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On the normality of $R(X)$

Dedicated to Professor Gorô Azumaya on his sixtieth birthday

By

Tomoharu AKIBA

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Introduction.

Throughout this paper, we understand by a ring a commutative ring with identity.

Let R be a ring, and let $R[X]$ be the polynomial ring of an indeterminate X over R. For an $f=f(X)\in R[X]$, we denote by $C(f)$, the ideal of R generated by the coefficients of f. Let $N=N(R)=\{f\in R[X]\mid C(f)=R\}$. Then N is a multiplicatively closed subset of $R[X]$, and we set $R(X)=R[X]_N$ (See [7]).

Let T be a ring containing R, and let S be the integral closure of R in T. Let us consider the problem:

(P) Is $S(X)$ the integral closure of $R(X)$ in $T(X)$?[1)]

In [5], Gilmer and Hoffmann gave the affirmative answer to (P) under an additional condition that $T[X]$ is quasi-normal.[2)] As be shown by an example in § 1, the answer to (P) is, in general, negative. We shall give a slight generalization of the result of Gilmer and Hoffmann. In § 2, we shall consider the case where $T=Q(R)$ (=total quotient ring of R). Our main result in § 2 is:

If R is a quasi-normal noetherian ring, then $R(X)$ is integrally closed in $T(X)$.

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§ **1.** We shall begin with an example.

Example. Let k be a field, and let U, V and W be indeterminates. Set $R=k[U, V, W]/(U^2)=k[u, v, w]$, where u, v and w are the canonical images of U, V and W in R, respectively. Let S be the integral closure of R in $T=Q(R)$.

Now we shall show that $S(X)$ is not the integral closure of $R(X)$ in $T(X)$. Take $\alpha=u/(vX+w)$. Since v and w are non-zero-divisors in R and, since $\alpha^2=0$, we see that $\alpha\in T(X)$ and integral over $R(X)$. So it is sufficient to show that

1) See Exercise 2 on page 415 in [4].

2) A ring is quasi-normal if it is integrally closed in its total quotient ring ([1], [2]).

$\alpha\notin S(X)$. Suppose the contrary. Then there are an $f\in N(R)$ and $g^*\in S[X]$ such that $u/(vX+w)=g^*/f$ (see Lemma 1.2 below). From the fact that $u^2=0$, it follows that $g^{*2}=0$ and, therefore g^* is of the form $(u/d)g$, where d is a non-zero-divisor of R and $g\in R[X]$. Thus we get $udf=u(vX+w)g$. Here we may take $d\in k[v, w]$ and $f, g\in k[v, w][X]$, as easily seen. Then we have that $df=(vX+w)g$, since every non-zero element of $k[v, w]$ is not a zero-divisor in R. On the other hand, the ring $k[v, w]$ is isomorphic to the polynomial ring $k[V, W]$, and, hence, $vX+w$ is a prime element of $k[v, w][X]$. Then it is clear that $C(f)\in(v, w)$, which contradicts the assumption that $f\in N(R)$.

The following lemma, which was suggested by Professor Nagata, is proved in a similar way as our proof of Theorem 3.2 in [1] and we omit the proof.

Lemma 1.1. *Let R be a ring, and let α be an element of $R(X)$ integral over $R[X]$. Then there exists an $h\in R[X]$ such that $\alpha-h$ is nilpotent in $R(X)$. In particular, if R is reduced, $R[X]$ is integrally closed in $R(X)$.*

The proof of the following lemma is found in [5].

Lemma 1.2. *Let R, S and T be as in Introduction and let N denote $N(R)$. Then*: (i) *$S[X]$ is the integral closure of $R[X]$ in $T[X]$*, (ii) $S(X)$ $(=S[X]_{N(S)})$ $=S[X]_N$, *and* (iii) *$S(X)$ is the integral closure of $R(X)$ in $T[X]_N$.*

The following proposition is a slight generalization of the result of Gilmer and Hoffmann, because the reducedness of T does not imply the quasi-normality of $T[X]$, in general, even if $T=Q(T)$. (See [1] and [3].)

Proposition 1.3. *Let R, S and T be as in Introduction. If T is reduced, then $S(X)$ is the integral closure of $R(X)$ in $T(X)$.*

Proof. Let α be an element of $T(X)$ integral over $R(X)$. Then there is an $f\in N(R)$ such that $f\alpha$ is integral over $R[X]$, and hence, integral over $T[X]$. Since $f\alpha\in T(X)$, $f\alpha$ is in $T[X]$ by virtue of Lemma 1.1. Then $f\alpha\in S[X]$ by Lemma 1.2, whence $\alpha\in S[X]_{N(R)}=S(X)$.

§ **2.** Throughout this section, we assume that $T=Q(R)$. Therefore, if R is noetherian, $T(X)$ coincides with $Q(R[X])$.

Lemma 2.1. *Assume that R is a quasi-normal noetherian ring. Let n be a non-zero nilpotent element of R, and let M be a maximal ideal of R which contains a non-zero-divisor. Then M does not contain* Ann(n), *where* Ann$(n)=\{r\in R \mid rn=0\}$.

Proof. Since R_M is reduced by virture of Proposition 1.1 in [2], it is clear that $M\not\supseteq \text{Ann}(n)$.

Now we state our main result.

Theorem 2.2. *If R is a quasi-normal noetherian ring, then $R(X)$ is integrally closed in $T(X)$, that is, $R(X)$ is quasi-normal.*

Proof. Let α be an element of $T(X)$ integral over $R(X)$. To show $\alpha \in R(X)$, we may assume that α is nilpotent by Lemma 1.1. Then, it is easy to see that we may restrict α to an element of the form n/f, where n is a non-zero nilpotent element of R and $f \in R[X]$ such that $C(f)$ contains a non-zero-divisor of R. Write $f=a_0+a_1X+\cdots+a_nX^n$ with $a_i \in R$ $(i=0, 1, \cdots, n)$. Since $C(f)=(a_0, a_1, \cdots, a_n)$, $(a_0, a_1, \cdots, a_n)+\mathrm{Ann}(n)=R$ by virtue of Lemma 2.1. Take $b \in \mathrm{Ann}(n)$ so that $(a_0, a_1, \cdots, a_n, b)=R$. Then, setting $g=b+a_0X+\cdots+a_nX^{n+1}$, we get $n/f=nX/g$ with $C(g)=R$, which implies that $\alpha=n/f \in R(X)$. Thus the proof is complete.

Corollary 2.3. *Let R be a (not necessarily noetherian) ring, and let S be the integral closure of R in T. If S is noetherian, then $S(X)$ is the integral closure of $R(X)$ in $T(X)$.*

Proof. By Theorem 2.2, $S(X)$ is integrally closed in $T(X)$. On the other hand, since $S(X)$ is integral over $R(X)$ by virtue of Lemma 1.2, the corollary follows.

A ring is called a Prüfer ring if every finitely generated ideal containing a non-zero-divisor is invertible. A Prüfer ring is quasi-normal (see [6]).

Proposition 2.4. *If R is a Prüfer ring, then $R(X)$ is integrally closed in $T(X)$.*

In order to prove the proposition, we need the following Lemma.

Lemma 2.5. *Let R be a ring, and let f be an element of $R[X]$ such that $C(f)$ is invertible. Then for any $g \in R[X]$, $C(fg)=C(f)C(g)$. (See Chap. IV in* [4].)

Proof of Proposition 2.4. Take an α in $T(X)$ integral over $R(X)$. As in the proof of Theorem 2.2, we may assume that α is nilpotent and is of the form n/f, where n is a nilpotent element of R and $f \in R[X]$ such that $C(f)$ contains a non-zero-divisor of R. Write $f=a_0+a_1X+\cdots+a_nX^n$, and therefore, $C(f)=(a_0, a_1, \cdots, a_n)$. Since $C(f)$ is invertible by our assumption, there are $b_0/s, b_1/s, \cdots, b_n/s$ in $C(f)^{-1}$ such that $\sum_i a_i(b_i/s)=1$, where $b_i \in R$ $(i=0, 1, \cdots, n)$ and s is a non-zero-divisor of R. Let $g=b_0+b_1X+\cdots+b_nX^n$, and let $h=fg$. Then, by Lemma 2.5, $C(h)=C(f)C(g)=(s)$. Hence there is an $h' \in R[X]$ such that $h=sh'$ and $C(h')=R$, that is, $h' \in N(R)$. Then $\alpha=n/f=ng/(fg)=(n/s)/h' \in R(X)$, since n/s is nilpotent and R is quasi-normal.

A Bezout ring is a ring such that every finitely generated ideal is principal. It is clear that a Bezout ring is a Prüfer ring. Hence we have:

Corollary 2.6. *If R is a Bezout ring, then $R(X)$ is integrally closed in $T(X)$.*

KYOTO UNIVERSITY

References

[1] T. Akiba, Integrally-closedness of polynomial rings, Japan. J. Math., **6**, 1980.
[2] T. Akiba and M. Nagata, On normality of a noetherian ring, J. Math. Kyoto Univ., **17**, 1977.
[3] J.W. Brewer, D.L. Costa, and K. McCrimmon, Seminormality and root closure in polynomial rings and algebraic curves, J. Algebra, **58**, 1979.
[4] R. Gilmer, Multiplicative Ideal Theory, Dekker, New York, 1972.
[5] R. Gilmer and J.F. Hoffmann, A characterization of Prufer domains in terms of polynomials, Pacific J. Math., **60**, 1975.
[6] M.D. Larsen and P.J. McCarthy, Multiplicative Theory of Ideals, Academic Press, New York and London, 1971.
[7] M. Nagata, Local Rings, John Wiley, New York, 1962; reprint ed. Krieger, Huntingon, 1974.