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The total energy decay of solutions for the wave equation with a dissipative term

By

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§ 1. Introduction and the result.

Let Ω be an open domain $\subset \boldsymbol{R}^n (n \geqq 1)$ exterior to a smooth bounded closed surface $\partial\Omega$. We shall consider the exterior initial-boundary value problem of the following type:

$$L[u]=u_{tt}(x, t)+a(x, t)u_t(x, t)-\Delta u(x, t)=0, \tag{1.1}$$

where $t \geqq 0$, $x=(x_1, x_2, \cdots, x_n) \in \Omega$, $u_{tt}=\dfrac{\partial^2 u}{\partial t^2}$, $u_t=\dfrac{\partial u}{\partial t}$, $\Delta u=\sum_{k=1}^n \dfrac{\partial^2 u}{\partial x_k^2}$ and $a(x, t)$ is non-negative;

$$u(x, 0)=f(x) \quad \text{and} \quad u_t(x, 0)=g(x), \tag{1.2}$$

where $f(x)$ and $g(x)$ are real-valued continuous functions with compact support contained in the ball of radius ρ centered at the origin and $f(x)$ belongs to class C^1;

$$u(x, t)=0 \text{ on } \partial\Omega \quad \text{or} \quad \frac{\partial u}{\partial n}(x, t)=0 \text{ on } \partial\Omega, \tag{1.3}$$

where $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$.

The assumptions on the dissipative term $a(x, t)$ of (1.1) will be stated precisely afterwards.

Let $u=u(x, t)$ be a real-valued smooth solution of (1.1), (1.2) and (1.3). We define the total enegy $E(t)$ and $E(0)$ for u as follows.

$$E(t)=\int_\Omega \{|u_t(x, t)|^2+|\nabla u(x, t)|^2\} dx$$

and

$$E(0)=\int_\Omega \{|u_t(x, 0)|^2+|\nabla u(x, 0)|^2\} dx$$

$$=\int_\Omega \{|g(x)|^2+|\nabla f(x)|^2\} dx=\|g\|^2+\|\nabla f\|^2,$$

where $|\nabla u|^2=\sum_{k=1}^{n}\left|\frac{\partial u}{\partial x_k}\right|^2$.

In this paper we shall study the order of decay of $E(t)$ as $t\to\infty$. Because of the dissipative term $a(x, t)$ $E(t)$ is expected to decay to 0 as $t\to\infty$.

Mochizuki [3] and Matsumura [2] obtained the following results for solutions of the initial value problem for the equation (1.1) in the entire $\boldsymbol{R}^n$ and (1.2).

Mochizuki's result: If $0\leqq a(x, t)\leqq C(1+|x|)^{-1-\delta}$ with positive constants C and δ, then $E(t)$ does not decay to 0 as $t\to\infty$.

Matsumura's result: If $a(x, t)\geqq 0$ and

$$\min_{|x|\leq t+\rho} a(x, t)\geqq(K+\varepsilon t)^{-1} \qquad \text{for all} \quad t\geqq 0$$

and

$$\max_{|x|\leq t+\rho} a_t(x, t)\leqq\varepsilon^2(2\gamma^2+6\gamma+3)(2+\gamma)^{-1}(K+\varepsilon t)^{-2} \qquad \text{for all} \quad t\geqq 0,$$

where K, ε and ρ are positive constants and $\gamma=(3\varepsilon-2+\sqrt{9\varepsilon^2-4\varepsilon+4})/2$, and if the initial data are supported in the ball $\{x; |x|<\rho\}$, then the total energy decays to 0 as $t\to\infty$ with the order $t^{-2/2+\gamma}$.

Now we state our assumptions on $a(x, t)$.

Assumption on $a(x, t)$: (1) *$a(x, t)$ is real, non-negative and differentiable in t (>0).*

(2) *For some $\delta>0$ $a(x, t)$ and $a_t(x, t)$ are bounded in $\Omega\times[\delta, \infty)$, and $ta(x, t)$ and $t^2a_t(x, t)$ are also bounded in $\Omega\times[0, \delta]$.*

(3) *$a(x, t)$ and $a_t(x, t)$ are continuous in $\Omega\times(0, \infty)$.*

(4) *There exist positive consiants t_0 and α $(0<\alpha\leqq 2)$ such that the following inequalities hold:*

i) $$t a(x, t)\geqq\alpha,$$

ii) $$(\alpha-1))\alpha-2)-(\alpha-1)t a(x, t)-t^2a_t(x, t)\geqq 0$$

for any (x, t) such that $t>t_0$ and $|x|\leqq t+\rho$.

Under these assumptions we shall investigate the order of decay of $E(t)$, and in §3 we shall prove the following result.

Theorem. *Let $a(x, t)$ satisfy the above assumptions, and let u be a real-valued smooth solution of* (1.1), (1.2) *and* (1.3). *Then for any $t>t_0$,*

$$E(t)\leqq\frac{C}{t^\alpha},$$

where C depends only on the initial data.

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§ 2 Some auxiliary results.

Note that ρ has been chosen such that the ball with radius ρ centered at the origin contains $\boldsymbol{R}^n\text{-}\Omega$ and the support of $f(x)$ and $g(x)$.

Lemma 2.1. *Let u be a solution of* (1.1), (1.2) *and* (1.3). *Then u is identically zero for $|x|>+t\rho$ $(t>0)$.*

The proof is simillar to the one in the case of the wave equation (see, e. g., [1], pp. 642-647), and is omitted.

We note that in the case of the Dirichlet boundary condition $u_t(x, t)$ as well as $u(x, t)$ is equal to 0 on $\partial\Omega$.

Lemma 2.2. *Let u be a solution of* (1.1), (1.2) *and* (1.3). *Then*

$$E(t)\leqq E(0)\,. \tag{2.1}$$

Proof. From $2u_t L[u]=0$, we have

$$\iint_{\Omega\times(0,t]} 2a(x, t)(u_t)^2 dx dt=\iint_{\Omega\times(0,t]} 2(u_t\Delta u-u_t u_{tt}) dx dt$$

$$=\iint_{\Omega\times(0,t]}\left\{2\sum_{k=1}^{n}(u_t u_{x_k})_{x_k}-\sum_{k=1}^{n}(u_{x_k})_t^2-(u_t)_t^2\right\}dx dt\,.$$

Noting that $u=0$ for $|x|>t+\rho$ as asserted by Lemma 2.1, and the boundary condition (1.3), and applying integration by parts, we have

$$\iint_{\Omega\times(0,t]} 2a(x, t)(u_t)^2 dx dt$$

$$=-\int_\Omega\{|\nabla u(x, t)|^2+|u_t(x, t)|^2\}dx+\int_\Omega\{|\nabla u(x, 0)|^2+|u_t(x, 0)|^2\}dx$$

$$=-E(t)+E(0)\,.$$

Thus (2.1) follows from $a(x, t)\geqq 0$.

Lemma 2.3. *Let u be a solution of* (1.1), (1.2) *and* (1.3). *Then for any $t>0$*

$$\int_\Omega u^2(x, t)dx\leqq 2E(0)t^2+2\|f\|^2\,. \tag{2.2}$$

Proof. Applying Schwarz' inequality to the equation

$$u(x, t)=\int_0^t u_t(x, \tau)d\tau+f(x)\,,$$

we have

$$u^2(x, t)=\left\{\int_0^t u_t(x, \tau)d\tau+f(x)\right\}^2 \tag{2.3}$$

$$\leqq 2\left[\left\{\int_0^t u_t(x, \tau)d\tau\right\}^2+f(x)^2\right]\leqq 2\left\{t\int_0^t u_t^2(x, \tau)d\tau+f(x)^2\right\}.$$

If we integrate the both sides of (2.3) over Ω, then we have

$$\int_\Omega u^2(x, t)dx \leqq 2\left\{t\int_\Omega dx\int_0^t u_t^2(x, \tau)d\tau + \int_\Omega f(x)^2 dx\right\}$$

$$= 2\left\{t\int_0^t d\tau\int_\Omega u_t^2(x, \tau)dx + \|f\|^2\right\}.$$

From Lemma 2.2 we have

$$\int_\Omega u_t^2(x, t)dx \leqq E(t) \leqq E(0).$$

Hence we obtain

$$\int_\Omega u^2(x, t)dx \leqq 2E(0)t^2 + 2\|f\|^2,$$

which proves the lemma.

Let α and t_0 be the constants in Assumption (4) on $a(x, t)$. Let $\phi(t)$ be a C^2-function depending only on t and be defined in $[0, \infty)$ such that

$$\phi(t) = \begin{cases} \frac{\alpha}{2}t^{\alpha-1} & \text{for } t \geqq t_0 \\ t^2 & \text{for } 0 \leqq t \leqq t_0/2. \end{cases}$$

Now we shall show an energy identity of the following form.

Lemma 2.4. *Let u be a solution of* (1.1), (1.2) *and* (1.3), *and $\phi(t)$ as above. Then for any $T > t_0$*

$$\begin{aligned}(2.4)\qquad & \frac{1}{2}T^\alpha E(T) + \frac{\alpha}{2}T^{\alpha-1}\int_\Omega u(x, T)u_t(x, T)dx \\ & + \iint_{\Omega\times[0, t_0]}\left(\phi - \frac{\alpha}{2}t^{\alpha-1}\right)(|\nabla u|^2 + |u_t|^2)dxdt \\ & + \iint_{\Omega\times[0, T]}(at^\alpha - 2\phi)|u_t|^2 dxdt \\ & + \frac{\alpha}{4}T^{\alpha-2}\int_\Omega \{a(x, T) - (\alpha-1)\}\, u(x, T)^2 dx \\ & + \frac{1}{2}\iint_{\Omega\times[0, T]}\{\phi_{tt} - (\phi a)_t\}\, u^2 dxdt = 0.\end{aligned}$$

Proof. We note that the following identities hold.

$$\begin{aligned}(2.5)\qquad t^\alpha u_t L[u] = & -\sum_{k=1}^n (t^\alpha u_t u_{x_k})_{x_k} + \sum_{k=1}^n \left\{\frac{1}{2}t^\alpha (u_{x_k})^2\right\}_t \\ & - \frac{\alpha}{2}t^{\alpha-1}|\nabla u|^2 + \frac{1}{2}\{t^\alpha (u_t)^2\}_t - \frac{1}{2}\alpha t^{\alpha-1}(u_t)^2 + t^\alpha a(u_t)^2,\end{aligned}$$

$$\begin{aligned}(2.6)\qquad \phi(t)uL[u] = & -\sum_{k=1}^n (\phi u u_{x_k})_{x_k} + \phi|\nabla u|^2 + (\phi u u_t)_t - \frac{1}{2}(\phi_t u^2)_t \\ & + \frac{1}{2}\phi_{tt}u^2 - \phi(u_t)^2 + \left(\frac{1}{2}\phi a u^2\right)_t - \frac{1}{2}(\phi a)_t u^2.\end{aligned}$$

Let $B=\{x;|x|<\rho'+T\}\cap\bar{\Omega}$, where ρ' $(>\rho)$ and $T(>t_0)$ are any fixed constants, and let $\partial B[0,T]$ denote the surface of the cylinder $B[0,T]=B\times[0,T]$ in $\bar{\Omega}\times[0,\infty)$. Let ∂B_x be the lateral surface of $B[0,T]$, and ∂B_T and ∂B_0 be the upper and the lower bases of $B[0,T]$, respectively. Let $n=(\xi_1,\xi_2,\cdots,\xi_n,\tau)$ be the outward unit normal to $\partial B[0,T]$ and $\frac{\partial}{\partial n}$ be the outward directional derivative to $\partial B[0,T]$. Then $\partial B[0,T]=\partial B_x\cup\partial B_0\cup\partial B_T$ and $n=(\xi_1,\xi_2,\cdots,\xi_n,0)$, $(0,0,\cdots,0,-1)$ and $(0,0,\cdots,0,1)$ on ∂B_x, ∂B_0, and ∂B_T, respectively.

Now we have by integrating by parts

$$\text{(2.7)}\qquad \iint_{B[0,T]}\left\{-\sum_{k=1}^{n}(t^{\alpha}u_tu_{x_k})_{x_k}-\sum_{k=1}^{n}(\phi uu_{x_k})_{x_k}\right\}dxdt$$

$$=-\int_{\partial B_x}\left(\sum_{k=1}^{n}t^{\alpha}u_tu_{x_k}\xi_k+\sum_{k=1}^{n}\phi uu_{x_k}\xi_k\right)dS$$

$$=-\int_{\partial B_x}\left(t^{\alpha}u_t\frac{\partial u}{\partial n}+\phi u\frac{\partial u}{\partial n}\right)dS=0\,,$$

where we have used the boundary condition (1.3) and $u_t=0$ on $\partial\Omega$, and we should note in view of Lemma 2.1 that $u(x,t)$ and all its derivatives vanish in $\{(x,t);|x|\geqq\rho'+T$ and $0\leqq t\leqq T\}$. Also we have

$$\text{(2.8)}\qquad \iint_{B[0,T]}\left[\sum_{k=1}^{n}\left\{\frac{1}{2}t^{\alpha}(u_{x_k})^2\right\}_t+\frac{1}{2}\{t^{\alpha}(u_t)^2\}_t+(\phi uu_t)_t-\frac{1}{2}(\phi_tu^2)_t+\left(\frac{1}{2}\phi au^2\right)_t\right]dxdt$$

$$=\left(\int_{\partial B_T}-\int_{\partial B_0}\right)\left\{\frac{1}{2}t^{\alpha}|\nabla u|^2+\frac{1}{2}t^{\alpha}(u_t)^2+\phi uu_t-\frac{1}{2}\phi_tu^2+\frac{1}{2}\phi au^2\right\}dS$$

$$=\frac{1}{2}T^{\alpha}\int_B\{|\nabla u(x,T)|^2+|u_t(x,T)|^2\}dx+\phi(T)\int_B u(x,T)u_t(x,T)dx$$

$$-\frac{1}{2}\phi_t(T)\int_B u^2(x,T)dx+\frac{1}{2}\phi(T)\int_B a(x,T)u^2(x,T)dx$$

$$=\frac{1}{2}T^{\alpha}E(T)+\phi(T)\int_{\Omega}u(x,T)u_t(x,T)dx-\frac{1}{2}\phi_t(T)\int_{\Omega}u^2(x,T)dx$$

$$+\frac{1}{2}\phi(T)\int_{\Omega}a(x,T)u^2(x,T)dx$$

$$=\frac{1}{2}T^{\alpha}E(T)+\frac{\alpha}{2}T^{\alpha-1}\int_{\Omega}u(x,T)u_t(x,T)dx$$

$$+\frac{\alpha}{4}T^{\alpha-2}\int_{\Omega}\{Ta(x,T)-(\alpha-1)\}u^2(x,T)dx\,.$$

In the above integrals dS denotes the surface element of $\partial B[0,T]$, and we have used the relations $\phi(0)=\phi_t(0)=0$ and $\lim_{t\to0}\int_B\phi(t)a(x,t)u^2(x,t)dx=0$, which follow from the definition of ϕ.

Integrating $(t^{\alpha}u_t+\phi u)L[u]=0$ over $\Omega\times[0,T]$ and taking accout of (2.5), (2.6), (2.7) and (2.8), we have

$$0=\frac{1}{2}T^{\alpha}E(T)+\frac{\alpha}{2}T^{\alpha-1}\int_{\Omega}u(x, T)u_t(x, T)dx$$

$$+\iint_{\Omega\times[0, t_0]}\left(\phi-\frac{\alpha}{2}t^{\alpha-1}\right)(|\nabla u|^2+|u_t|^2)dxdt$$

$$+\iint_{\Omega\times[0, T]}(at^{\alpha}-2\phi)(u_t)^2dxdt+\frac{\alpha}{4}T^{\alpha-2}\int_{\Omega}\{Ta(x, T)-(\alpha-1)\}u^2(x, T)dx$$

$$+\frac{1}{2}\iint_{\Omega\times[0, T]}\{\phi_{tt}-(\phi a)_t\}u^2dxdt,$$

where we should note $\phi(t)=\frac{\alpha}{2}t^{\alpha-1}$ for $t\geqq t_0$. Thus we have completed the proof.

Lemma 2.5. *Let u be a solution of* (1.1), (1.2) *and* (1.3). *Then for any $t\geqq t_0$*

$$t^{\alpha}E(t)+\alpha t^{\alpha-1}\int_{\Omega}u(x, t)u_t(x, t)dx+\frac{\alpha}{2}t^{\alpha-2}\int_{\Omega}u^2(x, t)dx\leqq C, \tag{2.9}$$

where C depends only on $E(0)$ and $\|f\|$.

Proof. We put

$$I_1=2\iint_{\Omega\times[0, t_0]}\left(\phi-\frac{\alpha}{2}t^{\alpha-1}\right)(|\nabla u|^2+|u_t|^2)dxdt,$$

$$I_2=2\iint_{\Omega\times[0, T]}(at^{\alpha}-2\phi)(u_t)^2dxdt=2\iint_{\Omega\times[0, t_0]}+2\iint_{\Omega\times[t_0, T]}$$
$$=J_1+J_2,$$

$$I_3=\frac{\alpha}{2}T^{\alpha-2}\int_{\Omega}\{Ta-(\alpha-1)\}u^2dx,$$

and

$$I_4=\iint_{\Omega\times[0, T]}\{\phi_{tt}-(\phi a)_t\}u^2dxdt=\iint_{\Omega\times[0, t_0]}+\iint_{\Omega\times[t_0, T]}$$
$$=K_1+K_2.$$

Let us compute I_k $(k=1, 2, 3, 4)$. We have from Lemma 2.2

$$|I_1|\leqq\int_0^{t_0}\left|\phi-\frac{\alpha}{2}t^{\alpha-1}\right|dt\int_{\Omega}(|\nabla u|^2+|u_t|^2)dx$$

$$\leqq E(0)\int_0^{t_0}\left|\phi-\frac{\alpha}{2}t^{\alpha-1}\right|dt\leqq C_1E(0),$$

and

$$|J_1|\leqq\int_0^{t_0}|at^{\alpha}-2\phi|dt\int_{\Omega}(u_t)^2dx\leqq E(0)\int_0^{t_0}|at^{\alpha}-2\phi|dt\leqq C_2E(0),$$

where the positive constants C_1 and C_2 are independedt of u. We have from Lemma 2.3

$$|K_1|\leqq\int_0^{t_0}|\phi_{tt}-(\phi a)_t|dt\int_{\Omega}u^2dx$$

$$\leqq 2\int_0^{t_0}\{(E(0)t^2+\|f\|^2)|\phi_{tt}-(\phi a)_t|\}dt \leqq C_3(E(0)+\|f\|^2),$$

where the positive constant C_3 depends only on t_0, bounds of $|a|$ and $|a_t|$, and ϕ. By Assumtion (4) we see that

$$J_2=2\iint_{\Omega\times[t_0,T]}(at-\alpha)t^{\alpha-1}(u_t)^2dxdt \geqq 0$$

and

$$I_3 \geqq \frac{\alpha}{2}t^{\alpha-2}\int_\Omega u^2(x,t)dx.$$

Since $\phi_{tt}-(\phi a)_t=\frac{\alpha}{2}\{(\alpha-1)(\alpha-2)-(\alpha-1)ta-t^2a_t\}$ for $t\geqq t_0$, by Assumption (4) we have

$$K_2 \geqq 0.$$

Thus it follows from Lemma 2.4 that

$$t^\alpha E(t)+\alpha t^{\alpha-1}\int_\Omega u(x,t)u_t(x,t)dx+\frac{\alpha}{2}t^{\alpha-2}\int_\Omega u^2(x,t)dx \leqq C$$
$$=C_1E(0)+C_2E(0)+C_3(E(0)+\|f\|^2),$$

which prove the lemma.

Lemma 2.6. *Let u be a solution of* (1.1), (1.2) *and* (1.3). *Then for any $t\geqq t_0$ and for appropriate positive constants A and B*

$$\int_\Omega u^2(x,t)dx=\|u(\cdot,t)\|^2 \leqq At^{2-\alpha}+B. \tag{2.10}$$

Proof. Noting that $t^\alpha E(t)\geqq 0$ and $\int_\Omega u(x,t)u_t(x,t)dx=\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2$, from Lemma 2.5 we obtain

$$t\frac{d}{dt}u(\cdot,t)\|^2+\|u(\cdot,t)\|^2=\frac{d}{dt}(t\|u(\cdot,t)\|^2) \leqq \frac{2}{\alpha}Ct^{2-\alpha}$$

for any $t>t_0$. Integrating both sides from t to t_0, we have

$$t\|u(\cdot,t)\|^2-t_0\|u(\cdot,t_0)\|^2<\frac{2C}{\alpha(3-\alpha)}(t^{3-\alpha}-t_0^{3-\alpha}).$$

Thus we have

$$\|u(\cdot,t)\|^2<\frac{2C}{\alpha(3-\alpha)}t^{2-\alpha}+\frac{1}{t}\left\{t_0\|u(\cdot,t_0)\|^2-\frac{2C}{\alpha(3-\alpha)}t_0^{3-\alpha}\right\}.$$

Here we put

$$A=\frac{2C}{\alpha(3-\alpha)}t^{2-\alpha} \quad\text{and}\quad B=2E(0)t_0^2+2\|f\|^2+\frac{2C}{\alpha(3-\alpha)}t_0^{2-\alpha}.$$

Then from Lemma 2.3 we can easily show

$$B>\frac{1}{t}\left\{t_0\|u(\cdot,t_0)\|^2-\frac{2C}{\alpha(3-\alpha)}t_0^{3-\alpha}\right\}.$$

Thus we completed the proof.

§3. Proof of the Theorem.

Now applying Lemma 2.5 and Lemma 2.6, we can give the proof of the Theorem.

Proof of the Theorem. Applying Lemma 2.6 and $\|u_t(\cdot, t)\| \leqq \sqrt{E(t)}$ to

$$\left|\int_\Omega u(x, t)u_t(x, t)dx\right| \leqq \|u(\cdot, t)\| \|u_t(\cdot, t)\|,$$

we get

$$\left|\int_\Omega u(x, t)u_t(x, t)dx\right| \leqq \sqrt{At^{2-\alpha}+B}\|u_t\| \leqq \sqrt{(At^{2-\alpha}+B)E(t)}.$$

Therefore from Lemma 2.5 we have

$$\begin{aligned} t^\alpha E(t) &\leqq \alpha t^{\alpha-1}\sqrt{(At^{2-\alpha}+B)E(t)}+C \\ &\leqq \alpha\sqrt{(At^\alpha+Bt^{2\alpha-2})E(t)}+C \leqq \alpha\sqrt{(A+B)t^\alpha E(t)}+C. \end{aligned}$$

So we have

$$\left(t^{\alpha/2}E(t)^{1/2}-\frac{\alpha\sqrt{A+B}}{2}\right)^2 \leqq \frac{\alpha^2(A+B)}{4}+C$$

and

$$t^\alpha E(t) \leqq \left(\frac{\alpha\sqrt{A+B}}{2}+\sqrt{\frac{\alpha^2(A+B)}{4}+C}\right)^2,$$

which was to be proved. Thus we have concluded the proof of the Theorem.

§4. Remarks and examples.

Our $a(x, t)$ is admitted to have a singularity like $t^{-\delta}$ $(0 \leqq \delta \leqq 1)$ at $t=0$ and behave like t^δ $(-1 \leqq \delta < 1)$ as $t \to \infty$ under our Assumptions on $a(x, t)$. The typical form of $a(x, t)$ is that of $\lambda(x)/t$ for all $t > t_0$, where t_0 is a suitable non-negative constant and $\lambda(x)$ is a bounded positive valued function of x. Hence the equation $L[u]=0$ includes the Euler-Poisson-Darboux equation as a special case. We remark the following. If $\min_{x\in\Omega} \lambda(x) \leqq 2$, then we can put $\alpha = \min_{x\in\Omega} \lambda(x)$ and get the energy decay with the order of $t^{-\alpha}$ for $0 < \alpha \leqq 2$. But if $\min_{x\in\Omega} \lambda(x) > 2$, then we cannot put $\alpha = \min_{x\in\Omega} \lambda(x)$, but at most $\alpha = 2$. The author obtained more detailed results on the decay problem concerning the Euler-Poisson-Darboux equation. These results will be given in a forthcoming paper.

Here we shall give several examples of $a(x, t)$. In the following examples we assume that $\lambda(x)$ is a smooth, bounded and positive-valued function of x.

Example 1. Let $a(x, t) = \lambda(x)/t^\varepsilon$ with $0 < \varepsilon < 1$. Then for any $\alpha < 1+\varepsilon$

$$E(t) \leqq \frac{C}{t^\alpha}.$$

Example 2. Let $a(x, t)=\lambda(x)$. Then we can take $\alpha=1$, and

$$E(t)\leqq\frac{C}{t}.$$

Example 3. Let $a(x, t)=\lambda(x)t^{\varepsilon}$ with $0<\varepsilon<1$. Then for $\alpha=1-\varepsilon$

$$E(t)\leqq\frac{C}{t^{\alpha}}.$$

Example 4. Let $a(x, t)=(1+|x|)^{-\varepsilon}(1+t)^{-1+\varepsilon}$ with $0\leqq\varepsilon\leqq1$. Then for any $\alpha<1$

$$E(t)\leqq\frac{C}{t^{\alpha}}.$$

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