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On the Cauchy problem for a semi-linear hyperbolic system

By

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Introduction

We consider the following system of semi-linear partial diffential equations for $u(x, t)$ and $v(x, t)$:

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}-\lambda\dfrac{\partial u}{\partial x}=au-uv\,, \\ \dfrac{\partial v}{\partial t}-\mu\dfrac{\partial v}{\partial x}=bv+uv\,, \qquad (t>0,\ -\infty<x<+\infty)\,, \end{cases}$$

with the initial data:

$$(2)\qquad \begin{cases} u(x,\ 0)=\phi(x)\,, \\ v(x,\ 0)=\psi(x)\,, \end{cases}$$

where λ, μ, a and b are real constants. And we suppose $\lambda\neq\mu$, for if $\lambda=\mu$, then the system (1) would reduce to a system of ordinary diffential equations. Furthermore we suppose

$$(3)\qquad \begin{cases} 0\leqq\phi(x)\leqq M\,,\quad 0\leqq\psi(x)\leqq M\,, \qquad (-\infty<x<+\infty)\,, \\ \phi(x)\,,\quad \psi(x)\in\mathfrak{B}^1(-\infty,\ +\infty)\,, \end{cases}$$

where $\mathfrak{B}^1(-\infty, +\infty)$ means the function spaces of all C^1 functions with bounded first derivatives defind over $(-\infty, +\infty)$ and M is a positive constant.

The system (1)-(2) has an ecological meaning, when a is positive and b is negative. That is, the system (1) can be considered as describing a development in time of two elements of prey $u(x, t)$ and predator $v(x, t)$ running on a straight line with the speed of λ and μ respectively. As to the constant a and b, we may consider them as a rate of natural multiplication of prey without predator and a rate of natural extinction of predator without prey respectively.

§ 1. Preliminary

By setting $\dfrac{b}{a}=\gamma$ and $a=\varepsilon$ in (1), we have $b=\gamma\varepsilon$ and the system (1) can be written as follows:

$$(4)\qquad \begin{cases} \dfrac{\partial u}{\partial t}-\lambda\dfrac{\partial u}{\partial x}=\varepsilon u-uv\,, \\ \dfrac{\partial v}{\partial t}-\mu\dfrac{\partial v}{\partial x}=\gamma\varepsilon v+uv\,. \end{cases}$$

If we put $\varepsilon=0$ in (4), then we get

$$(5)\qquad \begin{cases} \dfrac{\partial u}{\partial t}-\lambda\dfrac{\partial u}{\partial x}=-uv\,, \\ \dfrac{\partial v}{\partial t}-\mu\dfrac{\partial v}{\partial x}=uv\,. \end{cases}$$

The exact solutions of the system (5) with the initial data (2) can be seen in Hashimoto, H. [1] and Hirota, R. [2] etc. Now we put these exact solutions as $u_0(x, t)$ and $v_0(x, t)$. Then it is obvious that $u_0(x, t)$ and $v_0(x, t)$ are bounded together with their first derivatives with respect to x and t and non-negative over

$$(6)\qquad \Omega_T=(-\infty, +\infty)\times[0, T]\,, \qquad (T>0)\,.$$

(See, for example, Yoshikawa, A. and Yamaguti, M. [5]).

Under these properties, we can set as follows:

$$(7)\qquad \begin{cases} A=\max\{\sup\limits_{(x,t)\in\Omega_T} u_0(x, t),\ \delta \sup\limits_{(x,t)\in\Omega_T} v_0(x, t)\}\,, \\ \text{where } \delta=\max\{1, |\gamma|\}\,. \\ L=\max\left\{\dfrac{\delta}{2}(e^{2AT}-1),\ \dfrac{1}{2A}(e^{2AT}-1)\right\}\,. \\ P=\max\left\{\sup\limits_{(x,t)\in\Omega_T}\left|\dfrac{\partial u_0(x, t)}{\partial x}\right|,\ \sup\limits_{(x,t)\in\Omega_T}\left|\dfrac{\partial u_0(x, t)}{\partial t}\right|,\right. \\ \qquad\left.\sup\limits_{(x,t)\in\Omega_T}\left|\dfrac{\partial v_0(x, t)}{\partial x}\right|,\ \sup\limits_{(x,t)\in\Omega_T}\left|\dfrac{\partial v_0(x, t)}{\partial t}\right|\right\}\,. \\ Q_T=\sqrt{L(\delta+2L)}\,. \end{cases}$$

The purpose of this paper is to obtain the solutions $u(x, t)$ and $v(x, t)$ of the Cauchy problem (4)-(2) in the following form:

$$(8)\qquad \begin{cases} u(x, t)=\sum_{n=0}^{\infty}u_n(x, t)\varepsilon^n\,, \\ v(x, t)=\sum_{n=0}^{\infty}v_n(x, t)\varepsilon^n\,, \qquad (x, t)\in\Omega_T\,. \end{cases}$$

Here, $u_0(x, t)$ and $v_0(x, t)$ are solutions of the Cauchy problem (5)-(2), and $u_n(x, t)$ and $v_n(x, t)$ $(n\geqq 1)$ are solutions of the systems of linear partial diffential equations for $u_n(x, t)$ and $v_n(x, t)$:

$$(9)\qquad \begin{cases} \dfrac{\partial u_n}{\partial t}-\lambda\dfrac{\partial u_n}{\partial x}=u_{n-1}-(u_nv_0+\cdots+u_0v_n)\,, \\ \dfrac{\partial u_n}{\partial t}-\mu\dfrac{\partial u_n}{\partial x}=\gamma v_{n-1}+(u_nv_0+\cdots+u_1v_n)\,, \end{cases}$$

with the initial data

$$
(10) \qquad \begin{cases} u_n(x,\ 0)=0\ , \\ v_n(x,\ 0)=0\ , \quad (n \geqq 1)\ . \end{cases}
$$

Note that (9) can be obtained by substituting (8) into (4) and by arranging on the power of ε. The formula (8) indicates that the solutions of the Cauchy problem (4)-(2) are analytic functions of ε.

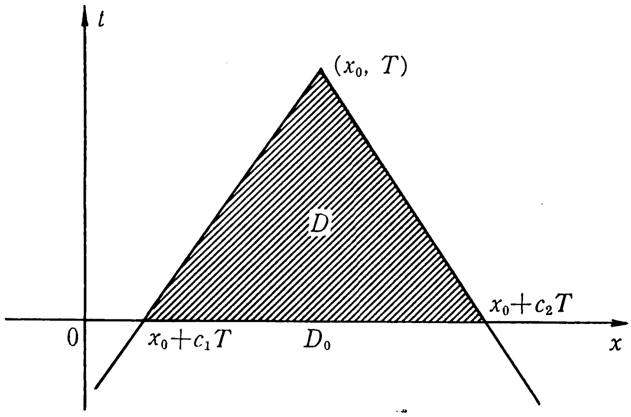
Here, we may give what is called Haar's inequality which is used in proving several lemmas is the next section. Suppose that the following system of linear partial differential equations is given:

$$
\begin{cases} \dfrac{\partial u_1}{\partial t} - c_1 \dfrac{\partial u_1}{\partial x} = a_{11}(x,\ t)u_1 + a_{12}(x,\ t)u_2 + b_1(x,\ t)\ , \\ \dfrac{\partial u_2}{\partial t} - c_2 \dfrac{\partial u_2}{\partial x} = a_{22}(x,\ t)u_1 + a_{22}(x,\ t)u_2 + b_2(x,\ t)\ , \end{cases}
$$

with the initial data:

$$
\begin{cases} u_1(x,\ 0)=\phi_1(x)\ , \\ u_2(x,\ 0)=\phi_2(x)\ , \end{cases}
$$

where c_1 and c_2 are constants such that $c_1 < c_2$. $a_{ij}(x,\ t)$ and $b_{ij}(x,\ t)$ $(1 \leqq i,\ j \leqq 2)$ are supposed to be continuous over $(-\infty,\ +\infty) \times [0,\ +\infty)$. Furthermore we suppose that $\phi_1(x)$ and $\phi_2(x)$ are continuous together with their first derivatives over $(-\infty,\ +\infty)$. Now, in the following figure let us put a closed domain inside a triangle as D and $[x_0+c_1T,\ x_0+c_2T]$ as D_0.



By setting

$$
a = \max_{1 \leq i,\, j \leq 2} \{ \sup_{(x,\, t) \in D} | a_{ij}(x,\ t) | \}\ ,
$$

$$
b = \max_{1 \leq i \leq 2} \{ \sup_{(x,\, t) \in D} | b_i(x,\ t) | \}\ ,
$$

$$
h = \max_{1 \leq i \leq 2} \{ \sup_{x \in D_0} | \phi_i(x) | \}\ ,
$$

Haar's inequality can be written in such a way that

$$(11)\qquad |u_1(x,\ t)|,\ |u_2(x,\ t)|\leqq he^{2at}+\frac{b}{2a}(e^{2at}-1)$$
$$\leqq he^{2aT}+\frac{b}{2a}(e^{2aT}-1),\qquad (x,\ t)\in D.$$

§2. The proof of several lemmas

Lemma 1. *The solutions $u_n(x, t)$ and $v_n(x, t)$ of the Cauchy problem* (9)-(10) *can be estimated over Ω_T as follows:*

$$(12)\qquad |u_n(x,\ t)|,\quad |v_n(x,\ t)|\leqq a_n Q_T^{2n-1},$$

where

$$(13)\qquad a_n=\begin{cases}1 & ;\ n=1,\ 2,\ 3,\\ a_{n-1}a_1+\cdots+a_2a_{n-2}; & n\geqq 4.\end{cases}$$

Proof. First we will show that (12) holds for $n=1$. If $n=1$ in (9) and (10), we have

$$(14)\qquad \begin{cases}\dfrac{\partial u_1}{\partial t}-\lambda\dfrac{\partial u_1}{\partial x}=(-v_0)u_1+(-u_0)v_1+u_0,\\ \dfrac{\partial v_1}{\partial t}-\mu\dfrac{\partial v_1}{\partial x}=v_0u_1+u_0v_1+\gamma v_0,\\ u_1(x,\ 0)=v_1(x,\ 0)=0.\end{cases}$$

Applying Haar's inequality (11) to the system of partial differential equations (14) for $u_1(x, t)$ and $v_1(x, t)$, we get

$$|u_1(x,\ t)|,\ |v_1(x,\ t)|\leqq\frac{A}{2A}(e^{2At}-1)\leqq\frac{1}{2}(e^{2At}-1)\leqq L\leqq Q_T=a_1Q_T,$$

where A, L and Q_T are given in (7). In the same way, we can also show that (12) holds for $n=2$, 3.

Secondly, if we suppose that (12) holds for n ($n\geqq 4$), then we will show that (12) holds also for $n+1$. For $n+1$, (9) and (10) can be written as follows

$$(15)\qquad \begin{cases}\dfrac{\partial u_{n+1}}{\partial t}-\lambda\dfrac{\partial u_{n+1}}{\partial x}=(-v_0)u_{n+1}+(-u_0)v_{n+1}+u_n-(u_nv_1+\cdots+u_1v_n),\\ \dfrac{\partial v_{n+1}}{\partial t}-\mu\dfrac{\partial v_{n+1}}{\partial x}=v_0u_{n+1}+u_0v_{n+1}+\gamma v_n+(u_nv_1+\cdots+u_1v_n),\end{cases}$$

$$(16)\qquad \begin{cases}u_{n+1}(x,\ 0)=0,\\ v_{n+1}(x,\ 0)=0.\end{cases}$$

Here, applying Haar's inequality (11) to the system (15) of linear partial differential equations for $u_{n+1}(x, t)$ and $v_{n+1}(x, t)$ with initial data (16), we get

$$|u_n-(u_nv_1+\cdots+u_1v_n)|\leqq(|u_n|+|u_n|\,|v_1|+|v_n|\,|u_1|)$$
$$+(|u_{n-1}|\,|v_2|+\cdots+|u_2|\,|v_{n-1}|)\leqq a_nQ_T^{2n-1}(\delta+2L)$$

$$+(a_{n-1}a_2+\cdots+a_2a_{n-1})Q_T^{2n}.$$

Therefore, by applying Haar's inequality (11) to (15) and (16), we get

$$|u_{n+1}(x,\ t)|,\ |v_{n+1}(x,\ t)|\leqq\{a_nQ_T^{2n-1}(\delta+2L)+(a_{n-1}a_2+\cdots$$

$$+a_2a_{n-1})Q_T\}\frac{1}{2A}(e^{2AT}-1)$$

$$\leqq a_na_1Q_T^{2n-1}L(\delta+2L)+(a_{n-1}a_2+\cdots+a_2a_{n-1})LQ_T^{2n}$$

$$\leqq(a_na_1+\cdots+a_2a_{n-1})Q_T^{2n+1}=a_{n+1}Q_T^{2n+1}.$$

Hence, we get the inequality (12) also for $n+1$. Therefore by induction we get the estimate (12) and the formula (13) for all $n\geqq 1$. Q. E. D.

Lemma 2. *The radius of convergence of a series* $\sum_{n=1}^{\infty}a_nz^n$ *generated by the sequence* $\{a_n\}$ *defined in* (13) *is equal to* $\frac{1}{3}$. *Here, z is a complex variable.*

Proof. By using (13) and adjusting formaly each coefficient of $z_n(n\geqq 1)$, we have

(17) $$\sum_{n=3}^{\infty}a_nz^n=\sum_{n=1}^{\infty}a_nz^n\cdot\sum_{n=2}^{\infty}a_nz^n.$$

Now, we set

(16) $$f(z)=\sum_{n=1}^{\infty}a_nz^n.$$

Then, by (17), we have

$$\{f(z)\}^2-(z+1)f(z)+z^2+z=0.$$

Solving this quadratic equation for $f(z)$ and observing that $f(0)=0$ by (18), we get

(19) $$f(z)=\frac{1}{2}(z+1-\sqrt{(1+z)(1-3z)}).$$

From this, because $z=-1$ and $z=\frac{1}{3}$ are singular points of $f(z)$, we can see that the radius of convergence of $f(z)=\sum_{n=1}^{\infty}a_nz^n$ is equal to $\frac{1}{3}$. Q. E. D.

Lemma 3. *For any* $T>0$, *if we choose arbitrarily* ε *as* $|\varepsilon|<\frac{1}{3Q_T^2}$, *then the right-hand sides of* (8) *converge uniformly over* Ω_T. *Here,* Ω_T *and* Q_T *are defined in* (6) *and* (7) *respectively.*

Proof. From (12) we get

$$|u_n(x,\ t)\varepsilon^n|\leqq a_nQ_T^{2n-1}\varepsilon^n=\frac{1}{Q_T}a_n(Q_T^2\varepsilon)^n.$$

Then we get

$$|\sum_{n=0}^{\infty} u_n(x, t)\varepsilon^n| \leqq u_0(x, t) + \frac{1}{Q_T}\sum_{n=1}^{\infty} a_n(Q_T^2\varepsilon)^n .$$

Therefore, Lemma 2 shows that $\sum_{n=0}^{\infty} u^n(x, t)\varepsilon^n$ converges uniformly over Ω_T. In the same manner we can see that $\sum_{n=0}^{\infty} v_n(x, t)\varepsilon^n$ also converges uniformly over Ω_T. Q. E. D.

Now, we will show that the right-hand sides of (8) are differentiable term by term with respect to x and t, and that the derivatives satisfy the following relations:

$$(20)\quad \begin{cases} \dfrac{\partial u(x, t)}{\partial x} = \sum_{n=0}^{\infty} \dfrac{\partial u_n(x, t)}{\partial x}\varepsilon^n, & \dfrac{\partial u(x, t)}{\partial t} = \sum_{n=0}^{\infty} \dfrac{\partial u_n(x, t)}{\partial t}\varepsilon^n, \\ \dfrac{\partial v(x, t)}{\partial x} = \sum_{n=0}^{\infty} \dfrac{\partial v_n(x, t)}{\partial x}\varepsilon^n, & \dfrac{\partial v(x, t)}{\partial t} = \sum_{n=0}^{\infty} \dfrac{\partial v_n(x, t)}{\partial t}\varepsilon^n . \end{cases}$$

Here, for $n \geqq 0$ we put as follows

$$(21)\quad \begin{cases} \dfrac{\partial u_n(x, t)}{\partial x} = \tilde{u}_n(x, t), & \dfrac{\partial v_n(x, t)}{\partial x} = \tilde{v}_n(x, t), \\ \dfrac{\partial u_n(x, t)}{\partial t} = \bar{u}_n(x, t), & \dfrac{\partial v_n(x, t)}{\partial t} = \bar{v}_n(x, t) . \end{cases}$$

By using (21) and differentiating both sides of (9) and (10) with respect to x, we get

$$(22)\quad \begin{cases} \dfrac{\partial \tilde{u}_n}{\partial t} - \lambda \dfrac{\partial \tilde{u}_n}{\partial x} = (-v_0)\tilde{u}_n + (-u_0)\tilde{v}_n + u_{n-1} - (u_n\tilde{v}_0 + \cdots + u_1\tilde{v}_{n-1}) \\ \qquad\qquad - (\tilde{u}_{n-1}v_1 + \cdots + \tilde{u}_0 v_n), \\ \dfrac{\partial \tilde{v}_n}{\partial t} - \mu \dfrac{\partial \tilde{v}_n}{\partial x} = v_0\tilde{u}_n + u_0\tilde{v}_n + \gamma\tilde{v}_{n-1} + (u_n\tilde{v}_0 + \cdots + u_1\tilde{v}_{n-1}) \\ \qquad\qquad + (\tilde{u}_{n-1}v_1 + \cdots + \tilde{u}_0 v_n) \quad (n \geqq 1), \end{cases}$$

with initial data

$$(23)\quad \begin{cases} \tilde{u}_n(x, 0) = 0, \\ \tilde{v}_n(x, 0) = 0, \quad (n \geqq 1). \end{cases}$$

In the same way, by differentiating both sides of (9) with respect to t, we get

$$(24)\quad \begin{cases} \dfrac{\partial \bar{u}_n}{\partial t} - \lambda \dfrac{\partial \bar{u}_n}{\partial x} = (-v_0)\bar{u}_n + (-u_0)\bar{v}_n + \bar{u}_{n-1} - (u_n\bar{v}_0 + \cdots + u_1\bar{v}_{n-1}) \\ \qquad\qquad - (\bar{u}_{n-1}v_1 + \cdots + \bar{u}_0 v_n), \\ \dfrac{\partial \bar{v}_n}{\partial t} - \mu \dfrac{\partial \bar{v}_n}{\partial x} = v_0\bar{u}_n + u_0\bar{v}_n + \gamma\bar{v}_{n-1} + (u_n\bar{v}_0 + \cdots + u_1\bar{v}_{n-1}) \\ \qquad\qquad + (\bar{u}_{n-1}v_1 + \cdots + \bar{u}_0 v_n) \quad (n \geqq 1). \end{cases}$$

As to the initial data for $\bar{u}_n(x, t)$ and $\bar{v}_n(x, t)$ $(n \geqq 1)$, by using (9), (10) and (23) we get

$$(25)\qquad \begin{cases} \bar{u}_1(x, 0)=u_0(x, 0)=\phi(x), \\ \bar{v}_1(x, 0)=\gamma v_0(x, 0)=\gamma\phi(x), \\ \bar{u}_n(x, 0)=\bar{v}_n(x, 0)=0, \qquad (n\geqq 2). \end{cases}$$

Under these preparation, we can prove three more lemmas:

Lemma 4. *For* $\bar{u}_n(x, t)$, $\bar{v}_n(x, t)$; $\tilde{u}_n(x, t)$, $\tilde{v}_n(x, t)$ *defined in* (21), *we get the following estimates over* Ω_T:

$$(26)\qquad |\bar{u}_n(x, t)|,\ |\bar{v}_n(x, t)|\leqq d_n M e^{2AT} Q_T^{2n-2}+P c_n Q_T^{2n}, \quad (n\geqq 1),$$

$$(27)\qquad |\tilde{u}_n(x, t)|,\ |\tilde{v}_n(x, t)|\leqq P c_n Q_T^{2n}, \qquad (n\geqq 1),$$

where

$$(28)\qquad \begin{cases} c_1=1, \quad c_n=2(a_n+a_{n-1}c_1+\cdots+a_1 c_{n-1})-c_{n-1}, & (n\geqq 2), \\ d_1=1, \quad d_n=2(a_1 d_{n-1}+\cdots+a_{n-1}d_1)-d_{n-1}, & (n\geqq 2), \end{cases}$$

M and a sequence $\{a_n\}$ are defined in (3) *and* (13) *respectively, and A, Q_T and P are defined in* (7).

Proof. To prove this by induction, first we will show that $\bar{u}_1(x, t)$ and $\bar{v}_1(x, t)$ satisfy (26) for $n=1$. By setting $n=1$ in (24) and (25), we have a system of linear partial differential equations for $\bar{u}_1(x, t)$ and $\bar{v}_1(x, t)$:

$$(29)\qquad \begin{cases} \dfrac{\partial \bar{u}_1}{\partial t}-\lambda\dfrac{\partial \bar{u}_1}{\partial x}=(-v_0)\bar{u}_1+(-u_0)\bar{v}_1+(\bar{u}_0-\bar{u}_0 u_1-\bar{u}_0\bar{v}_1), \\ \dfrac{\partial \bar{v}_1}{\partial t}-\mu\dfrac{\partial \bar{v}_1}{\partial x}=v_0\bar{u}_1+u_0\bar{v}_1+(\bar{v}_0+\bar{v}_0 u_1+\bar{u}_0\bar{v}_1), \end{cases}$$

with the initial data:

$$(30)\qquad \begin{cases} \bar{u}_1(x, 0)=\phi(x), \\ \bar{v}_1(x, 0)=\gamma\phi(x). \end{cases}$$

Then, from (7) we get

$$(31)\qquad \begin{cases} |\bar{u}_0-\bar{v}_0 u_1-\bar{u}_0 v_1|\leqq P(1+2L)\leqq P(\delta+2L), \\ |\gamma\bar{v}_0+\bar{v}_0 u_1+\bar{u}_0 v_1|\leqq P(|\gamma|+2L)\leqq P(\delta+2L), \end{cases}$$

where $\delta=\max\{1, |\gamma|\}$. Applying Haar's inequality (11) to (29) and (30), and by considering (31) we get

$$|\bar{u}_1(x, t)|\leqq Me^{2AT}+P(\delta+2L)\frac{1}{2A}(e^{2AT}-1)\leqq d_1 M e^{2AT}+c_1 P Q_T^2,$$

and in the same way, we get also

$$|\bar{v}_1(x, t)|\leqq d_1 M e^{2AT}+c_1 P Q_T^2,$$

where $c_1=d_1=1$.

Secondly, we will show that if (12) holds for n ($n\geqq 2$), then (12) holds also for $n+1$. Replacing n by $n+1$ in (24), we have

$$(32)\quad \begin{cases} \dfrac{\partial \bar{u}_{n+1}}{\partial t}-\lambda\dfrac{\partial \bar{u}_{n+1}}{\partial x}=(-v_0)\bar{u}_{n+1}+(-u_0)\bar{v}_{n+1}+\bar{u}_n \\ \qquad -(u_{n+1}\bar{v}_0+\cdots+u_1\bar{v}_n)-(\bar{u}_n v_1+\cdots+\bar{u}_0 v_{n+1}), \\ \dfrac{\partial \bar{v}_{n+1}}{\partial t}-\mu\dfrac{\partial \bar{v}_{n+1}}{\partial x}=v_0\bar{u}_{n+1}+u_0\bar{v}_{n+1}+\gamma\bar{v}_n \\ \qquad +(u_{n+1}\bar{v}_0+\cdots+u_1\bar{v}_n)+(\bar{u}_n v_1+\cdots+\bar{u}_0 v_n). \end{cases}$$

Ane by (3) and (7), we have

$$\begin{aligned} &|\bar{u}_n-(u_{n+1}\bar{v}_0+\cdots+u_1\bar{v}_n)-(\bar{u}_n v_1+\cdots+\bar{u}_0 v_{n+1})| \\ &\quad \leqq (|\bar{u}_n|+|u_1||\bar{v}_n|+|\bar{u}_n||v_1|)+(|u_{n+1}||\bar{v}_0|+\cdots+|u_2||\bar{v}_{n-1}|) \\ &\qquad +(|\bar{u}_{n-1}||v_2|+\cdots+|\bar{u}_0||v_{n+1}|) \\ &\quad \leqq (d_n M e^{2AT} Q_T^{2n-2}+c_n P Q_T^{2n})(1+2L)+2\{a_{n+1} P Q_T^{2n+1} \\ &\qquad +a_n Q_T^{2n-1}(d_1 M e^{2AT}+c_1 P Q_T^2)+\cdots+a_2 Q_T^3(d_{n-1} M e^{2AT} Q_T^{2n-4}+c_n P Q_T^{2n-2})\} \\ &\quad =(d_n M e^{2AT} Q_T^{2n-2}+c_n P Q_T^{2n})(1+2L) \\ &\qquad +2\{(a_{n+1}+a_n c_1+\cdots+a_2 c_{n-1}) P Q_T^{2n+1}+M e^{2AT}(a_n d_1+\cdots \\ &\qquad +a_2 d_{n-1}) Q_T^{2n-1}\}. \end{aligned}$$

In the same way, we get

$$\begin{aligned} &|\gamma\bar{v}_n+(u_{n+1}\bar{v}_0+\cdots+u_1\bar{v}_n)+(\bar{u}_n v_1+\cdots+\bar{u}_0 v_{n+1})| \\ &\quad \leqq (d_n M e^{2AT} Q_T^{2n-2}+c_n P Q_T^{2n})(|\gamma|+2L) \\ &\qquad +2\{(a_{n+1}+a_n c_1+\cdots+a_2 c_{n-1}) P Q_T^{2n+1} \\ &\qquad +M e^{2AT}(a_n d_1+\cdots+a_2 d_{n-1}) Q_T^{2n-1}\}. \end{aligned}$$

Under these circumstances, by applying Haar's inequality (11) to (32) we get

$$|\bar{u}_{n+1}(x, t)|,\ |\bar{v}_{n+1}(x, t)| \leqq d_{n+1} M e^{2AT} Q_T^{2n}+P c_{n+1} Q_T^{2n+2}.$$

Therefore by induction we can see that the estimate (26) hold for all $n \geqq 1$.

In the same manner, we can prove the estimate (27). Q. E. D.

Lemma 5. *Both the series* $\sum_{n=1}^{\infty} c_n z^n$ *and* $\sum_{n=1}^{\infty} d_n z^n$ *generated by* $\{c_n\}$ *and* $\{d_n\}$ *defined in* (28) *have* $\dfrac{1}{3}$ *as radius of convergence. Here, the variable* z *is complex number.*

Proof. We set $g(z)=\sum_{n=1}^{\infty} c_n z^n$. Then, by (28) we have

$$a_{n-1}c_1+\cdots+a_1 c_{n-1}=\frac{1}{2}(c_n+c_{n-1})-a_n.$$

By using (18) and the above relations, we get the following formal relation:

$$f(z)g(z)=a_1c_1z^2+(a_2c_1+a_1c_2)z^3+\cdots+(a_nc_1+a_{n-1}c_2+\cdots+a_1c_n)z^{n+1}+\cdots$$

$$=z_2+\left\{\frac{1}{2}(c_3+c_2)-a_3\right\}z_3+\cdots+\left\{\frac{1}{2}(c_{n+1}+c_n)-a_{n+1}\right\}z^{n+1}+\cdots$$

$$=z^2+\frac{1}{2}\sum_{n=3}^{\infty}c_nz^n+\frac{1}{2}z\sum_{n=2}^{\infty}c_nz^n-\sum_{n=3}^{\infty}a_nz^n$$

$$=\frac{1}{2}(1+z)g(z)-f(z)+\frac{1}{2}z\,.$$

By using $f(z)=\frac{1}{2}\{(z+1)-\sqrt{(1+z)(1-3z)}\}$ derived from (19), if we solve the linear equation for $g(z)$, we get

$$g(z)=\frac{-1}{\sqrt{(1+z)(1-3z)}}+1\,.$$

Since $g(z)$ has two singular points at $z=-1$ and $z=\frac{1}{3}$, the radius of convergence of the series $g(z)=\sum_{n=1}^{\infty}c_nz_n$ is equal to $\frac{1}{3}$.

Next, if we set $h(z)=\sum_{n=1}^{\infty}d_nz^n$, in the same way as $g(z)$, we get

$$h(z)=\frac{2z^2+z}{2\sqrt{(1+z)(1-3z)}}\,.$$

Hence, the radius of convergence of $\sum_{n=1}^{\infty}d_nz^n$ is also equal to $\frac{1}{3}$. Q. E. D.

Lemma 6. *For any* $T>0$, *if we choose arbitrarily* ε *as* $|\varepsilon|<\frac{1}{3Q_T^2}$, *then the right-hand sides of* (8) *are differentiable term by term with respect to* x *and* t, *and the right-hand sides of* (20) *converge uniformly over* Ω_T.

Proof. From (26) and (27), we get

$$\left|\sum_{n=1}^{\infty}\frac{\partial u_n(x,\ t)}{\partial t}\varepsilon^n\right|\leqq\sum_{n=1}^{\infty}|\bar{u}_n(x,\ t)\varepsilon^n|\leqq\frac{Me^{2AT}}{Q_T^2}\sum_{n=1}^{\infty}d_n(Q_T^2\varepsilon)^n+P\sum_{n=1}^{\infty}c_n(Q_T^2\varepsilon)^n\,.$$

in the same way, we get

$$\left|\sum_{n=1}^{\infty}\frac{\partial v_n(x,\ t)}{\partial t}\varepsilon^n\right|\leqq\frac{Me^{2AT}}{Q_T^2}\sum_{n=1}^{\infty}d_n(Q_T^2\varepsilon)^n+P\sum_{n=1}^{\infty}c_n(Q_T^2\varepsilon)^n\,,$$

$$\left|\sum_{n=1}^{\infty}\frac{\partial u_n(x,\ t)}{\partial x}\varepsilon^n\right|,\quad\left|\sum_{n=1}^{\infty}\frac{\partial v_n(x,\ t)}{\partial x}\varepsilon^n\right|\leqq P\sum_{n=1}^{\infty}c_n(Q_T^2\varepsilon)^n\,.$$

Considering the above estimates and using Lemma 5, we find that for any ε such that $|\varepsilon|<\frac{1}{Q_T^2}$, the right-hand sides of (20) converge uniformly over Ω_T. Therefore the right-hand sides of (8) are differentiable term by term with respect to x and t.

Q. E. D.

§ 3. The proof of Theorem

Here, we will summarize the contents mentioned in § 1 in the form of a theorem:

Theorem. *For any* $T>0$, *if we choose* ε *arbitrarily as* $|\varepsilon|<\frac{1}{3Q_T^2}$, *then the solutions of the Cauchy problem* (4)—(2) *can be expressed in the form* (8) *and the right-hand sides of* (8) *converge uniformly over* Ω_T. *Here,* Ω_T *and* Q_T *are defind in* (6) *and* (7) *respectively.*

Proof. By the initial data (10) and the fact that $u_0(x, t)$ and $v_0(x, t)$ are solutions of the Cauchy problem (5)-(2), we can easily see that the solutions $u(x, t)$ and $v(x, t)$ of the Cauchy problem (4)-(2) expreesed as (8) satisfy the initial data (2). And by Lemma 3, Lemma 6 and (9), we can see that $u(x, t)$ and $v(x, t)$ in (8) satisfy the equation (4). Consequently, we come to the conclusion that $u(x, t)$ and $v(x, t)$ in (8) are solutions of the Cauchy problem (4)-(2). Q. E. D.

Remark: In the theorem mentioned above, first we give $T>0$ arbitrarily and then according to T we choose $\varepsilon>0$ arbitrarily as $|\varepsilon|<\frac{1}{3Q_T^2}$. Even if, conversely, we first give $\varepsilon>0$ arbitrarily and then according to ε we choose $T>0$ as $|\varepsilon|<\frac{1}{3Q_T^2}$, the theorem mentioned above holds as well. The latter can be proved in the same manner as the former.

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