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On the conditions for the hyperbolicity of systems with double characteristic roots, I.[1]

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Contents

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§ 0. Introduction, notation and definitions

0.1° Introduction. In the case of the scalar operators of higher order, we know the satisfactory results on the necessary and sufficient condition for $\mathscr{E}$ well-posedness of the Cauchy problem, if the characteristic roots of those principal part are real and of constant multiplicity. (See M. Yamaguti [24], S. Mizohata-Y. Ohya [17], [18], H. Flaschka-G. Strang [5], and J. Chazarin [1], [2].)

On the other hand, in the case of the first order systems, we know the similar results under some assumptions with respect to the structure of the eigen-spaces of the principal symbols. Especially, when the multiplicities of the characteristic roots are at most double and constant, we have the necessary and sufficient condition for $\mathscr{E}$ well-posedness assuming that the dimensions of the null spaces of the principal symbols are constant. (See V. M. Petkov [19], [20] and H. Yamahara [25].) However, when the dimensions are not constant, the situation is fairly different from the previous case, even if we suppose that the multiplicities of the characteristic roots are constant. In this article, we shall try to make clear the complexity of the problem for the case of systems.

Let us consider the following Cauchy problem in an open set Ω in $\boldsymbol{R}^{n+1}$;

$$\begin{cases} (1) \quad Pu \equiv P_p u + Bu \equiv D_t u - \sum_{i=1}^{n} A^i D_{x_i} u + Bu = f, \\ (2) \quad u(t_0, x) = u_0(x), \end{cases}$$

where $A^i(t, x)$ and $B(t, x)$ are C^∞-matrices of order N and $A^i(t, x)$ is real $(1 \leqq i \leqq n)$. Throughout this paper, we are going to consider our problem under the following assumption;

Assumption 1. The characteristic roots $\tau = \lambda_j(t, x; \xi)$ of $\det P_p(t, x; \tau, \xi) = 0$ are real, at most double and of constant multiplicity.

We wished to give the necessary and sufficient condition for $\mathscr{E}$ well-posedness under the above assumption, but, up to now, we have not yet succeeded it. However, we can easily see the necessity of the following condition (L).

$$(L) \qquad {}^{co}P_p P_s {}^{co}P_p + (1/2i) {}^{co}P_p \{P_p, {}^{co}P_p\}|_{\tau=\lambda_j} = 0 \qquad \text{in} \quad \Omega \times \boldsymbol{R}^n \backslash \{O\},$$

for the double characteristic roots, where ${}^{co}P_p$ is the cofactor matrix of P_p and $P_s = B - (1/2) \sum_{i=0}^{n} P_{p(i)}^{(i)}$ is the subprincipal symbol.

In this article, we shall consider the consequence of the condition (L), the existence of "stably non-hyperbolic" operators with real characteristic roots and the additional conditions to (L) for $\mathscr{E}$ well-posedness.

Here, we shall treat the case when the dimension n of x-space is 1. In the section 1, we shall show that the condition (L) derives the smoothness of the eigen-vector along the characteristic curves. Moreover, we shall investigate when Y. Demay's sufficient condition can be satisfied. (See Y. Demay [3], [4].) In the section 2, as an application of the results in the section 1, we establish a theorem on the necessary condition for the weak hyperbolicity and obtain some examples of "stably non-hyperbolic" operators with real characteristic roots. On the other hand, in the section 3, we shall present some examples of the operators which satisfy the condition (L) but for which the Cauchy problems are not $\mathscr{E}$ well-posed, and we shall consider the additional conditions to (L) for the $\mathscr{E}$ well-posedness. There, we shall restrict ourselves to the case where the eigen-vectors of the principal symbol are piecewise smooth.

In the forthcoming paper [26],[2] in the sections 4 and 5, we shall generalize the results in the sections 1 and 3 to the case of $n\geqq 2$. However, the results in these sections will be a little more rough because there exist some difficulties proper to the higher dimension case. Nevertheless, if the size N of the system is two, we can avoid such difficulties. We shall announce the results in the case of $N=2$ in the appendix 2. The sections 4 and 5 correspond to 1 and 3, respectively.

In the appendix 1, we shall show the differences between the local $\mathscr{E}$ well-posedness "in the future" and that "in the past", between the local $\mathscr{E}$ well-posedness and the $\mathscr{E}$ well-posedness and between the $\mathscr{E}$ well-posedness and $\gamma^{(\infty)}$ well-posedness.

H. Uryu obtained a similar result as Y. Demay's without assuming that the multiplicities of the characteristic roots are constant. (See H. Uryu [23].) On the other hand, S. Tarama established a similar result as H. Yamahara's not assuming that the multiplicities of characteristic roots are constant but assuming that the dimensions of the eigen-spaces are always one. (See S. Tarama [21].) S. Tarama pointed out that the assumption on the dimensions of the eigen-spaces is rather important than that on the multiplicities of the characteristics. Both of them are concerned with the sufficiency. Moreover, many authors considered the problem for the cases when the multiplicities of the characteristic roots are changeable.

0.2° Notation.[3] As usual, we shall use the following; $\boldsymbol{N}$ is the set of the natural numbers. We denote the number of the elements of a set A by $\# A$. $K\Subset K'$ means that K is compact and $K\subset \mathring{K}'$. Here, $\mathring{K}$ and $\bar{K}$ is the open kernel and the closure of K. Ω is an open set in $\boldsymbol{R}_t^1\times\boldsymbol{R}_x^n$. We set $\Omega_{t_0}^+=\Omega\cap\{t\geqq t_0\}$, $\Omega_{t_0}^-=\Omega\cap\{t\leqq t_0\}$ and $\Omega_{t_0}=\Omega\cap\{t=t_0\}$. $T^*(\Omega)=\Omega\times\boldsymbol{R}^n$ is the cotangent space of Ω, regarding t as a parameter. $T^*(\Omega)\backslash\{O\}$ means $\Omega\times(\boldsymbol{R}^n\backslash\{O\})$. $\tilde{x}=(t, x)=(x_0, x)\in\Omega$, $\tilde{\xi}=(\tau, \xi)=(\xi_0, \xi)\in\boldsymbol{R}\times\boldsymbol{R}^n\subset\boldsymbol{C}\times\boldsymbol{R}^n$, $|\xi|=(\sum_{i=1}^n \xi_i^2)^{1/2}$.

$$\partial_t=\frac{\partial}{\partial t},\quad \partial_{x_i}=\frac{\partial}{\partial x_i},\quad D_t=-\sqrt{-1}\partial_t,\quad D_{x_i}=-\sqrt{-1}\partial_{x_i}\qquad (1\leqq i\leqq n).$$

2) We shall use the straight numbers in order to name the sections in this paper and the forthcoming one [26]. From now on, we only call the number of the section in order to indicate the section in [26].

3) Since we shall use the notes and definitions given in this paper in [26], we give here the general notation and definitions with respect to the dimension n of x-space.

$\sigma(P)=P(x,\xi)$ means the symbol of the (pseudo-)differential operator $P(x, D_x)$, that is, $P(x, D_x)u=\int e^{i(x-y)\cdot\xi} P(x,\xi)u(y)dy\bar{d}\xi$, $\bar{d}\xi=(2\pi)^{-n}d\xi$. $P\cdot Q$ is the product of $P(x, D_x)$ and $Q(x, D_x)$ as the operators and sometimes we express $P_1\cdots P_j$ by $\prod_{i=1}^{j} P_i$. $\sigma_k(P)$ means the symbol of the homogeneous part of order k of $P(x, D_x)$. Λ is the pseudo-differential operator with symbol $|\xi|$ when $|\xi|\geqq 1$. $U(x, y, D_y)=u_\phi(x, y, D_y)$ means the Fourier integral operator with the phase function $\phi(x,\xi)$ and the amplitude function $u(x,\xi)$, that is, $U(x, y, D_y)f=\int u(x,\xi)e^{i\{\phi(x,\xi)-y\cdot\xi\}}f(y)dy\bar{d}\xi$. $a^{(\alpha)}_{(\beta)}(\tilde{x},\tilde{\xi})=\partial^\alpha_{\tilde{\xi}}D^\beta_{\tilde{x}}a(\tilde{x},\tilde{\xi})$, $a^{(i)}_{(k)}(\tilde{x},\tilde{\xi})=\partial_{\xi_i}D_{x_k}a(\tilde{x},\tilde{\xi})$, $(0\leqq i, k\leqq n)$. $\{\ ,\ \}$ is the Poisson bracket, that is, $\{a, b\}=\sum_{i=0}^{n}\left(\frac{\partial a}{\partial\xi_i}\frac{\partial b}{\partial x_i}-\frac{\partial a}{\partial x_i}\frac{\partial b}{\partial\xi_i}\right)$. $\nabla u=\left(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right)$. $|u|_{q,K}=\sum_{|\alpha|\leqq q}\max_{x\in K}|D^\alpha u|$. $\|u\|_{q,\Omega}$ is the standard norm in $H^q(\Omega)$, Sobolev space of order q.

Let P_p be the principal part of $P=ID_t-A_1(t, x; D_x)+B(t, x)$, that is, $P_p=ID_t-A_1(t, x; D_x)$, where $A_1(t, x; D_x)=\sum_{i=1}^{n}A^i(t, x)D_{x_i}$. Let P_s be the subprincipal symbol of P, that is, $P_s=B(t, x)-\frac{1}{2i}\sum_{i=0}^{n}P^{(i)}_{p(i)}(x_0, x; \xi_0, \xi)$. Here, I is the identity matrix of order N, $A^i(t, x)$ and $B(t, x)$ are C^∞ matrices of order N, and $A^i(t, x)$ are real $(1\leqq i\leqq n)$. Let $\lambda_j(t, x; \xi)$ be a characteristic root of $\det P_p(t, x; \tau, \xi)=0$ $(1\leqq j\leqq s)$. $\lambda_{\max}(t, x)=\max_{\xi,j}|\lambda_j(t, x; \xi)|$, where ξ and j run over the unit sphere in $\boldsymbol{R}^n$ and $1\leqq j\leqq s$, respectively. $\bar{\lambda}_{\max}=\sup_{(t,x)}\lambda_{\max}(t, x)$, where (t, x) runs over Ω. Let $R_j(t, x; \xi)$ be $\operatorname{rank}P_p(t, x; \lambda_j(t, x; \xi), \xi)$, $(\xi\neq O)$, and let us set $L_j(t, x, \xi)={}^{co}P_pP_s{}^{co}P_p+\frac{1}{2i}{}^{co}P_p\{P_p, {}^{co}P_p\}|_{\tau=\lambda_j(t,x;\xi)}$, the so-called Levi matrix, where ${}^{co}P_p={}^t(\Delta_{ij})$, Δ_{ij} being the (i, j)-cofactor of $P_p(t, x; \tau, \xi)$.

Finally, we set the Gevrey classes. Let $\gamma^{(\kappa)}_h(\Omega)$ be the set of C^∞-functions defined in Ω whose derivatives satisfy the following:

$^\forall K$: a compact set in Ω, $^\exists C$: a positive constant such that $\sup_{x\in K}|f^{(\alpha)}(x)|\leqq Ch^{|\alpha|}(|\alpha|!)^\kappa$ for arbitrary α, where $f^{(\alpha)}(x)=\left(\frac{\partial}{\partial x}\right)^\alpha f(x)$.

We set $\gamma^{(\kappa)}(\Omega)=\bigcup_{h>0}\gamma^{(\kappa)}_h(\Omega)$, $\gamma^{\langle\kappa\rangle}(\Omega)=\bigcap_{h>0}\gamma^{(\kappa)}_h(\Omega)$ and $\gamma^{(\infty)}(\Omega)=\bigcup_{\kappa\geqq 1}\gamma^{(\kappa)}(\Omega)$ $(=\bigcup_{\kappa\geqq 1}\gamma^{\langle\kappa\rangle}(\Omega))$. $\gamma^{(\kappa)}(\Omega)$ is called the inductive Gevrey class of order κ and $\gamma^{\langle\kappa\rangle}(\Omega)$ does the projective Gevrey class of order κ, $(1\leqq\kappa\leqq\infty)$. (See V. Ya. Ivriĭ [8].) We remark that $\gamma^{(\infty)}(\Omega)\subsetneqq\mathscr{E}(\Omega)$.

0.3° Definitions

Let us consider the Cauchy problem (1)–(2):

$$\begin{cases} (1)\quad Pu\equiv D_tu-A_1(t, x; D_x)u+B(t, x)u=f(t, x), \\ (2)\quad u(t_0, x)=u_0(x), \end{cases}$$

where $A_1(t, x; D_x)=\sum_{i=1}^{n}A^i(t, x)D_{x_i}$, $A^i(t, x)$ and $B(t, x)$ are C^∞-matrices of order N, and $A^i(t, x)$ are real $(1\leqq i\leqq n)$.

In this paper, to make our view-point clear we adapt the following definitions which are slightly different from usual ones. Therefore, we define "our words", here.

Definition 1. ($\mathscr{E}$ well-posedness.) The Cauchy problem (1)–(2) is $\mathscr{E}$ well-posed[4] in Ω, if, for each t_0 such that $\Omega_{t_0} \neq \phi$, there exists the unique solution in $\mathscr{E}(\Omega_{t_0}^+)$ for each $u_0(x) \in \mathscr{E}(\Omega_{t_0})$ and each $f(t, x) \in \mathscr{E}(\Omega_{t_0}^+)$.

Remark. In this definition, Ω will be restricted to some special shapes for the uniqueness. However, we shall omit the proposition on the shape of Ω. (See the theorems in the sections 3 and 5.)

Definition 2. (Hyperbolicity.) P is hyperbolic in Ω, if the Cauchy problem (1)–(2) is $\mathscr{E}$ well-posed in Ω.

If P is hyperbolic in Ω, we can see, by virtue of Banach's closed graph theorem, that "the loss of regularity" on each compact set is finite.

Definition 3. (Loss of regularity.) Let P be hyperbolic in Ω and let K be an arbitrary compact set in Ω. The loss of regularity on K from t_0 is l, if the restriction of the solution on $K_{t_0}^+$ belongs at most to $\bigcap_{j=1}^{p-l-1} \mathscr{E}_t^j(H^{p-l-j}(K_t))$ for arbitrary $u_0(x)$ in $H_{loc}^p(\Omega_{t_0})$ and $f(t, x)=0$, $(p \gg l+1)$. Moreover, the loss of regularity in Ω is the supremum of the loss of regularity on K from t_0 for arbitrary compact set K and arbitrary t_0.

(See also V. Ya. Ivriĭ-P. V. Petkov [10] and V. Ya. Ivriĭ [9].)

We also use weaker notion than the hyperbolicity.

Definition 4. (Local $\mathscr{E}$ well-posedness.) The Cauchy problem (1)–(2) is locally $\mathscr{E}$ well-posed at $(t_0, x_0) \in \Omega$, if there exists a neighbourhood ω of (t_0, x_0) such that the Cauchy problem (1)–(2) has the unique solution in $\mathscr{E}(\omega_{t_0}^+)$ for each $u_0(x) \in \mathscr{E}(\overline{\Omega_{t_0}})$ and each $f(t, x) \in \mathscr{E}(\overline{\Omega_{t_0}^+})$.

Definition 5. (Local hyperbolicity.) P is locally hyperbolic at $(t_0, x_0) \in \Omega$ [in Ω], if the Cauchy problem (1)–(2) is locally $\mathscr{E}$ well-posed at (t_0, x_0) [at every point in Ω, respectively].

Remark. We shall present an operator which is locally hyperbolic in Ω but is not hyperbolic in Ω in the appendix 1.

Moreover, we introduce the notions on the stability of the local hyperbolicity with respect to the lower order term.

Definition 6. (Strong hyperbolicity.) P_p is strongly hyperbolic at (t_0, x_0) [in Ω], if P_p+B is locally hyperbolic at (t_0, x_0) [in Ω, respectively] for every lower order term $B(t, x)$.

Definition 7. (Weak hyperbolicity.) P_p is weakly hyperbolic at (t_0, x_0) [in Ω],

4) We should say "uniformly $\mathscr{E}$ well-posed", but we omit the word "uniformly".

if there exist two lower order terms $B(t, x)$ and $B'(t, x)$ such that P_p+B is locally hyperbolic at (t_0, x_0) [in Ω, respectively] but P_p+B' is not locally hyperbolic at (t_0, x_0) [in Ω, respectively].

For convenience sake, we introduce the following notion.

Definition 8. (Stable non-hyperbolicity.) P_p is stably non-hyperbolic at (t_0, x_0) [in Ω], if there exists no lower order term $B(t, x)$ such that P_p+B be locally hyperbolic at (t_0, x_0) [in Ω, respectively].

Finally, we introduce two more notions on hypersurface and domain.

Definition 9. (Space-like hypersurface.) A hypersurface T defined by $t=\psi(x) \in C^\infty$ is called space-like, if $\lambda_{\max}|\nabla\psi(x)|<1$ on T.

Definition 10. (Lense-shaped domain.) A domain ω is lense-shaped if we can find a constant t' such that ω is included in $\Omega_{t'}^+$ and $\{(t, x);\ |x-x^1|\leqq\bar{\lambda}_{\max}\cdot|t-t^1|,\ t'<t\leqq t^1\}$ is included in ω for arbitrary $(t^1, x^1)\in\omega$, where $\bar{\lambda}_{\max}=\sup_{(t,x)\in\omega}\lambda_{\max}(t, x)$.

§ 1. Consideration of the condition (L)

1.1° Necessity of the condition (L)

As well known, if P is locally hyperbolic in Ω, the characteristic roots of P_p must be real in Ω. (P. D. Lax [14] and S. Mizohata [16].)

In this paper, we consider local hyperbolicity of $P(t, x; D_t, D_x)$ under the following assumption.

Assumption 1. Each characteristic root $\tau=\lambda_j(t, x; \xi)$ of $\det P_p(t, x; \tau, \xi)=0$ is real, of constant multiplicity and at most double.

Let λ_j be double for $1\leqq j\leqq r$ and be simple for $r+1\leqq j\leqq s\ (=N-r)$.

Since the necessity of the Levi condition in V. M. Petkov [19], [20] and in H. Yamahara [25] is proved micro-locally, the proof remains true only under the assumption 1.

Their conditions are equivalent to the following (L) at the points where $R_j \equiv \operatorname{rank} P_p(t, x; \lambda_j(t, x; \xi), \xi)=N-1$.[5)]

$$\text{(L)}\qquad L_j(t, x, \xi)\equiv{}^{co}P_pP_s{}^{co}P_p+\frac{1}{2i}{}^{co}P_p\{P_p, {}^{co}P_p\}|_{\tau=\lambda_j(t,x;\xi)}=0,$$

$$\text{on}\quad \Omega\times\boldsymbol{R}^n\backslash\{O\},\quad (1\leqq j\leqq r).$$

The condition (L) makes sense only at the points where $R_j(t, x, \xi)=N-1$, because at every points where $R_j(t, x, \xi)=N-2$, $L_j(t, x, \xi)$ vanishes automatically $(1\leqq j\leqq r)$.

Proposition 1.1. (Necessity of the condition "$L_j(t_0)=0$".)

5) S. Tarama pointed out the equivalence of the condition (L) to H. Yamahara's one in [25] at the points where $R_j(t, x, \xi)=N-1$, $(1\leqq j\leqq r)$. (Unpublished.)

Under the assumption 1, *if* P *is locally hyperbolic at* (t_o, x_o),

$$L_j(t_o, x_o, \xi)=0 \quad \textit{for arbitrary} \quad \xi\in\boldsymbol{R}^n \quad \textit{and} \quad 1\leqq j\leqq r.$$

Corollary 1.2. (Necessity of the condition (L).)
Under the assumption 1, *if* P *is locally hyperbolic in* Ω, *the condition* (L) *must be satisfied.*

Under the assumption 1, if $R_j\equiv$constant, (L) is also the sufficient condition for the hyperbolicity of P in Ω. (See H. O. Kreiss [12], P. V. Petkov [19], [20] and H. Yamahara [25].)

However, if R_j is not constant, we need more precise consideration of the operators which satisfy the condition (L). In the cases of $n=1$ and of $n\geqq 2$, the situations are essentially different. From now on, in the sections 1, 2 and 3, we consider the case of $n=1$, that is,

$$P(t, x; D_t, D_x)=D_t-A^1(t, x)D_x+B(t, x), \quad (x=x_1).$$

Here, $R_j(t, x, \xi)$ is not independent of $\xi(\neq 0)$. We write $R_j(t, x)$ instead of $R_j(t, x, \xi)$.

1.2° Smoothness of the eigen-vectors

Let G^j be an arbitrary connected component of $\{(t, x)\in\Omega;\ R_j(t, x)=N-1\}$, $(1\leqq j\leqq r)$. G^j is open. We can take the real unit eigen-vector $\vec{e}_j(t, x)$ of $A^1(t, x)$ belonging to λ_j in $C^\infty(G^j)$, but, in general, $\vec{e}_j(t, x)$ cannot be extended in $C^\infty(\overline{G^j})$. However, for the operators which satisfy the condition (L), we have the following.

Theorem 1.3. (Smoothness of $\vec{e}_j(t, x)$ along the characteristic curves.)
In addition to the assumption 1, *suppose that* $P(t, x; D_t, D_x)$ *satisfies the condition* (L).

(i) *If* $\pi(s)$ *is a characteristic curve of* $\lambda_j(t, x)$ *such that* $\pi(s)\in G^j$ *when* $0\leqq s<s'$ *and* $\pi(s')\in\partial G^j$, *(that is,* $\pi(s')$ *is the first point along* $\pi(s)$ *where* $R_j=N-2$), *then,* $\vec{e}_j(t, x)$ *can be extended as a* C^∞*-real unit eigen-vector in* $\omega\cap\overline{G^j}$, *where* ω *is a neighbourhood of* $\pi(s')$ *in* Ω. *(See the figure* 1.)

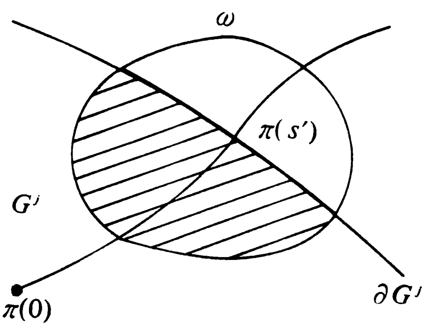

Figure 1.

(ii) *When* $R_j = N-1$ *in* $\omega \backslash \partial G^j$ *(if necessary, shrinking* ω*), we can take* $\check{e}_j^-(t, x)$ *in* $C^\infty(\omega \cap \overline{G^j})$ *and* $\check{e}_j^+(t, x)$ *in* $C^\infty(\omega \cap \overline{G^{j^c}})$ *by* (i). *if* $\lim_{s \to s'-0} \check{e}_j^-(\pi(s)) = \lim_{s \to s'+0} \check{e}_j^+(\pi(s))$,

$$\check{e}_j(s) = \begin{cases} \check{e}_j^-(\pi(s)), & s \leqq s', \\ \check{e}_j^+(\pi(s)), & s \geqq s', \end{cases}$$

is infinitely differentiable at $s = s'$.

Moreover, if $\check{e}_j^-(t, x)$ *and* $\check{e}_j^+(t, x)$ *coincide on* $\omega \cap \partial G^j$,

$$\check{e}^j(t, x) = \begin{cases} \check{e}_j^-(t, x), & (t, x) \in \omega \cap \overline{G^j}, \\ \check{e}_j^+(t, x), & (t, x) \in \omega \cap \overline{G^{j^c}}, \end{cases}$$

is, in fact, infinitely differentiable on $\omega \cap \partial G^j$, *that is,* $\check{e}_j(t, x)$ *belongs to* $C^\infty(\omega)$.

Proof. At first, we reduce $P(t, x; D_t, D_x)$ to a family of the systems of size 2×2 and the scalar operators. We use $\sim$ in order to express asymptotic equivalences.

Lemma 1.4. (Emblocking operator.)

For arbitrary (t, x) *in* Ω, *there exists a neighbourhood* θ *of* (t, x) *such that* $P(t, x; D_t, D_x)$ *has an "emblocking operator"* $\mathscr{B}(t, x; D_x)$ *which satisfies, in* θ, *the following;*

(i) (1.1) $$\mathscr{B}(t, x; \xi) \sim \sum_{k=0}^{\infty} \mathscr{B}_{-k}(t, x)\xi^{-k},$$

where the right-hand side is a formal series.

(ii) (1.2) $P(t, x; D_t, D_x) \cdot \mathscr{B}(t, x; D_x) \equiv \mathscr{B}(t, x; D_x) \cdot \mathscr{C}(t, x; D_t, D_x)$, mod. $S^{-\infty}$.

(1.3) $$\mathscr{C}(t, x; D_t, D_x) = \begin{pmatrix} \boxed{\mathscr{C}^1} & & & & & 0 \\ & \boxed{\mathscr{C}^2} & & & & \\ & & \ddots & & & \\ & & & \boxed{\mathscr{C}^r} & & \\ & & & & \mathscr{C}^{r+1} & \\ 0 & & & & & \ddots \quad \mathscr{C}^s \end{pmatrix},$$

where $\mathscr{C}^j(t, x; \tau, \xi) \sim \mathscr{C}_1^j(t, x; \tau, \xi) + \sum_{k=0}^{\infty} \mathscr{C}_{-k}^j(t, x)\xi^{-k}$, $\mathscr{C}^j$ *is of size* 2×2 *when* $1 \leqq j \leqq r$, *and is scalar when* $r+1 \leqq j \leqq s$. $\mathscr{C}_1^j(t, x; \tau, \xi) = \tau I_2 - \xi \mathscr{C}_1^{j'}(t, x)$ *and* $\mathscr{C}_1^{j'}(t, x)$ *has the double eigen-value* $\lambda_j(t, x)$ *when* $1 \leqq j \leqq r$, *and* $\mathscr{C}_1^j(t, x; \tau, \xi) = \tau - \lambda_j(t, x)\xi$ *when* $r+1 \leqq j \leqq s$. (I_2 *is the unit matrix of order* 2.)

Here, as $\mathscr{B}(t, x; D_x)$, *we take a pseudo-differential operator which has the asymptotic symbol given by the right-hand side of* (1.1).

(iii) $\mathscr{C}_1^j(t, x, \tau, \xi) + \mathscr{C}_0^j(t, x)$ *satisfies the condition* (L) *in* θ, *if and only if*

$P(t, x; \tau, \xi)$ satisfies the condition $L_j(t, x)(\equiv L_j(t, x, \xi)/\xi^{2N-2})=0$ in θ.

This lemma shall be proved for general dimension n as Lemma 4.6 in the section 4.

From now on, we write I instead of I_2.

Let us set $\mathscr{C}_1^j(t, x; \tau, \xi)=(\tau-\lambda_j(t, x)\xi)I-\xi\tilde{A}_j(t, x)$, that is, $\tilde{A}_j(t, x)=\mathscr{C}_1^{j\prime}(t, x)-\lambda_j(t, x)I$, and $\tilde{A}_j(t, x)=\begin{pmatrix} a_j(t, x) & b_j(t, x) \\ c_j(t, x) & d_j(t, x)\end{pmatrix}$, $(1\leq j\leq r)$. Since $\tilde{A}_j(t, x)$ has the double eigen-value 0, the followings are satisfied;

$$(1.4)\quad \begin{cases} a_j(t, x)+d_j(t, x)\equiv 0, \quad (a_j(t, x))^2+b_j(t, x)c_j(t, x)\equiv 0, \\ (\tilde{A}_j(t, x))^2\equiv 0 \quad \text{and} \quad {}^{co}\mathscr{C}_1^j(t, x; \tau, \xi)\equiv(\tau-\lambda_j(t, x)\xi)I+\tilde{A}_j(t, x)\xi. \end{cases}$$

The eigen-vector $\hat{e}_j(t, x)$ of $A^1(t, x)$ belonging to $\lambda_j(t, x)$ is given by $\mathscr{B}_0(t, x)\cdot \hat{e}'_j(t, x)$ where $\hat{e}'_j(t, x)$ has the form ${}^t(0,\dots, 0, \overset{2j-1}{e_1}, \overset{2j}{e_2}, 0,\dots, 0)$ and $\hat{e}(t, x)={}^t(e_1, e_2)$ is the eigen-vector of $\tilde{A}_j(t, x)$. At the points where $b_j(t, x)\neq 0$, $\hat{e}(t, x)$ is given by ${}^t(b_j, -a_j)$, that is, ${}^t(1, -a_j/b_j)$ and at the points where $c_j(t, x)\neq 0$, $\hat{e}(t, x)$ is given by ${}^t(a_j, c_j)$, that is, ${}^t(a_j/c_j, 1)$. Since $|b_j(t, x)|+|c_j(t, x)|\neq 0$ in G^j, at least one of ${}^t(1, -a_j/b_j)$ and ${}^t(a_j/c_j, 1)$ has the sense in G^j. We shall consider the behaviors of $-a_j(t, x)/b_j(t, x)$ and $a_j(t, x)/c_j(t, x)$ near ∂G^j.

From now on, we omit the suffix j and write $A(t, x)$ and $B(t, x)$ instead of $\tilde{A}_j(t, x)$ and $\mathscr{C}_0^j(t, x)$, respectively. Using $x_0=t$, $x_1=x$, $\xi_0=\tau$ and $\xi_1=\xi$, we calculate the left-hand side of the condition (L) with respect to $C_1(t, x; \tau, \xi)+C_0(t, x)$.

(1.5)

$$\{(\xi_0-\lambda\xi_1)I+\xi_1 A\}\left[B-\frac{1}{2}\sum_{i=0}^{1}\{(\xi_0-\lambda\xi_1)_{(i)}^{(i)}I-(\xi_1 A)_{(i)}^{(i)}\}\right]\{(\xi_0-\lambda\xi_1)I+\xi_1 A\}$$

$$+\frac{1}{2}\{(\xi_0-\lambda\xi_1)I+\xi_1 A\}\Big[\sum_{i=0}^{1}(\xi_0-\lambda\xi_1)^{(i)}I-(\xi_1 A)^{(i)}\}\{(\xi_0-\lambda\xi_1)_{(i)}I+(\xi_1 A)_{(i)}\}$$

$$-\sum_{i=0}^{1}\{(\xi_0-\lambda\xi_1)_{(i)}I-(\xi_1 A)_{(i)}\}\{(\xi_0-\lambda\xi_1)^{(i)}I+(\xi_1 A)^{(i)}\}\Big]\Big|_{\xi_0=\lambda\xi_1}$$

$$=(AA_{(0)}-\lambda AA_{(1)}+ABA)\xi_1^2.$$

Let us set $a_t=\frac{\partial}{\partial t}a(t, x)$, $a_x=\frac{\partial}{\partial x}a(t, x)$ and so on, and let $B(t, x)$ be $-\sqrt{-1}\begin{pmatrix}\alpha_o(t, x) & \beta_o(t, x)\\ \gamma_o(t, x) & \delta_o(t, x)\end{pmatrix}$.

By virtue of (1.4), (1.5)$=0$ becomes

$$(1.6)\quad (-a/b)_t-\lambda(-a/b)_x-\beta_o(-a/b)^2-(\alpha_o-\delta_o)(-a/b)+\gamma_o=0, \qquad \text{when } b\neq 0,$$

$$(1.7)\quad (a/c)_t-\lambda(a/c)_x-\gamma_o(a/c)^2-(\delta_o-\alpha_o)(a/c)+\beta_o=0, \qquad \text{when } c\neq 0.$$

Let us put $g=-a/b$, $h=a/c$, $\alpha=\mathrm{Re}\,\alpha_o$, $\beta=\mathrm{Re}\,\beta_o$, $\gamma=\mathrm{Re}\,\gamma_o$ and $\delta=\mathrm{Re}\,\delta_o$.
Since $-a/b$ and a/c are always real, we consider only real solutions of (1.6) and (1.7). The real parts of (1.6) and (1.7) become

(1.6′) $$g_t-\lambda g_x-\beta g^2-(\alpha-\delta)g+\gamma=0,$$

(1.7′) $$h_t-\lambda h_x-\gamma h^2-(\delta-\alpha)h+\beta=0.$$

(1.6′) and (1.7′) have the real solutions $g=-a(t, x)/b(t, x)$ and $h=a(t, x)/c(t, x)$ when $b(t, x)\neq 0$ and when $c(t, x)\neq 0$, respectively. Regarding g and h as the unknown functions, let us seek for the solutions of (1.6′) and (1.7′) which coincide with $-a(t, x)/b(t, x)$ when $b(t, x)\neq 0$ and with $a(t, x)/c(t, x)$ when $c(t, x)\neq 0$, respectively, in $G\cap\theta$. (1.6′) and (1.7′) are the ordinary differential equations along the characteristic curves of $\partial_t-\lambda(t, x)\partial_x$.

(1.6″) $$\dot g=\beta g^2+(\alpha-\delta)g-\gamma,$$

(1.7″) $$\dot h=\gamma h^2+(\delta-\alpha)h-\beta,$$

where $\dot a$ is $\frac{da}{ds}\left(\equiv\frac{da}{dt}\right)$, the derivative along the characteristic curve. Setting $M=\sup_{(t,x)\in\theta}\{|\alpha(t, x)|, |\beta(t, x)|, |\gamma(t, x)|, |\delta(t, x)|\}$, both (1.6″) and (1.7″) are majorized by the following;

(1.8) $$\dot H\leqq 2M(1+H^2).$$

Let us take $\theta_1=\text{Arctan}\,2$, $\theta_2=\text{Arctan}\,3$, $\varepsilon_0=(\theta_2-\theta_1)/4M$, $\bar t=s'-\varepsilon_0$ and $\pi(s'-\varepsilon_0)=(\bar t, \bar x)$. Since $\pi(s'-\varepsilon_0)$ is contained in G, one of $|-a(\bar t, \bar x)/b(\bar t, \bar x)|$ and $|a(\bar t, \bar x)/c(\bar t, \bar x)|$ has the sense and is smaller than or equal to 1. For example, we treat the case when $|-a(\bar t, \bar x)/b(\bar t, \bar x)|\leqq 1$. We can take a neighbourhood ζ of $\bar x$, such that $|-a(\bar t, x)/b(\bar t, x)|<2$ for $x\in\zeta$. Let ω in Theorem 1.3 be the intersection of $\{(t, x);\ |t-\bar t|<(\theta_2-\theta_1)/2M\}$ and the set covered by the family of the characteristic curves starting from ζ. The solution of (1.6′) which satisfies the relation $g(\bar t, x)=-a(\bar t, x)/b(\bar t, x)$ on $\{\bar t\}\times\zeta$ exists at least in ω and it is smaller than 3. Shrinking ζ, if necessary, ω contains the piece of ∂G which is covered by the characteristic curves starting from ζ. $g(t, x)$ belongs to $C^\infty(\omega)$ and coincide with $-a(t, x)/b(t, x)$ in $G\cap\omega$ by the uniqueness of the solution of the Cauchy problem. $(\sqrt{1+g^2})^{-1\,t}(1, g(t, x))$

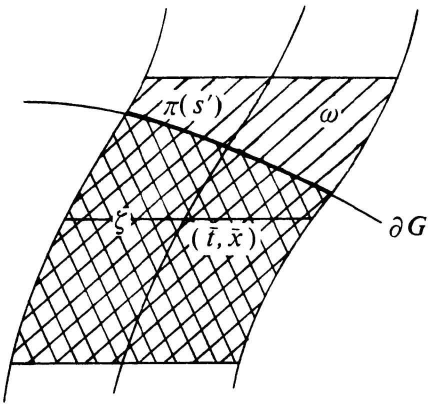

Figure 2.

gives the eigen-vector of $C'_1(t, x)$ in $G \cap \omega$. Thus the property (i) is proved.

From now on, we assume that $R(t, x)=N-1$ in $\omega \backslash \partial G$. By virtue of the property (i), we can take the unit eigen-vectors $\vec{e}^-(t, x)={}^t(e_1^-(t, x), e_2^-(t, x))$ and $\vec{e}^+(t, x)={}^t(e_1^+(t, x), e_2^+(t, x))$ in $C^\infty(\omega \cap \bar{G})$ and in $C^\infty(\omega \cap \overline{G^c})$, respectively (shrinking ω, if necessary). Suppose that $\vec{e}^-(\pi(s'))=\vec{e}^+(\pi(s'))$. By the above proof of (i), as $g \leq 3$ in $G \cap \omega$, we have the estimate $e_1^-(t, x) \geqq (\sqrt{10})^{-1}$ on $\partial G \cap \omega$. Then, there is a neighbourhood $\omega_0 (\subset \omega)$ of $\pi(s')$ such that $e_1^+(t, x) \neq 0$ in $\omega_0 \cap \overline{G^c}$ because of $e_1^+(\pi(s')) \geq 10^{-1/2}$. Since the dimension of the eigen-space of $C'_1(t, x)$ is one in $\omega \backslash \partial G$, $e_2^{\pm}(t, x)/e_1^{\pm}(t, x)$ coincide with $-a(t, x)/b(t, x)$ in $\omega_0 \cap G$ and in $\omega_0 \cap \mathring{G}^c$, respectively. Therefore,

$$g=\begin{cases} e_2^-(t, x)/e_1^-(t, x), & \text{in} \quad \omega_0 \cap \bar{G}, \\ e_2^+(t, x)/e_1^+(t, x), & \text{in} \quad \omega_0 \cap \overline{G^c}, \end{cases}$$

satisfies the equation (1.6″) in $\omega_0 \cap \bar{G}$ and in $\omega_0 \cap \overline{G^c}$ respectively and it is continuous at $\pi(s')$. This implies that g is the solution of (1.6″) on $\pi(s) \cap \omega_0$, and then, it is infinitely differentiable at $s=s'$.

Moreover, if $\vec{e}^-(t, x)=\vec{e}^+(t, x)$ on $\partial G \cap \omega_0$, g satisfies the equation (1.6′) except ∂G and it is continuous on ∂G. This implies that g is the solution of (1.6′) in ω_0, and then, it belongs to $C^\infty(\omega_0)$. On the other hand, if $|a(\bar{t}, \bar{x})/c(\bar{t}, \bar{x})| \leqq 1$, we have the same consequence by the similar way. Thus the property (ii) is also proved.

Q. E. D.

Remark 1. Of course, near the first point $\pi(s'')$ along $\pi(s)$ where $R_j=N-2$ ($s''<0$), we have the same results as (i) in Theorem 1.3 and this is already used in order to state the property (ii).

Remark 2. Y. Demay [3], [4] gave a sufficient condition for the hyperbolicity under the assumption 1;

(L′) Let $l_p=P_s{}^{co}P_p+\frac{1}{2i}\{P_p, {}^{co}P_p\}$ and $l'_p={}^{co}P_pP_s+\frac{1}{2i}\{{}^{co}P_p, P_p\}$. There are two symbols of the pseudo-differential operators S_j and S'_j of order $N-2$ such that $l_p \equiv P_pS_j$ and $l'_p \equiv S'_jP_p$ modulo $\{\tau-\lambda_j(t, x; \xi)\}$ near $\tau=\lambda_j(t, x; \xi)$, $(1 \leqq j \leqq r)$. (See also K. Kajitani [11].)

Obviously, the condition (L′) is stronger than (L) in some cases in algebraic sense.

The condition (L′) seems to be realized only for some special class of P_p. For example, if $n=1$, (L′) can be satisfied only when R_j is constant on each characteristic curve. In fact, by virtue of Theorem 1.3, we have a "Jordan's normalizer" $J(t, x; D_x)$ of $C^j(t, x; D_t, D_x)$ such that $JC^j=JD^j$ and $D^j=I(D_t-\lambda(t, x)D_x)+\begin{pmatrix} 0 & \varepsilon_j(t, x) \\ 0 & 0 \end{pmatrix} D_x+\cdots$ in $(\omega \cap \overline{G^j}) \times \boldsymbol{R} \backslash \{0\}$ (ω being a neighbourhood of $\pi(s)$). On the other hand, we can see the invariance of (L′) under the transformation NPN^{-1}, where $\sigma(N)(t, x; \xi)$ is an arbitrary regular matrix of degree zero. Then, if there exists S_j, $\varepsilon_j(t, x)$ must satisfy the equation of type $\frac{d}{ds}\varepsilon_j = a\varepsilon_j$ along the characteristic curve $\pi(s)$ starting from a point in G^j as far as $\pi(s)$ stays in $\overline{G^j}$. This implies that $\varepsilon_j|_{\pi(s)} \neq 0$ if and only if $\varepsilon_j|_{\pi(0)} \neq 0$, that is, $\pi(s)$ never crosses ∂G^j. Especially, if all of coefficients of P_p depend only on t and R_j is not constant for some j, the condition (L′) cannot be satisfied for any lower order term. (See Corollary 4.9 and the remark of Corollary 4.2.)

Moreover, (L′) is stronger than the necessary and sufficient condition even if each R_j is constant on every characteristic curve.

Example 1.

$$P_1 = ID_t - \begin{pmatrix} 0 & x(t^2+x^2) \\ 0 & 0 \end{pmatrix} D_x, \quad x \in \boldsymbol{R}^1.$$

Let B be $\begin{pmatrix} \alpha(t, x) & \beta(t, x) \\ \gamma(t, x) & \delta(t, x) \end{pmatrix}$. For P_1, (L) is equivalent to $\gamma = 0$ when $x \neq 0$. Then, γ must identically vanish because of the continuity of $\gamma(t, x)$. Conversely, $\gamma \equiv 0$ is sufficient for the (local) hyperbolicity of $P_1 + B$ in $\boldsymbol{R}^2$.

However, there is no $B(t, x)$ such that $P_1 + B$ satisfies the condition (L′).

On the other hand, (L′) covers a typical case when $R_j(t, x)$ changes. (See the remark 3.)

Remark 3. $\check{e}_j(t, x)$ is, in general, not smooth near ∂G^j which coincide a characteristic curve. It is neither guranteed by Theorem 1.3 nor necessary for the weak hyperbolicity.

Example 2. Let us take real functions $a(x)$, $b(x)$ and $c(x)$ which belong to $C^\infty(\boldsymbol{R})$ and satisfy the relation $a^2(x) + b(x)c(x) \equiv 0$ on $\boldsymbol{R}$.

$$P_2 = ID_t - \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix} D_x$$

has some $B(t, x)$ which satisfy (L′), (for example $B(t, x) \equiv 0$). Therefore, P_2 is weakly hyperbolic in $\boldsymbol{R}^2$ when $|b(x)| + |c(x)| \not\equiv 0$.

However, the eigen-vector $\check{e}(x)$, which is expressed by $\{a^2(x) + c^2(x)\}^{-1/2}\,{}^t(a(x), c(x))$, or $\{a^2(x) + b^2(x)\}^{-1/2}\,{}^t(b(x), -a(x))$ when $|b(x)| + |c(x)| \neq 0$, does not always belong to $C^\infty(\bar{G})$.

For example, let $a(x) = \varphi(x)\cos\frac{1}{x}\sin\frac{1}{x}$, $b(x) = -\varphi(x)\cos^2\frac{1}{x}$ and $c(x) = \varphi(x)\sin^2\frac{1}{x}$, where $\varphi(x)$ belongs to $C^\infty(\boldsymbol{R}^1)$, vanishes at the origin of infinite order and does not vanish in $\boldsymbol{R}\backslash\{0\}$. Then $\bar{G}$ is $G_+ = \{(t, x) \mid x \geqq 0\}$ or $G_- = \{(t, x) \mid x \leqq 0\}$, but $\check{e}(x) = {}^t\left(\cos\frac{1}{x}, \sin\frac{1}{x}\right)$ belongs neither to $C^0(G_+)$ nor to $C^0(G_-)$. Moreover, on the operator P_2, even if $R = 1$ in $\boldsymbol{R}^2\backslash\boldsymbol{R}_t \times \{0\}$ and $\check{e}(x)$ belongs to $C^\infty(\boldsymbol{R}^2\backslash\boldsymbol{R} \times \{0\}) \cap C^0(\boldsymbol{R}^2)$, in general, we cannot take $\check{e}$ in $C^\infty(\boldsymbol{R}^2)$. For example, let $a(x) = |x|\varphi(x)$, $b(x) = -\varphi(x)$ and $c(x) = x^2\varphi(x)$, where $\varphi(x)$ is the above one. Then

$\tilde{e}(x)=(1+x^2)^{-1/2\,t}(1,|x|)$ belongs to $C^\infty(\boldsymbol{R}^2\backslash\boldsymbol{R}\times\{0\})\cap C^0(\boldsymbol{R}^2)$ but does not belong to $C^1(\boldsymbol{R}^2)$.

It is interesting to compare the example 2 and the examples 6 and 7 in the section 2.

Remark 4. $\tilde{e}_j(t, x)$ is, in general, not smooth across ∂G^j even if ∂G^j does not coincide with any characteristic curve. It is neither guaranteed by Theorem 1.3 nor necessary for the weak hyperbolicity.

Example 3.

$$P_3=ID_t-\begin{pmatrix} 0 & \mu(t, x) \\ \nu(t, x) & 0 \end{pmatrix} D_x,$$

where $\mu(t, x)\begin{cases} >0, & \text{if } t>0, \\ =0, & \text{if } t\leqq 0, \end{cases}$

$\nu(t, x)\begin{cases} =0, & \text{if } t\geqq 0, \\ >0, & \text{if } t<0. \end{cases}$

$G^+=\{t>0\}$ and $G^-=\{t<0\}$. Let us take $\Omega=\boldsymbol{R}^2$. Since $\tilde{e}={}^t(1, 0)$ in G^+ and $\tilde{e}={}^t(0, 1)$ in G^-, we cannot take $\tilde{e}$ in $C^0(\boldsymbol{R}^2)$. However, for $t_0<0$, we can uniquely solve the Cauchy problem (1)–(2) step by step in $[t_0, 0]\times\boldsymbol{R}$ and $[0, \infty)\times\boldsymbol{R}$ if $P=P_3+B$ satisfies the condition (L), that is, P is hyperbolic in $\boldsymbol{R}^2$ under the condition (L). (Setting $B(t, x)=\begin{pmatrix}\alpha(t, x) & \beta(t, x)\\ \gamma(t, x) & \delta(t, x)\end{pmatrix}$, the condition (L) is equivalent to "$\beta(t, x)=0$ when $t\leqq 0$ and $\gamma(t, x)=0$ when $t\geqq 0$".) Here, the loss of regularity in $\boldsymbol{R}^2$ is 2. More precisely, for $t_0\geqq 0$, the loss from t_0 is 1, but, for $t_0<0$ and $t>0$, the loss from t_0 is 2. (Even if $t_0<0$, the loss from t_0 is 1 as far as $t\leqq 0$.)

1.3° Analyticity of the eigen-vectors

In spite of the remarks 3 and 4 of Theorem 1.3, in the case when $A^1(t, x)$ is real analytic, the eigen-vector $\tilde{e}_j(t, x)$ behaves more simply.

Let us set $\Omega^j_w=\{(t, x)\in\Omega\mid R_j(t, x)=N-1\}$ and $\Omega^j_s=\{(t, x)\in\Omega\mid R_j(t, x)=N-2\}$ $(=\Omega\backslash\Omega^j_w)$. Ω^j_w is open and Ω^j_s is closed in Ω.

Theorem 1.5. (Analyticity of the eigen-vectors.)

Suppose the real analyticity of $A^1(t, x)$ and the assumption 1. *Then, for each j, only one of the following arises,* $(1\leqq j\leqq r)$;

I) $\Omega^j_s=\Omega$,

II) $\overline{\Omega^j_w}=\Omega$, *that is,* $\Omega^j_s=\partial\Omega^j_w$.[6)]

Moreover, in the second case, if $P(t, x; D_t, D_x)$ satisfies the condition (L), *we can take the eigen-vector $\tilde{e}_j(t, x)$ in the real analytic class in Ω.*

Remark. Theorem 1.5 does not remain true in the case of $n\geqq 2$.

6) $\overline{\Omega^j_w}$ is the closure of Ω^j_w in Ω and $\partial\Omega^j_w$ is the boundary of Ω^j_w in Ω.

Example 4.

$$P_4 = ID_t - \begin{pmatrix} x_1x_2 & -x_1^2 \\ x_2^2 & -x_1x_2 \end{pmatrix} D_{x_1}, \quad x \in \mathbf{R}^n \quad (n \geqq 2).$$

Let Ω be $\mathbf{R}^n$. $\Omega_s = \{x_1 = x_2 = 0\}$. P_4 itself satisfies the condition (L). However, $\tilde{e} = (x_1^2 + x_2^2)^{-1/2\,t}(x_1, x_2)$ is not continuous on $\partial\Omega_w = \Omega_s$.

Besides, P_4 is weakly hyperbolic, because $P_4 + B$ satisfies the condition (L') for some $B(t, x)$, for example $B = \alpha(t, x)I$. (See the remark 3 of Theorem 1.3. It is interesting to compare this example and the example 5 in the section 2. See also the appendix 2.)

Proof of Theorem 1.5. If $A^1(t, x)$ is real analytic, we have only one of the cases I and II for each j $(1 \leqq j \leqq r)$, because Ω_s^j is the zero set of the real analytic function $\sum_{i,k} (\Delta_{i,k}^j)^2$, where $\Delta_{i,k}^j$ is the (i, k)-cofactor of $A^1(t, x) - \lambda_j(t, x)I$ and i and k run over $1 \leqq i, k \leqq N$.

We shall prove the latter half of the theorem. We use the same notation which appear in the proof of Theorem 1.3, omitting the suffix j. Let us set $A(t, x) = \mathscr{C}_1'(t, x) - \lambda(t, x)I = \begin{pmatrix} a(t, x) & b(t, x) \\ c(t, x) & -a(t, x) \end{pmatrix}$. $a(t, x)$, $b(t, x)$ and $c(t, x)$ are also real analytic and satisfy the condition (1.4):

$$(1.4) \qquad (a(t, x))^2 + b(t, x)c(t, x) \equiv 0.$$

In the second case, $(b(t, x))^2 + (c(t, x))^2 \neq 0$ in θ except the analytic set Ω_s. Then, $-a(t, x)/b(t, x)$ is meromorphic. Near the determinate pole of $-a(t, x)/b(t, x)$, $a(t, x)/c(t, x)(\equiv [-a(t, x)/b(t, x)]^{-1})$ is holomorphic. Therefore, if we show that $-a(t, x)/b(t, x)$ has not the indeterminate poles, at least one of $-a(t, x)/b(t, x)$ and $a(t, x)/c(t, x)$ has the sense everywhere.

We shall show the absence of the indeterminate pole by the reduction to absurdity.

Suppose that $-a(t, x)/b(t, x)$ has the indeterminate poles. Since the dimension of (t, x)-space is two, the indeterminate poles are the isolated points. Let (t', x') be one of them and $\pi(s)$ be the characteristic curve which pass (t', x'). We take a positive number ε_0 $(<(\theta_2 - \theta_1)/4M$, see the proof of Theorem 1.3.) sufficiently small such that $\pi(t' - \varepsilon_0)(= (\bar{t}, \bar{x}))$ is not an indeterminate pole and $\{t = \bar{t}\}$ is not con-

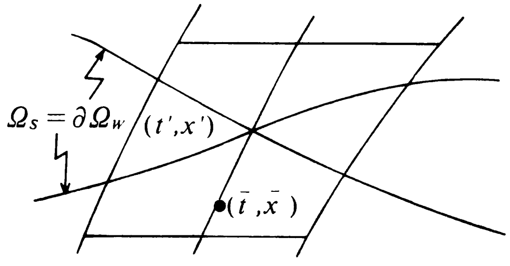

Figure 3.

tained in $\Omega_s(=\partial\Omega_w)$. Then one of $|-a(\bar{t}, \bar{x})/b(\bar{t}, \bar{x})|$ and $|a(\bar{t}, \bar{x})/c(\bar{t}, \bar{x})|$ has the sense and is smaller than or equal to 1 by the relation $[-a(t, x)/b(t, x)][a(t, x)/c(t, x)]\equiv 1$. (If $(\bar{t}, \bar{x})$ is on Ω_s, that is, $a(\bar{t}, \bar{x})=b(\bar{t}, \bar{x})=c(\bar{t}, \bar{x})=0$, we use $-a(\bar{t}, \bar{x})/b(\bar{t}, \bar{x})=\lim_{x\to\bar{x}} -a(\bar{t}, x)/b(\bar{t}, x)$ or $a(\bar{t}, \bar{x})/c(\bar{t}, \bar{x})=\lim_{x\to\bar{x}} a(\bar{t}, x)/c(\bar{t}, x)$. At least one of them is decided because $\{t=\bar{t}\}$ traverses $\partial\Omega_w$ and $(\bar{t}, \bar{x})$ is not an indeterminate pole.)

First we consider the case where $|-a(\bar{t}, \bar{x})/b(\bar{t}, \bar{x})|\leqq 1$. $-a(\bar{t}, x)/b(\bar{t}, x)$ is holomorphic near $\bar{x}$ as the function with one value x. By virtue of the proof of Theorem 1.3, the C^∞-solution[7] g of

$$g_t-\lambda g_x-\beta g^2-(\alpha-\delta)g+\gamma=0, \tag{1.6'}$$

exists in a neighbourhood ω of $(\bar{t}, \bar{x})$ which contains (t', x'). (See the figure 3.) Since g coincides with $-a(t, x)/b(t, x)$ in $\Omega_w\cap\omega$, (t', x') can not be an indeterminate pole. (It is a removable pole.)

In the case $|a(\bar{t}, \bar{x})/c(\bar{t}, \bar{x})|\leqq 1$, we also has the solution h of

$$h_t-\lambda h_x-\gamma h^2-(\delta-\alpha)h+\beta=0, \tag{1.7'}$$

in a neighbourhood ω which contains (t', x') and this implies that (t', x') is not an indeterminate pole of $a(t, x)/c(t, x)$. (It is a removable pole or a determinate pole of $-a(t, x)/b(t, x)$.)

In either case, (t', x') must not be an indeterminate pole of $-a(t, x)/b(t, x)$. This is contrary to the hypothesis.

Since at least one of $-a(t, x)/b(t, x)$ and $a(t, x)/c(t, x)$ has the sense at every point in θ, we have the local expression of $\vec{e}(t, x)$;

$$\vec{e}(t, x)=(a^2+b^2)^{-1/2}\,{}^t(b,\ -a) \quad\text{or}\quad =(a^2+c^2)^{-1/2}\,{}^t(a,\ c).$$

We can continue them in θ because they are real and normalized, the dimension of the eigen-space is one on Ω_w and Ω_w is dense in Ω. (The normal real eigen-vector is unique except the product of (-1).) Let us set $\vec{e}''_j=\mathscr{B}_0\vec{e}'_j$, where $\vec{e}'_j=(0,\dots,0,\overset{2j-1}{\vee}\!e_1,\overset{2j}{\vee}\!e_2,0,\dots,0)$ and $\vec{e}={}^t(e_1, e_2)$. Since $\vec{e}''_j$ is the real eigen-vector of $A^1(t, x)$ belonging to $\lambda_j(t, x)$, $\vec{e}_j(t, x)$ is given by $|\vec{e}''_j|^{-1}\vec{e}''_j$ in θ. Once again, we continue $\vec{e}_j(t, x)$ in Ω by the same way as the above. Q.E.D.

§ 2. Stably non-hyperbolic operators

In this section, we apply Theorem 1.3 to the necessity for the weak hyperbolicity. By virtue of Corollary 1.2 and Theorem 1.3 or Theorem 1.5, we have the following;

Theorem 2.1. (Necessity of the smoothness of the eigen-vectors.)
Under the assumption 1, *suppose that* P_p *is weakly hyperbolic in* Ω. *Then,*

7) In general, α, β, γ and δ are not holomorphic but only infinitely differentiable.

some of $R_j(t, x)$ *are not identically* $N-2$ $(1 \leqq j \leqq r)$. *Next, when* $R_j(t, x) \not\equiv N-2$ *(that is,* $\Omega^j_w \neq \phi$*), we can take the unit real eigen-vector* $\tilde{e}_j(t, x)$ *which satisfies the properties* (i) *and* (ii) *in Theorem* 1.3 (*or the property in Theorem* 1.5 *when the coefficients of* P_p *are all real analytic*).

By virtue of this theorem, we are in a position to show several examples of stably non-hyperbolic operators.

Example 5.[8] The following P_5 is stably non-hyperbolic in any neighbourhood of the origin.

$$P_5 = ID_t - \begin{pmatrix} tx & -t^2 \\ x^2 & -tx \end{pmatrix} D_x .$$

Here, λ is 0 and Ω_w is $\boldsymbol{R}^2 \backslash \{O\}$, but $\tilde{e}(t, x) = (t^2+x^2)^{-1/2}\,{}^t(t, x)$ does not belong to C^0 in any neighbourhood of the origin. This is contrary to Theorem 2.1. (See also Theorem 1.5.)

More precisely, P_5 is stably non-hyperbolic at the origin. This was shown by H. Yamahara through the direct proof. (Unpublished.) In this case, for $P_5 + B$, there is an open set ω' in any neighbourhood of the origin such that $\omega' \cap \{t=0\} \neq \phi$ and

$$(*) \qquad L = {}^{co}P_5 P_s {}^{co}P_5 + \frac{1}{2i} {}^{co}P_5 \{P_5, {}^{co}P_5\}|_{\tau=0} \neq 0 \quad \text{on} \quad \omega'.$$

Therefore, the local uniqueness of the Cauchy problem (1)–(2) in ω' with the initial line $\{t=0\}$ is guaranteed by the similar way as in W. Matsumoto [15]. Then, the local hyperbolicity at the origin implies the local hyperbolicity at $(0, x_o) \in \omega'$. However, this is contrary to $(*)$ by Proposition 1.1.

Even if all coefficients depend only on t, we have the following example.

Example 6. The following P_3 is stably non-hyperbolic in any neighbourhood of $(0, x_o)$ for each $x_o \in \boldsymbol{R}$.

$$P_6 = ID_t - \varphi(t) \begin{pmatrix} \cos\frac{1}{t}\sin\frac{1}{t} & -\cos^2\frac{1}{t} \\ \sin^2\frac{1}{t} & -\cos\frac{1}{t}\sin\frac{1}{t} \end{pmatrix} D_x ,$$

where $\varphi(t)$ belongs to $C^\infty(\boldsymbol{R}^1)$, vanishes at the origin of infinite order and does not vanish in $\boldsymbol{R}\backslash\{0\}$. Here, λ is 0 and $\Omega_s = \{0\} \times \boldsymbol{R}$. However, $\tilde{e}(t) = {}^t\left(\cos\frac{1}{t}, \sin\frac{1}{t}\right)$ does not belong to $C^0(\omega_0^+)$ for any neighbourhood ω of $(0, x_o)$. This is contrary to Theorem 2.1. (See the property (i) in Theorem 1.3.)

In this case, it is difficult to show the stable non-hyperbolicity at $(0, x_o)$. How-

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8) The examples 5 and 6 were presented in W. Matsumoto [15] concerning the local uniqueness in the Cauchy problem.

ever, if we require a little stronger uniqueness property of the solution in the definition of the local $\mathscr{E}$ well-posedness like V. M. Petkov [19], [20], we can see it easily.

P_5 and P_6 are stably non-hyperbolic even if we take $\Omega = \boldsymbol{R}^{2+}_0 (=\{(t, x) \in \boldsymbol{R}^2 \mid t \geqq 0\})$ or $=\boldsymbol{R}^{2-}_0$. On the other hand, the following P_7 is weakly hyperbolic in $\boldsymbol{R}^{2+}_0$ and in $\boldsymbol{R}^{2-}_0$ but it is stably non-hyperbolic in any neighbourhood of $(0, x_0)$, for each $x_0 \in \boldsymbol{R}^1$.

Example 7.

$$P_7 = ID_t - \varphi(t)\begin{pmatrix} |t| & -1 \\ t^2 & -|t| \end{pmatrix} D_x,$$

where $\varphi(t)$ is that in the example 6.

Here, $\lambda = 0$ and $\Omega_s = \{0\} \times \boldsymbol{R}$. $\check{e} = (1+t^2)^{-1/2t}(1, |t|)$ does not belong to $C^1(\omega)$ no matter how it belongs to $C^\infty(\boldsymbol{R}^2 \backslash \{0\} \times \boldsymbol{R}) \cap C^0(\boldsymbol{R}^2)$, where ω is an arbitrary neighbourhood of $(0, x_0)$ and x_0 is an arbitrary point in $\boldsymbol{R}$. This is contrary to Theorem 2.1. (See the property (ii) in Theorem 1.3.)

In this case, $P_7 + \begin{pmatrix} \alpha(t, x) & 0 \\ 1 & \alpha(t, x) \end{pmatrix}$ is hyperbolic in $\boldsymbol{R}^{2+}_0$, (especially, at each point on $\{t=0\}$). Here, $\alpha(t, x)$ is an arbitrary element in $C^\infty(\boldsymbol{R}^2)$. Therefore, P_7 is weakly hyperbolic <u>at</u> each point $(0, x_0)$ but it is stably non-hyperbolic <u>in</u> any neighbourhood of $(0, x_0)$.

In the examples 5, 6 and 7, P_i has no lower order term such that $P_i + B$ satisfy the condition (L), ($i=4$, 5 and 6). However, there is another sort of stably non-hyperbolic operators which have some lower order terms by which the condition (L) is satisfied.

Example 8.

$$P_8 = ID_t - \begin{pmatrix} 0 & \mu(t, x) \\ \nu(t, x) & 0 \end{pmatrix} D_x,$$

where
$$\mu(t, x)\begin{cases} >0, & \text{on } [(-\infty, 0) \cup (\bigcup_{i=1}^{\infty} (a_{2i+1}, a_{2i}))] \times \boldsymbol{R}, \\ =0, & \text{otherwise} \end{cases}$$

$$\nu(t, x)\begin{cases} >0, & \text{on } \bigcup_{i=1}^{\infty} (a_{2i}, a_{2i-1}) \times \boldsymbol{R} \\ =0, & \text{otherwise.} \end{cases}$$

Here, $\{a_i\}$ is a decreasing sequence, it converges to 0 and $a_1 = \infty$. The characteristic root λ of P_8 is 0 and always double, and $\Omega_s = \bigcup_i \{a_i\} \times \boldsymbol{R} \cup \{0\} \times \boldsymbol{R}$.

Let $B(t, x)$ be $\begin{pmatrix} \alpha(t, x) & \beta(t, x) \\ \gamma(t, x) & \delta(t, x) \end{pmatrix}$. The condition (L) is equivalent to

"$\gamma(t, x) = 0$ on $\operatorname{supp} \mu$ and $\beta(t, x) = 0$ on $\operatorname{supp} \nu$".

We shall consider the stable non-hyperbolicity of this example in detail in the next section.

It is remarkable that $e^{-ip\phi}(P_8+B)[f\hat{e}e^{ip\phi}]=O(1)$ under the condition (L), where $\hat{e}$ is the eigen-vector of $A=\begin{pmatrix}0 & \mu\\ \nu & 0\end{pmatrix}$, ϕ is an arbitrary real phase function and f is an arbitrary scalar element in $C^\infty(\Omega)$. (See V. M. Petkov [20].) The reason why P_8 is stably non-hyperbolic differs from those of P_5, P_6 and P_7. The condition (L) is also necessary for the local hyperbolicity in the Gevrey class $\gamma^{\langle\kappa\rangle}$, where $\kappa>2$.[9] Here, we seek the unique solution in $\mathscr{E}(\omega^+_{t_0})$ corresponding to the data $u_0(x)$ in $\gamma^{\langle\kappa\rangle}(\omega_{t_0})$ and the right-hand side $f(t, x)$ in $\gamma^{\langle\kappa\rangle}(\omega^+_{t_0})$. (See the definitions 4 and 5.) Therefore, P_5, P_6 and P_7 are also stably non-hyperbolic in the class $\gamma^{\langle\kappa\rangle}$, $(\kappa>2)$. On the other hand, P_8+B is hyperbolic in the class $\gamma^{(\infty)}$ under the condition (L).[10] We can prove this by the similar way as in the next section.

§3. On the sufficient conditions for the hyperbolicity

In this section, we show that we need some additional conditions to (L) for the hyperbolicity in Ω through the examples if the coefficients of P_p belong only to $C^\infty(\Omega)$. Moreover, we establish two theorems for the hyperbolicity under some additional conditions which refuse such examples.

Under the assumption 1, the operator P is reduced to a family of the systems of size 2×2 and the scalar operators. The simplest hyperbolic operator of size 2×2 is the following:

$$(3.1)\qquad P_0+B=ID_t-\begin{pmatrix}\lambda(t, x) & \varepsilon(t, x)\\ 0 & \lambda(t, x)\end{pmatrix}D_x+\begin{pmatrix}\alpha(t, x) & \beta(t, x)\\ \gamma(t, x) & \delta(t, x)\end{pmatrix},$$

where $\varepsilon(t, x)\equiv 0$ or $\gamma(t, x)\equiv 0$ in Ω.[11]

The former condition $\varepsilon(t, x)\equiv 0$ is equivalent to rank $P_0|_{\tau=\lambda}\equiv 0$, and the latter one $\gamma(t, x)\equiv 0$ is guaranteed by both of rank $P_0|_{\tau=\lambda}=1$ on a dense set in Ω and the condition (L) because the condition (L) is equivalent to $\varepsilon(t, x)\gamma(t, x)\equiv 0$.

Corresponding to the above, we introduce the following hypothesis.

(H.1) For each j, $(1\leqq j\leqq r)$, only one of the followings arises:

I) $\Omega\equiv\Omega^j_s(\equiv\{(t, x)\in\Omega\,|\,R_j(t, x)=N-2\})$,

II) $\Omega=\overline{\Omega^j_w}(\equiv$the closure of $\{(t, x)\in\Omega\,|\,R_j(t, x)=N-1\})$.

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9) This is provable by the modified method of S. Mizohata [16], using V. Ya. Ivrii's idea [8]. Here, we assume that the coefficients of P belong to $\gamma^{(1+\kappa/2)}$. However, this assumption will be relaxed. In the forthcoming paper, the precise proof will be given.

10) S. Tarama also showed this in much more general class. His proof is very different from ours. (See S. Tarama [29].) Here, we assume that all coefficients of P belong to $\gamma^{(\infty)}$. The solution u, in fact, belong to $\gamma^{(\infty)}$. A remark on the hyperbolicity in class $\gamma^{(\infty)}$ and in more wide classes will be announced in the forthcoming paper. (See [27] and [28].)

11) This is seen by the similar way as the proofs of V. M. Petkov [19], [20] and of H. Yamahara [25]. H. Yamahara assumed that, for all j, $R_j(t, x)\equiv N-1$ or $N-2$ in Ω, but we can relax "for all j" to "for each j", $(1\leqq j\leqq r)$.

In the case I, the highest two terms of $\mathscr{C}^j(t, x; D_t, D_x)$ in Lemma 1.4 has automatically the form (3.1) where $\varepsilon(t, x)\equiv 0$. However, in the case II, in general, we cannot transform $\mathscr{C}^j(t, x; D_t, D_x)$ smoothly to the form (3.1). (See the remarks 3 and 4 of Theorem 1.3.) Therefore, we introduce one more hypothesis:

(H.2) In the case II, $\mathring{e}_j(t, x)$ is taken in $C^\infty(\Omega)$.

Obviously, if P satisfies the hypotheses (H.1) and (H.2) P is hyperbolic, because P is transformed to the systems of type (3.1) with $\varepsilon\equiv 0$ or $\gamma\equiv 0$ and the scalar operators. (See Corollary 4.9.)

3.1° The case with real analytic coefficients

If all of the coefficients of P_p are real analytic, the hypotheses (H.1) and (H.2) are satisfied by virtue of Theorem 1.5. Therefore, we have the following theorem.

Theorem 3.1. (Hyperbolicity in the case with real analytic coefficients.)
Suppose that the assumption 1 *is satisfied and that all coefficients of* P_p *are real analytic and bounded in* Ω. *Then,* $P=P_p+B$ *is hyperbolic in* Ω *under the condition* (L).

Here, if the case I *arises for every* j, *the loss of regularity in* Ω *is* 0, *and if the case* II *arises for at least one of* j, *the loss is* 1 *in* Ω $(1\leqq j\leqq r)$.

Remark. For the local hyperbolicity in Ω, we need not the boundedness of the coefficients of P_p in Ω.

3.2° The case with C^∞-coefficients

In the case where the coefficients of P_p belong only to $C^\infty(\Omega)$, the situation is much more complicated. Here, we restrict ourselves to the cases where $\mathring{e}_j(t, x)$ can be extended on each $\overline{G^j}$ in C^∞-class, and we use only the modified Petkov's and Yamahara's techniques. (We do not use Demay's idea.) For such restriction, we introduce the following new assumption. (See the remark 3 of Theorem 1.3.)

Assumption 2. The boundary of each connected component of Ω^j_w lies on a family of some disjoint space-like curves $\{T^j_l\}_l$ $(1\leqq j\leqq r)$, in general as a subset.

Remark 1. The determination of such curves is not unique.

Remark 2. If all of coefficients of P_p depend only on t, the assumption 2 is automatically satisfied.

$\{T^j_l\}_l$ divides Ω into a family of open connected subdomains $\{\Omega^j_k\}_k$ and a closed set Σ^j, where $R_j(t, x)$ is constant on each Ω^j_k and where $\Sigma^j\supseteqq\bigcup_l T^j_l$ and $\mathring{\Sigma}^j=\phi$, $(1\leqq j\leqq r)$. (See the figure 4.)

Remark 3. $R_j(t, x)$ is free on Σ^j, that is, $R_j(t, x)$ is changeable only on Σ^j. Of course, $R_j(t, x)$ need not change on Σ^j. Under the assumption 2, $\mathring{e}_j(t, x)$ can be extended on each $\overline{\Omega^j_k}$ in C^∞-class by virtue of Theorem 1.3, but neither the hypothesis (H.1) nor (H.2) is, in general, satisfied. When one of the hypotheses (H.1) and (H.2) fails, $P(t, x; D_t, D_x)$ may not be hyperbolic in Ω even if the condition (L) is satisfied. We shall show this by the examples in the paragraphs 3.3° and

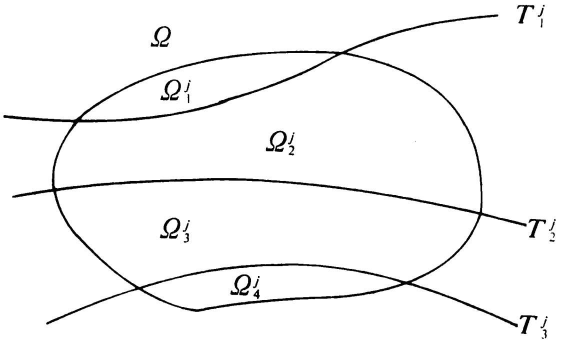

Figure 4.

3.4°. However, if we add the following assumption, P is hyperbolic under the condition (L).

Assumption 3. $\{T_l^j\}_l$ in the assumption 2 does not accumulate on arbitrary compact set K in Ω for each j, that is,
(4) There exists a positive constant δ_K such that

$$\operatorname{dist}(T_k^j \cap K, T_l^j \cap K) \geqq \delta_K, \quad \text{if} \quad k \neq l, \quad (1 \leqq j \leqq r).$$

Remark. Under the assumption 3, $\Sigma^j = \bigcup_l T_l^j$.

Theorem 3.2. (Hyperbolicity in the case with C^∞-coefficients.)

Suppose the assumptions 1, 2, 3 *and boundedness of the coefficients of* P_p. *Then, under the condition* (L), P *is hyperbolic in* Ω. *Here, the loss of regularity on* K *from* t_0 *is at most* $\max_j \#\{\Omega_k^j \mid \Omega_k^j \cap K_{t_0}^+ \neq \phi, R_j(t, x) = N-1$ *on* $\Omega_k^j\}$, *where* K *is lense-shaped.*

Remark 1. For the local hyperbolicity in Ω, we need not the boundedness of the coefficients of P_p in Ω.

Remark 2. The loss of regularity in Ω may be infinite, in general.

Example 9.

$$P_9 = ID_t - \begin{pmatrix} 0 & \mu(t, x) \\ \nu(t, x) & 0 \end{pmatrix} D_x,$$

where

$$\mu(t, x) \begin{cases} >0, & t + a_{2i} < x < t + a_{2i+1}, \quad (i \geqq 0), \\ =0, & \text{otherwise}, \end{cases}$$

$$\nu(t, x) \begin{cases} >0, & t + a_{2i-1} < x < t + a_{2i}, \quad (i \geqq 1), \\ =0, & \text{otherwise}. \end{cases}$$

Here, $a_0 = -\infty$ and $a_i = \sum_{k=1}^{i} \frac{1}{k}$.

Obviously, the characteristic root λ is 0 and double, and $\Omega_s=\{x=t+a_i\}_{i=1}^{\infty}=\{T_i\}_i$. P_9+B is hyperbolic in $\boldsymbol{R}^2$ under the condition (L) by virtue of the above theorem. The loss of regularity on $[0, \varepsilon]\times[x_\circ, a_n]$ is $l_n=\max\left\{j;\ \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{j}<\varepsilon\right\}-n+1$. ($x_\circ$ is an arbitrary point less than a_n. See the proof of Proposition 3.8. Then, the loss of regularity in $[0, \varepsilon]\times\boldsymbol{R}$ is infinite because $\sup_n l_n=\infty$.

Theorem 3.2 will be proved in the section 5 as Theorem 5.3 for general dimension. Its proof is long. However, if T_l^j are expressed by $\{a_l^j\}\times\boldsymbol{R}$ for all l and j, since we can uniquely solve the Cauchy problem on each $([\bar{a}_i, \bar{a}_{i+1}]\times\boldsymbol{R})\cap\Omega$, we can obtain the solution (1) and (2) in $\Omega_{t_\circ}^+$ solving the Cauchy problem step by step on $\Omega_{t_\circ}^+\cap\Omega_{\bar{a}_q}^-$, $\Omega_{\bar{a}_q}^+\cap\Omega_{\bar{a}_{q+1}}^-,\ldots$ and $\Omega_{\bar{a}_p}^+\cap\Omega_t^-$, where $\bigcup_{j,l}\{a_l^j\}=\{\bar{a}_1<\cdots<\bar{a}_r\}$ and $\bar{a}_{q-1}\leqq t_\circ<\bar{a}_q\leqq\bar{a}_p<t\leqq\bar{a}_{p+1}$.
This shows the conclusion of Theorem 3.2.

3.3° The case where the hypothesis (H.1) fails

In this paragraph, we present an example in which the hypothesis (H.1) fails.

Example 10.

$$P_{10}=\frac{\partial}{\partial t}-\begin{pmatrix}0 & \mu(t, x)\\ 0 & 0\end{pmatrix}\frac{\partial}{\partial x},$$

where $\mu(t, x)\begin{cases} >0, & \text{on } (a_{2i+1}, a_{2i})\times\boldsymbol{R}, \quad (i\geqq 1),\\ =0, & \text{otherwise,}\end{cases}$

$\{a_i\}$ decreases strictly and converges to 0 as i tends to infinity. Here, for convenience sake, we set $a_1=\infty$ and $a_2\leqq 1$.

Let Ω be $\boldsymbol{R}^2$. Here, $\lambda=0$, $T_i=\{a_i\}\times\boldsymbol{R}$, $T_\circ=\{0\}\times\boldsymbol{R}$, $\Omega_i=(a_{i+1}, a_i)\times\boldsymbol{R}$, $\Omega_\circ=(-\infty, 0)\times\boldsymbol{R}$, $\Omega_w=\bigcup_{i=1}^{\infty}(a_{2i+1}, a_{2i})\times\boldsymbol{R}$ and $\Omega_s=\bigcup_{i=1}^{\infty}[a_{2i}, a_{2i-1}]\times\boldsymbol{R}\cup(-\infty, 0]\times\boldsymbol{R}$. (See the figure 5.) Then $\Omega_s\neq\Omega$ and $\overline{\Omega_w}\neq\Omega$, that is, the hypothesis (H.1)

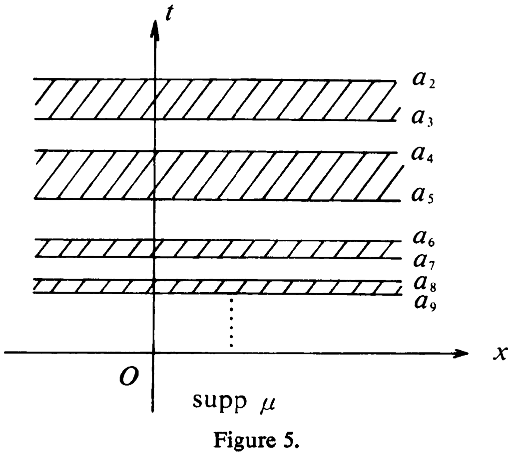

Figure 5.

fails. However, the hypothesis (H.2) is satisfied because we can take $\check{e}(t, x)={}^t(1, 0)$ in Ω.

Setting $B(t, x)=-\begin{pmatrix}\alpha(t, x) & \beta(t, x)\\ \gamma(t, x) & \delta(t, x)\end{pmatrix}$ the condition (L) is equivalent to the following:

$$(\mathrm{L}_1) \qquad \gamma\equiv 0 \quad \text{on} \quad \Omega_w.$$

Obviously, for $t_\circ>0$, the Cauchy problem (1)–(2) is uniquely solvable in $[t_\circ, \infty)\times\boldsymbol{R}$ under the condition (L) solving step by step in $[t_\circ, a_k]\times\boldsymbol{R}$, $[a_k, a_{k-1}]\times\boldsymbol{R}$,..., and $[a_2, \infty)\times\boldsymbol{R}$. ($a_k$ is the least element in $\{a_i\}$ greater than $t_\circ$.) On the other hand, for $t_\circ<0$, (1)–(2) is also uniquely solvable in $[t_\circ, 0]\times\boldsymbol{R}$ under the condition (L). Therefore, we need consider only the case when $t_\circ=0$.

Let us set $\varphi(t, x)=\exp\left\{-\int_0^t\alpha(\tau, x)d\tau\right\}$, $\psi(t, x)=\exp\left\{-\int_0^t\delta(\tau, x)d\tau\right\}$, $v=\begin{pmatrix}\varphi(t, x) & 0\\ 0 & \psi(t, x)\end{pmatrix}u(t, x)$ and $v_\circ(x)=\begin{pmatrix}\varphi(t_\circ, x) & 0\\ 0 & \psi(t_\circ, x)\end{pmatrix}u_\circ(x)$, then the Cauchy problem for $P_{10}+B$ is transformed to the following:

$$(3.2)\qquad \begin{cases}(\tilde{P}_{10}+\tilde{B})v=\dfrac{\partial}{\partial t}v-\begin{pmatrix}0 & \tilde{\mu}\\ 0 & 0\end{pmatrix}\dfrac{\partial}{\partial x}v-\begin{pmatrix}0 & \tilde{\beta}\\ \tilde{\gamma} & 0\end{pmatrix}v=\tilde{f},\\[2ex] v(t_\circ, x)=v_\circ(x),\end{cases}$$

where $\tilde{\mu}=\mu\varphi[\psi]^{-1}$, $\tilde{\beta}=\left\{\beta+\mu\int_0^t\delta'_x(\tau, x)d\tau\right\}\varphi[\psi]^{-1}$, $\tilde{\gamma}=\gamma\psi[\varphi]^{-1}$ and $\tilde{f}=\begin{pmatrix}\varphi & 0\\ 0 & \psi\end{pmatrix}f$.

Let us set

$$(3.3)\qquad \mathscr{M}_{2i-1}(a_{2i-1}, a_{2i}; x)=(b_{2i-1}^{jk}(x))_{j,k=1,2}$$
$$=I+\sum_{k=1}^{\infty}\int_{a_{2i}}^{a_{2i-1}}\tilde{B}(s_k, x)ds_k\int_{a_{2i}}^{s_k}\tilde{B}(s_{k-1}, x)ds_{k-1}\cdots\int_{a_{2i}}^{s_2}\tilde{B}(s_1, x)ds_1,$$

where $\tilde{B}(t, x)=\begin{pmatrix}0 & \tilde{\beta}(t, x)\\ \tilde{\gamma}(t, x) & 0\end{pmatrix}$, and let us set $d(x)=\#\{i\geqq 2;\ b_{2i-1}^{21}(x)\neq 0\}$. Here,

$$b_{2i-1}^{21}(x)=\int_{a_{2i}}^{a_{2i-1}}\tilde{\gamma}(s_1, x)ds_1$$
$$+\int_{a_{2i}}^{a_{2i-1}}\int_{a_{2i}}^{s_3}\int_{a_{2i}}^{s_2}\tilde{\gamma}(s_3, x)\tilde{\beta}(s_2, x)\tilde{\gamma}(s_1, x)ds_1ds_2ds_3+\cdots$$
$$+\int_{a_{2i}}^{a_{2i-1}}\int_{a_{2i}}^{s_{2k+1}}\int_{a_{2i}}^{s_{2k}}\cdots\int_{a_{2i}}^{s_2}\tilde{\gamma}(s_{2k+1}, x)\tilde{\beta}(s_{2k}, x)\cdots\tilde{\gamma}(s_1, x)ds_1ds_2\cdots ds_{2k+1}$$
$$+\cdots.$$

We set $R_\circ=\max\limits_{(t,x)}\{|\tilde{\beta}(t, x)|, |\tilde{\gamma}(t, x)|\}$, where (t, x) runs over $[0, a_2]\times K$.

Proposition 3.3. (Hyperbolicity of $P_{10}+B$.)

The necessary and sufficient condition for the hyperbolicity of $P=P_{10}+B$ in $\boldsymbol{R}^2$ [the local hyperbolicity of P near $(0, x_\circ)$, $x_\circ$ being arbitrary point in $\boldsymbol{R}$] is the condition (L_1) *and the following* (M_1).

(M_1) *For arbitrary compact set K in $\boldsymbol{R}_x$,*

$$d_K \equiv \sup_{x \in K} d(x) < \infty .$$

Here, if P is hyperbolic, the propagation speed is 0 *and the loss of regularity on $\boldsymbol{R} \times K$ from* 0 *is at most d_K+1. (More precisely, for example, if $a_2 \leqq (\sqrt{2}/R_\circ)\cdot e^{-(1/2)R_\circ}$, the loss on $\boldsymbol{R}\times K$ from* 0 *is exactly d_K+1.)*

Since the proof of the sufficiency suggests the necessity of the condition (M_1), first, we show the sufficiency.

Proof of sufficiency. (1. solvability.)

Under the conditions (L_1) and (M_1), we solve (3.2) from $t_\circ = s > 0$ to a_2 with the data $v_\circ(x)$ and the right-hand side 0 and then we make s tend to 0. From now on, we omit $\tilde{}$ of $\tilde{\mu}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{f}$.

Let us set

$$(3.4)_o \quad \mathscr{M}_{2i-1}(t_2, t_1) \equiv \mathscr{M}_{2i-1}(t_2, t_1; x)$$

$$= I + \sum_{k=1}^{\infty} \int_{t_1}^{t_2} \tilde{B}(s_k, x) ds_k \int_{t_1}^{s_k} \tilde{B}(s_{k-1}, x) ds_{k-1} \cdots \int_{t_1}^{s_2} \tilde{B}(s_1, x) ds_1$$

$$= I + \mathscr{B}_{2i-1}(t_2, t_1; x), \quad (a_{2i} \leqq t_1 < t_2 \leqq a_{2i-1}),$$

and

$$(3.4)_e \quad \mathscr{M}_{2i}(t_2, t_1) \equiv \mathscr{M}_{2i}(t_2, t_1; x, \partial_x)$$

$$= \begin{pmatrix} 0 & \int_{t_1}^{t_2} \mu(\tau, x) d\tau \\ 0 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 1 & \int_{t_1}^{t_2} \beta(\tau, x) d\tau \\ 0 & 1 \end{pmatrix}$$

$$= I + \mathscr{M}'_{2i}(t_2, t_1; x, \partial_x), \quad (a_{2i+1} \leqq t_1 < t_2 \leqq a_{2i}).$$

when $a_{i+1} \leqq t_\circ = s < t \leqq a_i$, the solution of (3.2) is given by

$$(3.5) \qquad v(t, x) = \mathscr{M}_i(t, s) v_\circ .$$

Therefore, the solution of (3.2) for $0 < a_{q+1} \leqq t_\circ = s < a_q \leqq a_{p+1} < t \leqq a_p$ is given by the following:

$$(3.6) \qquad v(t, x) = E(t, s) v_\circ = E(t, s; x, \partial_x) v_\circ(x)$$

$$= \mathscr{M}_p(t, a_{p+1}) \Big[\prod_{i=p+1}^{q-1} \mathscr{M}_i(a_i, a_{i+1}) \Big] \mathscr{M}_q(a_q, s) v_\circ .$$

Here, we establish a lemma on d_K. Let us set $\bar{d}(x) = \# \left\{ i \geqq 2 \,\middle|\, \left(\frac{\partial}{\partial x}\right)^k b_{2i-1}^{21}(x) \neq 0 \right.$ for some $\left. k \geqq 0 \right\}$, and $\bar{d}_K = \sup_{x \in K} \bar{d}(x)$.

Lemma 3.4. *If the compact set K has not the isolated points, we have the following*:

$$d_K = \bar{d}_K .$$

Proof. Show that $d_K \geqq \bar{d}_K$. Since $\bar{d}(x)$ is upper semi-continuous, there is a point $x_\circ \in K$ such that $\bar{d}(x_\circ)=l$. ($l=\bar{d}_K$ when $\bar{d}_K<\infty$ and l is an arbitrary large number when $\bar{d}_K=\infty$.) By the assumption, we can take $i(1),\dots, i(l)$ such that $\left(\frac{\partial}{\partial x}\right)^{k_{i(j)}} b^{21}_{2i(j)-1}(x_\circ)\neq 0$ ($k_{i(j)}\geqq 0$). On the other hand, if $\left(\frac{\partial}{\partial x}\right)^{k} b^{21}_{2i-1}(x_\circ)\neq 0$ ($k\geqq 0$), there is a neighbourhood U_{2i-1} of $x_\circ$ where $b^{21}_{2i-1}(x)\neq 0$ except at $\{x_\circ\}$. Since $x_\circ$ is not an isolated point, $K'=K\cap(\bigcap_{j=1}^{l} U_{2i(j)-1})\backslash\{x_\circ\}$ is not empty and $b^{21}_{2i(j)-1}(x)\neq 0$ in K'. This shows that $d_K\geqq \bar{d}_K$. On the other hand, obviously $d_K\leqq \bar{d}_K$. Therefore, we obtain the equality $d_K=\bar{d}_K$. Q.E.D.

From now on, we assume that K has not an isolated point. Let $D=\left\{2i-1 \,\middle|\, \left(\frac{\partial}{\partial x}\right)^{k} b^{21}_{2i-1}(x)\neq 0, \text{ for some } k\geqq 0,\ i\geqq 2\right\}$ be $\{i(1)<i(2)<\cdots<i(l)\}$, ($l\leqq d_K$). Since $\mathscr{M}'_i(t, x)$ is of type $\begin{pmatrix}0 & *\\ 0 & 0\end{pmatrix}$ and $\partial_x^\alpha \mathscr{B}_i(t, s)$ is upper triangle for $i\notin D$ and $\alpha\geqq 0$, $\mathscr{M}'_i\mathscr{B}_{n_1}\cdots\mathscr{B}_{n_m}\mathscr{M}'_{i'}=0$ ($n_k\notin D$, $1\leqq k\leqq m$). Then, by $(3.4)_o$, $(3.4)_e$ and (3.6), the solution $v(t, x)$ is expressed as the following;

$$(3.7)\quad v(t, x)=E(t, s)v_\circ$$

$$=\{I+\sum_{k=1}^{r}\sum_{i_1<\cdots<i_k}\mathscr{B}_{i_1}\cdots\mathscr{B}_{i_k}+\sum_{k=0}^{r}\sum_{i_1<\cdots<i_k}\sum_{j}\mathscr{B}_{i_1}\cdots\mathscr{B}_{*}\mathscr{M}'_j\mathscr{B}_{**}\cdots\mathscr{B}_{i_k}$$

$$+\sum_{k=0}^{r}\sum_{i_1<\cdots<i_k}\sum_{j_1<j_2}\mathscr{B}_{i_1}\cdots\mathscr{B}_{*}\mathscr{M}'_{j_1}\mathscr{B}_{**}\cdots\mathscr{B}_{***}\mathscr{M}'_{j_2}\mathscr{B}_{****}\cdots\mathscr{B}_{i_k}+\cdots$$

$$+\sum_{k=0}^{r}\sum_{i_1<\cdots<i_k}\sum_{j_1<\cdots<j_m}\mathscr{B}_{i_1}\cdots\mathscr{M}'_{j_1}\cdots\mathscr{M}'_{j_2}\cdots\mathscr{M}'_{j_m}\cdots\mathscr{B}_{i_k}\}v_\circ$$

$$=\{I+E_0(t, s)+E_1(t, s)+\cdots+E_m(t, s)\}v_\circ,$$

where $a_{q+1}\leqq s<a_q\leqq a_{p+1}<t\leqq a_p$, $r=\left[\frac{q+1}{2}\right]-\left[\frac{p}{2}\right]$, $m=\#([p+1, q-1]\cap D)+1 \leqq d_K+1$, i_l is odd and j_n is even. In the above formula, $i_1,\dots, i_{k-1}$ and i_k run under the restriction of $\{i_1,\dots, i_k\}\cap(j_n, j_{n+1})\cap D\neq\phi$ in $E_l(t, s)$ ($l\geqq 2$), and $\mathscr{B}_i$ and $\mathscr{M}'_j$ express $\mathscr{B}_i=\mathscr{B}_i(a_i, a_{i+1})$ and $\mathscr{M}'_j=\mathscr{M}'_j(a_j, a_{j+1})$ when $p<i, j<q$, $\mathscr{B}_p=\mathscr{B}_p(t, a_{p+1})$, $\mathscr{M}'_p=\mathscr{M}'_p(t, a_{p+1})$, $\mathscr{B}_q=\mathscr{B}_q(a_q, s)$ and $\mathscr{M}'_q=\mathscr{M}'_q(a_q, s)$.

Now, let us establish a lemma on the estimates of $\mathscr{B}_i$ and $\mathscr{M}'_j$. Here, $\mathscr{M}(t, s) \ll \mathscr{N}(t, s)$ means that each component of $\mathscr{N}(t, s)$ majorizes the corresponding one of $\mathscr{M}(t, s)$ as a differential operator. Let R be the maximum of the absolute values of the derivatives of $\mu(t, x)$, $\beta(t, s)$ and $\gamma(t, x)$ of order up to $2N+d_K+1$ on $[0, a_2]\times K$.

Lemma 3.5.

$$(3.8)\qquad \partial_x^\alpha\mathscr{B}_i(t, s)\ll\exp[R'(t-s)]\begin{pmatrix}R'^2(t-s)^2/2 & R'(t-s)\\ R'(t-s) & R'^2(t-s)^2/2\end{pmatrix}\sum_{a=0}^{\alpha}\partial_x^a,$$

for $0\leqq\alpha\leqq N.$

$$(3.9)\qquad \partial_t^\alpha\partial_x^\beta\mathscr{B}_i(t,s)\ll\alpha!\Big(\sum_{j=1}^{\alpha}R'^{j}\Big)\exp[R'(t-s)]\begin{pmatrix}1&1\\1&1\end{pmatrix}\sum_{b=0}^{\beta}\partial_x^b,$$

if $1\leqq\alpha\leqq N$ *and* $0\leqq\beta\leqq N$.

$$(3.10)\qquad \partial_x^\alpha\mathscr{B}_i(a_i,a_{i+1})\ll\exp[R'(a_i-a_{i+1})]\times\begin{pmatrix}R'^2(a_i-a_{i+1})^2/2 & R'(a_i-a_{i+1})\\ 0 & R'^2(a_i-a_{i+1})^2/2\end{pmatrix}\sum_{a=0}^{\alpha}\partial_x^a,$$

for $0\leqq\alpha\leqq N$, *if* $i\notin D$.

$$(3.11)\qquad \partial_x^\alpha\mathscr{M}'_j(t,s)\ll 2R'(t-s)\begin{pmatrix}0&1\\0&0\end{pmatrix}\sum_{a=0}^{\alpha+1}\partial_x^a,\qquad \textit{for}\ \ 0\leqq\alpha\leqq N.$$

$$(3.12)\qquad \partial_t^\alpha\partial_x^\beta\mathscr{M}'_j(t,s)\ll 2^{\alpha+1}R'\begin{pmatrix}0&1\\0&0\end{pmatrix}\sum_{b=0}^{\beta+1}\partial_x^b,\qquad \textit{for}\ \ 1\leqq\alpha\leqq N\ \textit{and}\ 0\leqq\beta\leqq N.$$

$$(3.13)\qquad \partial_t^\alpha\partial_x^\beta E_h(t,s)\ll C_\alpha(d_K,p,h,N)\begin{pmatrix}1&1\\1&1\end{pmatrix}\sum_{b=0}^{\beta+h}\partial_x^b,\qquad \textit{for}\ \ 0\leqq\alpha,\ \beta\leqq N.$$

Here, $R'=2^NR$ *and* $C_0(d_K, p, h, N)$ *tends to* 0 *when* p *tends to infinity* (*that is,* t *tends to* 0).

Proof. The (1, 1)-component $\bar{b}_i^{11}(t,s)$ of $\mathscr{B}_i(t,s)$ is given by

$$(3.14)\qquad \sum_{k=1}^{\infty}\int_s^t\beta(s_k,x)ds_k\int_s^{s_k}\gamma(\tau_k,x)d\tau_k\int_s^{\tau_k}\beta(s_{k-1},x)ds_{k-1}\times\int_s^{s_{k-1}}\gamma(\tau_{k-1},x)d\tau_{k-1}\cdots\int_s^{\tau_2}\beta(s_1,x)ds_1\int_s^{s_1}\gamma(\tau_1,x)d\tau_1.$$

Then, $\partial_x^\alpha\bar{b}_i^{11}(t,s)$ is majorized by the following as a differential operator;

$$(3.15)\qquad \begin{aligned}\partial_x^\alpha\bar{b}_i^{11}(t,s)&\ll\sum_{k=1}^{\infty}(2k+1)^\alpha R^{2k}\frac{(t-s)^{2k}}{(2k)!}\sum_{a=0}^{\alpha}\partial_x^a\\&\ll\sum_{k=1}^{\infty}(2^NR)^{2k}\frac{(t-s)^{2k}}{(2k)!}\sum_{a=0}^{\alpha}\partial_x^a\\&\ll\frac{R'^2(t-s)^2}{2}\exp[R'(t-s)]\sum_{a=0}^{\alpha}\partial_x^a.\end{aligned}$$

Moreover, for $1\leqq\alpha\leqq N$ and $0\leqq\beta\leqq N$,

$$(3.16)\qquad \partial_t^\alpha\partial_x^\beta\bar{b}_i^{11}(t,s)\ll\sum_{k=1}^{\infty}(2k+1)^\beta R^{2k}\sum_j j^{\alpha-j}\frac{(t-s)^{2k-j}}{(2k-j)!}\Big(\sum_{b=0}^{\beta}\partial_x^b\Big),$$

(j runs from 1 to min $\{2k,\alpha\}$.)

$$\ll \alpha! \sum_{j=1}^{\alpha} R'^j \sum_{k=[(j+1)/2]}^{\infty} \frac{\{R'(t-s)\}^{2k-j}}{(2k-j)!} \sum_{b=0}^{\beta} \partial_x^b$$

$$\ll \alpha!(\sum_{j=1}^{\alpha} R'^j) \exp [R'(t-s)] \sum_{b=0}^{\beta} \partial_\alpha^b.$$

Since the other components of $\mathscr{B}_i(t, x)$ are majorized by the same way, we can see (3.8) and (3.9). If $i \notin D$, the (2, 1)-component $\partial_x^\alpha \bar{b}_i^{21}(a_i, a_{i+1}) \equiv \partial_x^\alpha b_i^{21}(a_i, a_{i+1})$ of $\partial_x^\alpha \mathscr{B}_i(a_i, a_{i+1})$ vanishes. This shows (3.10).

(3.11) and (3.12) are easily seen.

Now, we show the estimate (3.13). If all of i_l do not belong to D, the following is obtained by (3.10);

$$(3.17) \qquad \partial_x^\alpha \prod_{l=1}^{k} \mathscr{B}_{i_l}(a_{i_l}, a_{i_l+1}) \ll (k+1)^\alpha R'^{2k} \exp [R'(a_{i_1} - a_{i_k+1})]$$

$$\times \prod_{l=1}^{k} (a_{i_l} - a_{i_l+1}) \begin{pmatrix} 2^{-k} & 1 \\ 0 & 2^{-k} \end{pmatrix} \sum_{a=0}^{\alpha} \partial_x^a, \quad (i_l \notin D,\ l=1,\dots, k).$$

Here, we have used $0 < a_i \leqq a_2 \leqq 1$ $(i \geqq 2)$.

Moreover, if i_l do not belong to D except i_1, we have the following estimate;

$$(3.18) \qquad \partial_x^\alpha \prod_{l=1}^{k} \mathscr{B}_{i_l}(a_{i_l}, a_{i_l+1}) \ll 2(k+1)^\alpha R'^{2k} \exp [R'(a_{i_1} - a_{i_k+1})]$$

$$\times \prod_{l=1}^{k} (a_{i_l} - a_{i_l+1}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sum_{a=0}^{\alpha} \partial_x^a, \quad (i_1 \in D \text{ and } i_l \notin D,\ l=2,\dots, k).$$

By (3.18), we see the following:

$$(3.19) \qquad \partial_x^\alpha \mathscr{B}_{i_1}(a_{i_1}, a_{i_1+1}) \cdots \mathscr{M}'_j(a_j, a_{j+1}) \cdots \mathscr{B}_{i_k}(a_{i_k}, a_{i_k+1})$$

$$\ll (k+2)^\alpha 2^{2-k} R'^{2k+1} \exp [R'(a_{i_1} - a_{i_k+1})]$$

$$\times (a_j - a_{j+1}) \prod_{l=1}^{k} (a_{i_l} - a_{i_l+1}) \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \sum_{a=0}^{\alpha+1} \partial_x^a,$$

$$(i_1 \in D \text{ and } i_l \notin D,\ l=2,\dots, k).$$

(3.7), (3.8), (3.18) and (3.19) bring us

$$(3.20) \qquad \partial_x^\alpha \mathscr{B}_{i_1} \cdots \mathscr{M}'_{j_1} \cdots \mathscr{M}'_{j_2} \cdots \mathscr{M}'_{j_h} \cdots \mathscr{B}_{i_k}$$

$$\ll 2^{2m}(k+h+1)^\alpha R'^{2k+h} \exp [R'(a_{i_1} - a_{i_k+1})]$$

$$\times \prod_{n=1}^{h} (a_{j_n} - a_{j_n+1}) \prod_{l=1}^{k} (a_{i_l} - a_{i_l+1}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sum_{a=0}^{\alpha+h} \partial_x^a,$$

because $\{i_1,\dots, i_k\}$ contains at most $m-1$ elements of D: $\{n_1,\dots, n_*\}$, $\mathscr{B}_p(t, a_{p+1})$ and $\mathscr{B}_q(a_q, s)$ may not be triangle even if p and q do not belong to D, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{m+1}$

$=2^m\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, and we regard the left-hand side of (3.20) as $\partial_x^\alpha[(\mathscr{B}_{i_1}\cdots)(\mathscr{B}_{n_1}\cdots)\cdots(\mathscr{B}_{n_*}\cdots)\mathscr{B}_{i_k}]$. By the inequality:

$$(3.21)\qquad \sum_{p\leqq i_1<\cdots<i_k\leqq q}\prod_{l=1}^{k}(a_{i_l}-a_{i_l+1})$$
$$=\sum_{i_1\geqq p}(a_{i_1}-a_{i_1+1})\sum_{i_2\geqq i_1+1}(a_{i_2}-a_{i_2+1})\cdots\sum_{i_k\geqq i_{k-1}+1}(a_{i_k}-a_{i_k+1})$$
$$\leqq\prod_{i=p}^{p+k-1}a_i,$$

we have the following by virtue of (3.20), (3.21), $m\leqq d_K+1$ and $a_2\leqq 1$;

$$(3.22)\qquad \partial_x^\alpha E_h(t,s)\ll 2^{2(d_K+1)}e^{R'}(2^N R')^h\prod_{i=p}^{p+h-1}a_i\sum_{k=\delta}^{r}(2^N R'^2)^k\prod_{i=p}^{p+k-1}a_i\Big(\sum_{a=0}^{\alpha+h}\partial_x^a\Big)$$
$$\ll C(d_K)(a_pR_1)^h\sum_{k=\delta}^{\infty}R_2^k\prod_{i=p}^{p+k-1}a_i\Big(\sum_{a=0}^{\alpha+h}\partial_x^a\Big),$$

where $C(d_K)=2^{2(d_K+1)}e^{R'}$, $R_1=2^NR'$, $R_2=2^NR'^2$, $\delta=1$ if $h=0$ and $\delta=0$ if $h\geqq 1$.

Let us take $p_\circ$ such that $a_{p_\circ}R_2\leqq\frac{1}{2}$ and $p_\circ>p$. Then,

$$(3.23)\qquad \sum_{k=\delta}^{\infty}R_2^k\prod_{i=p}^{p+k-1}a_i<\sum_{k=\delta}^{p_\circ-p}R_2^k\prod_{i=p}^{p+k-1}a_i+R_2^{p_\circ-p}\prod_{i=p}^{p_\circ-1}a_i\sum_{k=1}^{\infty}\Big(\frac{1}{2}\Big)^k$$
$$<\begin{cases}2a_pR_2\displaystyle\sum_{k=0}^{p_\circ-p-1}(a_pR_2)^k, & (\delta=1),\\[2ex] 2\displaystyle\sum_{k=0}^{p_\circ-p}(a_pR_2)^k, & (\delta=0).\end{cases}$$

Therefore, (3.22) is majorized as the following:

$$(3.24)\qquad \partial_x^\alpha E_h(t,s)\ll\begin{cases}C(d_K,h,N)(a_pR_2)\displaystyle\sum_{a=0}^{\alpha+h}\partial_x^a, & (h=0),\\[2ex] C(d_K,h,N)(a_pR_2)^h\displaystyle\sum_{a=0}^{\alpha+h}\partial_x^a, & (h\geqq 1).\end{cases}$$

Obviously, the right-hand side of (3.24) tends to 0 when a_p tends to 0 (that is, t tends to 0).

For $\partial_t^\alpha\partial_x^\beta E_h(t,s)$ $(\alpha\geqq 1)$, we can see (3.13) by the similar way as the above estimate, using (3.10), (3.12), and the relations

$$(3.25)\qquad \partial_t^\alpha\partial_x^\beta E_h(t,s)=\begin{cases}\partial_x^\beta[(\partial_t^\alpha\mathscr{B}_p(t,a_{p+1}))E_h(a_{p+1},s)], & \text{if } p \text{ is odd},\\ \partial_x^\beta[(\partial_t^\alpha\mathscr{M}'_p(t,a_{p+1}))E_h(a_{p+1},s)], & \text{if } p \text{ is even}.\end{cases}$$

Q. E. D.

Since $E(t,s)=I+\sum_{h=0}^{m}E_h(t,s)$ and $m\leqq d_K+2$, Lemma 3.5 implies that $E(t,s)$ converges to an element of $\mathscr{E}_t([0,a_2];\mathscr{L})$ as s tends to 0, where $\mathscr{L}$ is the space of the bounded operators on $\mathscr{E}_x(K)$. We set $E(t, 0)=\lim_{s\downarrow 0} E(t, s)$. Then, $E(t, 0) =\lim_{q\to\infty} E(t, a_q)$ is of order at most d_K+1 and $E(0, 0)=I$ because $C_\circ(d_K, p, h, 0)$ tends to 0 in (3.13) as p tends to infinity (that is, t tends to 0). Moreover, if $a_2 \leqq (\sqrt{2/R_\circ})e^{-R_\circ/2}$, b_{2i-1}^{jj} in (3.3) is positive by virtue of (3.8), ($j=1, 2$). Thus, each $\mathscr{M}_{n_1}\cdots\mathscr{M}_{*}\mathscr{M}_i\mathscr{M}_{*+1}\cdots\mathscr{M}_{n_j}$ has the true order 1 for $n_k \notin D_0=\{2i-1\geqq 3 \mid b_{2i-1}^{21}(x)\neq 0\}$, $(1\leqq k\leqq j)$ and even i. This implies that $E(t, 0)$ has the true order d_K+1 for sufficiently large t at some point $x_\circ$ in K where $d(x_\circ)=d_K$, that is, the loss of regularity is exactly d_K+1 on $\boldsymbol{R}\times K$ from 0.

The solution of (3.2) is expressed by the following:

$$v(t, x)=E(t, t_\circ; x, \partial_x)v_\circ(x)+\int_{t_\circ}^{t} E(t, s; x, \partial_x)\tilde{f}(s, x)ds, \qquad (0\leqq t_\circ<t\leqq a_2).$$

Q. E. D.

Proof of sufficiency. (2. Finite propagation speed.)

In order to see that the propagation speed is 0, we only need show that the backward Cauchy problem for $P^*=(P_{10}+B)^*$ on $[0, a_2]\times\boldsymbol{R}$ is solvable and that the backward propagation speed is zero. (See, for example, H. Kumano-go [13].) Under the condition (L_1), obviously, P^* has zero backward propagation speed in $(0, a_2]\times\boldsymbol{R}$, that is, on $[0, a_2]\times\boldsymbol{R}$. On the other hand, the backward solvability on $[0, a_2]\times\boldsymbol{R}$ can be seen by the same way as the proof of the forward solvability under the conditions (L_1) and (M_1) with respect to the backward direction. (We exchange a_{2i-1} and a_{2i} each other in the condition (M_1).) Therefore, we only need show the following lemma.

Lemma 3.6. (Invariance of the conditions (L_1) and (M_1) under the $*$-transformation.)

P^* satisfies the conditions (L_1) *and* (M_1) *with respect to the backward direction, if and only if P satisfies the conditions* (L_1) *and* (M_1).

Proof. With respect to the condition (L_1), the invariance under the $*$-transformation is easily seen. Now, we consider the condition (M_1). "$b_{2i-1}^{21}(x)=0$" means that the fundamental matrix $E(t, a_{2i})$ of the ordinary differential equation $\frac{d}{dt}u+Bu=0$ ($E(a_{2i}, a_{2i})=I$) is upper triangle at $t=a_{2i-1}$. Then, $(E(a_{2i-1}, a_{2i}))^{-1}$ is also upper triangle. By the way, $E(t, a_{2i})(E(a_{2i-1}, a_{2i}))^{-1}$ is the fundamental matrix of $\frac{d}{dt}u+Bu=0$ $(E(a_{2i-1}, a_{2i-1})=I)$ and, obviously, it is upper triangle at $t=a_{2i}$. This implies the following.

(3.26)

$$b_{2i-1}^{21\prime}(x)\equiv\int_{a_{2i-1}}^{a_{2i}} \gamma(s_1, x)ds_1+\cdots$$

$$+\int_{a_{2i-1}}^{a_{2i}}\int_{a_{2i-1}}^{s_{2k+1}}\int_{a_{2i-1}}^{s_{2k}}\cdots\int_{a_{2i-1}}^{s_2} \gamma(s_{2k+1}, x)\beta(s_{2k}, x)\cdots\gamma(s_1, x)ds_1\cdots ds_{2k}ds_{2k+1}+\cdots$$

$$=0.$$

The converse is also easily seen.

On the other hand, P^* is expressed by the following:

$P^*=I\partial_t-\begin{pmatrix}0 & 0\\ \mu & 0\end{pmatrix}\partial_x-\begin{pmatrix}0 & -\bar{\gamma}\\ \mu_x-\bar{\beta} & 0\end{pmatrix}$, ($\bar{a}$ means the complex conjugate of a).

Since $\begin{pmatrix}0 & 0\\ \mu & 0\end{pmatrix}$ is lower triangle, now, we need consider the (1, 2)-element of the backward fundamental matrix on $[a_{2i}, a_{2i-1}]\times \boldsymbol{R}$, where $\mu\equiv 0$. Then, we set

$$\begin{aligned}b_{2i-1}^{12*}=&\int_{a_{2i-1}}^{a_{2i}}\{-\bar{\gamma}(s_1, x)\}ds_1+\cdots\\ &+\int_{a_{2i-1}}^{a_{2i}}\int_{a_{2i-1}}^{s_{2k+1}}\int_{a_{2i-1}}^{s_{2k}}\cdots\int_{a_{2i-1}}^{s_2}\{-\bar{\gamma}(s_{2k+1}, x)\}\{-\bar{\beta}(s_{2k}, x)\}\cdots\\ &\{-\bar{\gamma}(s_1, x)\}ds_1\cdots ds_{2k}ds_{2k+1}+\cdots,\end{aligned}$$

and $d^*(x)=\#\{i\geqq 2\,|\,b_{2i-1}^{12*}(x)\neq 0\}$.

By virtue of the above consideration, $b_{2i-1}^{12*}(x)\equiv -\overline{b_{2i-1}^{21'}}=0$ if and only if $b_{2i-1}^{21}(x)=0$. This implies that $d(x)=d^*(x)$. Therefore, the following condition (M_1^*):

(M_1^*) For arbitrary compact set K in $\boldsymbol{R}_x$,

$$d_k^*\equiv\sup_{x\in K} d^*(x)<\infty,$$

holds good if and only if the condition (M_1) is satisfied. Q.E.D.

Proof of necessity.

Since the necessity of the condition (L_1) is already shown, we now show that $P=P_{10}+B$ is not locally hyperbolic at some point $(0, x_0)$ under the condition (L_1), supposing that the condition (M_1) fails.

Let $d_K=\infty$, for some compact K. We can find a sequence $\{x_l\}$ such that $d(x_l)\geqq l$ and which converges to x_0. We lead the contradiction, assuming that P is locally hyperbolic at $(0, x_0)$.

In the first place, we establish an apriori estimate of the solution of (3.2) for $t_0=0$.

Lemma 3.7. (Apriori estimate.)

If P is locally hyperbolic at $(0, x_0)$, there exist a positive integer N, a positive constant C and a compact set $S_0=[0, \varepsilon_0]\times[x_0-\varepsilon_0, x_0+\varepsilon_0]$ contained in ω, such that

$$|v(t, x)|_{0,S_0}\leqq C\{|v(0, x)|_{N,\varpi_0}+|\tilde{f}(t, x)|_{N,\varpi_0^+}\}. \tag{3.27}$$

This lemma easily follows through Banach's closed graph theorem.

Let us take p_0 which satisfies $a_{p_0}R\leqq 1$ and set $\varepsilon=\min\{\varepsilon_0, a_{p_0}\}$. We set $S=[0, \varepsilon]\times[x_0-\varepsilon, x_0+\varepsilon]$ and fix l such that $x_l\in(x_0-\varepsilon, x_0+\varepsilon)$ and $\#D(x_l)\geqq N+2$ where $D(x_l)=\{2i-1\,|\,a_{2i-1}\leqq\varepsilon$ and $b_{2i-1}^{21}(x_l;\, a_{2i-1}, a_{2i})\neq 0\}=\{i(1), i(2),\ldots\}$. Set $p=i(1)-1$ and $q=i(N)+1$. Both p and q are even.

From now on, we consider the solution of the following equation:

$$(3.28)\qquad \begin{cases} \{\tilde{P}_{10}(t, x; \partial_t, \partial_x)+\tilde{B}(t, x)\}v(t, x; \xi)=0, & a_{q+1}\leqq t\leqq a_p, \\ v(t, x; \xi)=\eta(t)\zeta(x)^t\Big(\int_{a_{q+1}}^t \beta(\tau, x)d\tau,\ 1\Big)e^{ix\xi}, & 0\leqq t\leqq a_{q+1}, \end{cases}$$

where $\eta(t)\in C^\infty([0, a_p])$, $\eta(t)=\begin{cases}1, & a_{q+1}-\varepsilon_1\leqq t\leqq a_p, \\ 0, & 0\leqq t\leqq a_{q+2}+\varepsilon_1,\end{cases}$ $(\varepsilon_1=(a_{q+1}-a_{q+2})/3)$, $\zeta(x)\in C_o^\infty([x_o-\varepsilon, x_o+\varepsilon])$ and $\zeta(x_l)=1$. Let us set

$$(3.29)\qquad f(t, x; \xi)=\begin{cases} 0, \quad (0\leqq t\leqq a_{q+2} \text{ and } a_{q+1}\leqq t\leqq a_p), \\ (\tilde{P}_{10}+\tilde{B})\eta(t)\zeta(x)^t\Big(\int_{a_{q+1}}^t \beta(\tau, x)d\tau,\ 1\Big)e^{ix\xi} \\ \quad =\eta'(t)\zeta(x)^t\Big(\int_{a_{q+1}}^t \beta(\tau, x)d\tau,\ 1\Big)e^{ix\xi} \\ \quad -\eta(t)\zeta(x)\gamma(t, x)\int_{a_{q+1}}^t \beta(\tau, x)d\tau\,{}^t(0, 1)e^{ix\xi}, \\ \qquad\qquad (a_{q+2}\leqq t\leqq a_{q+1}). \end{cases}$$

Then, the following estimate holds;

$$(3.30)\qquad |f(t, x; \xi)|_{N,\bar{\omega}}=O(\xi^N).$$

$f(t, x; \xi)$ belongs to $C^\infty(\bar{\omega})$ because $\operatorname{supp}\eta'(t)$ and $\operatorname{supp}\eta(t)$ are respectively contained in $[a_{q+2}+\varepsilon_1, a_{q+1}-\varepsilon_1]$ and in $[a_{q+2}+\varepsilon_1, a_p]$, and because $\gamma(t, x)$ vanishes on $t=a_{q+1}$ of infinite order.

On the other hand, $v(t, x)$ is expressed by

$$(3.31)\qquad v(t, x; \xi)=E(t, a_{q+1})^t(0, \zeta(x))e^{ix\xi}, \quad a_{q+1}\leqq t\leqq a_p,$$

especially, near $t=a_{q+1}$,

$$(3.32)\qquad =\begin{cases} \zeta(x)^t\Big(\int_{a_{q+1}}^t \beta d\tau,\ 1\Big)e^{ix\xi}, \quad a_{q+1}-\varepsilon_1\leqq t\leqq a_{q+1}, \\ \zeta(x)^t\Big(i\xi\int_{a_{q+1}}^t \mu d\tau+\int_{a_{q+1}}^t \beta d\tau,\ 1\Big)e^{ix\xi} \\ \quad +\zeta'(x)^t\Big(\int_{a_{q+1}}^t \mu d\tau,\ 0\Big)e^{ix\xi}, \quad a_{q+1}\leqq t\leqq a_q. \end{cases}$$

Then, $v(t, x; \xi)$ is connected smoothly across $t=a_{q+1}$, because $\mu(t, x)$ vanishes on $t=a_{q+1}$ of infinite order.

Now, let us consider the behaviour of $v(a_p, x_l; \xi)$ with respect to ξ. $E(a_p, a_{q+1}; x_l, \partial_x)$ is of order $N+1$ and $\sigma_{N+1}(E(a_p, a_{q+1}, x_l, \partial_x))$ is expressed by the following:

$$(3.33)\qquad \sigma_{N+1}(E(a_p, a_{q+1}, x_l, \partial_x))$$
$$\equiv \sum_{j_1,\dots,j_{N-1}} \mathcal{M}_p^1\mathcal{M}_{i(1)}\cdots\mathcal{M}_{j_1}^1\cdots\mathcal{M}_{i(2)}\cdots\mathcal{M}_{j_2}^1\cdots\mathcal{M}_{i(k)}\cdots\mathcal{M}_{j_k}^1\cdots\mathcal{M}_{i(N)}\mathcal{M}_q^1,$$
$$(\text{mod. order } N),$$

where $i(k)\in D(x_l)$, $(1\leqq k\leqq N)$, j_l is even $(1\leqq l\leqq N-1)$, $\mathscr{M}_j^1=\int_{a_{j+1}}^{a_j}\mu(\tau, x)d\tau\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\partial_x$ and $\cdots$ expresses the product of $\mathscr{M}_i(a_i, a_{i+1}; x)$ for odd $i\notin D(x_l)$. Let us set $\mathscr{M}_i(a_i, a_{i+1}; x)=(b_i^{kl}(x))_{k,l=1,2}$ for odd i and $m_j(x)=\int_{a_{j+1}}^{a_j}\mu(\tau, x)d\tau$ for even j. Then, we have

$$\sigma_1(\mathscr{M}_{i(k)}\cdots\mathscr{M}_j^1\cdots)=\begin{pmatrix}0 & b_{i(k)}^{11}\\ 0 & b_{i(k)}^{21}\end{pmatrix}m_j(\prod_{i_1} b_{i_1}^{11})(\prod_{i_2} b_{i_2}^{22})\xi, \tag{3.34}$$

where i_1 and i_2 run over $i(k)-2\leq i_1\leq j-1$ and $j+1\leq i_2\leq i(k+1)+2$ respectively, because $\mathscr{M}_j^1$ is of type $\begin{pmatrix}0 & *\\ 0 & 0\end{pmatrix}$ and $\mathscr{M}_i$ is upper triangle when i is odd and is not contained in $D(x_l)$. Therefore,

$$\begin{aligned}&\sigma_{N+1}(E(a_p, a_{q+1}; x_l, \partial_x))\\ &=m_p m_q b_{i(N)}^{21}\prod_{k=1}^{N-1}[b_{i(k)}^{21}\sum_j m_j(\prod_{i_1} b_{i_1}^{11})(\prod_{i_2} b_{i_2}^{22})]\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\xi^{N+1},\end{aligned} \tag{3.35}$$

where i_1, i_2 and j run over $i(k)+2\leqq i_1\leqq j-1$, $j+1\leqq i_2\leqq i(k+1)-2$ and $i(k)+1\leqq j\leqq i(k+1)-1$, respectively. Here, the right-hand side of (3.35) is not zero because $m_j>0$ by virtue of $\mu(t, x)>0$ in (a_{j+1}, a_j) and because $b_i^{kk}\geqq 1-|\tilde{b}_i^{kk}|\geqq 1-(a_pR)^2/2\geqq\frac{1}{2}$ by (3.8) and by the choice of a_p and because $b_{i(k)}^{21}\neq 0$. Therefore, we obtain the following;

$$\begin{aligned}|v(a_p, x_l)|&=|E(a_p, a_{q+1}; x_l, \partial_x)^t(0, \zeta(x)e^{ix\xi})|\\ &\geqq c_0\xi^{N+1}, \qquad \text{for sufficiently large } \xi,\end{aligned} \tag{3.36}$$

where $c_0>0$.

(3.30) and (3.36) are contrary to Lemma 3.7. Q.E.D.

3.4° The case where the hypothesis (H.2) fails

Secondly, we present an example for which the hypothesis (H.2) is not satisfied. It is the example 8 presented in the section 2.

Example 8.

$$P_8=I\frac{\partial}{\partial t}-\begin{pmatrix}0 & \mu(t, x)\\ \nu(t, x) & 0\end{pmatrix}\frac{\partial}{\partial x},$$

where
$$\mu(t, x)\begin{cases}>0, & \text{in } \bigcup_{i=1}^{\infty}(a_{2i+1}, a_{2i})\times\boldsymbol{R},\\ =0, & \text{otherwise,}\end{cases}$$

$$\nu(t, x)\begin{cases}>0, & \text{in } \bigcup_{i=1}^{\infty}(a_{2i}, a_{2i-1})\times\boldsymbol{R}\cup(-\infty, 0)\times\boldsymbol{R},\\ =0, & \text{otherwise,}\end{cases}$$

and $\{a_i\}$ is the same one in the example 9.

Remark that $\mu\nu\equiv 0$ and then $\lambda=0$ is the double root of $\det P_8(t, x; \lambda, 1)=0$. Let Ω be $\boldsymbol{R}^2$. Here, $T_i=\{a_i\}\times\boldsymbol{R}$, $T_0=\{0\}\times\boldsymbol{R}$, $\Omega_i=(a_{i+1}, a_i)\times\boldsymbol{R}$, $\Omega_\circ=(-\infty, 0)\times\boldsymbol{R}$ and $\Omega_w=\boldsymbol{R}^2\setminus\bigcup_{i=0}^{\infty} T_i$. Then, $\overline{\Omega_w}=\boldsymbol{R}^2$, that is, the hypothesis (H.1) is satisfied. However, $\tilde{e}(t, x)$ is extended on each $\bar{\Omega}_i$ as the following:

$$\tilde{e}(t, x)=\begin{cases} {}^t(1, 0) & \text{on} \quad \operatorname{supp}\mu, \\ {}^t(0, 1) & \text{on} \quad \operatorname{supp}\nu, \end{cases}$$

then, the hypothesis (H.2) fails.

The condition (L) is equivalent to the following:

$$(\mathrm{L}_2) \qquad \begin{cases} \gamma=0, & \text{on} \quad \operatorname{supp}\mu, \\ \beta=0, & \text{on} \quad \operatorname{supp}\nu. \end{cases}$$

In this case, similarly as in the example 10, the Cauchy problem of P_8+B is uniquely solvable in $[t_\circ, \infty)\times\boldsymbol{R}$ for $t_\circ>0$ and in $[t_\circ, 0]\times\boldsymbol{R}$ for $t_\circ<0$ under the condition (L_2). However, for arbitrary lower order term $B(t, x)$ such that P_8+B satisfies the condition (L_2), P_8+B is not locally hyperbolic at $(0, x_\circ)$. ($x_\circ$ is an arbitrary point in $\boldsymbol{R}$.)

Proposition 3.8. (Stable non-hyperbolicity of P_8.)
P_8 is stably non-hyperbolic near $(0, x_\circ)$, where $x_\circ$ is an arbitrary point in $\boldsymbol{R}$.

Proof. To see the above proposition, we only need to show that P_8+B is not locally hyperbolic at $(0, x_\circ)$ under the condition (L_2). Let $B(t, x)$ be $\begin{pmatrix}\alpha(t, x) & \beta(t, x)\\ \gamma(t, x) & \delta(t, x)\end{pmatrix}$. The Cauchy problem for P_8+B is transformed to the following by the similar way as in the reduction with respect to $P_{10}+B$;

$$(3.37) \qquad \begin{cases} \tilde{P}v\equiv(\tilde{P}_8+\tilde{B})v\equiv\dfrac{\partial}{\partial t}v-\begin{pmatrix} 0 & \tilde{\mu} \\ \tilde{\nu} & 0\end{pmatrix}\dfrac{\partial}{\partial x}v-\begin{pmatrix} 0 & \tilde{\beta} \\ \tilde{\gamma} & 0\end{pmatrix}v=\tilde{f}, \\ v(t_\circ, x)=v_\circ(x). \end{cases}$$

Here, since $\operatorname{supp}\tilde{\mu}$, $\operatorname{supp}\tilde{\nu}$, $\operatorname{supp}\tilde{\beta}$ and $\operatorname{supp}\tilde{\gamma}$ are invariant under this transformation, the condition (L_2) is also invariant.

We set $m_i(t, s; x)=\int_s^t \tilde{\mu}(\tau, x)d\tau$, $b_i(t, s; x)=\int_s^t \tilde{\beta}(\tau, x)d\tau$, $\mathscr{M}_i^1(t, s; x, \partial_x)=\begin{pmatrix}0 & m_i(t, s; x)\\ 0 & 0\end{pmatrix}\partial_x$, $\mathscr{M}_i^0(t, s; x)=\begin{pmatrix}1 & b_i(t, s; x)\\ 0 & 1\end{pmatrix}$, $\mathscr{M}_i(t, s; x, \partial_x)=\mathscr{M}_i^1(t, s; x, \partial_x)+\mathscr{M}_i^0(t, s; x)$ for $a_{i+1}\leqq s<t\leqq a_i$ when i is even, and $m_i(t, s; x)=\int_s^t \tilde{\nu}(\tau, x)d\tau$, $b_i(t, s; x)=\int_s^t \tilde{\gamma}(\tau, x)d\tau$, $\mathscr{M}_i^1(t, s; x, \partial_x)=\begin{pmatrix}0 & 0\\ m_i(t, s; x) & 0\end{pmatrix}\partial_x$, $\mathscr{M}_i^0(t, s; x)=\begin{pmatrix}1 & 0\\ b_i(t, s; x) & 1\end{pmatrix}$, $\mathscr{M}_i(t, s; x, \partial_x)=\mathscr{M}_i^1(t, s; x, \partial_x)+\mathscr{M}_i^0(t, s; x)$ for $a_{i+1}\leqq s<t\leqq a_i$ when i is odd. For $a_{q+1}\leqq t_\circ=s<a_q\leqq a_{p+1}<t\leqq a_p$, the fundamental matrix $E(t, s)=E(t, s; x, \partial_x)$ of the Cauchy problem (3.37) is given by

$$(3.38) \qquad E(t, s)=\mathscr{M}_p(t, a_{p+1})\prod_{i=p+1}^{q-1}\mathscr{M}_i(a_i, a_{i+1})\mathscr{M}_q(a_q, s).$$

The order of $E(t, s)$ is $q-p+1$, and moreover, $\sigma_{q-p+1}(E(t, s))$ has the following form:

$$(3.39)\quad \sigma_{q-p+1}(E(a_p, a_{q+1}))=\prod_{i=p}^{q} \mathscr{M}_i^1(a_i, a_{i+1})=\prod_{i=p}^{q} m_i(a_i, a_{i+1})\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\xi^{q-p+1},$$

if p and q are even. Since $m_i(a_i, a_{i+1})>0$, $q-p+1$ is the true order and this is independent of the choice of $B(t, x)$ and x. Therefore, it is shown by the same way as the proof of the necessity in Proposition 3.3 that P_8+B is not locally hyperbolic at $(0, x_0)$. Q.E.D.

The examples 8 and 10 suggest that we need introduce some additional conditions to (L) for the hyperbolicity when the principal part has the coefficients in C^∞. They must be very complicated because the roles of the components in the lower order term are different each other. For example, in the example 8, if we relax the requirement on μ and ν as the following:

$$\operatorname{supp}\mu=\overline{\bigcup_{i=1}^{\infty}(a_{2i+1}, a_{2i})}\times \boldsymbol{R} \quad\text{and}\quad \operatorname{supp}\nu=\overline{[\bigcup_{i=1}^{\infty}(a_{2i}, a_{2i-1})\cup(-\infty, 0)]}\times \boldsymbol{R},$$

P_8 may not be stably non-hyperbolic and we can obtain the necessary and sufficient condition for the hyperbolicity of P_8+B but it is very complicated.

Appendices

§ A.1. Differences between some notions on the hyperbolicity

In this section, we show the differences between some notions on the hyperbolicity through some examples.

A.1.1° Forward and backward local hyperbolicities

For the present, in order to speak precisely, we add the word "forward" to the terminologies defined in the definitions 4, 5, 6, 7 and 8. (For example, "forward locally hyperbolic".) We say that the Cauchy problem (1)–(2) is backward locally $\mathscr{E}$ well-posed at (t_0, x_0) if we have the unique solution $u(t, x)$ in $\mathscr{E}(\omega_{t_0}^-)$ for arbitrary $u_0(x)$ in $\mathscr{E}(\overline{\Omega_{t_0}})$ and arbitrary $f(t, x)$ in $\mathscr{E}(\overline{\Omega_{t_0}^-})$, where ω is a neighbourhood of (t_0, x_0). Moreover, we say that P is backward locally hyperbolic in Ω, if the Cauchy problem (1)–(2) for P is backward locally $\mathscr{E}$ well-posed at every point in Ω. We show that the forward local hyperbolicity in Ω does not imply the backward local hyperbolicity in Ω.

We consider the operator P_8. For convenience sake, we change the variable t to $-t$.

Example A.

$$P_A=ID_t-\begin{pmatrix} 0 & \mu(t, x) \\ \nu(t, x) & 0 \end{pmatrix}D_x,$$

$$\text{where} \quad \mu(t, x)\begin{cases} >0, & \text{in} \quad \bigcup_i (a_{2i}, a_{2i+1}) \times \boldsymbol{R} \cup (0, \infty) \times \boldsymbol{R}, \\ =0, & \text{otherwise,} \end{cases}$$

$$\nu(t, x)\begin{cases} >0, & \text{in} \quad \bigcup_i (a_{2i-1}, a_{2i}) \times \boldsymbol{R}, \\ =0, & \text{otherwise.} \end{cases}$$

Here, $\{a_i\}$ is a strictly increasing sequence and it converges to 0. We regard a_1 as $-\infty$.

Proposition A.1. *P_A+B is forward locally hyperbolic in Ω under the condition* (L) *but P_A is backward stably nonhyperbolic in Ω, if $\Omega \cap \{t=0\} \neq \phi$.*

Proof. The former is obviously seen because, for each $(t_\circ, x_\circ)$ in $\boldsymbol{R}^2$, there exists a neighbourhood ω such that $R(t, x) \equiv \text{rank}\begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix} = 1$ in $\omega_{t_\circ}^+ \backslash \omega_{t_\circ}$.

The latter is proved in Proposition 3.8. Q.E.D.

A.1.2° Hyperbolicity and local hyperbolicity

From now on, we omit the word "forward" again. Now, we show that the local solvability at every point does not imply the semi-global solvability.

Proposition A.2. *P_A+B is locally hyperbolic in Ω under the condition* (L) *but it is not hyperbolic in Ω for any lower order term B if $\Omega \cap \{t=0\} \neq \phi$.*

Proof. The former is the result in Proposition A.1 and the latter is provable by the same way as the proof of Proposition 3.8. Q.E.D.

A.1.3° $\mathscr{E}$-hyperbolicity and $\gamma^{(\infty)}$-hyperbolicity

Many authors set $\gamma^{(\infty)} = \mathscr{E}$ when they treated $\gamma^{(\kappa)}$ well-posedness. However, in this article, we set $\gamma^{(\infty)} = \bigcup_{\kappa \geq 1} \gamma^{(\kappa)}$. Obviously, $\gamma^{(\infty)} \subsetneqq \mathscr{E}$.

In order to make clear the word, we use the terminology "$\mathscr{E}$-hyperbolic" in stead of "hyperbolic" in the definition 2. We say that the Cauchy problem (1)–(2) is $\gamma^{(\kappa)}$ well-posed in Ω, if we have the unique solution $u(t, x)$ in $\mathscr{E}(\Omega_{t_\circ}^+)$ for arbitrary $u_\circ(x)$ in $\gamma^{(\kappa)}(\Omega_{t_\circ})$ and arbitrary $f(t, x)$ in $\gamma^{(\kappa)}(\Omega_{t_\circ}^+)$ and for every $t_\circ$. We say that P is $\gamma^{(\kappa)}$-hyperbolic in Ω, if the Cauchy problem for P is $\gamma^{(\kappa)}$ well-posed in Ω. Therefore, if P is $\mathscr{E}$-hyperbolic, it is also $\gamma^{(\kappa)}$-hyperbolic. However, the converse is not true.

Proposition A.3. *P_8 is $\mathscr{E}$-stably non-hyperbolic in Ω, if $\Omega \cap \{t=0\} \neq \phi$. However, $P = P_8 + B$ is $\gamma^{(\infty)}$-hyperbolic under the condition* (L) *if the coefficients of P belong to $\gamma^{(\infty)}(\Omega)$, where the solution $u(t, x)$ also belongs to $\gamma^{(\infty)}(\Omega_{t_\circ}^+)$.*

Proof. The former is given in Proposition 3.8 and the latter is provable by the same way as the proof of Proposition 3.3. (See also S. Tarama [29].)

Q.E.D.

In the forthcoming paper, we shall discuss the best possible space of functions where the Cauchy problem (1)–(2) is well-posed under the condition (L). (See [27] and [28].)

§ A.2. Formulation in case of $N=2$

The assumptions 1 and 2 restrict the number n of the dimension of x-space related to the order N of the system. Especially, in case of $N=2$, n must be 1. However, in such case, we can have the similar results as Theorem 1.3, 1.5, 3.1, and 3.2 for arbitrary n under the following formulation.

Assumption 1.A. The equation $\det P_p(t, x; \tau, \xi)=0$ has a double root $\lambda(t, x; \xi)$.

$\lambda(t, x; \xi)$ is automatically real because the coefficients of P_p are all real. Moreover, it has the form $\sum_{i=1}^{n} \lambda^i(t,x)\xi_i$. Let us set $A^i(t,x)-\lambda^i(t,x)I=\begin{pmatrix} a_i(t, x) & b_i(t, x) \\ c_i(t, x) & d_i(t, x)\end{pmatrix}$, then a_i, b_i, c_i and d_i satisfy the following:

$$\text{(A.1)} \qquad d_i(t, x)\equiv -a_i(t, x), \quad (a_i(t, x))^2+b_i(t, x)c_i(t, x)\equiv 0.$$

Under the assumption 1.A, Proposition 1.1 and Corollary 1.2 are also valid. Put $\tilde{R}(t, x)=\sum_{i=1}^{n}(|b_i(t, x)|+|c_i(t, x)|)$. When $\sum_{i=1}^{n}|b_i(t_o, x_o)|\neq 0$, P_p is expressed by $ID_t-\sum_{i=1}^{n}\lambda_i(t, x)D_{x_i}-\begin{pmatrix}\tilde{a}(t, x) & 1 \\ \tilde{c}(t, x) & -\tilde{a}(t, x)\end{pmatrix}\cdot\sum_{i=1}^{n} b_i(t, x)D_{x_i}$ in a neighbourhood ω of (t_o, x_o), where $\tilde{a}(t, x)$ and $\tilde{c}(t, x)$ belong to $C^\infty(\omega)$, and when $\sum_{i=1}^{n}|c_i(t_o, x_o)|\neq 0$, P_p is expressed by $ID_t-\sum_{i=1}^{n}\lambda_i(t, x)D_{x_i}-\begin{pmatrix}\tilde{a}'(t, x) & \tilde{b}'(t, x) \\ 1 & -\tilde{a}'(t, x)\end{pmatrix}\cdot\sum_{i=1}^{n} c_i(t, x)D_{x_i}$ in a neighbourhood ω' of (t_o, x_o), where $\tilde{a}'(t, x)$ and $\tilde{b}'(t, x)$ belong to $C^\infty(\omega')$. These are derived from the assumption 1.A and the fact that $\boldsymbol{C}[\xi_1,\dots, \xi_n]$ is a unique factorization ring. Therefore, we can take the real unit eigen-vector $\tilde{e}(t, x)$ on the region where $\tilde{R}\neq 0$. Here, $\tilde{e}$ is independent of ξ. Let G be an arbitrary connected component of $\{(t, x)\in\Omega;\ \tilde{R}(t, x)\neq 0\}$. We have the following theorem corresponding to Theorem 1.3.

Theorem 2.1.A. (Smoothness of $\tilde{e}(t, x)$ along the bicharacteristic curves.)

In addition to the assumption 1.A, *suppose that the condition* (L) *is satisfied. Then,*

(i) *If $\pi(s)$ is a bicharacteristic curve belonging to $\lambda(t, x; \xi)$ such that $\pi(s)\in G\times\boldsymbol{R}^n\backslash\{O\}$ when $0\leqq s<s'$ and $\pi(s')\in\partial G\times\boldsymbol{R}^n\backslash\{O\}$, then $\tilde{e}(t, x)$ can be extended as the real unit eigen-vector on $\omega\cap\bar{G}$ in C^∞-class, where ω is a neighbourhood of $\{\pi(s);\ 0\leqq s\leqq s'\}$ in Ω.*

(ii) *When $\tilde{R}\neq 0$ in $\omega\backslash\partial G$ (if necessary, shrinking ω) we can take $\tilde{e}^-(t, x)$ in $C^\infty(\omega\cap\bar{G})$ and $\tilde{e}^+(t, x)$ in $C^\infty(\omega\cap G^c)$ by* (i), *If $\lim_{s\to s'-0}\tilde{e}^-(\pi(s))=\lim_{s\to s'+0}\tilde{e}^+(\pi(s))$,*

$$\tilde{e}=\begin{cases}\tilde{e}^-(\pi(s)), & (s\leqq s'),\\ \tilde{e}^+(\pi(s)), & (s\geqq s'),\end{cases}$$

is infinitely differentiable at $s=s'$.

Moreover, if $\tilde{e}^-(t, x)$ and $\tilde{e}^+(t, x)$ coincide on $\omega\cap\partial G$,

$$\tilde{e}(t, x)=\begin{cases}\tilde{e}^-(t, x), & (t, x)\in\omega\cap G,\\ \tilde{e}^+(t, x) & (t, x)\in\omega\cap\overline{G^c},\end{cases}$$

is, in reality, infinitely differentiable on $\omega \cap \partial G$, *that is,* $\tilde{e}(t, x)$ *belongs to* $C^\infty(\omega)$.

By Theorem 2.1.A and Corollary 1.2, we have the following theorem on the necessary condition for the weak hyperbolicity.

Theorem 2.2.A. (Necessity of the smoothness of $\tilde{e}(t, x)$.)

Suppose the assumption 1.A. *If* P_p *is weakly hyperbolic in* Ω, *we can take the unit eigen-vector* $\tilde{e}(t, x)$ *which satisfies the properties* (i) *and* (ii) *in Theorem* 2.1.A.

In order to obtain a similar result as Theorem 1.5, we need some additional assumptions besides the assumption 1.A. (See the example 4.)

Theorem 2.3.A. (Analiticity of $\tilde{e}(t, x)$.)

Suppose the assumption 1.A *and that the coefficients of* P_p *are all real analytic. Then, only one of the following two cases arises;*

(I) $\tilde{R} \equiv 0$ *in* Ω.

(II) $\tilde{R} \neq 0$ *in* Ω *except an analytic set. Here, if the zero set of* $\tilde{R}(t, x)$ *does not contain the projection of any bicharacteristic curve to* Ω *and the condition* (L) *is satisfied, we can take* $\tilde{e}(t, x)$ *in the real analytic class on* Ω.

Remark. In the situation under the assumption 1.A, the bicharacteristic curves are independent of ξ, then the projection of them to Ω is a family of curves. Under the additional assumption in the case II, the projection of the bicharacteristic curves may cross or contact the zero set of $\tilde{R}(t, x)$.

Theorem 2.3.A brings us the following theorem.

Theorem 2.4.A. ((Local) hyperbolicity in the case with real analytic coefficients.)

Under the assumption 1.A, *suppose that all coefficients of* P_p *are real analytic and that the zero set of* $\tilde{R}(t, x)$ *does not contain the projection of any bicharacteristic curve to* Ω *when* $\tilde{R}(t, x) \not\equiv 0$. *Then, the condition* (L) *is necessary and sufficient for the* (*local*) *hyperbolicity of* P *in* Ω.

Here, the loss of regularity in Ω *is* 0 *in the case I and is* 1 *in the case II.*

Remark. Of course, for the hyperbolicity, we assume the boundedness of the coefficients of P_p in Ω.

For the case with C^∞-coefficients, we introduce an assumption corresponding to the assumption 2.

Assumption 2.A. The boundary of each connected component of the non-zero set of $\tilde{R}(t, x)$ lies on some disjoint space-like hypersurfaces $\{T_l\}$, in general as a subset.

$\{T_l\}$ divides Ω into a family of the open connected subdomains $\{\Omega_k\}$ and a closed set Σ, where $\tilde{R}(t, x) \equiv 0$ or $\equiv 1$ on each Ω_k, $\Sigma \supseteq \bigcup_l T_l$ and $\mathring{\Sigma} = \phi$.

Theorem 2.5.A. ((Local) hyperbolicity in the case with C^∞-coefficients.)

Under the assumptions 1.A, 2.A *and* 3, P *is* (*locally*) *hyperbolic in* Ω, *if and* *only if the condition* (L) *holds good in* Ω.

Here, the loss of regularity on K *from* t_0 *is at most* $\#\{\Omega_k;\ \Omega_k \cap K_{t_0}^{+} \neq \phi$ *and* $\tilde{R}(t, x) \neq 0$ *on* $\Omega_k\}$.

Remark 1. Of course, for the hyperbolicity, we assume the boundedness of the coefficients of P_p.

Remark 2. Under the assumptions in Theorem 2.5.A, there are $\tilde{a}^k(t, x)$, $\tilde{b}^k(t, x)$ and $\tilde{c}^k(t, x)$ in $C^\infty(\overline{\Omega_k})$ such that $P_p \equiv I(D_t - \sum_{i=1}^{n} \lambda^i(t, x)D_{x_i}) - \tilde{A}^k(t, x) \sum_{i=1}^{n} (b_i(t, x) - c_i(t, x))D_{x_i}$ on $\overline{\Omega_k}$, where $\tilde{A}^k(t, x) = \begin{pmatrix} \tilde{a}^k(t, x) & \tilde{b}^k(t, x) \\ \tilde{c}^k(t, x) & -\tilde{a}^k(t, x) \end{pmatrix}$ and $\tilde{A}^k(t, x) \equiv 0$ or $\neq 0$ on $\overline{\Omega_k}$.

The theorems in this section are provable by the same ways as the proofs in the sections 1 and 5.

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