_1(\iota_K\varphi)(X_2, \dots, X_{p+1}).$$

The spaces $\mathscr{D}^p(K, K_0^0)$. We denote by $\mathscr{D}^p(K, K_0^0)$ the space of all functions φ in $\mathscr{T}^p(K, K_0^0)$ which take values in $\wedge^p(\mathfrak{k}_1^c)^*$. Clearly we have $\iota_K\mathscr{D}^p(\tilde{M}) = \mathscr{D}^p(K, K_0^0)$. Therefore there are unique differential operators d_K: $\mathscr{D}^p(K, K_0^0) \to \mathscr{D}^{p+1}(K, K_0^0)$, δ_K: $\mathscr{D}^p(K, K_0^0) \to \mathscr{D}^{p-1}(K, K_0^0)$ and Δ_K: $\mathscr{D}^p(K, K_0^0) \to \mathscr{D}^p(K, K_0^0)$ such that $d_K\iota_K = \iota_K d$, $\delta_K\iota_K = \iota_K\delta$ and $\Delta_K\iota_K = \iota_K\Delta$.

The differential operators $P, \bar{P}, Q$ **and** $\bar{Q}$. Let $\mathscr{F}(K)$ denote the space of all differentiable functions $K \to \boldsymbol{C}$. Let $\{e_1, \dots, e_n\}$ be a basis of $\mathfrak{k}_1^+$ having the properties in Lemma 6.1. Then the formal sums $P = \sum_i e_i\bar{e}_i$, $\bar{P} = \sum_i \bar{e}_i e_i$, $Q = \sum_i e_i e_i$ and $\bar{Q} = \sum_i \bar{e}_i\bar{e}_i$ may be regarded as differential operators of $\mathscr{F}(K)$ to itself.

We shall now give the explicit expressions of $d_K f$, $\delta_K\varphi$, $\Delta_K f$ and of $\Delta_K\varphi$, where $f \in \mathscr{D}^0(K, K_0^0)$ and $\varphi \in \mathscr{D}^1(K, K_0^0)$. Clearly we have

$$(d_K f)(X) = Xf \qquad \text{for all } X \in \mathfrak{k}_1^c.$$

Using the function φ, let us define functions $\bar{\vartheta}_K\varphi$ and $\vartheta_K\varphi$ respectively as follows:

$$\bar{\vartheta}_K\varphi = -\sum_i \bar{e}_i\varphi(e_i) \text{ and } \vartheta_K\varphi = -\sum_i e_i\varphi(e_i).$$

Then we have $\bar{\vartheta}_K\varphi$, $\vartheta_K\varphi\in\mathscr{D}^0(K, K_0^0)$. (The operators $\bar{\vartheta}_K$ and ϑ_K respectively correspond to the operators $\bar{\vartheta}$ and ϑ(cf. [9]) in the Kählerian manifold M.)

Lemma 6.3. (1) $\delta_K\varphi=\bar{\vartheta}_K\varphi+\vartheta_K\varphi$.

(2) $\Delta_K f=-2Pf=-2\bar{P}f$.

(3) $(\Delta_K\varphi)(e_j)=-2P\varphi(e_j)=-2\bar{P}\varphi(e_j)+\frac{n}{2(n+2)}\varphi(e_j)$.

We also note that $\Delta_K\bar{\vartheta}_K\varphi=\bar{\vartheta}_K\Delta_K\varphi$.

The spaces $\mathscr{T}^p(K, K_0)$ and $\mathscr{D}^p(K, K_0)$. We denote by $\mathscr{T}^p(K, K_0)$ (resp. $\mathscr{D}^p(K, K_0)$) the space of all functions in $\mathscr{T}^p(K, K_0^0)$ (resp. in $\mathscr{D}^p(K, K_0^0)$) which satisfy the equality

$$\varphi(za)(X_1, \dots, X_p)=\varphi(z_1)(Ad(a)X_1, \dots, Ad(a)X_p)$$
$$\text{for all } z\in K,\ a\in K_0 \text{ and } X_1, \dots, X_p\in\mathfrak{k}_1^c.$$

Clearly we have $\iota_K\mathscr{T}^p(M)=\mathscr{T}^p(K, K_0)$ and $\iota_K\mathscr{D}^p(M)=\mathscr{D}^p(K, K_0)$. And the space $\sum_p\mathscr{D}^p(K, K_0)$ is closed under the actions of d_K, δ_K and of Δ_K.

The spaces $\mathscr{C}^p(M')$ and $\mathscr{C}^p(K, K')$ ($M'=M$ or M, $K'=K_0^0$ or K_0). In § 5 $\mathscr{C}^p(M)$ denoted the space of all real valued (anti-symmetric) p-forms on M, and, for any real number λ, $\mathscr{C}^p(M)_{(\lambda)}$ denoted the space $\{\varphi\in\mathscr{C}^p(M) \mid \Delta\varphi=\lambda\varphi\}$. Analogously we use the notations $\mathscr{C}^p(\tilde{M})$ and $\mathscr{C}^p(K, K')$ to denote the real forms in $\mathscr{D}^p(\tilde{M})$ and the real functions in $\mathscr{D}^p(K, K')$ respectively. And we use the notations $\mathscr{C}^p(\tilde{M})_{(\lambda)}$ and $\mathscr{C}^p(K, K')_{(\lambda)}$ to denote the spaces $\{\varphi\in\mathscr{C}^p(\tilde{M}) \mid \Delta\varphi=\lambda\varphi\}$ and $\{\varphi\in\mathscr{C}^p(K, K') \mid \Delta_K\varphi=\lambda\varphi\}$ respectively. Clearly we have $\iota_K\mathscr{C}^p(\tilde{M})_{(\lambda)}=\mathscr{C}^p(K, K_0^0)_{(\lambda)}$ and $\iota_K\mathscr{C}^p(M)_{(\lambda)}=\mathscr{C}^p(K, K_0)_{(\lambda)}$.

6.3. The spaces $\mathscr{C}^0(\tilde{M})_{(+)}$ and $\mathscr{C}^0(M)_{(+)}$. In this paragraph we show that $\mathscr{C}^0(\tilde{M})_{(+)}=\mathscr{C}^0(M)_{(+)}=\mathscr{P}(M)$. For this purpose we first prove the following

Lemma 6.4. *Every $f\in\mathscr{C}^0(K, K_0^0)_{(+)}$ satisfies the equation*

$$XYZf=S(Y, Z)Xf \qquad \textit{for all } X, Y, Z\in\mathfrak{k}_1.$$

Let $f\in\mathscr{C}^0(K, K_0^0)_{(+)}$. For any X, Y, $Z\in\mathfrak{k}_1^c$, we put $\Phi(X, Y, Z)=XYZf-S(Y, Z)Xf$. Then we have $\Phi(X, Y, Z)=\Phi(X, Z, Y)=\Phi(Y, X, Z)$. (The equality $\Phi(X, Y, Z)=\Phi(Y, X, Z)$ follows from (3) of Lemma 5.2 and the equality $XYZf=YXZf-R(X, Y)Zf$.) Therefore to prove Lemma 6.4 it suffiecs to prove that $A_{ijk}=A_{\bar{i}jk}=0$, where $A_{ijk}=\Phi(e_i, e_j, e_k)$

and $A_{ijk}=\Phi(\bar{e}_i, e_j, e_k)$. By Lemma 6.1 A_{ijk} and A_{ijk} may be expressed respectively as follows:

$$A_{ijk}=e_ie_je_kf,$$

$$A_{ijk}=\bar{e}_ie_je_kf+\frac{1}{2(n+2)}\delta_{jk}e_if.$$

Let dK be an invariant volume element on K. We define an equivalence relation $\sim$ in $\mathscr{F}(K)$ as follows: Let $f_1, f_2\in\mathscr{F}(K)$. Then $f_1\sim f_2$ if and only if $\int f_1dK=\int f_2dK$. It is well known that $Xf_1\sim 0$ for any $f_1\in\mathscr{F}(K)$ and $X\in\mathfrak{k}^c$.

Lemma 6.5. $\sum_{i,j,k}A_{ijk}\bar{A}_{ijk}\sim-\frac{n}{32(n+2)^2}f^2+\frac{1}{4(n+2)}Qf\cdot\overline{Qf}.$

Proof. We have

$$a=\sum_{i,j,k}A_{ijk}\bar{A}_{ijk}=\sum_{i,j,k}\bar{e}_i(e_ie_je_kf\cdot\bar{e}_j\bar{e}_kf)-\sum_{i,j,k}\bar{e}_ie_ie_je_kf\cdot\bar{e}_j\bar{e}_kf,$$

whence

$$a\sim-\sum_{i,j,k}\bar{e}_ie_ie_je_kf\cdot\bar{e}_j\bar{e}_kf.$$

We have

$$\begin{aligned}\sum_i\bar{e}_ie_ie_je_kf&=\sum_i\bar{e}_ie_je_ke_if\\&=\sum_i e_j\bar{e}_ie_ke_if+\sum_i[\bar{e}_i, e_j]e_ke_if\\&=\sum_i e_je_k\bar{e}_ie_if+\sum_i e_j[[\bar{e}_i, e_k], e_i]f\\&\quad+\sum_i e_k[[\bar{e}_i, e_j], e_i]f+\sum_i[[\bar{e}_i, e_j], e_k]e_if\end{aligned}$$

Therefore using Lemma 6.1 and the fact that $\bar{P}f=-\frac{1}{4}f$ ((2) of Lemma 6.3), we obtain

$$\sum_i\bar{e}_ie_ie_je_kf=\frac{n}{4(n+2)}e_je_kf-\frac{1}{4(n+2)}\delta_{jk}Qf.$$

Hence

$$a\sim-\frac{n}{4(n+2)}\sum_{j,k}e_je_kf\cdot\bar{e}_j\bar{e}_kf+\frac{1}{4(n+2)}Qf\cdot\overline{Qf}.$$

Similarly we obtain

$$\sum_{j,k}e_je_kf\cdot\bar{e}_j\bar{e}_kf\sim\frac{1}{2(n+2)}\sum_k e_kf\cdot\bar{e}_kf\sim\frac{1}{8(n+2)}f^2.$$

Therefore Lemma 6.5 follows. Q. E. D.

In the same manner we can prove the following

Lemma 6.6. $\sum_{i,j,k} A_{ijk}\overline{A_{ijk}} \sim -3\left(-\frac{n}{32(n+2)^2}f^2+\frac{1}{4(n+2)}Qf\,\overline{Qf}\right).$

From Lemmas 6.5 and 6.6 it follows that

$$3\sum_{i,j,k} A_{ijk}\bar{A}_{ijk}+\sum_{i,j,k} A_{ijk}\bar{A}_{ijk}\sim 0,$$

meaning that $A_{ijk}=A_{ijk}=0$. We have thereby completed the proof of Lemma 6.4.

Proposition 6.7. $\mathscr{C}^0(\tilde{M})_{(+)}=\mathscr{C}^0(M)_{(+)}=\mathscr{P}(M)$.

Proof. We show that $\mathscr{C}^0(M)_{(+)}=\mathscr{P}(M)$. (Similarly we can show that $\mathscr{C}^0(\tilde{M})_{(+)}=\mathscr{P}(M)$.) From Lemmas 6.2 and 6.4 we see that every $f\in\mathscr{C}^0(M)_{(+)}$ satisfies the equation

$$\nabla_X\nabla_Y\nabla_Z f=S(Y, Z)Xf \qquad \text{for all } X, Y, Z\in T_p.$$

Let $\mathscr{P}'(M)$ denote the space of all solutions of this equation. Then we have $\mathscr{P}(M)+\boldsymbol{R}\subset\mathscr{P}'(M)$ by (2) of Proposition 5.6. Hence $\dim\mathscr{P}'(M)\geqq\dim\mathfrak{m}+1$. Converesely we assert that $\dim\mathscr{P}'(M)\leqq\dim\mathfrak{m}+1$. For this purpose we first show that $\alpha=\nabla\nabla f$ is a cross section of $\mathfrak{n}$ for any $f\in\mathscr{P}'(M)$. For all $X, Y, Z\in T_p$, we have

$$\nabla_Z\nabla_W\nabla_X\nabla_Y f-\nabla_W\nabla_Z\nabla_X\nabla_Y f=\alpha(Z, S(X, Y)W)-\alpha(W, S(X, Y)Z).$$

On the other hand it follows from the Ricci formula that this is equal to

$$-\alpha(R(Z, W)X, Y)-\alpha(X, R(Z, W)Y).$$

Therefore we know from Proposition 5.5 that α is a cross section of $\mathfrak{n}$. Now fix a point p of M. Then the assignment $f\to(f(p), (\nabla f)_p, (\nabla\nabla f)_p)$ gives an injective linear map of $\mathscr{P}'(M)$ to $\boldsymbol{R}\times T^*_p\times\mathfrak{n}_p$, whence $\dim\mathscr{P}'(M)\leqq\dim\mathfrak{m}+1$. This proves our assertion. Therefore we obtain $\mathscr{P}'(M)=\mathscr{P}(M)+\boldsymbol{R}$, and hence $\mathscr{P}(M)\subset\mathscr{C}^0(M)_{(+)}\subset\mathscr{P}(M)+\boldsymbol{R}$. It is now clear that $\mathscr{P}(M)=\mathscr{C}^0(M)_{(+)}$. (Another proof of Proposition 6.7 which uses Lemma 5.11 and Proposition 5.12 is given as follows: Let $f\in\mathscr{P}'(M)$ and V the vector field on M dual to df. Then we have $(L_V\nabla)(X, Y)=F(X, V)Y$ for all $X, Y\in T_p$ (cf. the proof of Proposition 5.12). Thus we get $df\in\mathscr{A}_G(M)$ and hence $f\in\mathscr{P}(M)$.)

Q. E. D.

§ 7. Rigidity for the canonical isometric imbeddings of the Grassmann manifolds $G^{2,n}(\boldsymbol{R})$, $n\geqq 3$.

The main aim of this section is to show that the two spaces $\mathscr{A}(M)$ and $\mathscr{A}_E(M)$ coincide for the Grassmann manifold $M=G^{2,n}(\boldsymbol{R})$ with $n\geqq 3$.

7. 1. Some lemmas on the space $\mathscr{A}(K, K_0)$. Let $\mathscr{A}(K, K_0)$ denote the image of $\mathscr{A}(M)$ by the isomorphism $\iota_K: \mathscr{C}^1(M)\to\mathscr{C}^1(K, K_0)$. For any $\varphi\in\mathscr{C}^1(K, K_0)$ we define functions A_{ij} and B_{ij} on K respectively as follows:

$$A_{ij}=e_i\varphi(e_j)+e_j\varphi(e_i)-\frac{2}{n}\delta_{ij}\sum_k e_k\varphi(e_k),$$

$$B_{ij}=\bar{e}_i\varphi(e_j)+e_j\varphi(\bar{e}_j)-\bar{e}_j\varphi(e_i)-e_i\varphi(\bar{e}_j).$$

Lemma 7. 1. *φ is in $\mathscr{A}(K, K_0)$ if and only if $A_{ij}=B_{ij}=0$.*

Proof. If $\varphi\in\mathscr{C}^1(M)$, we see from Proposition 5. 5 that φ is in $\mathscr{A}(M)$ if and only if $\Psi D\varphi=0$. Now take any function φ in $\mathscr{C}^1(K, K_0)$. We define a function $\alpha: K\to S^2(\mathfrak{k}_1^c)^*$ by $\alpha(X, Y)=X\varphi(Y)+Y\varphi(X)$ for all $X, Y\in\mathfrak{k}_1^c$, and a function $\Lambda: K\to S^2(\mathfrak{k}_1^c)^*\otimes\wedge^2(\mathfrak{k}_1^c)^*$ by

$$\begin{aligned}\Lambda(X, Y, Z, W)=&\alpha(S(X, Y)Z, W)-\alpha(Z, S(X, Y)W)\\&-\alpha(X, R(Z, W)Y)-\alpha(R(Z, W)X, Y)\\&\qquad\text{for all } X, Y, Z, W\in\mathfrak{k}_1^c.\end{aligned}$$

Then it follows from the remark above and Lemma 6. 2 that φ is in $\mathscr{A}(K, K_0)$ if and only $\Lambda=0$. An easy calculation using Lemma 6. 1 proves that $\Lambda=0$ if and only if

$$\alpha(e_i, e_j)-\frac{1}{n}\delta_{ij}\sum_k\alpha(e_k, e_k)=0,$$

$$\alpha(\bar{e}_i, e_j)-\alpha(e_i, \bar{e}_j)=0,$$

i. e., $A_{ij}=B_{ij}=0$. Q. E. D.

For any real number λ we put $\mathscr{A}(K, K_0)_{(\lambda)}=\mathscr{A}(K, K_0)\cap\mathscr{C}^1(K, K_0)_{(\lambda)}$. In what follows we consider a fixed function φ in $\mathscr{A}(K, K_0)_{(\lambda)}$. Let us define function f and u on K respectively by

$$f=\sum_i\bar{e}_i\varphi(e_i)=-\vartheta_K\varphi,$$

$$u=\sum_i e_i\varphi(e_i).$$

Note that $f\in\mathscr{D}^0(K, K_0^0)$.

Lemma 7. 2. (1) $P\varphi(e_j) = -\frac{\lambda}{2}\varphi(e_j)$,

$$\bar{P}\varphi(e_j) = (\frac{n}{4(n+2)} - \frac{\lambda}{2})\varphi(e_j).$$

(2) $Pf = \bar{P}f = -\frac{\lambda}{2}f$.

These facts follow immediately from Lemma 6. 3 and the fact that $\Delta_K \bar{\vartheta}_K \varphi = \bar{\vartheta}_K \Delta_K \varphi$.

Lemma 7. 3. (1) $(\frac{n}{2(n+2)} - \frac{\lambda}{2})\varphi(e_j) + e_j f - \frac{2}{n}\bar{e}_j u = 0$.

(2) $(\frac{n}{2(n+2)} - \lambda)f - \frac{2}{n}\bar{Q}u = 0$.

(3) $(\frac{n}{2(n+2)} - \frac{\lambda}{2})u + Qf - \frac{2}{n}Pu = 0$.

Lemma 7. 4. (1) $(\frac{\lambda}{2} + \frac{n-2}{4(n+2)})\varphi(e_j) - e_j\bar{f} + \bar{e}_j u + Q\varphi(\bar{e}_j) = 0$.

(2) $(\frac{\lambda}{2} + \frac{n-2}{4(n+2)})(f + \bar{f}) + \bar{Q}u + Q\bar{u} = 0$.

(3) $(\frac{\lambda}{2} + \frac{n-2}{4(n+2)})u + Pu = 0$.

The six equalities above can be obtained by succesively calculating the functions $\sum_i \bar{e}_i A_{ij}$, $\sum_{i,j} \bar{e}_j \bar{e}_i A_{ij}$, $\sum_{i,j} e_j \bar{e}_i A_{ij}$, $\sum_i e_i B_{ij}$, $\sum_{i,j} \bar{e}_j e_i B_{ij}$ and $\sum_{i,j} e_j e_i B_{ij}$. For the calculations we remark the following points:

$$\begin{aligned}\sum_i \bar{e}_i e_j \varphi(e_i) &= \sum_i e_j \bar{e}_i \varphi(e_i) + \sum_i [\bar{e}_i, e_j]\varphi(e_i) \\ &= e_j f + \sum_i \varphi([[\bar{e}_i, e_j], e_i]) \\ &= e_j f + \frac{n}{4(n+2)}\varphi(e_j).\end{aligned}$$

$$\begin{aligned}\sum_i e_i \bar{e}_j \varphi(e_i) &= \bar{e}_j u + \sum_i \varphi([[e_i, \bar{e}_j], e_i]) \\ &= \bar{e}_j u + \frac{n-2}{4(n+2)}\varphi(e_j).\end{aligned}$$

$$\begin{aligned}\sum_j \bar{e}_j Q\varphi(\bar{e}_j) &= \sum_{i,j} e_i \bar{e}_j e_i \varphi(\bar{e}_j) + \sum_{i,j} [\bar{e}_j, e_i] e_i \varphi(\bar{e}_j) \\ &= \sum_{i,j} e_i e_i \bar{e}_j \varphi(\bar{e}_j) + 2\sum_{i,j} e_i [\bar{e}_j, e_i]\varphi(\bar{e}_j) + \sum_{i,j} [[\bar{e}_j, e_i], e_i]\varphi(\bar{e}_j) \\ &= Q\bar{u} + 2\sum_{i,j} e_i \varphi([[\bar{e}_j, e_i], \bar{e}_j]) + \sum_{i,j} [[\bar{e}_j, e_i], e_i]\varphi(\bar{e}_j) \\ &= Q\bar{u} + \frac{n-2}{4(n+2)}\bar{f}.\end{aligned}$$

Furthermore calculating the functions $\sum_i e_i A_{ij}$ and $\sum_i e_j e_i A_{ij}$, we get the following

Lemma 7.5. (1) $Q\varphi(e_j) = -\frac{n-2}{n} e_j u.$

(2) $Qu=0.$

Lemma 7.6. $Q\bar{Q}u = (\frac{\lambda}{n+2} + \frac{n-4}{8(n+2)})u.$

Proof. Since $Qu=0$, we have

$$\begin{aligned} Q\bar{Q}u &= Q\bar{Q}u - \bar{Q}Qu \\ &= \sum_j e_j(e_j\bar{Q}u - \bar{Q}e_j u) + \sum_j (e_j\bar{Q}e_j u - \bar{Q}e_j e_j u). \end{aligned}$$

We have

$$e_j\bar{Q}u - \bar{Q}e_j u = 2\sum_k \bar{e}_k[e_j, \bar{e}_k]u + \sum_k [[e_j, \bar{e}_k], \bar{e}_k]u,$$

and

$$[e_j, \bar{e}_k]u = \sum_i e_i \varphi([[e_j, \bar{e}_k], e_i]) + \sum_i [[e_j, \bar{e}_k], e_i]\varphi(e_i).$$

Hence

$$e_j\bar{Q}u - \bar{Q}e_j u = -\frac{1}{4}\bar{e}_j u.$$

In the same manner we obtain

$$\sum (e_j\bar{Q}e_j u - \bar{Q}e_j e_j u) = \frac{n-6}{4(n+2)}\bar{P}u.$$

We have thus proved the equality

$$Q\bar{Q}u = -\frac{1}{4}Pu + \frac{n-6}{4(n+2)}\bar{P}u.$$

By (3) of Lemma 7.4 we have $Pu = -(\frac{\lambda}{2} + \frac{n-2}{4(n+2)})u$, and as is easily verified, $\bar{P}u = Pu + \frac{n}{2(n+2)}u$. Therefore Lemma 7.6 follows.

Q. E. D.

7.2. The eigenvalues of the operator $\Delta_K \mid \mathscr{A}(K, K_0)$

Proposition 7.7. $\delta_K \mathscr{A}(K, K_0)_{(\lambda)} = 0$ *if* $\lambda \neq \frac{1}{2}$.

Proof. Let $\varphi \in \mathscr{A}(K, K_0)_{(\lambda)}$. By (2) of Lemma 7.3 and (2) of

Lemma 7.4 we have $(\lambda-\frac{1}{2})(f+\bar{f})=0$. Since $\delta_K\varphi=-(f+\bar{f})$ by (1) of Lemma 6.3, we see that $\delta_K\varphi=0$ if $\lambda\neq\frac{1}{2}$.

Q. E. D.

Proposition 7.8. *If* $\mathscr{A}(K, K_0)_{(\lambda)}\neq 0$, *then* $\lambda=\frac{1}{2}$ *or* 1 *or* 2κ *or* 4κ $(\kappa=\frac{n}{4(n+2)})$.

Proof. Let φ be any non-zero function in $\mathscr{A}(K, K_0)_{(\lambda)}$. By (3) of Lemma 7.3 and (3) of Lemma 7.4 we have

$$Qf=\left(\frac{n-2}{2n}\lambda-\frac{n-1}{2n}\right)u.$$

By (2) of Lemma 7.3 we have

$$\bar{Q}u=\left(\frac{n^2}{4(n+2)}-\frac{n}{2}\lambda\right)f.$$

Hence we obtain

$$Q\bar{Q}u=\left(\frac{n^2}{4(n+2)}-\frac{n}{2}\lambda\right)\left(\frac{n-2}{2n}\lambda-\frac{n-1}{2n}\right)u.$$

This together with Lemma 7.6 gives the equality $(\lambda-\frac{1}{2})(\lambda-1)u=0$. Therefore we have $\lambda=\frac{1}{2}$ or 1 if $u\neq 0$. Now assume that $u=0$. Then we have $(\lambda-\frac{n}{2(n+2)})f=0$ by (2) of Lemma 7.3. Hence $\lambda=\frac{n}{2(n+2)}=2\kappa$ if $f\neq 0$. Assume further that $f=0$. Then we have $(\lambda-\frac{n}{n+2})\varphi(e_j)=0$ by (1) of Lemma 7.3, showing that $\lambda=\frac{n}{n+2}=4\kappa$.

Q. E. D.

Lemma 7.9. $Q\bar{Q}\varphi(e_j)=\frac{1}{16}\varphi(e_j)$ *for all* $\varphi\in\mathscr{A}(K, K_0)_{(4\kappa)}$.

Let $\varphi\in\mathscr{A}(K, K_0)_{(4\kappa)}$. Then we see from the proof of Proposition 7.8 just above that $u=f=0$. Therefore we have $Q\varphi(e_j)=0$ by (1) of Lemma 7.5. Now the proof of Lemma 7.9 can be carried out by calculating the difference $Q\bar{Q}\varphi(e_j)-\bar{Q}Q\varphi(e_j)$, which is quite similar to the proof of Lemma 7.6.

Proposition 7.10. $\mathscr{A}(K, K_0)_{(4\kappa)}=0$.

Proof. Let $\varphi\in\mathscr{A}(K, K_0)_{(4\kappa)}$. Since $u=f=0$, we see from (1) of Lemma 7.4 that

$$\frac{3n-2}{4(n+2)}\varphi(e_j)+Q\varphi(\bar{e}_j)=0.$$

Hence we also obtain

$$\frac{3n-2}{4(n+2)}\varphi(\bar{e}_j)+\bar{Q}\varphi(e_j)=0.$$

From these two equalities follows that

$$Q\bar{Q}\varphi(e_j)=\frac{1}{16}\left(\frac{3n-2}{n+2}\right)^2\varphi(e_j).$$

On the other hand we have $Q\bar{Q}\varphi(e_j)=\frac{1}{16}\varphi(e_j)$ by Lemma 7.9. Therefore we get $\varphi=0$. Q. E. D.

Proposition 7.11. $\mathscr{A}(K, K_0)_{(+)}\subset d_K\iota_K\mathscr{P}(M)$.

Proof. Let $\varphi\in\mathscr{A}(K, K_0)_{(+)}$. Since $\varDelta_K f=\frac{1}{2}f$, we have $\bar{e}_ie_je_kf+\frac{1}{2(n+2)}\delta_{jk}e_if=0$ by Lemma 6.4. Hence

$$\bar{e}_iQf+\frac{n}{2(n+2)}e_if=0.$$

Furthermore we have $u=-4Qf$ (see the proof of Proposition 7.8). Therefore we obtain

$$\bar{e}_iu=\frac{2n}{n+2}e_if.$$

Consequently we know from (1) of Lemma 7.3 that

$$\varphi(e_j)=-4e_jf.$$

Now let f' (resp. f'') denote the real (resp. imaginary) part of f. By Proposition 6.7 we have $\mathscr{C}^0(K, K_0^0)_{(+)}=\mathscr{C}^0(K, K_0)_{(+)}=\iota_K\mathscr{P}(M)$. Hence we see that both f' and f'' belong to $\iota_K\mathscr{P}(M)$. Let $d_Kf''\cdot I_0$ denote the function $\mathscr{C}^1(K, K_0^0)$ defined by $(d_Kf''\cdot I_0)(X)=(d_Kf'')(I_0X)$ for all $X\in\mathfrak{k}_1$, where the element I_0 in the centre of $\mathfrak{k}_0$ should be regarded as an endomorphism of $\mathfrak{k}_1$ in a natural manner. (Similary the group K_0 will be regarded as a subgroup of $GL(\mathfrak{k}_1)$.) Then the equality $\varphi(e_j)=-4\,e_jf$, obtained above, means that

$$\varphi=-4(d_Kf'+d_Kf''\cdot I_0).$$

Let us show that $d_Kf''\cdot I_0=0$, which will prove the proposition. We first remark that both d_Kf'' and $d_Kf''\cdot I_0$ belong to $\mathscr{C}^1(K, K_0)$. Therefore we have $(I_0X)_{za}f''=(aI_0X)_zf''=(I_0aX)_zf''$ for all $z\in K$, $a\in K_0$ and $X\in\mathfrak{k}_1$. Hence $(aI_0a^{-1}X-I_0X)f''=0$ for all $a\in K_0$ and $X\in\mathfrak{k}_1$. However we know that $K_0/K_0^0\cong Z_2$ and that $aI_0a^{-1}X=-I_0X$ for all $a\in K_0-K_0^0$ and $X\in\mathfrak{k}_1$. Therefore we obtain $(I_0X)f''=0$ for all $X\in\mathfrak{k}_1$ or $d_Kf''\cdot I_0=0$. (More precisely we have $f''=0$.) Q. E. D.

7. 3. The spaces $\mathscr{A}(M, \boldsymbol{f})$ and $\mathscr{A}(\tilde{M}, \tilde{\boldsymbol{f}})$.

Theorem 7. 12. *Consider the canonical isometric imbedding $\boldsymbol{f}$ of the real Grassmann manifold $M=G^{2,n}(\boldsymbol{R})$, $n\geqq 3$, into the Euclidean space $\mathfrak{m}$. Then the two spaces $\mathscr{A}(M, \boldsymbol{f})$ and $\mathscr{A}_E(M, \boldsymbol{f})$ coincide.*

Proof. By Theorem 3. 8 the canonical imbedding $\boldsymbol{f}$ is elliptic, and by Proposition 6. 7 the two spaces $\mathscr{C}^0(M)_{(\frac{1}{2})}$ and $\mathscr{P}(M)$ coincide. Furthermore from Propositions 7. 7, 7. 8, 7. 10 and 7. 11 we know that the eigenvalues of the operator $\Delta|\mathscr{A}(M)$ are 2κ, $\frac{1}{2}$ and 1, and that $\mathscr{A}(M)_{(2\kappa)}=\mathscr{C}^1(M)_{(2\kappa)}\cap\delta^{-1}(0)=\mathscr{A}_K(M)$, $\mathscr{A}(M)_{(\frac{1}{2})}=d\mathscr{P}(M)$ and $\delta\mathscr{A}(M)_{(1)}=0$. We have thus seen that the R space M satisfies conditions $(C_1)\sim(C_4)$ stated in 5. 5. Therefore the two spaces $\mathscr{A}(M)$ and $\mathscr{A}_E(M)$ coincide by Proposition 5. 10. Q. E. D.

Let us now consider the isometric immersion $\tilde{\boldsymbol{f}}=\boldsymbol{f}\circ\tilde{\omega}$ of the hermitian symmetric space $\tilde{M}$ into the Euclidean space $\mathfrak{m}$. (Of course this immersion is essentially different from the canoical isometric imbedding of $\tilde{M}$ into $\sqrt{-1}\,\mathfrak{k}$ given in 3. 3.) Clearly $\tilde{\boldsymbol{f}}$ is elliptic and of infinite type. Let I be the complex structure on $\tilde{M}$ induced from the element I_0 in the centre of $\mathfrak{k}_0$. For any $f\in\mathscr{P}(M)$ let $df\cdot I$ denote the 1-form on $\tilde{M}$ defined by $(df\cdot I)(X)=(df)(IX)$ for all $X\in T(\tilde{M})$, and let $\mathscr{B}(\tilde{M})$ denote the subspace of $\mathscr{C}^1(\tilde{M})_{(\frac{1}{2})}$ consisting of all the 1-forms $df\cdot I$ $(f\in\mathscr{P}(M))$.

The notations being as above, we shall prove the following

Theorem 7. 13. *The space $\mathscr{A}(\tilde{M}, \tilde{\boldsymbol{f}})$ is calculated as follows:*

$$\mathscr{A}(\tilde{M}, \tilde{\boldsymbol{f}})=\mathscr{A}_E(\tilde{M}, \tilde{\boldsymbol{f}})+\mathscr{B}(\tilde{M}).$$

Let us put $\mathscr{A}(\tilde{M}, \tilde{\boldsymbol{f}})_{(\lambda)}=\mathscr{A}(\tilde{M}, \tilde{\boldsymbol{f}})\cap\mathscr{C}^1(\tilde{M})_{(\lambda)}$ and $\mathscr{A}_E(\tilde{M}, \tilde{\boldsymbol{f}})_{(\lambda)}=\mathscr{A}_E(\tilde{M}, \tilde{\boldsymbol{f}})\cap\mathscr{C}^1(\tilde{M})_{(\lambda)}$. Clearly we have $\mathscr{A}_E(\tilde{M}, \tilde{\boldsymbol{f}})=\mathscr{A}_E(M)$. Hence $\mathscr{A}_E(\tilde{M}, \tilde{\boldsymbol{f}})_{(2\kappa)}=\mathscr{A}_K(M)$ and $\mathscr{A}_E(\tilde{M}, \tilde{\boldsymbol{f}})_{(\frac{1}{2})}=d\mathscr{P}(M)$. Since $\tilde{\boldsymbol{f}}$ is elliptic, $\mathscr{A}(\tilde{M}, \tilde{\boldsymbol{f}})$ is of finite dimension. Furthermore we deduce from the arguments in §5 that $\Delta\mathscr{A}(\tilde{M}, \tilde{\boldsymbol{f}})\subset\mathscr{A}(\tilde{M}, \tilde{\boldsymbol{f}})$. It follows that $\mathscr{A}(\tilde{M}, \tilde{\boldsymbol{f}})=\sum\mathscr{A}(\tilde{M}, \tilde{\boldsymbol{f}})_{(\lambda)}$.

Let $\mathscr{A}(K, K_0^0)$ denote the image of $\mathscr{A}(\tilde{M}, \tilde{\boldsymbol{f}})$ by the isomorphism

$\iota_K: \mathscr{C}^1(\tilde{M}) \to \mathscr{C}^1(K, K_0^0)$. For any $\varphi \in \mathscr{C}^1(K, K_0^0)$ we define functions $A_{ij} = A_{ij}(\varphi)$ and $B_{ij} = B_{ij}(\varphi)$ on K by the same formulas as before. Then we see that φ is in $\mathscr{A}(K, K_0^0)$ if and only if $A_{ij}(\varphi) = B_{ij}(\varphi) = 0$. Now the space $\iota_K \mathscr{B}(\tilde{M})$ consists of all the functions $d_K f \cdot I_0$ $(f \in \iota_K \mathscr{P}(M))$. We assert that $\iota_K \mathscr{B}(\tilde{M}) \subset \mathscr{A}(K, K_0^0)$, i. e., $\mathscr{B}(\tilde{M}) \subset \mathscr{A}(\tilde{M}, \tilde{f})$. Indeed take any function f in $\iota_K \mathscr{P}(M)$ and put $\varphi = d_K f \cdot I_0$. Then we have $A_{ij}(d_K f) = 0$, because $d_K f \in d_K \iota_K \mathscr{P}(M) \subset \mathscr{A}(K, K_0^0)$. Hence $A_{ij}(\varphi) = \sqrt{-1}\, A_{ij}(d_K f) = 0$. Furthermore we have $\bar{e}_i \varphi(e_j) + e_j \varphi(\bar{e}_i) = \sqrt{-1}\, \bar{e}_i e_j f - \sqrt{-1}\, e_j \bar{e}_i f = 0$. Hence $B_{ij}(\varphi) = 0$. Therefore we see that $\varphi \in \mathscr{A}(K, K_0^0)$, proving our assertion.

Put $\mathscr{A}(K, K_0^0)_{(\lambda)} = \mathscr{A}(K, K_0^0) \cap \mathscr{C}^1(K, K_0^0)_{(\lambda)}$. Then from the arguments in 7. 1 and 7. 2 we deduce the following facts: 1°. If $\mathscr{A}(K, K_0^0)_{(\lambda)} \neq 0$, then $\lambda = 2\kappa$ or $\frac{1}{2}$ or 1; 2°. $\delta_K \mathscr{A}(K, K_0^0)_{(2\kappa)} = 0$ and $\delta_K \mathscr{A}(K, K_0^0)_{(1)} = 0$; 3°. $\mathscr{A}(K, K_0^0)_{(\frac{1}{2})} \subset d_K \iota_K \mathscr{P}(M) + \iota_K \mathscr{B}(\tilde{M})$. Consequently we know that the eigenvalues of the operator $\Delta | \mathscr{A}(\tilde{M}, \tilde{f})$ are 2κ, $\frac{1}{2}$ and 1, and that $\mathscr{A}(\tilde{M}, \tilde{f})_{(2\kappa)} = \mathscr{C}^1(\tilde{M})_{(2\kappa)} \cap \delta^{-1}(0) = \mathscr{A}_K(M)$, $\mathscr{A}(\tilde{M}, \tilde{f})_{(\frac{1}{2})} = d\mathscr{P}(M) + \mathscr{B}(\tilde{M})$, and $\delta \mathscr{A}(\tilde{M}, \tilde{f})_{(1)} = 0$. Moreover since $\mathscr{C}^0(\tilde{M})_{(\frac{1}{2})} = \mathscr{P}(M)$ and $\delta \mathscr{A}(\tilde{M}, \tilde{f})_{(1)} = 0$, we deduce from the proof of Proposition 5. 10 that $\mathscr{A}(\tilde{M}, \tilde{f})_{(1)} = \mathscr{A}_E(\tilde{M}, \tilde{f})_{(1)}$. We have thus seen that $\mathscr{A}(\tilde{M}, \tilde{f}) = \mathscr{A}_E(\tilde{M}, \tilde{f}) + \mathscr{B}(\tilde{M})$, and have completed the proof of Theorem 7. 13.

Appendix

The non-linear equations of isometric imbeddings and the theorem of Janet-Cartan

1. Algebraic preliminaries. Let V be an n-dimensional vector space over a field K of characteristic zero.

By a *curvature like tensor* on V we mean a covariant tensor $C \in \wedge^2 V^* \otimes \wedge^2 V^*$ which satisfies the 1st Bianchi's identity, i. e.,

$$\underset{(x_1, x_2, x_3)}{\mathfrak{S}} C(x_1, x_2, x_3, x_4) = 0 \qquad \text{for all } x_1, \dots, x_4 \in V,$$

where $\underset{(x_1, x_2, x_3)}{\mathfrak{S}}$ stands for the cyclic sum with respect to the vectors x_1, x_2 and x_3. We denote by $K(V)$ the subspace of $\wedge^2 V^* \otimes \wedge^2 V^*$ consisting of all curvature like tensors. We also denote by $K'(V)$ the subspace of $V^* \otimes K(V)$ consisting of all covariant tensors $C \in V^* \otimes K(V)$ which satisfy the 2nd Bianchi's identity, i. e.,

$$\underset{(x_1, x_2, x_3)}{\mathfrak{S}} C(x_1, x_2, x_3, x_4, x_5) = 0 \qquad \text{for all } x_1, \dots, x_5 \in V.$$

For any $l\geqq 2$ we define a subspace $K^{(l)}(V)$ of $\otimes^{l+4}V^*$ by

$$K^{(l)}(V)=S^lV^*\otimes K(V)\cap S^{l-1}V^*\otimes K'(V).$$

Clearly we have

$$K^{(l)}(V)=S^2V^*\otimes K^{(l-2)}(V)\cap V^*\otimes K^{(l-1)}(V)\quad (l\geqq 2),$$

where we put $K^{(0)}(V)=K(V)$ and $K^{(1)}(V)=K'(V)$.

For any $l\geqq 0$ we define a linear map

$$\varDelta^{(l)}:\ \otimes^{l+4}V^*\rightarrow\otimes^{l+4}V^*$$

inductively as follows:

$$\begin{aligned}(\varDelta^{(0)}X)(x_1, x_2, x_3, x_4)= &\ X(x_1, x_3, x_2, x_4)-X(x_1, x_4, x_2, x_3)\\ &-X(x_2, x_3, x_1, x_4)+X(x_2, x_4, x_1, x_3),\end{aligned}$$

where $X\in\otimes^4V^*$ and $x_1, \ldots, x_4\in V$. And

$$x\,\lrcorner\,\varDelta^{(l)}X=\varDelta^{(l-1)}(x\,\lrcorner\,X)\quad (l\geqq 1),$$

where $X\in\otimes^{l+4}V^*$ and $x\in V$. We easily see that the map $\varDelta^{(l)}$ maps $S^{l+2}V^*\otimes S^2V^*$ into $K^{(l)}(V)$.

Let $\mathfrak{n}$ be a subspace of S^2V^*. Then it is easy to see that the kernel of the map $\varDelta^{(l)}:\ S^{l+2}V^*\otimes\mathfrak{n}\rightarrow K^{(l)}(V)$ coincides with the subspace $p^{l+2}(\mathfrak{n})$ of $S^{l+2}V^*\otimes\mathfrak{n}$, or in other words, the sequences

$$0\rightarrow p^{l+2}(\mathfrak{n})\xrightarrow{inj.}S^{l+2}V^*\otimes\mathfrak{n}\xrightarrow{\varDelta^{(l)}}K^{(l)}(V)$$

are exact for all $l\geqq 0$.

The notations being as above, we shall prove the following

Proposition 1. *Assume that the subspace* $\mathfrak{h}=\mathfrak{n}+\wedge^2V^*$ *of* $\otimes^2V^*$ *is involutive. Then the map* $\varDelta^{(l)}:\ S^{l+2}V^*\otimes\mathfrak{n}\rightarrow K^{(l)}(V)$ *is surjective for all* $l\geqq 0$*. Hence the sequence*

$$0\rightarrow p^{l+2}(\mathfrak{n})\rightarrow S^{l+2}V^*\otimes\mathfrak{n}\rightarrow K^{(l)}(V)\rightarrow 0$$

is exact for all $l\geqq 0$

Let p and k be any integers with $0\leqq p\leqq k$. We define a linear map $\delta_p:\ \otimes^kV^*\rightarrow\otimes^kV^*$ by

$$(\delta_pX)(x_1, \ldots, x_k)=\sum_{i=1}^{p+1}(-1)^iX(x_1, \ldots, \hat{x}_i, \ldots, x_{p+1}, x_i, \ldots, x_k),$$

where $X\in\otimes^kV$ and $x_1, \ldots, x_k\in V$. We note that the map $\varDelta^{(0)}$ may be expressed as the composition of the maps $\delta_1:\ \otimes^4V^*\rightarrow\wedge^2V^*\otimes\otimes^2V^*$

and $\partial_2: \otimes^4 V^* \to \otimes^2 V^* \otimes \wedge^2 V^*$.

For any $k \geqq 0$ and $l \geqq -1$ we put $\mathscr{A}^{k,l} = \wedge^k V^* \otimes \mathfrak{h}^{(l)}$, where we promise that $\mathfrak{h}^{(0)} = \mathfrak{h}$ and $\mathfrak{h}^{(-1)} = V^*$. It is easy to see that $\delta_k(\mathscr{A}^{k,l}) \subset \mathscr{A}^{k+1,l-1}$ and $\delta_{k+1}\delta_k C = 0$ for all $C \in \mathscr{A}^{k,l}$. Thus the system $\{\mathscr{A}^{k,l}, \delta_k\}$ gives a complex, which is nothing but the Spencer complex associated with the subspace $\mathfrak{h}$ of $\otimes^2 V^*$. Since $\mathfrak{h}$ is involutive, we know that the sequence

$$\mathscr{A}^{k-1,l+1} \xrightarrow{\delta_{k-1}} \mathscr{A}^{k,l} \xrightarrow{\delta_k} \mathscr{A}^{k+1,l-1}$$

is exact for any k and any $l \geqq 0$ (see [10]).

We put $\tilde{\mathscr{A}}^{k,l} = \wedge^k V^* \otimes p^l(\mathfrak{n})$. By Theorem 1.6 we see that the assignment $X \to X + \partial_{k+l} X$ gives an isomorphism of $\tilde{\mathscr{A}}^{k,l}$ onto $\mathscr{A}^{k,l}$ $(l \geqq 1)$. Clearly we have $\delta_k \partial_{k+l} = \partial_{k+l} \delta_k$ $(l \geqq 2)$. It follows that $\delta_k(\tilde{\mathscr{A}}^{k,l}) \subset \tilde{\mathscr{A}}^{k+1,l-1}$ $(l \geqq 2)$ and the sequence

$$\tilde{\mathscr{A}}^{k-1,l+1} \xrightarrow{\delta_{k-1}} \tilde{\mathscr{A}}^{k,l} \xrightarrow{\delta_k} \tilde{\mathscr{A}}^{k+1,l-1}$$

is exact for any k and any $l \geqq 2$.

Let us now prove Proposition 1.

The case where $l = 0$. Let $C \in K(V)$. We have $C \in \wedge^2 V^* \otimes \mathfrak{h}$, because $\wedge^2 V^* \subset \mathfrak{h}$. We have $\delta_2 C = 0$. Since the sequence

$$V^* \otimes \mathfrak{h}^{(1)} \xrightarrow{\delta_1} \wedge^2 V^* \otimes \mathfrak{h} \xrightarrow{\delta_2} \wedge^3 V^* \otimes V^*$$

is exact, there is a $C_1 \in V^* \otimes \mathfrak{h}^{(1)}$ such that $\delta_1 C_1 = C$. By Theorem 1.6 there is a unique $X \in V^* \otimes p^1(\mathfrak{n})$ such that $C_1 = X + \partial_2 X$. Hence $\delta_1 X + \delta_1 \partial_2 X = C$. This means that $\delta_1 X = 0$ and $\delta_1 \partial_2 X = C$, because $\delta_1 X \in \wedge^2 V^* \otimes \mathfrak{n}$ and $\delta_1 \partial_2 X$, $C \in \wedge^2 V^* \otimes \wedge^2 V^*$. Since $\delta_1 X = 0$, we have $X \in S^2 V^* \otimes \mathfrak{n}$. And we have $\varDelta^{(0)} X = \delta_1 \partial_2 X = C$.

The case where $l = 1$. Let $C \in K'(V)$. Since $x \rfloor C \in K(V)$ for all $x \in V$ and since the map $\varDelta^{(0)}: S^2 V^* \otimes \mathfrak{n} \to K(V)$ is surjective, we see that there is an $X \in V^* \otimes S^2 V^* \otimes \mathfrak{n}$ such that $x \rfloor C = \varDelta^{(0)}(x \rfloor X)$, i. e., $C = \varDelta^{(1)} X$. Since $\delta_1 X \in \wedge^2 V^* \otimes p^1(\mathfrak{n})$, it follows that $C_1 = \delta_1 X + \partial_3 \delta_1 X \in \wedge^2 V^* \otimes \mathfrak{h}^{(1)}$. A direct calculation proves that $\delta_2 \delta_1 X = \delta_2 \partial_3 \delta_1 X = 0$ and hence $\delta_2 C_1 = 0$. Since the sequence

$$V^* \otimes \mathfrak{h}^{(2)} \xrightarrow{\delta_1} \wedge^2 V^* \otimes \mathfrak{h}^{(1)} \xrightarrow{\delta_2} \wedge^3 V^* \otimes \mathfrak{h}$$

is exact, there is a $C_2 \in V^* \otimes \mathfrak{h}^{(2)}$ such that $\delta_1 C_2 = C_1$. Now C_2 can be expressed as $C_2 = Y + \partial_3 Y$ with a unique $Y \in V^* \otimes p^2(\mathfrak{n})$. Then we have $\delta_1 Y + \delta_1 \partial_3 Y = \delta_1 X + \partial_3 \delta_1 X = \delta_1 X + \delta_1 \partial_3 X$. Therefore putting $Z = X - Y$, we see that $Z \in V^* \otimes S^2 V^* \otimes \mathfrak{n}$ and $\delta_1 Z + \delta_1 \partial_3 Z = 0$. This last equality means that $\delta_1 Z = \delta_1 \partial_3 Z = 0$, because $\delta_1 Z \in \wedge^2 V^* \otimes V^* \otimes \mathfrak{n}$ and $\delta_1 \partial_3 Z \in \wedge^2 V^* \otimes V^* \otimes \wedge^2 V^*$. Since $\delta_1 Z = 0$, we have $Z \in S^3 V^* \otimes \mathfrak{n}$. And we have $\varDelta^{(1)} Z = \varDelta^{(1)}(X - Y) =$

$\Delta^{(l)}X=C$. (Note that $\Delta^{(l)}Y=0$, since $Y\in V^*\otimes p^2(\mathfrak{n})$).

The case where $l\geqq 2$. We prove our assertion by induction on the integer l. Assume that the map $\Delta^{(l-1)}$: $S^{l+1}V^*\otimes\mathfrak{n}\to K^{(l-1)}$ is surjective. Let $C\in K^{(l)}(V)$. Since $x\lrcorner C\in K^{(l-1)}$ for all $x\in V$, there is an $X\in V^*\otimes S^{l+1}V^*\otimes\mathfrak{n}$ such that $x\lrcorner C=\Delta^{(l-1)}(x\lrcorner X)$, i. e., $C=\Delta^{(l)}X$. We assert that $\delta_1 X\in\wedge^2 V^*\otimes p^l(\mathfrak{n})$. Indeed for any x, $y\in V$ we have $x\lrcorner y\lrcorner\delta_1 X\in S^l V^*\otimes\mathfrak{n}$ and $\Delta^{(l-2)}(x\lrcorner y\lrcorner\delta_1 X)=x\lrcorner y\lrcorner\Delta^{(l)}\delta_1 X$. Furthermore $\Delta^{(l)}\delta_1 X=\delta_1\Delta^{(l)}X=\delta_1 C=0$. It follows that $\Delta^{(l-2)}(x\lrcorner y\lrcorner\delta_1 X)=0$ or $x\lrcorner y\lrcorner\delta_1 X\in p^l(\mathfrak{n})$. Hence $\delta_1 X\in\wedge^2 V^*\otimes p^l(\mathfrak{n})$, proving our assertion. Now we have $\delta_2\delta_1 X=0$. As we have already remarked, the sequence

$$V^*\otimes p^{l+1}(\mathfrak{n})\xrightarrow{\delta_1}\wedge^2 V^*\otimes p^l(\mathfrak{n})\xrightarrow{\delta_2}\wedge^3 V^*\otimes p^{l-1}(\mathfrak{n})$$

is exact. Hence there is a $Y\in V^*\otimes p^{l+1}(\mathfrak{n})$ such that $\delta_1 X=\delta_1 Y$. If we put $Z=X-Y$, we have $Z\in V^*\otimes S^{l+1}V^*\otimes\mathfrak{n}$ and $\delta_1 Z=0$. These facts imply that $Z\in S^{l+2}V^*\otimes\mathfrak{n}$. Since $\Delta^{(l)}Y=0$, we obtain $\Delta^{(l)}Z=\Delta^{(l)}(X-Y)=\Delta^{(l)}X=C$.

We have thereby proved Proposition 1. Q. E. D.

Corollary. $\dim K^{(l)}(V)=\frac{l+1}{l+3}\cdot\frac{n(n-1)}{2}\cdot{}_nH_{l+2}\,(l\geqq 0)$.

Proof. Let $\mathfrak{n}=S^2V^*$. Then we have $\mathfrak{h}=\otimes^2 V^*$, which is clearly involutive. Therefore by Proposition 1 we obtain the exact sequence:

$$0\to p^{l+2}(S^2V^*)\to S^{l+2}V^*\otimes S^2V^*\to K^{(l)}(V)\to 0.$$

Furthermore we have $p^{l+2}(S^2V^*)\cong\mathfrak{h}^{(l+2)}\cong S^{l+3}V^*\otimes V^*$. From these follows the corollary. (Note that $\dim S^kV^*={}_nH_k=\frac{(n+k-1)!}{k!\,(n-1)!}$.)

Q. E. D.

We shall now prove Proposition 2. 3, as we promised. Let $\mathfrak{n}$ be an r-dimensional subspace of S^2V^*. Let us consider the exact sequence:

$$0\to p^l(\mathfrak{n})\to S^lV^*\otimes\mathfrak{n}\to K^{(l-2)}(V)\quad(l\geqq 2).$$

We have $\mathfrak{h}^{(l)}\cong p^l(\mathfrak{n})$. And by Corollary to Proposition 1 we have

$$\begin{aligned}m^{(l)}&=\dim S^lV^*\otimes\mathfrak{n}-\dim K^{(l-2)}(V)\\&=\left(r-N+\frac{2N}{l+1}\right)\cdot{}_nH_l.\end{aligned}$$

First consider the case when $r\geqq N=\frac{1}{2}n(n-1)$. Then $m^{(l)}>0$ and hence $\mathfrak{h}^{(l)}\neq 0$ for all $l\geqq 2$ except for $r=n=0$ or $r=0$, $n=1$. Next consider the case when $r<N$. Then $m^{(l)}>0$ and hence $\mathfrak{h}^{(l)}\neq 0$ if $2\leqq l\leqq\frac{N+r}{N-r}$. Therefore Proposition 2. 3 follows.

2. The equation $\Phi(\boldsymbol{f})=g$ and its prolongations. Let (M, g) be an n-dimesional Riemannian manifold, and let $\boldsymbol{R}^m$ be the m-dimensional Euclidean space $(m\geqq n)$. Let us consider the equation of isometric immersions of (M, g) into $\boldsymbol{R}^m$

$$\Phi(\boldsymbol{f})=g,$$

where $\Phi(\boldsymbol{f})=<d\boldsymbol{f}, d\boldsymbol{f}>$ and the unknown $\boldsymbol{f}$ is a local differentiable map of M to $\boldsymbol{R}^m$. Let ∇ be the covariant differentiation associated with the Riemannian metric g, and let R be its curvature. The *Riemann-Christoffel curvature* C is the covariant tensor field on M defined by

$$C(x_1, x_2, x_3, x_4)=-g(R(x_1, x_2)x_3, x_4).$$

(Hereafter $x, x_1, x_2, \ldots,$ will mean any vectors to M at any point p or at a given point p.)

Proposition 2. *Every solution of the equation $\Phi(\boldsymbol{f})=g$ satisfies the following equations*:

(1) $<\nabla_{x_1}\boldsymbol{f}, \nabla_{x_2}\boldsymbol{f}>=g(x_1, x_2)$.

(2) $<\nabla_{x_1}\nabla_{x_2}\boldsymbol{f}, \nabla_{x_3}\boldsymbol{f}>=0$.

(3) $<\nabla_x\nabla_{x_1}\nabla_{x_2}\boldsymbol{f}, \nabla_{x_3}\boldsymbol{f}>+<\nabla_{x_1}\nabla_{x_2}\boldsymbol{f}, \nabla_x\nabla_{x_3}\boldsymbol{f}>=0$.

(4) $<\nabla_{x_1}\nabla_{x_3}\boldsymbol{f}, \nabla_{x_2}\nabla_{x_4}\boldsymbol{f}>-<\nabla_{x_1}\nabla_{x_4}\boldsymbol{f}, \nabla_{x_2}\nabla_{x_3}\boldsymbol{f}>=C(x_1, x_2, x_3, x_4)$.

(5) $<\nabla_x\nabla_{x_1}\nabla_{x_3}\boldsymbol{f}, \nabla_{x_2}\nabla_{x_4}\boldsymbol{f}>+<\nabla_{x_1}\nabla_{x_3}\boldsymbol{f}, \nabla_x\nabla_{x_2}\nabla_{x_4}\boldsymbol{f}>$
$-<\nabla_x\nabla_{x_1}\nabla_{x_4}\boldsymbol{f}, \nabla_{x_2}\nabla_{x_3}\boldsymbol{f}>-<\nabla_{x_1}\nabla_{x_4}\boldsymbol{f}, \nabla_x\nabla_{x_2}\nabla_{x_3}\boldsymbol{f}>$
$=(\nabla_x C)(x_1, x_2, x_3, x_4)$.

The equation (4) is the so-called *Gaussian equation* for the isometric imbedding, which can be derived from the equation (3) and the Ricci formula: $\nabla_{x_1}\nabla_{x_2}\nabla_{x_3}\boldsymbol{f}=\nabla_{x_2}\nabla_{x_1}\nabla_{x_3}\boldsymbol{f}-(R(x_1, x_2)x_3)\boldsymbol{f}$. Let P denote the equation (1), and $P^{(1)}$(resp. $P^{(2)}$) the system of equation, (1) and (2) (resp. (1), (2) and (3)). Analogously let Q (resp. $Q^{(1)}$) denote the system of equations (1), (2) and (4) (resp. (1)$\sim$(5)). Note that $P^{(1)}$ (resp. $P^{(2)}$) is the first (resp. second) prolongation of P, and $Q^{(1)}$ is the first prolongation of Q. In the following we shall be mainly concerned with the system Q.

Let $J^k(M, m)$ be the vector bundle of all k-jets of local differentiable maps of M into $\boldsymbol{R}^m$. As usual the systems P, $P^{(1)}$, etc. may be represented by subvarieties of $J^1(M, m)$, $J^2(M, m)$, etc. We wish to have the exact expressions of the subvarieties. For this purpose we first give the expressions of the jet bundles $J^k(M, m)$ in terms of the covariant differentiation.

Let us consider the vector bundles:

$$T^k(M, m)=\sum_{l=0}^{k}\otimes^l T(M)^*\otimes\boldsymbol{R}^m.$$

Every element ω of $T^k(M, m)$ may be expressed as $\omega=(p\,;\,\omega_0, \ldots, \omega_k)$, where p is the origin of ω and ω_l is the l-th component of with respect to the decomposition above. Now let $\boldsymbol{u}$ be a local differentiable map of M to $\boldsymbol{R}^m$ defined on a neighborhood at a point $p\in M$. Then $(\nabla^l\boldsymbol{u})_p \in\otimes^l T(M)^*_p\otimes\boldsymbol{R}^m$, and the assignment

$$j^k_p\boldsymbol{u}\to(p\,;\,\boldsymbol{u}(p),\ (\nabla\boldsymbol{u})_p, \ldots, (\nabla^k\boldsymbol{u})_p)$$

gives an injective homomorphism of $J^k(M, m)$ into $T^k(M, m)$. Thus we may identify $J^k(M, m)$ with a subbundle of $T^k(M, m)$. Clearly we have $J^0(M, m)=T^0(M, m)$ and $J^1(M, m)=T^1(M, m)$. $\boldsymbol{u}$ being as above, we have $\nabla_{x_1}\nabla_{x_2}\boldsymbol{u}=\nabla_{x_2}\nabla_{x_1}\boldsymbol{u}$, $\nabla_{x_1}\nabla_{x_2}\nabla_{x_3}\boldsymbol{u}=\nabla_{x_2}\nabla_{x_1}\nabla_{x_3}\boldsymbol{u}-(R(x_1, x_2)x_3)\boldsymbol{u}$, and $\nabla_{x_1}\nabla_{x_2}\nabla_{x_3}\boldsymbol{u}=\nabla_{x_1}\nabla_{x_3}\nabla_{x_2}\boldsymbol{u}$. Hence $J^2(M, m)$ consists of all $(p\,;\,\omega_0, \omega_1, \omega_2)\in T^2(M, m)$ such that

$$\omega_2(x_1, x_2)=\omega_2(x_2, x_1),$$

and $J^3(M, m)$ consists of all $(p\,;\,\omega_0, \omega_1, \omega_2, \omega_3)\in T^3(M, m)$ such that

$$\omega_2(x_1, x_2)=\omega_2(x_2, x_1),$$

$$\omega_3(x_1, x_2, x_3)=\omega_3(x_2, x_1, x_3)-\omega_1(R(x_1, x_2)x_3),$$

$$\omega_3(x_1, x_2, x_3)=\omega_3(x_1, x_3, x_2).$$

We are now in a position to give the exact expressions of the subvarieties P, $P^{(1)}$, etc. First the subvariety P of $J^1(M, m)$ consists of all $(p\,;\,\omega_0, \omega_1)\in J^1(M, m)$ such that

$$<\omega_1(x_1),\ \omega_1(x_2)>=g(x_1, x_2).$$

It is clear that $\rho^1_{-1}(P)=M$ and P is a fibred submanifold of the vector bundle $J^1(M, m)$ over M. (In general $\rho^k_l(0\leqq l<k)$ denotes the projection of $J^k(M, m)$ onto $J^l(M, m)$, and ρ^k_{-1} the projection of $J^k(M, m)$ onto M.) Next the subvariety $P^{(1)}$ of $J^2(M, m)$ consists of all $(p\,;\,\omega_0, \omega_1, \omega_2)\in J^2(M, m)$ such that $(p\,;\,\omega_0, \omega_1)\in P$ and

$$<\omega_2(x_1, x_2), \omega_1(x_3)>=0,$$

and the subvariety $P^{(2)}$ of $J^3(M, m)$ consists of all $(p; \omega_0, \omega_1, \omega_2, \omega_3) \in J^3(M, m)$ such that $(p; \omega_0, \omega_1, \omega_2) \in P^{(1)}$ and

$$<\omega_3(x, x_1, x_2), \omega_1(x_3)>+<\omega_2(x_1, x_2), \omega_2(x, x_3)>=0.$$

It is easy to see that $\rho_1^2(P^{(1)})=P$ and $P^{(1)}$ is a subbundle of the vector bundle $(\rho_1^2)^{-1}(P)$ over P. Let $\alpha=(p; \omega_0, \omega_1) \in P$. Then $\omega_1(T(M)_p)$ is an n-dimensional subspace of $\boldsymbol{R}^m$, and let N_α denote the orthogonal complement of $\omega_1(T(M)_p)$ in $\boldsymbol{R}^m$. The union $N=\bigcup_\alpha N_\alpha$ forms a vector bundle over P. We note that the vector bundle $P^{(1)}$ over P is naturally isomorphic with the tensor product $S^2T(M)^*\otimes N$. ($S^2T(M)^*\otimes N$ stands for the tensor product $F\otimes N$, where F is the vector bundle over P induced from $S^2T(M)^*$ by the map $\rho_{-1}^1: P\to M$.)

Finally the subvariety Q of $J^2(M, m)$ consists of all $(p; \omega_0, \omega_1, \omega_2) \in P^{(1)}$ such that

$$<\omega_2(x_1, x_3), \omega_2(x_2, x_4)>-<\omega_2(x_1, x_4), \omega_2(x_2, x_3)>=C(x_1, x_2, x_3, x_4),$$

and the subvariety $Q^{(1)}$ of $J^3(M, m)$ consists of all $(p; \omega_0, \omega_1, \omega_2, \omega_3) \in P^{(2)}$ such that $(p; \omega_0, \omega_1, \omega_2) \in Q$ and

$$\begin{aligned}&<\omega_3(x, x_1, x_3), \omega_2(x_2, x_4)>+<\omega_2(x_1, x_3), \omega_3(x, x_2, x_4)>\\ &-<\omega_3(x, x_1, x_4), \omega_2(x_2, x_3)>-<\omega_2(x_1, x_4), \omega_3(x, x_2, x_3)>\\ &=(\nabla_x C)(x_1, x_2, x_3, x_4).\end{aligned}$$

3. The equation Q and the theorem of Janet-Cartan. Let $\beta=(p; \omega_0, \omega_1, \omega_2) \in P^{(1)}$. We put $\alpha=\rho_1^2(\beta)=(p; \omega_0, \omega_1)$, and define a linear map $\Theta_\beta: N_\alpha \to S^2T(M)_p^*$ by

$$\Theta_\beta(w)(x_1, x_2)=<w, \omega_2(x_1, x_2)>,$$

where $w\in N_\alpha$. We say that β is *non-degenerate* if the map Θ_β is injective.

Let $\beta\in P^{(1)}$ be non-degenerate. Then we denote by $\mathfrak{n}_\beta$ the image of N_α by the map Θ_β, and define a subspace $\mathfrak{h}_\beta$ of $\otimes^2T(M)_p^*$ by $\mathfrak{h}_\beta=\mathfrak{n}_\beta+\wedge^2T(M)_p^*$.

Proposition 3. *Let $\beta\in P^{(1)}$ be non-degenerate. Then $\mathfrak{h}_\beta$ is involutive if and only if there is a basis $\{e_1, \dots, e_n\}$ of $T(M)_p$ such that the $\frac{1}{2}n(n-1)$ vectors $\omega_2(e_i, e_j)$ $(1\leqq i\leqq j\leqq n-1)$ are linearly independent.*

This fact follows easily from Corollary 1 to Proposition 1.7.

In what follows we assume that

$$\frac{1}{2}n(n+1) \leqq m \leqq \frac{1}{2}n(n+3).$$

We denote by $P_{\#}^{(1)}$ the subset of $P^{(1)}$ composed of all $\beta \in P^{(1)}$ such that β is non-degenerate and such that $\mathfrak{h}_\beta$ is involutive. Then we see from Proposition 1. 7 or Proposition 3 that $P_{\#}^{(1)}$ is an open dense subset of $P^{(1)}$ and $\rho_1^2(P_{\#}^{(1)}) = P$. Putting $Q_\# = P_{\#}^{(1)} \cap Q$, we shall show that $Q_\#$ is an involutive equation.

For each $p \in M$ we put $K_p = K(T(M)_p)$, the space of curvature like tensors on $T(M)_p$. Then the union $K = \bigcup_p K_p$ form a vector bundle over M. Let $\tilde{K}$ denote the vector bundle over P induced from K by the map $\rho_{-1}^1: P \to M$. For any $\beta \in P_{\#}^{(1)}$ we define a covariant tensor $\Omega(\beta)$ on $T(M)_p$ by

$$\Omega(\beta)(x_1, x_2, x_3, x_4) = \langle \omega_2(x_1, x_3), \omega_2(x_2, x_4) \rangle - \langle \omega_2(x_1, x_4), \omega_2(x_2, x_3) \rangle,$$

Then $\Omega(\beta) \in K_p$, and the assignment $\beta \to (\alpha, \Omega(\beta))$ gives a map of $P_{\#}^{(1)}$ to $\tilde{K}$, which we denote by $\tilde{\Omega}$. Let $\tilde{C}$ denote the cross section of $\tilde{K}$ corresponding to the Riemann-Christoffel curvature C which is a cross section of K. Then we know that $Q_\#$ is the inverse image of $\tilde{C}$ by the map $\tilde{\Omega}$.

For any $\beta \in P_{\#}^{(1)}$ we define a linear map Δ_β: $S^2T(M)^*{}_p \otimes \mathfrak{n}_\alpha \to K_p$ by

$$\begin{aligned}
&\Delta_\beta(\xi)(x_1, x_2, x_3, x_4) \\
&= \langle \xi(x_1, x_3), \omega_2(x_2, x_4) \rangle + \langle \omega_2(x_1, x_3), \xi(x_2, x_4) \rangle \\
&\quad - \langle \xi(x_1, x_4), \omega_2(x_2, x_3) \rangle - \langle \omega_2(x_1, x_4), \xi(x_2, x_3) \rangle.
\end{aligned}$$

where $\xi \in S^2T(M)_p^* \otimes N_\alpha$. Note that Δ_β may be regarded as the differential at β of the map $\tilde{\Omega}: P_{\#}^{(1)} \cap (\rho_1^2)^{-1}(\alpha) \to \tilde{K}_\alpha$. The isomorphism Θ_β: $N_\alpha \to \mathfrak{n}_\beta$ naturally induces an isomorphism of $S^2T(M)_p^* \otimes N_\alpha$ onto $S^2T(M)_p^* \otimes \mathfrak{n}_\beta$, which we denote by X_β:

$$X_\beta(\xi)(x_1, x_2, x_3, x_4) = \langle \xi(x_1, x_2), \omega_2(x_3, x_4) \rangle,$$

where $\xi \in S^2T(M)_p^* \otimes N_\alpha$. Then we have

$$\Delta_\beta = \Delta^{(0)} \circ X_\beta.$$

Lemma 4. *$P_{\#}^{(1)}$ is a fibred manifold over $\tilde{K}$ with $\tilde{\Omega}$ as projection.*

Proof. The fact that $\tilde{\Omega}(P_{\#}^{(1)}) = \tilde{K}$ was already shown in E. Cartan [3]. Therefore it suffices to show that $\Delta_\beta(S^2T(M)_p^* \otimes N_\alpha) = K_p$ for all $\beta \in P_{\#}^{(1)}$. However this follows from Proposition 1 (for $l=0$), because $\mathfrak{h}_\beta$ is involutive and $\Delta_\beta = \Delta^{(0)} \circ X_\beta$. Q. E. D.

By Lemma 4 we have

Proposition 5. *$\rho_1^2(Q_\sharp)=P$, and $Q_\sharp$ is a fibred submanifold of the fibred manifold $P_\sharp^{(1)}$ over P.*

Let $\beta\in Q_\sharp$. We denote by $\mathfrak{g}_\beta$ the symbol of $Q_\sharp$ at β, which is nothing but the kernel of the map Δ_β.

Proposition 6. *The symbol $\mathfrak{g}_\beta$ is involutive.*

Proof. Since $\mathfrak{h}_\beta$ is involutive, so is $\mathfrak{h}_\beta^{(2)}$ (as a subspace of $T(M)_p^*\otimes\mathfrak{h}_\beta^{(1)}$). Therefore it follows from Theorem 1.6 that $p^2(\mathfrak{n}_\beta)$ is involutive (as a subspace of $T(M)_p^*\otimes p^1(\mathfrak{n}_\beta)$). We have $p^1(\mathfrak{n}_\beta)=T(M)_p^*\otimes\mathfrak{n}_\beta$, and the isomorphism X_β: $S^2T(M)_p^*\otimes N_\alpha\rightarrow S^2T(M)_p^*\otimes\mathfrak{n}_\beta$ maps $\mathfrak{g}_\beta$ onto $p^2(\mathfrak{n}_\beta)$. Thus we see that $\mathfrak{g}_\beta$ is involutive. Q. E. D.

Let us consider the first prolongation $\mathfrak{g}_\beta^{(1)}$ of $\mathfrak{g}_\beta$:

$$\mathfrak{g}_\beta^{(1)}=S^3T(M)_p^*\otimes N_\alpha\cap T(M)_p^*\otimes\mathfrak{g}_\beta.$$

Proposition 7. *The dimension of $\mathfrak{g}_\beta^{(1)}$ is constant.*

Proof. From the proof of Proposition 6 we see that $\mathfrak{g}_\beta^{(1)}\cong p^3(\mathfrak{n}_\beta)$. From the exact sequence (for $l=1$) in Proposition 1, it follows that $\dim p^3(\mathfrak{n}_\beta)$ is constant. Q. E. D.

We put

$$Q_\sharp^{(1)}=Q^{(1)}\cap(\rho_2^3)^{-1}(Q_\sharp).$$

Proposition 8. $\rho_2^3(Q_\sharp^{(1)})=Q_\sharp$.

Proof. Let $\beta\in Q_\sharp$. We define a linear map Δ'_β: $S^3T(M)_p^*\otimes N_\alpha\rightarrow K^{(1)}(T(M)_p)$ by

$$\begin{aligned}&\Delta'_\beta(\xi)(x, x_1, x_2, x_3, x_4)\\&=<\xi(x, x_1, x_3), \omega_2(x_2, x_4)>+<\omega_2(x_1, x_3), \xi(x, x_2, x_4)>\\&\quad-<\xi(x, x_1, x_4), \omega_2(x_2, x_3)-<\omega_2(x_1, x_4), \xi(x, x_2, x_3)>,\end{aligned}$$

where $\xi\in S^3T(M)_p^*\otimes N_\alpha$. The isomorphism $\Theta_\beta: N_\alpha\rightarrow\mathfrak{n}_\beta$ naturally induces an isomorphism of $S^3T(M)_p^*\otimes N_\alpha$ onto $S^3T(M)_p^*\otimes\mathfrak{n}_\beta$, which we denote by X'_β:

$$X'_\beta(\xi)(x, x_1, x_2, x_3, x_4)=<\xi(x, x_1, x_2), \omega_2(x_3, x_4)>,$$

where $\xi\in S^3T(M)_p^*\otimes N_\alpha$. Then we have $\Delta'_\beta=\Delta^{(1)}\circ X'_\beta$. By Proposition 1 (for $l=1$), we see that Δ'_β is surjective. Hence there is a $\eta\in S^3T(M)_p^*\otimes N_\alpha$

such that $\Delta'_p(\eta)=(\nabla C)_p$. Let us define an element $\bar{\omega}_3$ of $\otimes^3 T(M)^*_p \otimes \omega_1(T(M)_p)$ by

$$<\bar{\omega}_3(x, x_1, x_2), \omega_1(x_3)>+<\omega_2(x_1, x_2), \omega_2(x, x_3)>=0.$$

Then we easily see that $(p; \omega_0, \omega_1, \omega_2, \bar{\omega}_3+\eta)$ belongs to $Q^{(1)}_\sharp$.

Q. E. D.

By Proposition 5~8 we have proved the following

Theorem 9. *Assume that* $\frac{1}{2}n(n+1)\leqq m\leqq \frac{1}{2}n(n+3)$. *Then the equation* $Q_\sharp$ *is involutive.*

For the definition of an involutive equation, see [6].

By virtue of Theorem 9 we know that every Riemannian manifold M of dimension n can be locally isometrically imbedded in the Euclidean space of dimension $\frac{1}{2}n(n+1)$, where everything should be considered in the real analytic category. This is the theorem of Janet-Cartan. A recent paper of J. Gasqui proves Theorem 9 in somewhat different fashion (see [5]).

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