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# On the Cauchy problem for some non-kowalewskian equations with distinct characteristic roots

By

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## 1. Introduction.

Consider a linear partial differential operator

$$P(x;D_x,D_t)=D_t^m+a_1(x;D_x)D_t^{m-1}+\cdots+a_m(x;D_x),\ (x,t)\in\boldsymbol{R}^l\times[0,T]\equiv\Omega \tag{1.1}$$

where $a_i(x;D_x)$ $(1\leqq i\leqq m)$ is a linear partial differential operator in $\boldsymbol{R}^l$.

It is said that $P(x;D_x,D_t)$ defined by (1.1) is non-kowalewskian if

$$\max_{1\leq j\leq m}\ \text{order}\ a_j(x;D_x)/j\equiv b>1\,. \tag{1.2}$$

Denote the homogeneous part of order $jb$ of $a_j(x;D_x)$ by $a_j^0(x;D_x)$.

$$P^0(x;\xi,\tau)=\tau^m+a_1^0(x;\xi)\tau^{m-1}+\cdots+a_m^0(x;\xi) \tag{1.3}$$

is said to be the principal symbol of $P(x;D_x,D_t)$. $D_t=-i\dfrac{\partial}{\partial t}$, $D_x=-i\dfrac{\partial}{\partial x}$.

Consider the forward and backward Cauchy problem

$$\begin{cases} P(x;D_x,D_t)u(x,t)=f(x,t) & \text{on}\quad \Omega \\ D_t^ju(x,t_0)=g_j(x), \quad j=0,1,\cdots,m-1 & \text{for any}\quad t_0\in[0,T]\,. \end{cases} \tag{1.4}$$

As is well known, it is necessary for the forward and backward Cauchy problem (1.4) to be $H^\infty$-wellposed that the characteristic equation in $\tau$ $P^0(x;\xi,\tau)=0$ has the only real roots for any $(x,\xi)\in\boldsymbol{R}^l\times\boldsymbol{R}^l$. (cf. Petrowskii [4] and Mizohata [3]). As a corollary it follows from $H^\infty$-wellposedness that $b=\max\{\text{order}\ a_j/j\,;1\leqq j\leqq m\}$ is an integer if we assume that $b>1$.

Denote the characteristic roots by $\lambda_j(x,\xi)$, i.e.

$$P^0(x;\xi,\tau)=\prod_{j=1}^m(\tau-\lambda_j(x,\xi))\,. \tag{1.5}$$

From now on we only consider the case where $b=2$.

We shall give sufficient conditions for the forward and backward Cauchy problem to have a unique solution in $L^2(\boldsymbol{R}^l)$.

We assume the following conditions.

Condition (A). The characteristic roots $\lambda_j(x, \xi)$ are non-zero, real and distinct for $(x, \xi)\in \boldsymbol{R}^l\times\boldsymbol{R}^l\backslash 0$, more precisely,

$$\inf_{\substack{1\leq j\leq m \\ (x,\xi)\in R^l\times S^{l-1}}} |\lambda_j(x, \xi)|>0\,, \tag{1.6}$$

$$\inf_{\substack{j\neq k \\ (x,\xi)\in R^l\times S^{l-1}}} |\lambda_j(x, \xi)-\lambda_k(x, \xi)|>0\,. \tag{1.7}$$

Condition (B). For each $j$,

$$H_{\lambda_j}\varphi_j(x, \xi)=h_j(x, \xi) \tag{1.8}$$

has a $C^\infty$ bounded real solution $\varphi_j(x, \xi)$ homogeneous of degree 0 in $\xi$. Here

$$H_f g=\{f, g\}=\sum_{j=1}^{l}\left(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j}-\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}\right) \tag{1.9}$$

denotes the Poisson bracket and $H_f$ the Hamilton field,

$$\begin{aligned} h_j(x, \xi)=&\,l_j(x, \xi)(\mathrm{Im}\, M_1'(x, \xi))r_j(x, \xi)-\{\lambda_j(x, \xi),\, l_j(x, \xi)\}\,r_j(x, \xi) \\ &-\frac{1}{2i}\sum_{|\alpha|=1}\Big[l_{j(\alpha)}(x, \xi)(\lambda_j(x, \xi)I-M_2(x, \xi))r_j^{(\alpha)}(x, \xi) \\ &-l_j^{(\alpha)}(x, \xi)(\lambda_j(x, \xi)I-M_2(x, \xi))r_{j(\alpha)}(x, \xi)\Big]\,, \end{aligned} \tag{1.10}$$

$$M_2(x, \xi)=\begin{pmatrix} 0 & & |\xi|^2 & & 0 \\ & \ddots & & \ddots & \\ & & \ddots & & |\xi|^2 \\ -a_m^0(x, \xi/|\xi|)|\xi|^2 & \cdots & & \cdots & -a_1^0(x, \xi/|\xi|)|\xi|^2 \end{pmatrix}, \tag{1.11}$$

$$M_1(x, \xi)=\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ -a_m^1(x, \xi/|\xi|)|\xi| & \cdots & \cdots & -a_1^1(x, \xi/|\xi|)|\xi| \end{pmatrix}, \tag{1.12}$$

$a_j^1(x, \xi)$ is the homogeneous part of degree $2j-1$ of $a_j(x, \xi)$,

$$M_1'(x, \xi)=M_1(x, \xi)-\frac{1}{2i}\sum_{j=1}^{l}\frac{\partial^2}{\partial x_j\partial \xi_j}M_2(x, \xi)\,, \tag{1.13}$$

$l_j(x, \xi)$ (resp. $r_j(x, \xi)$) is a left (resp. right) null vector of $\lambda_j(x, \xi)I-M_2(x, \xi)$ which is homogeneous of degree 0 in $\xi$ such that $l_j(x, \xi)r_k(x, \xi)=\delta_{jk}$(Kronecker's delta) and $f_{(\beta)}^{(\alpha)}(x, \xi)=(iD_\xi)^\alpha D_x^\beta f(x, \xi)$.

For the global existence theorem in $C^\infty$ class for (1.8), we refer the reader to Duistermaat-Hörmander [1, Theorems 6.4.2 and 6.4.3.].

Our result is the following

**Theorem 1.1.** *Assume that the conditions* (A) *and* (B) *hold. For* $f(t)=f(x, t) \in C^1([0, T]; H^0(\boldsymbol{R}^l))$ *and* $(g_0(x), \cdots, g_{m-1}(x)) \in H^{2m}(\boldsymbol{R}^l) \times H^{2(m-1)}(\boldsymbol{R}^l) \times \cdots \times H^2(\boldsymbol{R}^l)$, *the forward and backward Cauchy problem* (1.4) *has a unique solution*

$$u(t)=u(x, t) \in C^0([0, T]; H^{2m}) \cap C^1([0, T]; H^{2(m-1)}) \cap \cdots \cap C^{m-1}([0, T]; H^2) \tag{1.14}$$

*and energy inequality*

$$|||u(t)|||^2 \leqq C(T)\left\{|||u(t_0)|||^2+\left|\int_{t_0}^t \|f(s)\|^2 ds\right|\right\}, \qquad t, t_0 \in [0, T] \tag{1.15}$$

*holds where*

$$|||u(t)|||^2=\sum_{j=1}^m \|D_t^{j-1}u(t)\|^2_{2(m-j)} \tag{1.16}$$

*and* $\|\cdot\|_k$ *is* $H^k(\boldsymbol{R}^l)$*-norm.*

As a special case, consider an operator with constant leading coefficients as 2-evolution, that is, an operator whose principal part $P^0(x; D_x, D_t)$ defined by (1.3) has constant coefficients. In this case Condition (B) reduces to a more explicit condition as follows.

Condition (B′)

$$\varphi_j(x, \xi)=\int_0^{\langle \frac{\nabla_\xi \lambda_j}{|\nabla_\xi \lambda_j|}, x\rangle} l_j(\xi) \operatorname{Im} M_1\left(x-t\frac{\nabla_\xi \lambda_j}{|\nabla_\xi \lambda_j|}, \frac{\xi}{|\nabla_\xi \lambda_j|}\right) r_j(\xi) dt \tag{1.17}$$

is a bounded function on $\boldsymbol{R}^l \times \boldsymbol{R}^l \backslash 0$, $j=1, \cdots, m$.

As a corollary of Theorem 1.1 we have the following

**Theorem 1.2.** *Let* $P(x; D_x, D_t)$ *be an operator with constant leading coefficients as 2-evolution. Assume that the conditions* (A) *and* (B′) *hold. Then the same assertion as Theorem* 1.1 *holds.*

## 2. Reduction to a system and its diagonalization.

Let $P(x; D_x, D_t)$ be a differential operator;

$$P(x; D_x, D_t)=D_t^m+a_1(x; D_x)D_t^{m-1}+\cdots+a_m(x; D_x) \quad \text{on } \Omega \tag{2.1}$$

where

$$a_j(x; D_x)=\sum_{|\alpha| \leq 2j} a_{\alpha j}(x) D_x^\alpha, \quad a_{\alpha j}(x) \in \mathcal{B}^\infty(\boldsymbol{R}^l). \qquad (\text{i.e. } b=2 \text{ in } (1.2)) \tag{2.2}$$

Put

$$a_j^s(x; \xi)=\sum_{|\alpha|=2j-s} a_{\alpha j}(x)\xi^\alpha, \qquad s=0, 1, \cdots, 2j. \tag{2.3}$$

We consider the Cauchy problem

$$\begin{cases} P(x; D_x, D_t)u(x, t)=f(x, t) & \text{on } \Omega \\ D_t^j u(x, t_0)=g_j(x), \quad j=0, 1, \cdots, m-1, & t_0 \in [0, T]. \end{cases} \tag{2.4}$$

We put

(2.5) $$u_j(x,\ t)=(\Lambda^2+1)^{m-j}D_t^{j-1}u(x,\ t),\qquad j=1,\ \cdots,\ m,$$

(2.6) $$U(x,\ t)={}^t(u_1(x,\ t),\ \cdots,\ u_m(x,\ t)).$$

Then we have a system of the following form

(2.7) $$\begin{cases} D_tU(x,\ t)=M(x\,;D_x)U(x,\ t)+F(x,\ t) \\ U(x,\ t_0)=G(x). \end{cases}$$

Here $M(x\,;D_x)=M_2+M_1+M_0$ is a pseudodifferential operator of order 2, $M_j$ is a pseudodifferential operator of homogeneous order $j$ $(j=1,\ 2)$ and $M_0$ is a pseudodifferential operator of order 0. The symbol $\sigma(M_j)=M_j(x,\ \xi)$ of $M_j(x\,;D_x)$ has the following form

(2.8) $$M_2(x\,;\xi)=\begin{pmatrix} 0 & |\xi|^2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & |\xi|^2 \\ -a_m^0(x,\xi/|\xi|)|\xi|^2 & \cdots & & -a_1^0(x\,;\xi/|\xi|)|\xi|^2 \end{pmatrix},$$

(2.9) $$M_1(x\,;\xi)=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ -a_m^1(x\,;\xi/|\xi|)|\xi| & \cdots & -a_1^1(x\,;\xi/|\xi|)|\xi| \end{pmatrix}.$$

(2.10) $$F(x,\ t)={}^t(0,\ \cdots,\ 0,\ f),$$

(2.11) $$G(x)={}^t((\Lambda^2+1)^{m-1}g_0(x),\ (\Lambda^2+1)^{m-2}g_1(x),\ \cdots,\ g_{m-1}(x)).$$

The Condition (A) implies that the system (2.7) is diagonalizable as follows.

**Proposition 2.1.** *Under the Condition* (A) *there exist a diagonal pseudodifferential operator* $\mathcal{D}(x\,;D_x)$ *of order* 2 *and a pseudodifferential operator* $N(x\,;D_x)$ *of order* 0 *such that*

(2.12) $$N(x\,;D_x)(D_t-M(x\,;D_x))\equiv(D_t-\mathcal{D}(x\,;D_x))N(x\,;D_x),\qquad (\text{mod.}\ S^0)$$

(2.13) $$|\det N(x\,;\xi)|\geqq\delta>0\qquad for\quad (x\,;\xi)\in \boldsymbol{R}^l\times\boldsymbol{R}^l.$$

*Proof.* At first consider the equation

(2.14) $$N(x\,;D_x)M(x\,;D_x)\equiv\mathcal{D}(x\,;D_x)N(x\,;D_x)\qquad (\text{mod.}\ S^1).$$

We put

$$N(x\,;\xi)=N_0(x\,;\xi)+N_{-1}(x\,;\xi),$$

$$\mathcal{D}(x\,;\xi)=\mathcal{D}_2(x\,;\xi)+\mathcal{D}_1(x\,;\xi),$$

$N_j(x\,;\xi)$, $\mathcal{D}_j(x,\ \xi)$ are homogeneous of degree $j$ in $\xi$.

Then (2.14) implies that

(2.15) $$N_0(x\,;\xi)M_2(x\,;\xi)=\mathcal{D}_2(x\,;\xi)N_0(x\,;\xi).$$

Since

(2.16) $$\det(\tau I - M_2(x\,;\xi)) = P^0(x\,;\xi,\tau) = \prod_{j=1}^{m}(\tau - \lambda_j(x,\xi)),$$

we have

(2.17) $$\mathcal{D}_2(x\,;\xi) = \begin{pmatrix} \lambda_1(x,\xi) & & 0 \\ & \ddots & \\ 0 & & \lambda_m(x,\xi) \end{pmatrix}$$

and

(2.18) $$N_0(x\,;\xi) = \begin{pmatrix} l_1(x,\xi) \\ \vdots \\ l_m(x,\xi) \end{pmatrix}.$$

Here $l_j(x,\xi)$ is a left nullvector of $\lambda_j(x,\xi)I - M_2(x\,;\xi)$ which is homogeneous of degree 0 in $\xi$ such that

(2.19) $$|\det N_0(x\,;\xi)| \geqq \delta > 0 \qquad \text{for} \quad (x,\xi) \in \boldsymbol{R}^l \times \boldsymbol{R}^l.$$

Next, consider the equation (2.12) (mod. $S^0$), that is,

(2.20) $$N_0(x\,;D)M_2(x\,;D) + (N_0(x\,;D)M_1(x\,;D) + N_{-1}(x\,;D)M_2(x\,;D))$$
$$\equiv \mathcal{D}_2(x\,;D)N_0(x\,;D) + (\mathcal{D}_2(x\,;D)N_{-1}(x\,;D) + \mathcal{D}_1(x\,;D)N_0(x,D)) \quad (\text{mod. } S^0).$$

It follows from (2.20) that

(2.21) $$\sum_{|\alpha|=1} N_0^{(\alpha)}(x\,;\xi)M_{2(\alpha)}(x\,;\xi) + (N_0(x\,;\xi)M_1(x\,;\xi) + N_{-1}(x\,;\xi)M_2(x\,;\xi))$$
$$= \sum_{|\alpha|=1} \mathcal{D}_2^{(\alpha)}(x,\xi)N_{0(\alpha)}(x\,;\xi) + (\mathcal{D}_2(x\,;\xi)N_{-1}(x\,;\xi) + \mathcal{D}_1(x\,;\xi)N_0(x\,;\xi)).$$

We put $N_{-1}(x\,;\xi)N_0^{-1}(x\,;\xi) = \tilde{N}_{-1}(x,\xi) = (\tilde{n}_{ij}(x,\xi))$, then we have

(2.22) $$\tilde{N}_{-1}(x\,;\xi)\mathcal{D}_2(x\,;\xi) - \mathcal{D}_2(x\,;\xi)\tilde{N}_{-1}(x\,;\xi)$$
$$= \mathcal{D}_1(x,\xi) - \Big\{N_0(x\,;\xi)M_1(x\,;\xi)N_0^{-1}(x\,;\xi)$$
$$- \sum_{|\alpha|=1}(\mathcal{D}_2^{(\alpha)}(x\,;\xi)N_{0(\alpha)}(x\,;\xi) - N_0^{(\alpha)}(x\,;\xi)M_{2(\alpha)}(x\,;\xi))N_0^{-1}(x\,;\xi)\Big\}$$

We put $R_1(x,\xi) = (r_{ij}(x,\xi))$ where

(2.23) $$R_1(x,\xi) = N_0(x\,;\xi)M_1(x\,;\xi)N_0^{-1}(x\,;\xi)$$
$$- \sum_{|\alpha|=1}(\mathcal{D}_2^{(\alpha)}(x\,;\xi)N_{0(\alpha)}(x\,;\xi) - N_0^{(\alpha)}(x\,;\xi)M_{2(\alpha)}(x\,;\xi))N_0^{-1}(x\,;\xi).$$

Then we choose $\mathcal{D}_1(x\,;\xi)$ such that

(2.24) $$\mathcal{D}_1(x\,;\xi) = \text{diagonal of } R_1(x,\xi).$$

Define

(2.25) $$\tilde{n}_{ij}(x\,;\xi) = \begin{cases} (\lambda_i(x,\xi) - \lambda_j(x,\xi))^{-1} r_{ij}(x\,;\xi) & (i \neq j) \\ 0 & (i = j). \end{cases}$$

Then $\mathscr{D}_1(x;\xi)$ and $N_{-1}(x;\xi)=\tilde{N}_{-1}(x;\xi)N_0(x;\xi)$ satisfy (2.21). This completes the proof.

## 3. Condition (C); $\mathscr{D}^*(x, D)\equiv\mathscr{D}(x, D)$ (mod. $S^0$).

In this section under the condition (A) we analyse the condition (B). We start from the following

**Proposition 3.1.** *Let $P(x, D)$ be a scalar pseudodifferential operator on $\boldsymbol{R}^l$ with symbol $\sigma(P)=p(x, \xi)=p_2(x, \xi)+p_1(x, \xi)+\cdots$ ($p_j$ is homogeneous of degree $j$ in $\xi$). Denote by $P^*(x, D)$ formally adjoint operator to $P(x, D)$. Then we have*

$$P^*(x, D)\equiv P(x, D) \qquad (\text{mod. } S^0) \tag{3.1}$$

*if and only if*

$$p_2(x, \xi) \text{ and } p_1'(x, \xi) \text{ are real-valued functions} \tag{3.2}$$

*where $p_1'(x, \xi)$ is the subprincipal symbol of $P(x, D)$, i.e.*

$$p_1'(x, \xi)=p_1(x, \xi)-\frac{1}{2i}\sum_{j=1}^{l}\frac{\partial^2}{\partial x_j\partial\xi_j}p_2(x, \xi). \tag{3.3}$$

*Proof.* By well known formula for pseudodifferential operators we have

$$\begin{aligned}\sigma(P^*)&=\sum_{|\alpha|\geq 0}\frac{(-1)^{|\alpha|}}{\alpha!}\overline{p_{(\alpha)}^{(\alpha)}(x, \xi)} \\ &=\overline{p_2(x, \xi)}+\Big(\overline{p_1(x, \xi)}+\sum_{|\alpha|=1}\frac{(-1)^{|\alpha|}}{\alpha!}\overline{p_{2(\alpha)}^{(\alpha)}(x, \xi)}\Big)+\cdots.\end{aligned} \tag{3.4}$$

Thus $P^*(x, D)\equiv P(x, D)$ (mod. $S^0$) holds if and only if

$$\begin{cases}\overline{p_2(x, \xi)}=p_2(x, \xi) \\ \overline{p_1(x, \xi)}+\sum_{|\alpha|=1}\dfrac{(-1)^{|\alpha|}}{\alpha!}\overline{p_{2(\alpha)}^{(\alpha)}(x, \xi)}=p_1(x, \xi),\end{cases} \tag{3.5}$$

that is,

$$\begin{cases}\operatorname{Im} p_2(x, \xi)=0, \\ \operatorname{Im} p_1'(x, \xi)=\operatorname{Im}\Big(p_1(x, \xi)-\dfrac{1}{2}\sum_{|\alpha|=1}p_{2(\alpha)}^{(\alpha)}(x, \xi)\Big)=0.\end{cases} \tag{3.6}$$

(Q. E. D.)

Now we back to section 2 and analyse the condition (B). We calculate the subprincipal symbol of $\mathscr{D}(x, D)$.

**Lemma 3.2.** *We have*

$$\begin{aligned}\mathscr{D}_1'(x, \xi)&=\mathscr{D}_1(x, \xi)-\frac{1}{2}\sum_{|\alpha|=1}\mathscr{D}_{2(\alpha)}^{(\alpha)}(x, \xi) \\ &=\textit{diagonal of }\Big\{N_0(x, \xi)M_1'(x, \xi)N_0^{-1}(x, \xi) \\ &\quad-\frac{1}{2}\sum_{|\alpha|=1}[(\mathscr{D}_2(x, \xi)N_{0(\alpha)}(x, \xi)N_0^{-1(\alpha)}(x, \xi)\end{aligned} \tag{3.7}$$

$$-N_{0(\alpha)}(x, \xi)M_2(x, \xi)N_0^{-1(\alpha)}(x, \xi))$$

$$-(\mathscr{D}_2(x, \xi)N_0^{(\alpha)}(x, \xi)N_{0(\alpha)}^{-1}(x, \xi)-N_0^{(\alpha)}(x, \xi)M_2(x, \xi)N_{0(\alpha)}^{-1}(x, \xi))$$

$$-2(\mathscr{D}_2^{(\alpha)}(x, \xi)N_{0(\alpha)}(x, \xi)-\mathscr{D}_{2(\alpha)}(x, \xi)N_0^{(\alpha)}(x, \xi))N_0^{-1}(x, \xi)]\Big\}$$

*where* $M_1'(x, \xi)$ *is the subprincipal symbol of* $M(x, D)$ *defined by* (1.13).

*Proof.* Using the identities

$$N_0(x, \xi)M_2(x, \xi)N_0(x, \xi)^{-1}=\mathscr{D}_2(x, \xi),$$

and

$$\mathscr{D}_1(x, \xi)=\text{diagonal of } R_1(x, \xi),$$

where

$$R_1(x, \xi)=N_0(x, \xi)M_1(x, \xi)N_0^{-1}(x, \xi)$$
$$+\sum_{|\alpha|=1}(N_0^{(\alpha)}(x, \xi)M_{2(\alpha)}(x, \xi)-\mathscr{D}_2^{(\alpha)}(x, \xi)N_{0(\alpha)}(x, \xi))N_0^{-1}(x, \xi),$$

we have the above result after elementary but tedious calculus. (Q. E. D.)

From the above lemma we have

**Lemma 3.3.** *Under the condition* (A),

$$\text{Im}\,\mathscr{D}_1'(x, \xi)=0 \tag{3.8}$$

*holds if and only if*

$$l_j(x, \xi)(\text{Im}\,M_1'(x, \xi))r_j(x, \xi)-\frac{1}{2i}\sum_{|\alpha|=1}[l_{j(\alpha)}(x, \xi)(\lambda_j(x, \xi)I-M_2(x, \xi))r_j^{(\alpha)}(x, \xi) \tag{3.9}$$
$$-l_j^{(\alpha)}(x, \xi)(\lambda_j(x, \xi)I-M_2(x, \xi))r_{j(\alpha)}(x, \xi)]$$
$$-\{\lambda_j(x, \xi), l_j(x, \xi)\}r_j(x, \xi)=0, \qquad j=1, \cdots, m.$$

**Remark 3.4.** The condition (3.9) is invariant for the choice of the null vectors satisfying $l_j(x, \xi)r_k(x, \xi)=\delta_{jk}$ except the last term $\{\lambda_j(x, \xi), l_j(x, \xi)\}r_j(x, \xi)$.

We replace the null vectors $l_j(x, \xi)$, $r_j(x, \xi)$ by

$$\tilde{l}_j(x, \xi)=\exp(\varphi_j(x, \xi))l_j(x, \xi), \qquad \tilde{r}_j(x, \xi)=\exp(-\varphi_j(x, \xi))r_j(x, \xi),$$

$$N_0(x, \xi)=\begin{pmatrix} \exp(\varphi_1(x, \xi))l_1(x, \xi) \\ \cdots\cdots\cdots\cdots \\ \exp(\varphi_m(x, \xi))l_m(x, \xi) \end{pmatrix}, \qquad l_j(x, \xi)r_k(x, \xi)=\delta_{jk}, \tag{3.10}$$

then we have

$$\{\lambda_j(x, \xi), \tilde{l}_j(x, \xi)\}\tilde{r}_j(x, \xi) \tag{3.11}$$
$$=\{\lambda_j(x, \xi), \varphi_j(x, \xi)\}+\{\lambda_j(x, \xi), l_j(x, \xi)\}r_j(x, \xi)$$
$$=H_{\lambda_j}\varphi_j(x, \xi)+\{\lambda_j(x, \xi), l_j(x, \xi)\}r_j(x, \xi).$$

Thus we have proved

**Lemma 3.5.** *Under the condition* (A)

$$\text{(3.8)} \qquad \operatorname{Im} \mathscr{D}_1'(x, \xi)=0$$

*holds if and only if there exists a* $C^\infty$ *real-valued solution homogeneous of degree* 0 *in* $\xi$ *for the equation*

$$\text{(3.12)} \qquad H_{\lambda_j}\varphi_j(x, \xi)=h_j(x, \xi), \qquad j=1, \cdots, m,$$

*where* $h_j(x, \xi)$ *is defined by* (1.10).

Boundedness of a solution of (3.12) needs for $N(x, D)$ to satisfy the condition (2.13).

As a conclusion of this section we have proved the following

**Proposition 3.6.** *The conditions* (A) *and* (B) *imply the condition* (C); $\mathscr{D}^*(x, D) \equiv \mathscr{D}(x, D)$ (mod. $S^0$) *and the condition* $|\det N(x, \xi)| \geqq \delta > 0$ *for* $(x, \xi) \in \boldsymbol{R}^l \times \boldsymbol{R}^l$.

## 4. Energy inequality.

In this section we derive an energy inequality for solutions of the equation (1.4). ($b=2$). Let

$$\text{(4.1)} \qquad P(x; D, D_t)=D_t^m+a_1(x, D)D_t^{m-1}+\cdots+a_m(x, D), \qquad D=D_x,$$

be an operator satisfying the conditions (A) and (B). Consider the equation

$$\text{(4.2)} \qquad P(x; D, D_t)u(x, t)=f(x, t), \qquad (x, t)\in\Omega.$$

As in section 2, we reduce (4.2) to a system

$$\text{(4.3)} \qquad L(x; D, D_t)U(x, t)=D_tU(x, t)-M(x, D)U(x, t)=F(x, t)$$

where

$$\text{(4.4)} \qquad U(x, t)={}^t((\Lambda^2+1)^{m-1}u(x, t), (\Lambda^2+1)^{m-2}D_tu(x, t), \cdots, D_t^{m-1}u(x, t)),$$

and

$$\text{(4.5)} \qquad M(x, D)\equiv\begin{pmatrix} 0 & \Lambda^2 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & \Lambda^2 \\ -a_m^0(x, D')|D|^2 & \cdots & & \cdots & -a_1^0(x, D')|D|^2 \end{pmatrix}$$

$$+\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ -a_m^1(x, D')|D| & \cdots & \cdots & -a_1^1(x, D')|D| \end{pmatrix} \quad (\text{mod. } S^0).$$

At first we derive an energy inequality for solutions of (4.3). Let $U(x, t)=U(t)$ be a solution of (4.3) with

(4.6) $$F(x, t)=F(t)\in C^0([0, T]; \prod^m H^0(\boldsymbol{R}^l))$$

such that

(4.7) $$U(t)\in C^0([0, T]; \prod^m H^2(\boldsymbol{R}^l))\cap C^1([0, T]; \prod^m H^0(\boldsymbol{R}^l))$$

In section 2 we have diagonalized (4.3) as follows:

(4.8) $$(D_t-\mathscr{D}(x, D))N(x, D)U(x, t)=B(x, D)U(x, t)+N(x, D)F(x, t).$$

Here $\mathscr{D}$ is a diagonal pseudodifferential operator of order 2, B and $N$ pseudodifferential operators of order 0 such that $|\det \sigma(N)(x, \xi)|\geqq\delta>0$.

If $U(t)\in C^0([0, T];\prod^m H^2)\cap C^1([0, T];\prod^m H^0)$, then $NU(t)\in C^0([0, T];\prod^m H^2)\cap C^1([0, T];\prod^m H^0)$. We set

(4.9) $$V(x, t)=N(x, D)U(x, t).$$

It follows from (4.8) that

(4.10) $$\frac{d}{dt}\|V(t)\|^2=2\operatorname{Re}\left(\frac{d}{dt}V(t), V(t)\right)$$
$$=2\operatorname{Re}(i\mathscr{D}V(t), V(t))+2\operatorname{Re}(iBU(t)+iNF(t), V(t)).$$

By virtue of the condition (C) we have

(4.11) $$|\operatorname{Re}(i\mathscr{D}V(t), V(t))|\leqq\text{const}\|V(t)\|^2.$$

Thus we have

(4.12) $$\frac{d}{dt}\|V(t)\|^2\leqq\text{const}(\|V(t)\|^2+\|U(t)\|^2)+\|NF(t)\|^2.$$

We set

(4.13) $$[U(t)]^2=\|NU(t)\|^2+\beta\|(\Lambda^2+1)^{-1}U(t)\|^2, \quad (\beta>0 \text{ sufficiently large}).$$

Then $[U(t)]$ defines an equivalent $L^2$-norm to $\|U(t)\|$, uniformly in $t\in[0, T]$.

Operate $(\Lambda^2+1)^{-1}$ to (4.3) we have

(4.14) $$\frac{d}{dt}(\Lambda^2+1)^{-1}U(t)=i(\Lambda^2+1)^{-1}MU(t)+i(\Lambda^2+1)^{-1}F(t).$$

It follows from (4.14) that

(4.15) $$\frac{d}{dt}\|(\Lambda^2+1)^{-1}U(t)\|^2\leqq\text{const}\|U(t)\|^2+\|(\Lambda^2+1)^{-1}F(t)\|^2.$$

From (4.12) and (4.15) it follows that

(4.16) $$\frac{d}{dt}[U(t)]^2\leqq\gamma[U(t)]^2+[F(t)]^2 \quad (\gamma>0).$$

This implies that

(4.17) $$[U(t)]^2\leqq C(T)\left\{[U(t_0)]^2+\left|\int_{t_0}^t [F(s)]^2 ds\right|\right\}.$$

Thus we have proved the following

**Proposition 4.1.** *Assume that the conditions* (A) *and* (B) *hold for* (4.3). *For* $F(t)\in C^0([0, T]; \prod^m H^0)$ *and solutions* $U(t)\in C^0([0, T]; \prod^m H^2)\cap C^1([0, T]; \prod^m H^0)$ *of* (4.3) *the energy inequality*

$$\|U(t)\|^2\leqq C(T)\left\{\|U(t_0)\|^2+\left|\int_{t_0}^t \|F(s)\|^2 ds\right|\right\} \tag{4.18}$$

*holds where* $C(T)$ *is a positive constant independent of* $U(t)$ *and* $F(t)$.

In view of (4.2), (4.3) and (4.4) we have the following

**Proposition 4.2.** *Assume that the conditions* (A) *and* (B) *hold for* (4.1). *For* $f(t)\in C^0([0, T]; H^0)$ *and solutions* $u(t)\in C^0([0, T]; H^{2m})\cap C^1([0, T]; H^{2(m-1)})\cap\cdots\cap C^{m-1}([0, T]; H^2)$ *of* (4.2) *the energy inequality*

$$|||u(t)|||^2\leqq C(T)\left\{|||u(t_0)|||^2+\left|\int_{t_0}^t \|f(s)\|^2 ds\right|\right\} \tag{4.19}$$

*holds where*

$$|||u(t)|||^2=\sum_{j=1}^m \|D_t^{j-1}u(t)\|^2_{2(m-j)}\,. \tag{4.20}$$

## 5. Proof of Theorem 1.1.

As in section 4 we define an inner product $(\,,\,)_{\mathcal{H}}$ and a norm $\|\cdot\|_{\mathcal{H}}$ equivalent to the usual $L^2(R^l)$-inner product and $L^2(\boldsymbol{R}^l)$-norm as follows:

$$\begin{aligned}(U(t), V(t))_{\mathcal{H}}=&(N(x, D)U(t), N(x, D)V(t))\\ &+c_0((\Lambda^2+1)^{-1}U(t), (\Lambda^2+1)^{-1}V(t))\\ &\text{for large positive } c_0 \text{(fixed)},\end{aligned} \tag{5.1}$$

$$\|U(t)\|_{\mathcal{H}}=\sqrt{(U(t), U(t))_{\mathcal{H}}} \quad \text{for} \quad U(t), V(t)\in C^0([0, T]; \prod^m H^0)\,. \tag{5.2}$$

By virtue of (2.13) there exist positive constants $c_1(T)$, $c_2(T)$ such that

$$c_1(T)\|U(t)\|\leqq\|U(t)\|_{\mathcal{H}}\leqq c_2(T)\|U(t)\| \quad \text{for} \quad t\in[0, T]\,. \tag{5.3}$$

We define the Hilbert space $\mathcal{H}=\prod^m H^0(\boldsymbol{R}^l)$ with norm $\|\cdot\|_{\mathcal{H}}$. We have reduced (2.4) to a system (2.7). We take for the domain of definition $D(M)$ of $M(x, D)$ the Sobolev space $\prod^m H^2(\boldsymbol{R}^l)$.

**Lemma 5.1.** *Assume that the conditions* (A) *and* (B) *hold. Then there exist a constant* $\beta$ *and a positive constant* $\delta_0$ *such that*

$$\|(\lambda I-iM(x, D))U\|^2_{\mathcal{H}}\geqq(|\lambda|-\beta)^2\|U\|^2_{\mathcal{H}}+\delta_0\|U\|^2_2 \tag{5.4}$$

*holds for real* $\lambda(|\lambda|>\beta)$ *and* $U(x)\in\prod^m H^2$ *which shows that* $(\lambda I-iM)$ *is one-to-one from* $\prod^m H^2$ *to* $\prod^m H^0$ *and the image* $(\lambda I-iM)\prod^m H^2$ *is closed in* $\prod^m H^0$.

*Poof.* For $U(x)\in\prod^{m}H^2(\boldsymbol{R}^l)$ and real $\lambda$ we have

$$\|(\lambda I-iM)U\|_{\mathscr{H}}^2=\lambda^2\|U\|_{\mathscr{H}}^2-2\lambda\,\mathrm{Re}(iMU,\,U)_{\mathscr{H}}+\|MU\|_{\mathscr{H}}^2, \tag{5.5}$$

$$\begin{aligned}2\,\mathrm{Re}(iMU,\,U)_{\mathscr{H}}&=i\{(MU,\,U)_{\mathscr{H}}-(U,\,MU)_{\mathscr{H}}\}\\&=i\{NMU,\,NU)-(NU,\,NMU)\}\\&\quad+ic_0\{((\Lambda^2+1)^{-1}MU,\,(\Lambda^2+1)^{-1}U)\\&\quad-((\Lambda^2+1)^{-1}U,\,(\Lambda^2+1)^{-1}MU)\}\\&=i\{(\mathscr{D}NU,\,NU)-(NU,\,\mathscr{D}NU)\}\\&\quad+i\{(BU,\,NU)-(NU,\,BU)\}\\&\quad+ic_0\{((\Lambda^2+1)^{-1}MU,\,(\Lambda^2+1)^{-1}U)\\&\quad-((\Lambda^2+1)^{-1}U,\,(\Lambda^2+1)^{-1}MU)\}\,.\end{aligned} \tag{5.6}$$

By virtue of the condition (C) we have

$$\begin{aligned}&|(\mathscr{D}NU,\,NU)-(NU,\,\mathscr{D}NU)|\\&\quad=|((\mathscr{D}-\mathscr{D}^*)NU,\,NU)|\leqq\gamma_1\|NU\|^2\qquad(\gamma_1>0)\,.\end{aligned} \tag{5.7}$$

Thus (5.6) and (5.7) imply that

$$|2\,\mathrm{Re}(iMU,\,U)_{\mathscr{H}}|\leqq\gamma\|U\|_{\mathscr{H}}^2\qquad(\gamma>0)\,. \tag{5.8}$$

Therefore for large $\lambda$ we have

$$\|(\lambda I-iM)U\|_{\mathscr{H}}^2\geqq(|\lambda|-\beta_1)^2\|U\|_{\mathscr{H}}^2+\|MU\|_{\mathscr{H}}^2\,. \tag{5.9}$$

By the definition and the condition (A) we have

$$\begin{aligned}\|MU\|_{\mathscr{H}}^2&=\|NMU\|^2+c_0\|(\Lambda^2+1)^{-1}MU\|^2\\&\geqq\|\mathscr{D}NU\|^2-c_1\|U\|^2\\&\geqq\delta_1\|NU\|_2^2-c_2\|U\|^2\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\|NU\|_2^2&\geqq c_3\|(\Lambda^2+1)NU\|^2\qquad(c_3>0)\\&\geqq c_3\|N'(\Lambda^2+1)U\|^2-c_4\|U\|^2,\end{aligned} \tag{5.11}$$

where $N'$ is a pseudodifferential operator of order 0 such that

$$|\det\sigma(N')(x,\,\xi)|\geqq\delta_2>0\,. \tag{5.12}$$

From (5.11) and (5.12) it follows that

$$\begin{aligned}\|NU\|_2^2&\geqq\delta_3\|(\Lambda^2+1)U\|^2-c_5\|U\|^2\\&\geqq\delta_0\|U\|_2^2-c_5\|U\|^2\qquad(\delta_0>0)\,.\end{aligned} \tag{5.13}$$

(5.10) and (5.13) imply that

$$\|MU\|_{\mathscr{H}}^2\geqq\delta_0\|U\|_2^2-c_5\|U\|^2\,. \tag{5.14}$$

(5.4) follows from (5.9) and (5.14). (Q. E. D.)

**Lemma 5.2.** *The formally adjoint operator* $L^*(x; D, D_t)=D_t-M^*(x, D)$ *satisfies the conditions* (A) *and* (B) *for some diagonalizer* $\tilde{N}(x, D)$. *More precisely we have*

$$\tilde{N}(x, D)L^*(x; D, D_t)\equiv(D_t-\mathcal{D}^*(x, D))\tilde{N}(x, D) \qquad (\text{mod. } S^0) \tag{5.15}$$

*and*

$$|\det\sigma(\tilde{N})(x, \xi)|\geqq\delta'>0 \qquad for \quad (x, \xi)\in \boldsymbol{R}^l\times\boldsymbol{R}^l. \tag{5.16}$$

From Lemmas 5.1 and 5.2 we have

**Lemma 5.3.** *For* $V(x)\in\prod^m H^2(\boldsymbol{R}^l)$ *and real* $\lambda(|\lambda|\geqq\beta')$ *we have*

$$\|(\lambda I-iM^*)V\|_{\mathcal{K}}^2\geqq(|\lambda|-\beta')^2\|V\|_{\mathcal{K}}^2+\delta_0'\|V\|_2^2 \qquad (\delta_0'>0) \tag{5.17}$$

*where*

$$\|V\|_{\mathcal{K}}^2=\|\tilde{N}(x, D)V\|^2+c\|(\Lambda^2+1)^{-1}V\|^2 \qquad (c\text{: large positive constant}) \tag{5.18}$$

*which is an equivalent norm to* $\prod^m H^0$*-norm* $\|\cdot\|$.

**Lemma 5.4.** *The image* $(\lambda I-iM)\prod^m H^2$ *is dense in* $\prod^m H^0$ *for large* $|\lambda|$, $\lambda\in\boldsymbol{R}$.

*Proof.* Suppose that the image is not dense in $\prod^m H^0$. Then there exists a $V(x)\in\prod^m H^0$, $V\neq0$ such that

$$((\lambda I-iM)U, V)=0 \qquad \text{for all} \quad U\in\prod^m H^2, \tag{5.19}$$

a fortiori for all $U\in\prod^m\mathcal{D}$. This implies that

$$(\lambda I+iM^*)V=0. \tag{5.20}$$

It follows from (5.20) that $M^*V\in\prod^m H^0$. Denote by $\psi(\xi)$ a $C^\infty(\boldsymbol{R}^l)$ function such that

$$\psi(\xi)=\begin{cases}1 & \text{for} \quad |\xi|\leqq1\\ 0 & \text{for} \quad |\xi|\geqq2\end{cases} \tag{5.21}$$

and

$$0\leqq\psi(\xi)\leqq1.$$

Define $\psi_n(\xi)=\psi\left(\frac{\xi}{n}\right)$, $\psi_n^{(\nu)}(\xi)=\left(\frac{\partial}{\partial\xi}\right)^\nu\psi_n(\xi)$ and

$$\psi_n(D)f(x)=(2\pi)^{-l}\int e^{ix\xi}\psi_n(\xi)\hat{f}(\xi)d\xi. \tag{5.22}$$

It follows from (5.21) that $\psi_n(D)V(x)$ and $\psi_n(D)M^*V$ belong to $H^\infty$. Applying the inequality (5.17) we have

$$\begin{aligned}0&=\|\psi_n(D)(\lambda I+iM^*)V\|_{\mathcal{K}}^2 \\ &=\|(\lambda I+iM^*)\psi_n(D)V-i[M^*, \psi_n(D)]V\|_{\mathcal{K}}^2\end{aligned} \tag{5.23}$$

$$\geqq \frac{1}{2}\|(\lambda I+iM^*)\phi_n(D)V\|_{\mathcal{K}}^2-\|[M^*,\ \phi_n(D)]V\|_{\mathcal{K}}^2$$

$$\geqq \frac{1}{2}(|\lambda|-\beta')^2\|\phi_n(D)V\|_{\mathcal{K}}^2+\frac{\delta_0'}{2}\|\phi_n(D)V\|_2^2-c\|[M^*,\ \phi_n(D)]V\|^2 .$$

Expanding the commutator we have

$$[M^*,\ \phi_n(D)]V(x)=\sum_{1\leq|\nu|\leq 2}\frac{(-1)^{|\nu|}}{\nu!}D_x^\nu M^*(x,\ D)\phi_n^{(\nu)}(D)V(x)+R_2(V) \tag{5.24}$$

and

$$\|R_2(V)\|\leqq \text{const}\ n^{-1}\|V\| . \tag{5.25}$$

The order of $D_x^\nu M^*$ is 2, thus we have

$$\|[M^*,\ \phi_n(D)]V(x)\|^2\leqq \text{const}\sum_{1\leq|\nu|\leq 2}\|\phi_n^{(\nu)}(D)V\|_2^2+\text{const}\ n^{-2}\|V\|^2 . \tag{5.26}$$

From (5.23) and (5.26) it follows that

$$0\geqq(|\lambda|-\beta')^2\|\phi_n(D)V\|^2+\delta'\|\phi_n(D)V\|_2^2 -\text{const}\sum_{1\leq|\nu|\leq 2}\|\phi_n^{(\nu)}(D)V\|_2^2-\text{const}\ n^{-2}\|V\|^2 . \tag{5.27}$$

More generally we have

$$0\geqq(|\lambda|-\beta')^2\|\phi_n^{(\nu)}(D)V\|^2+\delta'\|\phi_n^{(\nu)}(D)V\|_2^2 -\text{const}\sum_{|\nu|+1\leq|\nu'|\leq 2}\|\phi_n^{(\nu')}(D)V\|_2^2-\text{const}\ n^{-2}\|V\|^2 . \tag{5.28}$$

For large positive $R$, it follows from (5.27) and (5.28) that

$$0\geqq(|\lambda|-\beta')^2\|\phi_n(D)V\|^2+\delta'\|\phi_n(D)V\|_2^2 -\text{const}\sum_{|\nu|=1}\|\phi_n^{(\nu)}(D)V\|_2^2 -\text{const}\sum_{|\nu|=2}\|\phi_n^{(\nu)}(D)V\|_2^2 -\text{const}\ n^{-2}\|V\|^2 . \tag{5.29$_1$}$$

$$0\geqq(|\lambda|-\beta')^2\sum_{|\nu|=1}R\|\phi_n^{(\nu)}(D)V\|^2+\delta'\sum_{|\nu|=1}R\|\phi_n^{(\nu)}(D)V\|_2^2 -\text{const}\sum_{|\nu|=2}R\|\phi_n^{(\nu)}(D)V\|_2^2 -\text{const}\ n^{-2}R\|V\|^2 . \tag{5.29$_2$}$$

$$0\geqq(|\lambda|-\beta')^2\sum_{|\nu|=2}R^2\|\phi_n^{(\nu)}(D)V\|^2+\delta'\sum_{|\nu|=2}R^2\|\phi_n^{(\nu)}(D)V\|_2^2 -\text{const}\ n^{-2}R^2\|V\|^2 . \tag{5.29$_3$}$$

Summing up these inequalities we have

$$0\geqq(|\lambda|-\beta')^2\sum_{0\leq|\nu|\leq 2}R^{|\nu|}\|\phi_n^{(\nu)}(D)V\|^2 -\text{const}(1+R+R^2)n^{-2}\|V\|^2 +\delta'\|\phi_n(D)V\|_2^2 \tag{5.30}$$

$$+(\delta' R-\text{const})\sum_{|\nu|=1}\|\psi_n^{(\nu)}(D)V\|_2^2$$

$$+(\delta' R^2-\text{const } R-\text{const})\sum_{|\nu|=2}\|\psi_n^{(\nu)}(D)V\|_2^2 .$$

We choose a constant $R$ such that

$$\begin{cases} \delta' R-\text{const}>0\,, \\ \delta' R^2-\text{const } R-\text{const}>0\,. \end{cases}$$

Since $\|\psi_n(D)V\|\to\|V\|$ as $n\to\infty$, there exists a positive constant $n_0$ such that

$$\|\psi_n(D)V\|^2\geqq\frac{1}{2}\|V\|^2 \qquad \text{for} \quad n\geqq n_0 .$$

Thus for $n\geqq n_0$, we have

(5.31)
$$0\geqq\frac{1}{2}(|\lambda|-\beta')^2\|V\|^2-\text{const}(1+R+R^2)n^{-2}\|V\|^2 +(|\lambda|-\beta')^2\sum_{1\leq|\nu|\leq 2}R^{|\nu|}\|\psi_n^{(\nu)}(D)V\|^2 .$$

If $\|V\|\neq 0$ and $|\lambda|$ is large, then the second member of (5.31) is positive which is contradiction. (Q. E. D.)

From Lemmas 5.1 and 5.4 we have the following fundamental

**Proposition 5.5.** *Under the conditions* (A) *and* (B) *there exists a constant* $\beta$ *such that for any real* $\lambda$ *with* $|\lambda|>\beta$ *the operator* $(\lambda I-iM)$ *defines a one-to-one mapping of* $\prod^m H^2$ *onto* $\prod^m H^0$, *i.e., the resolvent* $(\lambda I-iM)^{-1}$ *exists for any* $|\lambda|>\beta$, $\lambda\in\boldsymbol{R}^1$ *and the inequality*

(5.32)
$$\|(\lambda I-iM)^{-n}\|_{\mathcal{L}(\prod^m H^0,\prod^m H^0)}\leqq\frac{c}{(|\lambda|-\beta)^n} \qquad (n=1,2,\cdots)$$

*holds where* $c$ *is a positive constant independent of* $\lambda$ *and* $n$.

**Corollary of Proposition 5.5.** *If* $U(x)\in\prod^m H^0(\boldsymbol{R}^l)$ *such that* $MU(x)\in\prod^m H^0$, *then* $U(x)\in\prod^m H^2(\boldsymbol{R}^l)$, *i.e.,*

(5.33)
$$D(M)=\prod^m H^2=\{U(x)\in\prod^m H^0\,;\,MU(x)\in\prod^m H^0\}\,.$$

Proposition 5.5 implies immediately the existence of a unique solution of the Cauchy problem (1.4) by applying the Hille-Yosida theorem. (Q. E. D.)

## 6. Examples.

In this section we give some examples of operators satisfying the conditions (A) and (B) (or (B')). At first consider the first order operators in $t$.

**Example 6.1.** (Takeuchi [5], [6])

(6.1)
$$P(x\,;D_x,D_t)=D_t+D_x^2+a(x)D_x\,, \qquad x\in\boldsymbol{R}^1\,, \quad t\in[0,T]\,.$$

In this case we can choose $\varphi(x, \xi)$ in condition (B′) such that

$$\varphi(x, \xi)=\varphi(x)=-\frac{1}{2}\int_0^x \mathrm{Im}\, a(y)dy\,, \tag{6.2}$$

$$N(x, \xi)=N(x)=e^{\varphi(x)} \qquad \text{(in (3.10))}. \tag{6.3}$$

If we assume the following condition (B′):

$$\int_0^x \mathrm{Im}\, a(y)dy \qquad \text{is a bounded function,} \tag{6.4}$$

then the Theorem 1.1 holds for this operator ($m=1$). The following equality holds:

$$\exp\Big(-\frac{1}{2}\int_0^x \mathrm{Im}\, a(y)dy\Big)(D_t+D_x^2+a(x)D_x) \tag{6.5}$$

$$\equiv(D_t+D_x^2+\mathrm{Re}\, a(x)D_x)\exp\Big(-\frac{1}{2}\int_0^x \mathrm{Im}\, a(y)dy\Big) \qquad (\mathrm{mod.}\, S^0).$$

**Example 6.2.**

$$P(x\,; D_x, D_t)=D_t+a(x)D_x^2+b(x)D_x\,, \qquad x\in \boldsymbol{R}^1, \quad t\in[0, T]. \tag{6.6}$$

In this case we assume the following conditions:

Condition (A): $a(x)$ is a real-valued function such that

$$M\geqq|a(x)|\geqq\delta>0\,, \qquad x\in \boldsymbol{R}^1.$$

Condition (B): $\displaystyle\int_0^x \frac{\mathrm{Im}\, b(y)}{a(y)}dy$ is a bounded function.

(Under the condition (A) this is equivalent to the condition: $\int_0^x \mathrm{Im}\, b(y)dy$ is bounded). Then Theorem 1.1 holds for this operator ($m=1$). We choose $\varphi(x, \xi)$ and $N(x, \xi)$ as follows:

$$\varphi(x, \xi)=\varphi(x)=-\frac{1}{2}\Big\{\int_0^x \frac{\mathrm{Im}\, b(y)}{a(y)}dy+\log|a(x)|\Big\}, \tag{6.7}$$

$$N(x, \xi)=N(x)=e^{\varphi(x)}=\frac{1}{\sqrt{|a(x)|}}\exp\Big(-\frac{1}{2}\int_0^x \frac{\mathrm{Im}\, b(y)}{a(y)}dy\Big). \tag{6.8}$$

The following equality holds:

$$N(x)(D_t+a(x)D_x^2+b(x)D_x) \tag{6.9}$$

$$\equiv(D_t+D_x a(x)D_x+\mathrm{Re}\, b(x)D_x)N(x) \qquad (\mathrm{mod.}\, S^0).$$

**Example 6.3.**

$$P(x\,; D_x, D_t)=D_t+|D_x|^2+\sum_{j=1}^{l} b_j(x)D_j\,, \quad x\in \boldsymbol{R}^l, \quad t\in[0, T], \quad D_j=-i\frac{\partial}{\partial x_j}. \tag{6.10}$$

We choose $\varphi(x, \xi)$ in the condition (B′) as follows:

$$\varphi(x, \xi)=-\frac{1}{2}\int_0^{\langle\frac{\xi}{|\xi|}, x\rangle}\sum_{j=1}^{l}\mathrm{Im}\, b_j\Big(x-t\frac{\xi}{|\xi|}\Big)\frac{\xi_j}{|\xi|}dt\,, \tag{6.11}$$

$$N(x, \xi)=\exp(\varphi(x, \xi)). \tag{6.12}$$

Condition (B′): $\varphi(x, \xi)$ defined by (6.11) is bounded implies the conclusion of Theorem 1.1 ($m=1$). The following equality holds:

$$\exp(\varphi(x, D_x))\Big(D_t+|D_x|^2+\sum_{j=1}^{l} b_j(x)D_j\Big) \tag{6.13}$$

$$\equiv\Big(D_t+|D_x|^2+\sum_{j=1}^{l}\operatorname{Re} b_j(x)D_j\Big)\exp(\varphi(x, D_x)) \qquad (\text{mod. } S^0).$$

**Example 6.4.**

$$\begin{cases} P(x; D_x, D_t)=D_t+A(x, D_x), \\ A(x, D_x)=\sum_{i,j=1}^{l} a_{ij}(x)D_iD_j+\sum_{j=1}^{l} b_j(x)D_j, \qquad x\in \boldsymbol{R}^l, \quad t\in[0, T]. \end{cases} \tag{6.14}$$

We assume the condition (A):

$$\begin{cases} a_{ij}(x) \text{ are real-valued functions satisfying } a_{ij}(x)=a_{ji}(x) \\ \text{and } \Big|\sum_{i,j=1}^{l} a_{ij}(x)\xi_i\xi_j\Big|\geqq\delta|\xi|^2, \qquad (\delta>0). \end{cases} \tag{6.15}$$

Our procedure is interpreted as follows: At first we transform $A(x, D)$ by $N(x, D)=\exp(\varphi(x, D))$ where $\varphi(x, D)$ is still to be determined:

$$\exp(\varphi(x, D))A(x, D)\equiv\tilde{A}(x, D)\exp(\varphi(x, D)), \qquad (\text{mod. } S^0). \tag{6.16}$$

Here the symbol of $\tilde{A}(x, D)$ has the following form:

$$\sigma(\tilde{A})=a_2(x, \xi)+a_1(x, \xi)+i\{a_2(x, \xi), \varphi(x, \xi)\}, \tag{6.17}$$

$$\begin{cases} a_2(x, \xi)=\sum_{i,j=1}^{l} a_{ij}(x)\xi_i\xi_j, \\ a_1(x, \xi)=\sum_{j=1}^{l} b_j(x)\xi_j. \end{cases} \tag{6.18}$$

Next we calculate the formal adjoint $\tilde{A}^*$ to $\tilde{A}$. For real-valued $\varphi(x, \xi)$,

$$\sigma(\tilde{A}^*)=a_2(x, \xi)+\overline{a_1(x, \xi)}+\frac{2}{i}\sum_{i,j=1}^{l}\frac{\partial a_{ij}(x)}{\partial x_i}\xi_j-i\{a_2(x, \xi), \varphi(x, \xi)\}. \tag{6.19}$$

We decompose $\tilde{A}(x, D)$ as follows:

$$\tilde{A}(x, D)=\frac{\tilde{A}+\tilde{A}^*}{2}+i\frac{\tilde{A}-\tilde{A}^*}{2i}, \tag{6.20}$$

where

$$\sigma\Big(\frac{\tilde{A}+\tilde{A}^*}{2}\Big)=a_2(x, \xi)+\operatorname{Re} a_1(x, \xi)+\frac{1}{i}\sum_{i,j=1}^{l}\frac{\partial a_{ij}(x)}{\partial x_i}\xi_j, \tag{6.21}$$

$$\sigma\Big(\frac{\tilde{A}-\tilde{A}^*}{2i}\Big)=\{a_2(x, \xi), \varphi(x, \xi)\}+\operatorname{Im} a_1(x, \xi)+\sum_{i,j=1}^{l}\frac{\partial a_{ij}(x)}{\partial x_i}\xi_j. \tag{6.22}$$

Finally we choose $\varphi(x, \xi)$ homogeneous of degree 0 such that condition (B) holds;

$$\{a_2(x, \xi), \varphi(x, \xi)\}+\sum_{j=1}^{l}\operatorname{Im} b_j(x)\xi_j+\sum_{i,j=1}^{l}\frac{\partial a_{ij}(x)}{\partial x_i}\xi_j=0. \tag{6.23}$$

Then the Theorem 1.1 ($m=1$) and the following equality hold:

$$\text{(6.24)}\qquad \exp(\varphi(x, D))\Big(D_t+\sum_{i,j=1}^{l} a_{ij}(x)D_iD_j+\sum_{j=1}^{l} b_j(x)D_j\Big)$$

$$\equiv\Big(D_t+\sum_{i,j=1}^{l} D_i a_{ij}(x)D_j+\sum_{j=1}^{l} \operatorname{Re} b_j(x)D_j\Big)\exp(\varphi(x, D)), \qquad (\text{mod. } S^0).$$

Now we give an example of differential operators of order 2 in $t$ (i.e. $m=2$).

**Example 6.5.**

$$\text{(6.25)}\qquad P(x; D_x, D_t)=D_t^2-|D_x|^4+\sum_{j=1}^{l} b_j(x)D_jD_t+\sum_{|\alpha|=3} c_\alpha(x)D_x^\alpha,$$

$$x\in \boldsymbol{R}^l, \quad t\in[0, T].$$

$$\text{(6.26)}\qquad P^0(\xi, \tau)=\tau^2-|\xi|^4=(\tau-|\xi|^2)(\tau+|\xi|^2).$$

It follows from (6.26) that the condition (A) is satisfied. In the notations of section 1, $a_1^0(x, \xi)=0$, $a_2^0(x, \xi)=-|\xi|^4$, $a_1^1(x, \xi)=\sum_{j=1}^{l} b_j(x)\xi_j$, $a_2^1(x, \xi)=\sum_{|\alpha|=3} c_\alpha(x)\xi^\alpha$, $\lambda_1(\xi)=|\xi|^2$, $\lambda_2(\xi)=-|\xi|^2$. Consider the equation

$$P(x; D, D_t)u(x, t)=f(x, t).$$

Putting $U(x, t)={}^t((\Lambda^2+1)u(x, t), D_tu(x, t))$, we have

$$\text{(6.27)}\qquad D_tU(x, t)=M(x, D)U(x, t)+F(x, t),$$

where

$$\text{(6.28)}\qquad \begin{cases} M(x, D)\equiv M_2(D)+M_1(x, D) \qquad (\text{mod. } S^0), \\ \sigma(M_2)=\begin{bmatrix} 0 & |\xi|^2 \\ |\xi|^2 & 0 \end{bmatrix}, \\ \sigma(M_1)=\begin{bmatrix} 0 & 0 \\ -\sum_{|\alpha|=3} c_\alpha(x)\Big(\frac{\xi}{|\xi|}\Big)^\alpha |\xi| & -\sum_{j=1}^{l} b_j(x)\frac{\xi_j}{|\xi|}|\xi| \end{bmatrix}. \end{cases}$$

We take a diagonalizer $N(x, D)$ as follows.

$$\text{(6.29)}\qquad \begin{cases} N(x, D)=N_0(x, D)+N_{-1}(x, D), \\ \sigma(N_0)=\begin{bmatrix} e^{\varphi_1(x,\xi)} & e^{\varphi_1(x,\xi)} \\ -e^{\varphi_2(x,\xi)} & e^{\varphi_2(x,\xi)} \end{bmatrix} \\ \text{where real-valued functions } \varphi_j(x, \xi) \text{ are still to be determined,} \\ \sigma(N_{-1})=\begin{bmatrix} -\frac{1}{4}\Big(a_2^1\Big(x, \frac{\xi}{|\xi|}\Big)-a_1^1\Big(x, \frac{\xi}{|\xi|}\Big)\Big)e^{\varphi_1(x,\xi)} & \frac{1}{4}\Big(a_2^1\Big(x, \frac{\xi}{|\xi|}\Big)-a_1^1\Big(x, \frac{\xi}{|\xi|}\Big)\Big)e^{\varphi_1(x,\xi)} \\ \frac{1}{4}\Big(a_2^1\Big(x, \frac{\xi}{|\xi|}\Big)+a_1^1\Big(x, \frac{\xi}{|\xi|}\Big)\Big)e^{\varphi_2(x,\xi)} & \frac{1}{4}\Big(a_2^1\Big(x, \frac{\xi}{|\xi|}\Big)+a_1^1\Big(x, \frac{\xi}{|\xi|}\Big)\Big)e^{\varphi_2(x,\xi)} \end{bmatrix}|\xi|^{-1}. \end{cases}$$

Then we have

$$N(x,\ D)M(x,\ D)\equiv\mathcal{D}(x,\ D)N(x,\ D)\qquad(\text{mod.}\ S^0)\,, \tag{6.30}$$

where

$$\begin{cases} \mathcal{D}(x,\ D)=\mathcal{D}_2(D)+\mathcal{D}_1(x,\ D)=\begin{bmatrix}\tilde{\lambda}_1(x,\ D) & 0\\ 0 & \tilde{\lambda}_2(x,\ D)\end{bmatrix}, \\ \sigma(\tilde{\lambda}_1)=|\xi|^2-\dfrac{1}{2}\Big(\sum_{|\alpha|=3}c_\alpha(x)\Big(\dfrac{\xi}{|\xi|}\Big)^\alpha+\sum_{j=1}^{l}b_j(x)\dfrac{\xi_j}{|\xi|}\Big)|\xi|-\dfrac{2}{i}\sum_{j=1}^{l}\xi_j\dfrac{\partial\varphi_1}{\partial x_j}\,, \\ \sigma(\tilde{\lambda}_2)=-|\xi|^2+\dfrac{1}{2}\Big(\sum_{|\alpha|=3}c_\alpha(x)\Big(\dfrac{\xi}{|\xi|}\Big)^\alpha-\sum_{j=1}^{l}b_j(x)\dfrac{\xi_j}{|\xi|}\Big)|\xi|+\dfrac{2}{i}\sum_{j=1}^{l}\xi_j\dfrac{\partial\varphi_2}{\partial x_j}\,. \end{cases} \tag{6.31}$$

We decompose $\sigma(\tilde{\lambda}_1)$ and $\sigma(\tilde{\lambda}_2)$ as follows:

$$\begin{aligned}\sigma(\tilde{\lambda}_1)=&\Big[|\xi|^2-\frac{1}{2}\Big(\sum_{|\alpha|=3}\operatorname{Re}c_\alpha(x)\Big(\frac{\xi}{|\xi|}\Big)^\alpha+\sum_{j=1}^{l}\operatorname{Re}b_j(x)\frac{\xi_j}{|\xi|}\Big)|\xi|\Big]\\ &+2i\Big[\sum_{j=1}^{l}\frac{\xi_j}{|\xi|}\frac{\partial\varphi_1}{\partial x_j}-\frac{1}{4}\Big(\sum_{|\alpha|=3}\operatorname{Im}c_\alpha(x)\Big(\frac{\xi}{|\xi|}\Big)^\alpha+\sum_{j=1}^{l}\operatorname{Im}b_j(x)\frac{\xi_j}{|\xi|}\Big)\Big]|\xi|\,,\end{aligned} \tag{6.32}$$

$$\begin{aligned}\sigma(\tilde{\lambda}_2)=&\Big[-|\xi|^2+\frac{1}{2}\Big(\sum_{|\alpha|=3}\operatorname{Re}c_\alpha(x)\Big(\frac{\xi}{|\xi|}\Big)^\alpha-\sum_{j=1}^{l}\operatorname{Re}b_j(x)\frac{\xi_j}{|\xi|}\Big)|\xi|\Big]\\ &-2i\Big[\sum_{j=1}^{l}\frac{\xi_j}{|\xi|}\frac{\partial\varphi_2}{\partial x_j}-\frac{1}{4}\Big(\sum_{|\alpha|=3}\operatorname{Im}c_\alpha(x)\Big(\frac{\xi}{|\xi|}\Big)^\alpha-\sum_{j=1}^{l}\operatorname{Im}b_j(x)\frac{\xi_j}{|\xi|}\Big)\Big]|\xi|\,.\end{aligned} \tag{6.33}$$

We choose the functions $\varphi_j(x,\ \xi)$ such that

$$\sum_{j=1}^{l}\frac{\xi_j}{|\xi|}\frac{\partial\varphi_1}{\partial x_j}-\frac{1}{4}\Big(\sum_{|\alpha|=3}\operatorname{Im}c_\alpha(x)\Big(\frac{\xi}{|\xi|}\Big)^\alpha+\sum_{j=1}^{l}\operatorname{Im}b_j(x)\frac{\xi_j}{|\xi|}\Big)=0\,, \tag{6.34}$$

and

$$\sum_{j=1}^{l}\frac{\xi_j}{|\xi|}\frac{\partial\varphi_2}{\partial x_j}-\frac{1}{4}\Big(\sum_{|\alpha|=3}\operatorname{Im}c_\alpha(x)\Big(\frac{\xi}{|\xi|}\Big)^\alpha-\sum_{j=1}^{l}\operatorname{Im}b_j(x)\frac{\xi_j}{|\xi|}\Big)=0 \tag{6.35}$$

hold, that is,

$$\begin{aligned}\varphi_1(x,\ \xi)=&\frac{1}{4}\int_0^{\langle\frac{\xi}{|\xi|},x\rangle}\Big\{\sum_{|\alpha|=3}\operatorname{Im}c_\alpha\Big(x-t\frac{\xi}{|\xi|}\Big)\Big(\frac{\xi}{|\xi|}\Big)^\alpha\\ &+\sum_{j=1}^{l}\operatorname{Im}b_j\Big(x-t\frac{\xi}{|\xi|}\Big)\frac{\xi_j}{|\xi|}\Big\}dt\,,\end{aligned} \tag{6.36}$$

and

$$\begin{aligned}\varphi_2(x,\ \xi)=&\frac{1}{4}\int_0^{\langle\frac{\xi}{|\xi|},x\rangle}\Big\{\sum_{|\alpha|=3}\operatorname{Im}c_\alpha\Big(x-t\frac{\xi}{|\xi|}\Big)\Big(\frac{\xi}{|\xi|}\Big)^\alpha\\ &-\sum_{j=1}^{l}\operatorname{Im}b_j\Big(x-t\frac{\xi}{|\xi|}\Big)\frac{\xi_j}{|\xi|}\Big\}dt\,.\end{aligned} \tag{6.37}$$

The condition (B) is as follows: Functions defined by (6.36) and (6.37) are bounded. Then Theorem 1.1 ($m=2$) holds.

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## Remarks added in proof.

**Remark 1.** Condition (A) implies that

$$\sum_{k=1}^{l} \xi_k \frac{\partial \lambda_j}{\partial \xi_k}(x, \xi) = 2\lambda_j(x, \xi) \neq 0 \quad \text{on} \quad T^*\boldsymbol{R}^l \backslash 0$$

by Euler's identity, i. e., $\nabla_\xi \lambda_j(x, \xi) \neq 0$ on $T^*\boldsymbol{R}^l \backslash 0$. Thus any integral curve of the Hamilton field $H_{\lambda_j}$ is regular and defined on $\boldsymbol{R}^1$ by virtue of the homogeneity of $\lambda_j(x, \xi)$ (homogeneous of degree 2 in $\xi$), $(1 \leqq j \leqq m)$.

**Remark 2.** Condition (B) is stated more explicitely as follows. (cf. Duistermaat-Hörmander [1]).

(B-1) No complete integral curve of the Hamilton field $H_{\lambda_j}$ is contained in a compact subset of $T^*\boldsymbol{R}^l \backslash 0$, $(1 \leqq j \leqq m)$,

(B-2) for every compact subset $K$ of $T^*\boldsymbol{R}^l \backslash 0$ there exists a compact subset $K'$ of $T^*\boldsymbol{R}^l \backslash 0$ such that every compact interval on an integral curve (of the Hamilton field $H_{\lambda_j}$) with end points in $K$ contained in $K'$, $(1 \leqq j \leqq m)$.

From conditions (B-1) and (B-2) it follows that

$$H_{\lambda_j} \varphi_j(x, \xi) = h_j(x, \xi) \tag{1.8}$$

has a real-valued $C^\infty(T^*\boldsymbol{R}^l \backslash 0)$ solution $\varphi_j(x, \xi)$ for any real-valued $C^\infty(T^*\boldsymbol{R}^l \backslash 0)$ function $h_j(x, \xi)$, $(1 \leqq j \leqq m)$.

(B-3) For a function $h_j(x, \xi)$ defined by (1.10) which is $C^\infty(T^*\boldsymbol{R}^l \backslash 0)$ real-valued, *bounded* on $\boldsymbol{R}^l \times S^{l-1}$ and homogeneous of degree 1 in $\xi$, (1.8) has a real-valued, *bounded* $C^\infty(T^*\boldsymbol{R}^l \backslash 0)$ solution $\varphi_j(x, \xi)$ homogeneous of degree 0 in $\xi$, $(1 \leqq j \leqq m)$.

In the case where the operator $P(x; D_x, D_t)$ defined by (1.1) and (1.2) with $b=2$ has the principal part $P^0(x; D_x, D_t)$ defined by (1.3) with constant coefficients, conditions (B-1) and (B-2) are automatically satisfied.