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On a lattice property of the space $\Gamma_{ho} \cap \Gamma_{he}$

By

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Introduction

In the Hilbert space $\Gamma=\Gamma(R)$ of square integrable real differentials on a Riemann surface R, let Γ_h (resp. Γ_{he}) be the subspace of Γ which consists of harmonic (resp. exact harmonic) differentials. The orthogonal complement of Γ_{he}^* in Γ_h is denoted by Γ_{ho}. The space of harmonic measure Γ_{hm} is defined as follows: $\omega\in\Gamma_{hm}$ if and only if for every $\varepsilon>0$ and every compact set E there exist a canonical region G $(\supset E)$ and a harmonic function w_G which is constant on each boundary component of G such that $\|\omega-\mathrm{d}w_G\|_G<\varepsilon$. The subspace Γ_{hm} is the orthogonal complement of the space Γ_{hse}^* in Γ_h, where Γ_{hse} consists of harmonic semiexact differentials. The subspace $\Gamma_{ho}\cap\Gamma_{he}$ clearly includes Γ_{hm}, and $\Gamma_{ho}\cap\Gamma_{he}=\Gamma_{hm}$ for finite bordered Riemann surfaces, but R. Accola showed an example of Riemann surface of infinite genus for which the equality does not hold.

Now for HP-functions u and v we denote by $u\wedge v$ (resp. $u\vee v$) the greatest harmonic minorant (resp. the least harmonic majorant) of u and v. A subspace $\Gamma_x\subset\Gamma_{he}$ forms a vector lattice if $\mathrm{d}u$ and $\mathrm{d}v\in\Gamma_x$ imply $\mathrm{d}(u\wedge v)$ and $\mathrm{d}(u\vee v)\in\Gamma_x$. We say that a subspace $\Gamma_x\subset\Gamma_{he}$ has a lattice property if $\mathrm{d}f\in\Gamma_x$ implies $\mathrm{d}(f\wedge c)\in\Gamma_x$ for every real constant c. The space Γ_{he} forms the vector lattice, hence it has a lattice property. It is pointed out in [4] that the space Γ_{hm} has the lattice property.

Here we shall show that $\Gamma_{ho}\cap\Gamma_{he}$ has also the lattice property. Some related subjects shall be investigated.

1. We shall show first that Γ_{hm} forms the vector lattice. This implies that Γ_{hm} has the lattice property.

Proposition 1. *Let u and v be harmonic functions such that* $\mathrm{d}u$ *and* $\mathrm{d}v$ *belong to Γ_{hm}. Then* $\mathrm{d}(u\wedge v)$ *and* $\mathrm{d}(u\vee v)$ *belong to Γ_{hm}.*

Proof. It is sufficient to show $\mathrm{d}(u\wedge v)\in\Gamma_{hm}$. Let $\{R_n\}$ be a canonical regular exhaustion of R. For a given $\varepsilon>0$, there exists an R_m such that $\|\mathrm{d}u\|_{R-R_m}<\varepsilon$ and $\|\mathrm{d}v\|_{R-R_m}<\varepsilon$. Further, there exist an integer N $(>m)$ and harmonic functions u_n and v_n in R_n $(n>N)$ such that u_n and v_n are constant on each boundary component of R_n, $\|\mathrm{d}(u-u_n)\|_{R_n}<\varepsilon$, and $\|\mathrm{d}(v-v_n)\|_{R_n}<\varepsilon$. We have a continuous extension $\hat{u}_n$

(resp. $\hat{v}_n$) of u_n (resp. v_n) which is constant on each component of $R-R_n$. Then

$$\|d(u-\hat{u}_n)\|_R<2\varepsilon, \quad \text{and} \quad \|d(v-\hat{v}_n)\|_R<2\varepsilon.$$

If $u\not\equiv v$, a closed set $\{p\in R;\ u(p)=v(p)\}$ consists of analytic arcs. We denote $\mathbf{A}_r=\{p\in R;\ u(p)-v(p)<r\}$, $\mathbf{B}_r=\{p\in R;\ -r<u(p)-v(p)\}$ and $\mathbf{G}_r=\mathbf{A}_r\cap\mathbf{B}_r$. We can take $r>0$ such that $\|du\|_{G_r}<\varepsilon$ and $\|dv\|_{G_r}<\varepsilon$. Then, for $n>N$,

$$\|d\hat{u}_n\|_{G_r}<3\varepsilon, \quad \text{and} \quad \|d\hat{v}_n\|_{G_r}<3\varepsilon.$$

There exists an integer N' ($>N$) such that for $n>N'$

$$|u_n-u|<r/4 \quad \text{and} \quad |v_n-v|<r/4 \qquad \text{on} \quad R_m,$$

because u_n and v_n converge respectively to u and v in the sense of Dirichlet norm on a region which contains $\bar{R}_n$. Since $u_n<v_n$ on $\mathbf{A}_{-r/2}\cap R_m$ and $u_n>v_n$ on $\mathbf{B}_{-r/2}\cap R_m$, it follows that

$$\begin{aligned}
&\|d\min(\hat{u}_n,\hat{v}_n)-d\min(u,v)\| \\
&\le \|du_n-du\|_{A_{-r/2}\cap R_m}+\|dv_n-dv\|_{B_{-r/2}\cap R_m}+\|d\hat{u}_n\|_{G_r\cup(R-R_m)} \\
&\quad +\|d\hat{v}_n\|_{G_r\cup(R-R_m)}+\|du\|_{G_r\cup(R-R_m)}+\|dv\|_{G_r\cup(R-R_m)} \\
&<18\varepsilon.
\end{aligned}$$

Hence we can take sequences $\{\hat{u}_n\}$ and $\{\hat{v}_n\}$ such that

$$\lim_{n\to\infty}\|d\min(\hat{u}_n,\hat{v}_n)-d\min(u,v)\|=0.$$

Let $w_{i,n}$ be a harmonic function on R_i ($i>n$) such that $w_{i,n}=\min(\hat{u}_n,\hat{v}_n)$ on ∂R_i and $\hat{w}_{i,n}$ be a continuous extension of $w_{i,n}$ which is constant on each component of $R-R_i$. Since $w_{j,n}-\hat{w}_{i,n}$ ($j>i$) is a Dirichlet potential in R_j, we have

$$\|d(w_{j,n}-\hat{w}_{i,n})\|_{R_j}^2\le\|dw_{i,n}\|_{R_i}^2-\|dw_{j,n}\|_{R_j}^2.$$

It follows that $\{w_{i,n}\}$ converges to a harmonic function w_n in R and dw_n belongs to Γ_{hm}. We can show that $\min(\hat{u}_n,\hat{v}_n)=w_n+g_n$, where g_n is a Dirichlet potential (cf. [3]). Hence we have

$$\|dw_n-d(u\wedge v)\|\le\|d\min(\hat{u}_n,\hat{v}_n)-d\min(u,v)\|.$$

It follows that $d(u\wedge v)\in\Gamma_{hm}$, q.e.d.

2. Next we show the lattice property of the space $\Gamma_{ho}\cap\Gamma_{he}$.

Proposition 2. *Let f be a harmonic function in R such that $df\in\Gamma_{ho}\cap\Gamma_{he}$. Then for every real constant c, $d(f\wedge c)$ belongs to $\Gamma_{ho}\cap\Gamma_{he}$.*

Proof. We may assume $c=0$. Let $\mathbf{G}_r=\{p\in R; f(p)>r\}$, for real r. Take $r<0$ and set

$$h_r=\begin{cases} 1 & \text{on} \quad G_0 \\ 0 & \text{on} \quad R-G_r \\ 1-f/r & \text{on} \quad G_r-G_0 . \end{cases}$$

For $g \in HBD$ (bounded harmonic Dirichlet function) $g_r = gh_r$ is a bounded Dirichlet function. By the orthogonal decomposition:

$$\Gamma = \Gamma_{co} \dotplus \Gamma_e^* \qquad \text{(cf. [2])},$$

(1)
$$\begin{aligned} 0 &= (\mathrm{d}f,\ \mathrm{d}g_r^*) = (\mathrm{d}f,\ \mathrm{d}g_r^*)_{G_r} \\ &= (\mathrm{d}f,\ \mathrm{d}g^*)_{G_0} + (\mathrm{d}f,\ \mathrm{d}g^*)_{G_r-G_0} - \frac{1}{r}(\mathrm{d}f,\ \mathrm{d}(fg)^*)_{G_r-G_0} . \end{aligned}$$

We have

$$\begin{aligned} &|(\mathrm{d}f,\ \mathrm{d}(fg)^*)_{G_r-G_0}| \\ &= \left| \iint_{G_r-G_0} \left(-\frac{\partial f}{\partial x} \frac{\partial (fg)}{\partial y} + \frac{\partial f}{\partial y} \frac{\partial (fg)}{\partial x} \right) \mathrm{d}x\mathrm{d}y \right| \\ &\leq \iint_{G_r-G_0} \left| f\left(-\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) \right| \mathrm{d}x\mathrm{d}y \\ &\leq -\frac{r}{2} \iint_{G_r-G_0} \left\{ \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 + \left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2 \right\} \mathrm{d}x\mathrm{d}y \\ &= -\frac{r}{2} \{ \|\mathrm{d}f\|_{G_r-G_0}^2 + \|\mathrm{d}g\|_{G_r-G_0}^2 \} . \end{aligned}$$

It follows that

$$\lim_{r\to 0} \left| \frac{1}{r} (\mathrm{d}f,\ \mathrm{d}(fg)^*)_{G_r-G_0} \right| \leq \lim_{r\to 0} \frac{1}{2} \{ \|\mathrm{d}f\|_{G_r-G_0}^2 + \|\mathrm{d}g\|_{G_r-G_0}^2 \} = 0.$$

Since $\lim_{r\to 0} (\mathrm{d}f,\ \mathrm{d}g^*)_{G_r-G_0} = 0$, we have $(\mathrm{d}f,\ \mathrm{d}g^*)_{G_0} = 0$ from (1), hence $(\mathrm{d}f,\ \mathrm{d}g^*)_{R-G_0} = 0$, because $(\mathrm{d}f,\ \mathrm{d}g^*) = 0$. By the orthogonal decomposition $\Gamma_e = \Gamma_{he} \dotplus \Gamma_{eo}$, we have

$$(\mathrm{d}(f \wedge 0),\ \mathrm{d}g^*) = (\mathrm{d}\min(f,\ 0),\ \mathrm{d}g^*) = (\mathrm{d}f,\ \mathrm{d}g^*)_{R-G_0} = 0.$$

Since every harmonic Dirichlet function h is approximated by HBD-functions in the sense of Dirichlet norm, we have $(\mathrm{d}(f \wedge 0),\ \mathrm{d}h^*) = 0$. It follows that $\mathrm{d}(f \wedge 0) \in \Gamma_{ho}$ and get the conclusion, q.e.d.

3. Let $X = \Gamma_x + \Gamma_{eo}$ ($\Gamma_x \subset \Gamma_h$) and for compact set F on R, $X_F = \{\omega \in X;\ \omega = 0 \text{ on } F\}$. Then X and X_F are closed subspaces of Γ. We denote by X^F the orthogonal complement of X_F in X and by $\omega_x^F = \omega^F$ the orthogonal projection of $\omega \in X$ to X^F.

Lemma 1. (Yamaguchi [4]) *Let $\Gamma_x \subset \Gamma_{he}$ have the lattice property and $\omega \in \Gamma_x \dotplus \Gamma_{eo}$. Let W^F be the Dirichlet function such that $dW^F = \omega_x^F$ (cf. [3]). If $W^F \leq c$ on F, then $W^F \leq c$ on R.*

Proof. We have a representation $W^F = W + W_0$, where $dW \in \Gamma_x$ and W_0 is a Dirichlet potential. There exists a Green potential P such that $|W_0| \leq P$ (cf. [3]). We see that

$$\min(W+W_0, c) = \min(W, c-W_0) + W_0 \leq \min(W, c+P) + W_0,$$

$$\min(W+W_0, c) \geq \min(W-P, c) = \min(W, c+P) - P.$$

Hence we get Royden's decomposition: $\min(W^F, c) = W \wedge c + P_0$, where P_0 is a Dirichlet potential. By the assumption we have $\mathrm{d}\min(W^F, c) \in X$ and $\min(W^F, c) = W^F$ on F. Therefore $\mathrm{d}(\min(W^F, c) - W^F) \in X_F$, and

$$\|\mathrm{d}\min(W^F, c)\|^2 = \|\mathrm{d}(\min(W^F, c) - W^F)\|^2 + \|\mathrm{d}W^F\|^2 \geq \|\mathrm{d}W^F\|^2.$$

On the other hand, clearly, $\|\mathrm{d}\min(W^F, c)\| \leq \|\mathrm{d}W^F\|$. It follows that $\min(W^F, c) = W^F$ and $W^F \leq c$ on R, q.e.d.

Let $\prod_x$ (resp. $\prod_{eo}$, $\prod_{eo}^*$) be the orthogonal projection from Γ to Γ_x ($\subset \Gamma_h$) (resp. Γ_{eo}, Γ_{eo}^*). We assume that $\omega \in \Gamma$ is supported in interior of F. For $\Gamma_x \subset \Gamma_{he}$ W_x denotes a harmonic function such that $\mathrm{d}W_x = \prod_x(\omega)$, and W_0 denotes a Dirichlet potential such that $\mathrm{d}W_0 = \prod_{eo}(\omega)$.

Proposition 3. *Let $\Gamma_x \subset \Gamma_{he}$ have the lattice property. Then W_x and $W_{x^\perp \cap he}$ are bounded, where $\Gamma_{x^\perp}$ is the orthogonal complement of Γ_x in Γ_h.*

Proof. By the assumption

$$\prod_x(\omega) + \prod_{eo}(\omega) = -\prod_{x^\perp}(\omega) - \prod_{eo}^*(\omega) \qquad \text{on} \quad R-F.$$

It follows that for any $\sigma \in X_F$

$$\begin{aligned}(\prod_x(\omega) + \prod_{eo}(\omega), \sigma) &= (\prod_x(\omega) + \prod_{eo}(\omega), \sigma)_{R-F} \\ &= (-\prod_{x^\perp}(\omega) - \prod_{eo}^*(\omega), \sigma)_{R-F} = (-\prod_{x^\perp}(\omega) - \prod_{eo}^*(\omega), \sigma) = 0.\end{aligned}$$

Hence we have $(\prod_x(\omega) + \prod_{eo}(\omega))_x^F = \prod_x(\omega) + \prod_{eo}(\omega)$. Now we may assume that W_0 is bounded on F. If W_0 is unbounded on F, we consider $\omega' = \mathrm{d}(W_x + W_0')$ for ω, where $W_0' = W_0$ on $R-F$, and $W_0' = \min(W_0, \max_{\partial F} W_0)$ on F. By Lemma 1 we know that $W_x + W_0$ is bounded. Since Γ_{he} and $\{0\}$ have the lattice property, $W_{he} + W_0$ and W_0 are bounded. By the fact $\prod_{x^\perp \cap he}(\omega) = \prod_{he}(\omega) - \prod_x(\omega)$, we know that $W_{x^\perp \cap he}$ is bounded, q.e.d.

Remark. Let W_x and W_y be bounded. If $\Gamma_x \perp \Gamma_y$, then W_{x+y} is bounded. If $\Gamma_x \subset \Gamma_y$, then $W_{x^\perp \cap y}$ is bounded. By these operations we can find subspaces Γ_x such that W_x is bounded.

Let p and q be two distinct points on R and $U_{p,q}$ (resp. $V_{p,q}$) be a harmonic function such that $\mathrm{d}U_{p,q} \in \Gamma_x$ (resp. $\mathrm{d}V_{p,q} \in \Gamma_{x^\perp}$),

$$(\mathrm{d}U, \mathrm{d}U_{p,q}) = U(p) - U(q) \qquad \text{for any} \quad \mathrm{d}U \in \Gamma_x,$$

(resp. $(dV, dV_{p,q})=V(p)-V(q)$ for any $dV \in \Gamma_{x^\perp} \cap \Gamma_{he}$). We can construct a differential $\omega \in \Gamma$ such that ω has compact support and $(dU, \omega)=U(p)-U(q)$ for any harmonic Dirichlet function U. Therefore we have

Corollary 1. *If Γ_x has the lattice property, then $U_{p,q}$ and $V_{p,q}$ are bounded.*

Let $HX^\perp=\{U;\ dU \in \Gamma_{x^\perp} \cap \Gamma_{he}\}$ and $HBX^\perp=\{U \in HX^\perp;\ U$ is bounded$\}$. Since the points p and q can be chosen arbitrarily, we have the following.

Corollary 2. *If Γ_x has the lattice property, $HBX^\perp$ is dense in $HX^\perp$. In other words, by the usual notations for null classes of Riemann surfaces, $O_{HBX^\perp}=O_{HX^\perp}$.*

Particularly we can take Γ_{hm} and $\Gamma_{ho} \cap \Gamma_{he}$ for Γ_x. Let $KD=\{U;\ dU \in \Gamma_{hm}^\perp \cap \Gamma_{he}\}$, $KD'=\{U;\ dU \in (\Gamma_{ho} \cap \Gamma_{he})^\perp \cap \Gamma_{he}\}$, $KBD=\{U \in KD;\ U$ *is bounded*$\}$, *and* $KBD'=\{U \in KD';\ U$ *is bounded*$\}$.

Corollary 3. $O_{KBD}=O_{KD}$ *and* $O_{KBD'}=O_{KD'}$.

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