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A class of imperfect prime ideals having the equality of ordinary and symbolic powers

By

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1. Introduction.

Given a polynomial ring R over a field, we are interested in prime ideals $\mathfrak{p}\subset R$ having the following property:
(A) $\mathfrak{p}^n=\mathfrak{p}^{(n)}$ for every positive integer n, where $\mathfrak{p}^{(n)}$ denotes the n-th symbolic power of $\mathfrak{p}$, i.e. the $\mathfrak{p}$-primary component of $\mathfrak{p}^n$.

In [5, Theorem 1], Hochster proved that (A) is equivalent to each of the following properties:

(B) $gr_{\mathfrak{p}}(R):=\bigoplus_{n=0}^{\infty}\mathfrak{p}^n/\mathfrak{p}^{n+1}$, the associated graded ring of R with respect to $\mathfrak{p}$, is a domain.
(C) The Rees ring $R[T, \mathfrak{p}T^{-1}]$, the subring of $R[T, T^{-1}]$ generated over R by the indeterminate T and the elements aT^{-1} with $a\in\mathfrak{p}$, is a unique factorization domain.

On the other hand, Samuel had conjectured that a unique factorization domain is a Cohen-Macaulay ring. Thus, it may be possible that (A) or (B) implies the Cohen-Macaulay property of $gr_{\mathfrak{p}}(R)$ because, by [6, Theorem 4.11], the Cohen-Macaulay property of $gr_{\mathfrak{p}}(R)$ is equivalent to the Cohen-Macaulay property of $R[T, \mathfrak{p}T^{-1}]$. If we have a prime ideal $\mathfrak{p}\subset R$ with (A) then we can construct either a Cohen-Macaulay graded domain or a counter-example to Samuel's conjecture.

Until now, beside some solitary examples, only two classes of prime ideals $\mathfrak{p}$ with (A) in polynomial rings over a field have been known:
1) $\mathfrak{p}$ is a complete intersection prime (see, e.g., [5, (2.1)]).
2) $\mathfrak{p}$ is generated by the $r\times r$ minors of an $r\times s$ matrix of indeterminates, $r\geqq s$ (see [5, (2.2)], [14] or [2]).

By all known prime ideals $\mathfrak{p}$ with (A) $gr_{\mathfrak{p}}(R)$ is always a Choen-Macaulay domain. Note that Nagata had raised the question of whether the zero-graded part of a positively graded Cohen-Macaulay ring is a Cohen-Macaulay ring [10, Question 3]. So one might also expect that (A) implies the Cohen-Macaulay property of $R/\mathfrak{p}$, the zero-graded part of $gr_{\mathfrak{p}}(R)$. But, like Nagata's question which was negatively answered in [10], that is not true. The first counter-example for that was shown by Hochster [5, (2.3)], and an another can be found in [13]. However, in these two examples, using [12, Lemma, p. 740], one can easily see that the local ring of $R/\mathfrak{p}$ at the origin is a Buchsbaum ring. Here we have to emphasize that the Buchsbaum rings generalize the Cohen-Macaulay rings in a quite natural way. (See [11] or [12] for definition and further informations; notice that in [11] one used the term of I-rings instead of Buchsbaum rings.)

Recall that an ideal $\mathfrak{a}\subset R$ is perfect (i.e. $\mathrm{dh}_R R/\mathfrak{a}=\mathrm{grade}\,\mathfrak{a}$) if and only if $R/\mathfrak{a}$ is a Cohen-Macaulay ring. We will give, in every polynomial ring $k[X]$ of $2r+2$ indeterminates over an arbitrary field, $r\geqq 2$, an imperfect homogeneous prime ideal P of dimension $r+2$ having the equality of ordinary and symbolic powers such that $gr_P(k[X])$ is a Cohen-Macaulay domain and $k[X]_{(X)}/(P)$ is a non-Buchsbaum ring of depth 3.

2. Statements about P.

Let $X=\{x_{ij}\,;\, i=1, 2 \text{ and } j=1, \cdots, r\}\cup\{x_1, x_2\}$ be a set of indeterminates. Let

$$p_{ij}=x_{1i}x_{2j}-x_{1j}x_{2i}$$

$$q_{ij}=x_1x_{2i}x_{2j}-x_2x_{1i}x_{1j}$$

for all $i, j=1, \cdots, r$. We define P to be the ideal in $k[X]$ generated by all elements p_{ij} and q_{ij}. P has the following geometrical meaning:

Proposition 1. *Let u be an indeterminate. Let Q be the ideal in $k[X, u]$ generated by the 2×2 minors of the matrix*

$$\begin{pmatrix} x_{11} \cdots x_{1r} & x_1 & u \\ x_{21} \cdots x_{2r} & u & x_2 \end{pmatrix}.$$

Then Q is a prime ideal and P is the defining prime ideal of the projection of the algebraic variety in $k^{2r+2}\times k$ determined by Q on the first factor (*i.e. $Q\cap k[X]=P$; see* [7, Chap. IV, § 2]).

Let A denote the local ring $k[X]_{(X)}$ and $\mathfrak{m}$ its maximal ideal. Let $H^i_{\mathfrak{m}}(M)$ denote the i-th local cohomology group of a finitely generated A-module M. Let X_1 and X_2 denote the sets $\{x_{11}, \cdots, x_{1r}\}$ and $\{x_{21}, \cdots, x_{2r}\}$, respectively. Then, considering the ring structure of $A/(P)$ we obtain:

Proposition 2. *$H^i_{\mathfrak{m}}(A/(P))=0$ for $i\neq 3$, $r+2$, and $H^3_{\mathfrak{m}}(A/(P))\cong H^2_{\mathfrak{m}}(A/(X_1, X_2))$.*

Since $H^2_{\mathfrak{m}}(A/(X_1, X_2))$ is isomorphic to the injective hull of k over $k[[x_1, x_2]]$ ([3, p. 67]), which is not a vector space over k, $A/(P)$ is not a Buchsbaum ring by [12, Corollary 1.1]. Moreover, by [3, Corollary 3.10], from Proposition 2 we also get depth $A/(P)=3$.

Let $Y=\{y_{ij}\,;\, 1\leqq i<j\leqq r\}$ and $Z=\{z_{ij}\,;\, 1\leqq i\leqq j\leqq r\}$ be sets of indeterminates. Let $y_{ii}=0$, $y_{ji}=-y_{ij}$, and $z_{ji}=z_{ij}$ for all $i=1, \cdots, r$ and $i<j\leqq r$. Let

$$a_{ijl}=x_{1i}y_{jl}-x_{1j}y_{il}+x_{1l}y_{ij}$$

$$b_{ijl}=x_{2i}y_{jl}-x_{2j}y_{il}+x_{2l}y_{ij}$$

$$c_{ijlm}=y_{im}y_{jl}-y_{jm}y_{il}+y_{lm}y_{ij}$$

$$d_{ijlm}=y_{jl}z_{im}-y_{jm}z_{il}-y_{lm}z_{ij}$$

$$f_{ijl}=x_{1i}z_{jl}-x_{1j}z_{il}-x_1x_{2l}y_{ij}$$

$$g_{ijl}=x_{2i}z_{jl}-x_{2j}z_{il}-x_2x_{1l}y_{ij}$$

$$h_{ijlm}=z_{im}z_{jl}-z_{il}z_{jm}-x_1x_2y_{lm}y_{ij}$$

for all $i, j, l, m=1, \cdots, r$. Let I denote the ideal in $k[X, Y, Z]$ generated by all elements $p_{ij}, q_{ij}, a_{ijl}, b_{ijl}, c_{ijlm}, d_{ijlm}, f_{ijl}, g_{ijl}, h_{ijlm}$. Using the same technique employed in [4], we can show that I is a perfect prime ideal. Thus we get:

Proposition 3. *$gr_P(k[X])\cong k[X, Y, Z]/I$ and it is a Gorenstein domain.*

As we already mentioned at the beginning of § 1, the fact that $P^n=P^{(n)}$ for every positive integer n is only a consequence of Proposition 3.

3. Proofs of the Propositions.

Proof of Proposition 1. Let v be a new indeterminate. Let Q_1 denote the ideal in $k[X, u, v]$ generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_{11} \cdots x_{1r} & x_1 & u \\ x_{21} \cdots x_{2r} & v & x_2 \end{pmatrix}.$$

By [4, Theorem 1], Q_1 is a prime ideal with $\operatorname{ht} Q_1=r+1$. Let Q_2 denote the ideal $(Q_1, u-v, x_1-x_{2r}, x_{1r}-x_{2(r-1)}, \cdots, x_{12}-x_{21})$. Then $k[X, u, v]/Q_2$ is isomorphic to the coordinate ring of the Veronese variety $V_{2,r+1}$; see [1, § 4]. Hence Q_2 is a prime ideal and

$$\begin{aligned}\operatorname{ht} Q_2&=\dim k[X, u, v]-\dim V_{2,r+2}\\&=2(r+2)-2=2r+2\,.\end{aligned}$$

Since Q_1, Q_2 are homogeneous prime ideals with $\operatorname{ht} Q_2/Q_1=\operatorname{ht} Q_2-\operatorname{ht} Q_1=r+1$ and Q_2/Q_1 is generated by the $r+1$ elements $u-v, x_1-x_{2r}, x_{1r}-x_{2(r-1)}, \cdots, x_{12}-x_{21}$, we can conclude that $Q_1\subset(Q_1, u-v)\subset(Q_1, u-v, x_1-x_2)\subset \cdots \subset Q_2$ is a chain of prime ideals. From this it follows especially that $k[X, u, v]/(Q_1, u-v)$ is a domain of dimension $r+2$. But $k[X, u, v]/(Q_1, u-v)\cong k[X, u]/Q$. Hence Q is a prime ideal with

$$\begin{aligned}\operatorname{ht} Q&=\dim k[X, u]-\dim k[X, u, v]/(Q_1, u-v)\\&=2r+3-(r+2)=r+1\,.\end{aligned}$$

As a consequence of this, $\operatorname{ht} Q\cap k[X]\leqq r$. Further, it can be easily checked that

$P\subseteqq Q\cap k[X]$. Thus, to prove $Q\cap k[X]=P$ it suffices to show that P is a prime ideal with ht $P=r$. For that we have the following relations:

$$x_{11}p_{ij}=x_{1i}p_{1j}-x_{1j}p_{1i}$$

$$x_{11}q_{ij}=x_{1i}q_{1j}-x_{1}x_{2j}q_{1i}\,.$$

From these relations we see that $Pk[X, x_{11}^{-1}]$ can be generated by the elements $p_{12}, \cdots, p_{1r}, q_{11}$. On the other hand, eliminating $x_{22}, \cdots, x_{2r}, x_2$ by the help of these elements we also see that $k[X, x_{11}^{-1}]/(p_{12}, \cdots, p_{1r}, q_{11})\cong k[X_1, x_{21}, x_1, x_{11}^{-1}]$. Hence, $Pk[X, x_{11}^{-1}]$ must be a prime ideal of height r and x_2 is not a zerodivisor on $Pk[X, x_{11}^{-1}]$. Let P' denote the inverse image of $Pk[X, x_{11}^{-1}]$ in $k[X]$. Then P' is also a prime ideal with ht $P'=r$ and x_2 is not a zerodivisor on P', i.e. $P':x_2=P'$. Further, since $x_{11}^{n}P'\subseteqq P$ for some large n, $P'\subseteqq(P, x_2):x_{11}^{n}$. Note that (P, x_2) has the primary decomposition $(P, x_2, (X_2)^2)\cap(P, x_1, x_2)$, where $(P, x_2, (X_2)^2)$ is a (X_2, x_2)-primary ideal and (P, x_1, x_2) is a prime ideal ([4, Theorem 1]), and that x_{11} is not a zerodivisor on (X_2, x_2) and (P, x_1, x_2). So $(P, x_2):x_{11}^{n}=(P, x_2)$. Hence, $P'\subseteqq(P, x_2)$ or $P'=P+x_2(P':x_2)=P+x_2P'$. Now, applying Nakayama's lemma we get $P'=P$, which shows that P is a prime ideal with ht $P=r$. The proof for Proposition 1 is completed.

To prove Proposition 2 and Proposition 3 we prepare some lemmas. Let R be an arbitrary local ring with the maximal ideal $\mathfrak{q}$. Then we have two well-known lemmas about Cohen-Macaulay R-modules:

Lemma 4. *A finitely generated R-module M is Cohen-Macaulay if and only if $H_{\mathfrak{q}}^{i}(M)=0$ for all $i=0, \cdots,$ dim $M-1$.*

Proof. It follows immediately from [3, Corollary 3.10].

Lemma 5. *Let $0\to M'\to M\to M''\to 0$ be an exact sequence of finitely generated R-modules. Then*

(i) *M' is Cohen-Macaulay if M, M'' are Cohen-Macaulay with* dim $M''\geqq$ dim $M-1$.

(ii) *M is Cohen-Macaulay if M', M'' are Cohen-Macaulay with* dim $M''=$ dim M.

Proof. Notice that dim $M=\max\{\dim M', \dim M''\}$ by [8, (12.D) and (12.H)]. Then we easily get the statements of Lemma 5 from Lemma 4 by considering the local cohomology sequence

$$\cdots\longrightarrow H_{\mathfrak{q}}^{i-1}(M'')\longrightarrow H_{\mathfrak{q}}^{i}(M')\longrightarrow H_{\mathfrak{q}}^{i}(M)\longrightarrow H_{\mathfrak{q}}^{i}(M'')\longrightarrow\cdots$$

The following lemma will play an important role in the proofs of Proposition 2 and Proposition 3:

Lemma 6. *Let P_j denote the ideal (P, x_{1j}, x_{2j}) in A, $j=1, \cdots, r$. Then $P_j/(P)$ is a Cohen-Macaulay A-module of dimension $r+2$.*

Proof. By permutation it suffices to show Lemma 6 for $j=r$. If $r=1$, $P_1/(P)=(x_{11}, x_{21})/(q_{11})$ and the statement follows immediately from Lemma 5(i) by considering the exact sequence

$$0 \longrightarrow (x_{11}, x_{21})/(q_{11}) \longrightarrow A/(q_{11}) \longrightarrow A/(x_{11}, x_{21}) \longrightarrow 0 .$$

Let $r>1$. Note that $(P, x_{11})/(P)\cong A/((P): x_{11})=A/(P)$ and $(P_r, x_{11})/P_r\cong A/(P_r: x_{11})=A/P_r$. We construct the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_r/(P) & \longrightarrow & P_1\cap P_r/(P) & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow & \overset{\alpha}{\searrow} & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A/(P) & \longrightarrow & P_1/(P) & \longrightarrow & P_1/(P, x_{11}) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A/P_r & \longrightarrow & (P_r, x_{11}, x_{21})/P_r & \longrightarrow & (P_r, x_{11}, x_{21})/(P_r, x_{11}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where α is induced by the multiplication with x_{11} and E denotes the module $P_1\cap(P_r, x_{11})/(P, x_{11})$. It can be easily seen that $E\cong(P_r, x_{11}): x_{21}/(P, x_{11}): x_{21}$ and that

$$(P_r, x_{11})=(P_r, x_{11}, x_{21})\cap(X_1, x_1, x_{2r})\cap(X_1, x_{21}^2, x_{22}, \cdots, x_{2r})$$

$$(P, x_{11})=(P, x_{11}, x_{21})\cap(X_1, x_1)\cap(X_1, x_{21}^2, x_{22}, \cdots, x_{2r});$$

hence

$$(P_r, x_{11}): x_{21}=(X_1, x_1, x_{2r})\cap(X_1, X_2)$$

$$(P, x_{11}): x_{21}=(X_1, x_1)\cap(X_1, X_2),$$

therefore $E\cong(X_1, x_1, x_{2r})\cap(X_1, X_2)/(X_1, x_1)\cap(X_1, X_2)\cong A/((X_1, x_1)\cap(X_1, X_2)): x_{2r}=A/(X_1, X_2)$, which is a Cohen-Macaulay module of dimension $r+1$. Further, by induction we may also assume that $(P_r, x_{11}, x_{21})/P_r$ is a Cohen-Macaulay module of dimension $r+1$. Thus, applying Lemma 4 to E and $(P_r, x_{11}, x_{21})/P_r$, we get the following commutative diagram:

$$\begin{array}{ccccc}
 & & & & H_{\mathfrak{m}}^{i-1}((P_r, x_{11}, x_{21})/P_r)=0 \\
 & & & & \downarrow \\
0=H_{\mathfrak{m}}^{i-1}(E) & \longrightarrow & H_{\mathfrak{m}}^{i}(P_r/(P)) & \longrightarrow & H_{\mathfrak{m}}^{i}(P_1\cap P_r/(P)) \\
 & & \downarrow \beta_i & \overset{\alpha_i}{\searrow} & \downarrow \\
 & & H_{\mathfrak{m}}^{i}(A/(P)) & \longrightarrow & H_{\mathfrak{m}}^{i}(P_1/(P))
\end{array}$$

for all $i=0, \cdots, r+1$. This diagram shows that α_i and β_i are injective for all $i=0, \cdots, r+1$. Now we consider the commutative diagram:

$$\begin{array}{ccccc}
H^i_{\mathfrak{m}}(P_r/(P)) & \xrightarrow{\alpha_i} & H^i_{\mathfrak{m}}(P_1/(P)) & & \\
& \searrow^{\beta_i} & & \searrow^{\gamma_i} & \\
\wr\Vert & & H^i_{\mathfrak{m}}(A/(P)) & \xrightarrow{x_{11}} & H^i_{\mathfrak{m}}(A/(P)) \\
& \nearrow & & \nearrow & \\
H^i_{\mathfrak{m}}(P_r/(P)) & \xrightarrow{x_{11}} & H^i_{\mathfrak{m}}(P_r/(P)) . & &
\end{array}$$

Like β_i, γ_i is also injective, hence $x_{11}\beta_i=\gamma_i\alpha_i$ is injective, too. From this we can conclude that x_{11} is not a zerodivisor of $H^i_{\mathfrak{m}}(P_r/(P))$, or, since every element of $H^i_{\mathfrak{m}}(P_r/(P))$ is annihilated by some power of x_{11}, $H^i_{\mathfrak{m}}(P_r/(P))=0$ for $i=0, \cdots, r+1$. Therefore $P_r/(P)$ is a Cohen-Macaulay module by Lemma 4, where $\dim P_r/(P)=r+2$ is evident.

Now we prove Proposition 2.

Proof of Proposition 2. By Lemma 6, $P_1/(P)$ is a Cohen-Macaulay module of dimension $r+2$. Hence, using the local cohomology sequence of the middle row of the first diagram in the proof of Lemma 6, we easily see that

$$H^i_{\mathfrak{m}}(A/P))\cong\begin{cases} 0, & \text{if } i=0 \\ H^{i-1}_{\mathfrak{m}}(P_1/(P, x_{11})), & \text{if } i=1, \cdots, r+1 . \end{cases}$$

On the other hand, since

$$P_1/(P, x_{11})\cong A/(P, x_{11}): x_{21}=A/(X_1, x_1)\cap(X_1, X_2)$$

$$A/(X_1, X_2)\cong(X_1, X_2, x_1)/(X_1, X_2)\cong(X_1, x_1)/(X_1, x_1)\cap(X_1, X_2)$$

we have the following exact sequence

$$0 \longrightarrow A/(X_1, X_2) \longrightarrow P_1/(P, x_{11}) \longrightarrow A/(X_1, x_1) \longrightarrow 0 .$$

Hence, applying Lemma 4 to the Cohen-Macaulay modules $A/(X_1, X_2)$ and $A/(X_1, x_1)$, we also get

$$H^{i-1}_{\mathfrak{m}}(P_1/(P, x_{11}))\cong\begin{cases} 0, & \text{if } i\neq 3, r+2 \\ H^2_{\mathfrak{m}}(A/(X_1, X_2)) & \text{if } i=3 . \end{cases}$$

From the above two relations of local cohomology groups, Proposition 2 is clear.

Remark. Let A' be the local ring $k[X]_{(X_1, X_2)}$ and $\mathfrak{m}'$ its maximal ideal. Using the same method as above we can show that $H^i_{\mathfrak{m}'}(A'/(P))=0$ for $i\neq 1, r$ and $H^1_{\mathfrak{m}'}(A'/(P))\cong A'/\mathfrak{m}'$; hence $A'/(P)$ is a Buchsbaum ring by [12, Corollary 1.1].

The following simple but useful lemma is due to [4, § 5]:

Lemma 7. *Let* $\mathfrak{a}$ *be an ideal and* x *an element such that* $\sqrt{\mathfrak{a}}\subseteqq(\mathfrak{a}, x)$. *Then* $\mathfrak{a}$ *is radical, i.e.* $\sqrt{\mathfrak{a}}=\mathfrak{a}$, *in the following cases:*

(i) *There exists an ideal* $\mathfrak{b}\supseteqq\sqrt{\mathfrak{a}}$ *such that* $x\mathfrak{b}=\mathfrak{a}$ *and* $\mathfrak{b}: x=\mathfrak{b}$.

(ii) $\sqrt{\mathfrak{a}} : x=\sqrt{\mathfrak{a}}$ *and* $\bigcap_{n=1}^{\infty}(\mathfrak{a}, x^{n})=\mathfrak{a}$.

Proof. From the assumption $\sqrt{\mathfrak{a}}\subseteqq(\mathfrak{a}, x)$ we get $\sqrt{\mathfrak{a}}=\mathfrak{a}+x(\sqrt{\mathfrak{a}} : x)$. In the first case, $x(\sqrt{\mathfrak{a}} : x)\subseteqq x(\mathfrak{b}: x)=x\mathfrak{b}\subseteqq\mathfrak{a}$, and in the second case, $\sqrt{\mathfrak{a}}=\mathfrak{a}+x\sqrt{\mathfrak{a}}$ $=\mathfrak{a}+x^{2}\sqrt{\mathfrak{a}}=\cdots\subseteqq\bigcap_{n=1}^{\infty}(\mathfrak{a}, x^{n})=\mathfrak{a}$.

The proof of Proposition 3 begins, properly speaking, with the proof of the following lemma, which is of independent interest because it gives a new class of (generically) perfect prime ideals (see [4, § 0]):

Lemma 8. *Let*

$$F_{ijl}=x_{1i}z_{jl}-x_{1j}z_{il}$$

$$G_{ijl}=x_{2i}z_{jl}-x_{2j}z_{il}$$

$$H_{ijlm}=z_{il}z_{jm}-z_{im}z_{jl}$$

for all $i, j, l, m=1, \cdots, r$. *Let* J *denote the ideal in* $k[X, Z]$ *generated by all elements* $p_{ij}, q_{ij}, F_{ijl}, G_{ijl}, H_{ijlm}$. *Then* J *is a perfect prime ideal with* $\operatorname{ht} J=\binom{r+2}{2}-2$.

Proof. The case $r=1$ is trivial. Let $r>1$. We introduce some notations. Let Z_j denote the set $\{z_{1j}, \cdots, z_{rj}\}$ for all $j=1, \cdots, r$. Let $j_1, \cdots, j_h$, $h<r$, be an arbitrary family of integers with $1\leqq j_1\leqq\cdots\leqq j_h\leqq r$. We denote by $J(j_1, \cdots, j_h)$ the ideal in $k[X, Z]$ generated by $J, Z_{j_1}, \cdots, Z_{j_h}$ and all elements x_{1j}, x_{2j} with $j=j_1, \cdots, j_h$. By induction we may assume that $J(j_1, \cdots, j_h)$ are perfect prime ideals of height

$$\left[\binom{r-h+2}{2}-2\right]+\left[r+\cdots+(r-h+1)+2h\right]=\binom{r+2}{2}+h-2.$$

Using these induction hypotheses, we claim:

(1) (J, z_{1r}) is an unmixed radical ideal of height $\binom{r+2}{2}-1$.

To get (1), we have to consider a large class of ideals. Let s, t be arbitrary integers with $r\geqq s\geqq t\geqq 1$. Let $J_{s,t}$ denote the ideal in $k[X, Z]$ generated by $J, Z_{s+1}, \cdots, Z_r$ and the elements $z_{1s}, \cdots, z_{ts}$. We show that $J_{s,t}$ is a radical ideal. Of course, $J_{1,1}$ is a prime ideal because $J_{1,1}=(P, Z)$. Suppose $s>1$ and let

$$z=\begin{cases} z_{1(s-1)}, & \text{if } t=s \\ z_{(t+1)s}, & \text{if } t<s. \end{cases}$$

Notice that

$$(J_{s,t},\ z)=\begin{cases} J_{s-1,1}, & \text{if } t=s \\ J_{s,t+1}, & \text{if } t<s. \end{cases}$$

Then by induction on the number of elements in the set $\{z_{1s}, \cdots, z_{ts}\}\cup Z_{s+1}\cup\cdots\cup Z_r$, we may assume that $(J_{s,t}, z)$ is a radical ideal. Hence $\sqrt{J_{s,t}}\subseteq(J_{s,t}, z)$. On the other hand, if we define

$$J'_{s,t}=\begin{cases} J(s, \cdots, r), & \text{if } t=s \\ J(1, \cdots, t, s, \cdots r), & \text{if } t<s, \end{cases}$$

then it can be checked that $zJ'_{s,t}\subseteq J_{s,t}$. Further, by the induction hypotheses on $J(s, \cdots, r)$ and $J(1, \cdots t, s, \cdots, r)$, we see that $J'_{s,t}\supseteq\sqrt{J_{s,t}}$ and $J'_{s,t}: z=J'_{s,t}$. Thus, by Lemma 2 (i), $J_{s,t}$ is a radical ideal. Especially, $J_{1,r}=(J, z_{1r})$ is a radical ideal. From this we have

$$(2)\qquad (J, z_{1r})=(P, Z)\cap J(1)\cap J(r).$$

Hence, using the induction hypotheses on $J(1)$ and $J(r)$, we see that (J, z_{1r}) is an unmixed ideal of height $\binom{r+2}{2}-1$. Thus (1) is proved.

Next we will show the following facts:

(3) $\sqrt{J}$ has only one associated prime of height $\binom{r+2}{2}-2$.

(4) z_{1r} is not a zerodivisor on $\sqrt{J}$.

Note that we have the following relations:

$$x_{11}p_{ij}=x_{1i}p_{1j}-x_{1j}p_{1i}$$

$$x_{11}q_{ij}=x_{1i}q_{1j}-x_1x_{2j}p_{1i}$$

$$x_{11}F_{ijl}=x_{1i}F_{1jl}-x_{1j}F_{1il}$$

$$x_{11}G_{ijl}=x_{2i}F_{1jl}-x_{2j}F_{1il}-z_{1l}p_{ij}$$

$$x_{11}H_{ijlm}=x_{im}F_{1jl}-z_{jm}F_{1il}-z_{1l}F_{ijm}$$

for all $i, j, l, m=1, \cdots, r$. From these relations we see that $Jk[X, Z, x_{11}^{-1}]$ can be generated by the elements p_{1i}, q_{11}, F_{1ij} with $i=2, \cdots, r$ and $j=1, \cdots, r$. Eliminating $x_{21}, \cdots, x_{2r}, x_2, z_{12}, \cdots, z_{rr}$ by the help of these elements we then get an isomorphism $k[X, Z, x_{11}^{-1}]/(J)\cong k[X_1, x_{21}, x_1, z_{11}, x_{11}^{-1}]$. Hence $Jk[X, Z, x_{11}^{-1}]$ is a prime ideal of height $\binom{r+2}{2}-2$ and $x_{12}, \cdots, x_{2r}, z_{1r}$ are not zerodivisors on $Jk[X, Z, x_{11}^{-1}]$. Let J' denote the inverse image of $Jk[X, Z, x_{11}^{-1}]$ in $k[X, Z]$. Then J' is also a prime ideal of height $\binom{r+2}{2}-2$ and $x_{11}, \cdots, x_{2r}, z_{1r}$ are not zerodivisors on J'. Note that the same facts also hold if we replace x_{11} by an arbitrary element of the set $X_1\cup X_2$. We easily see that J' is the only associated prime of J which does not contain X_1, X_2. Thus, $\sqrt{J}=J'\cap\sqrt{(J, X_1, X_2)}$. On the other hand, it is not hard to see from [1, § 4, Corollary] that (J, X_1, X_2) is a prime ideal of height $\binom{r+2}{2}+2r>\binom{r+2}{2}-2$ and that z_{1r} is not a zero-divisor on (J, X_1, X_2). So $\sqrt{J}$ has only one associated prime of height $\binom{r+2}{2}-2$ and z_{1r} is not a zerodivisor on $\sqrt{J}$. Hence (3) and (4) are just proved.

Now, from (1) and (4) we conclude that $\sqrt{J}=J$, by Lemma 7 (ii), and that J is unmixed, by [8, (15.E), Lemma 4 and Lemma 5]. Hence by (3), J is a prime ideal with $\operatorname{ht} J=\binom{r+2}{2}-2$. It remains to show the perfection of J or, equivalently, the Cohen-Maculay property of $k[X, Z]/J$.

Let B denote the local ring $k[X, Z]_{(X,Z)}$. In order to show the Cohen-Macaulay property of $k[X, Z]/J$ we have only to show the Cohen-Macaulay property of $B/(J)$ (see [9]) or, equivalently, the Cohen-Macaulay property of $B/(J, z_{1r})$. For that consider the following exact sequence

$$0 \longrightarrow (J(1))/(J, z_{1r}) \longrightarrow B/(J, z_{1r}) \longrightarrow B/(J(1)) \longrightarrow 0 .$$

Using the relation (2), by induction we know that $B/(J(1))$ is Cohen-Macaulay and $\dim B/(J(1))=\dim B/(J, z_{1r})$. Hence, by Lemma 5 (ii), it suffices to show that $(J(1))/(J, z_{1r})$ is Cohen-Macaulay.

Let us consider the exact sequence

$$0 \longrightarrow (J(r))/(J_{r,r}) \longrightarrow B/(J_{r,r}) \longrightarrow B/(J(r)) \longrightarrow 0 .$$

$B/(J(r))$ is Cohen-Macaulay like $B/(J(1))$. Further, since $J_{r,r}$ is a radical ideal by the proof of (1), it can be checked that

$$J_{r,r}=(P, Z)\cap J(r) . \tag{5}$$

Hence $\dim B/(J(r))=\dim B/(J_{r,r})$ and $(J(r))/(J_{r,r})\cong(P, Z, J(r))/(P, Z)\cong(P_r)/(P)$, which is a Cohen-Macaulay module by Lemma 6. Thus, $(J(r))/(J_{r,r})$ is Cohen-Macaulay by Lemma 5 (i). Note that $(J_{r,r}, J(1))=(J_{r,r}, Z_1, x_{11}, x_{21})$ has a similar structure like $J_{r,r}$. So using the same method as above, we can also show that $B/(J_{r,r}, J(1))$ is Cohen-Macaulay. Now, by Lemma 5 (ii), the exact sequence

$$0 \longrightarrow (J_{r,r}, J(1))/(J_{r,r}) \longrightarrow B/(J_{r,r}) \longrightarrow B/(J_{r,r}, J(1)) \longrightarrow 0$$

implies that $(J_{r,r}, J(1))/(J_{r,r})$ is Cohen-Macaulay. On the other hand, using the relations (2) and (5), we have

$$\begin{aligned}(J_{r,r}, J(1))/(J_{r,r})&\cong(J(1))/(J_{r,r}\cap J(1))\\ &\cong(J(1))/(P, Z)\cap J(1)\cap J(r))=(J(1))/(J_{1,r}) .\end{aligned}$$

Hence $(J(1))/(J_{1,r})$ is Cohen-Macaulay, as required. This completes the proof of Lemma 8.

Lemma 8 is used to prove the following lemma, which is, like Lemma 8, of independent interest.

Lemma 9. *Let s ($\leqq r$) be a positive integer. Let I_s denote the ideal in $k[X, Y, Z]$ generated by I and all elements y_{ij} with $i, j=1, \cdots, s$. Then I_s is a perfect prime ideal with* $\operatorname{ht} I_s=r^2+s-1$.

Proof. The case $s=r$ follows immediately from Lemma 8 because $I_r=(J, Y)$. Let $s<r$. By induction on s we may assume that I_{s+1} is a perfect prime ideal with $\operatorname{ht} I_{s+1}=r^2+s$.

Let t ($\leqq s$) be an arbitrary positive integer. Let $I_{s,t}$ denote the ideal in $k[X, Y, Z]$ generated by I_s and the elements $y_{1(s+1)}, \cdots, y_{t(s+1)}$. We prove that $I_{s,t}$ is a radical ideal. Note that $I_{s,s}=I_{s+1}$ is already a prime ideal. We may assume that $t<s$ and that, by induction on t, $(I_{s,t}, y_{(t+1)(s+1)})=I_{s,t+1}$ is radical; hence $\sqrt{I_{s,t}} \subseteqq (I_{s,t}, y_{(t+1)(s+1)})$. Let $I'_{s,t}$ denote the ideal in $k[X, Y, Z]$ generated by I_s and all elements $x_{1i}, x_{2i}, y_{ij}, z_{ij}$ with $i=1, \cdots, t$ and $j=1, \cdots, r$. Since $I'_{s,t}$ has a similar structure like I_s, by induction on r to the statement of Lemma 9 (the case $r=1$ is trivial) we may assume that $I'_{s,t}$ is a perfect prime ideal with

$$\operatorname{ht} I'_{s,t}=[(r-t)^2+(s-t)-1]+[2t+2rt-t^2]=r^2+s-t-1\,.$$

By this assumption we get $I'_{s,t} \supseteqq \sqrt{I_{s,t}}$ and $I'_{s,t} : y_{(t+1)(s+1)}=I'_{s,t}$. On the other hand, it can be easily checked that $y_{(t+1)(s+1)} I'_{s,t} \subseteqq I_{s,t}$. Hence, by Lemma 7 (i), $I_{s,t}$ is a radical ideal.

Since $(I_s, y_{1(s+1)})=I_{s,1}$ is the last member of the class of the ideals $I_{s,t}$, we have just shown that $(I_s, y_{1(s+1)})$ is a radical ideal. From this we can easily verify that $(I_s, y_{1(s+1)})=I_{s+1} \cap I'_{s,1}$. Note that I_{s+1}, $I'_{s,1}$, and $(I_{s+1}, I'_{s,1})=I'_{s+1,1}$ are perfect prime ideals of heights r^2+s, r^2+s, and r^2+s+1, respectively, by the induction hypotheses. Then, applying [4, Proposition 18], we see that $(I_s, y_{1(s+1)})$ is perfect. Thus it is clear that I_s is perfect if $y_{1(s+1)}$ is not a zerodivisor on I_s. Hence, to complete the roof of Lemma 9, we have only to show that I_s is a prime ideal with $\operatorname{ht} I_s=r^2+s-2$, because the fact that $y_{1(s+1)}$ is not a zerodivisor on I_s is then an immediate consequence of this.

Consider the following relations:

$$x_{12}p_{ij}=x_{1i}p_{2j}-x_{1j}p_{2i}$$

$$x_{12}q_{ij}=x_{1i}q_{2j}-x_1x_{2j}p_{2i}$$

$$x_{12}a_{ijl}=x_{1i}a_{2jl}-x_{1j}a_{2il}+x_{1l}a_{2ij}$$

$$x_{12}b_{ijl}=x_{22}a_{ijl}+y_{jl}p_{2i}-y_{il}p_{2j}+y_{ij}p_{2l}$$

$$x_{12}c_{ijlm}=y_{il}a_{2lm}-y_{il}a_{2jm}+y_{jl}a_{2im}-y_{2m}a_{ijl}+x_{1m}c_{2ijl}$$

$$x_{12}d_{ijlm}=y_{lm}f_{2ij}-y_{jm}f_{2li}+y_{jl}f_{2mi}+z_{2i}a_{jlm}-x_1x_{2i}c_{2jlm}$$

$$x_{12}f_{ijl}=x_{1i}f_{2jl}-x_{1j}f_{2il}-x_1x_{21}a_{2ij}$$

$$x_{12}g_{ijl}=x_{22}f_{ijl}+z_{jl}p_{2i}-z_{il}p_{2j}+y_{ij}q_{2l}$$

$$x_{12}h_{ijlm}=z_{im}f_{2jl}-z_{jm}f_{2il}-z_{2l}f_{ijm}+x_1y_{ij}g_{lm2}+x_1x_{2l}d_{mij2}\,.$$

We see that $I_s k[X, Y, Z, x_{12}^{-1}]$ can be generated by the elements p_{2j}, q_{22}, a_{2ij}, f_{2ij} with $i, j=1, \cdots, r$ and the elements y_{ij} with $i, j=1, \cdots, s$. It follows by eliminating $x_{21}, x_{23}, \cdots, x_{2r}, x_2, y_{ij} \in Y \backslash \{y_{2(s+1)}, \cdots, y_{2r}\}$, $z_{ij} \in Z \backslash \{z_{22}\}$ that $k[X, Y, Z, x_{12}^{-1}]/(I_s) \cong k[X_1, x_{22}, x_1, y_{2(s+1)}, \cdots, y_{2r}, z_{22}, x_{12}^{-1}]$. Hence $I_s k[X, Y, Z, x_{12}^{-1}]$ is a prime ideal of height r^2+s-1. Thus $y_{1(s+1)}$ is not a zerodivisor on $I_s k[X, Y, Z, x_{12}^{-1}]$. Let I_s' denote the inverse image of $I_s k[X, Y, Z, x_{12}^{-1}]$ in $k[X, Y, Z]$. Then I_s' is also a prime ideal with $\operatorname{ht} I_s'=r^2+s-1$ and $I_s' : y_{1(s+1)} = I_s'$. Since $x_{12}^n I_s' \subseteq I_s$ for some large n, $I_s' \subseteq (I_s, y_{1(s+1)}) : x_{12}^n$. But $(I_s, y_{1(s+1)}) = I_{s+2} \cap I_{s,1}'$, and it is not hard to see from the induction hypotheses on I_{s+1}, $I_{s,1}'$ that x_{12} is not a zerodivisor on $I_{s+1}, I_{s,1}'$. Hence, $(I_s, y_{1(s+1)}) : x_{12}^n = (I_s, y_{1(s+1)})$. So we get $I_s' \subseteq (I_s, y_{1(s+1)})$ or $I_s' = I_s + y_{1(s+2)}(I_s' : y_{1(s+1)}) = I_s + y_{1(s+1)} I_s'$. Now, applying Nakayama's lemma we have $I_s' = I_s$, which shows that I_s is a prime ideal with $\operatorname{ht} I_s = r^2+s-1$. Thus the proof of Lemma 9 is completed.

Proof of Proposition 3. Note that $I=I_1$ is a perfect prime ideal by Lemma 9. Then, by [5, § 0, Proposition], it suffices to show that $gr_P(k[X]) \cong k[X, Y, Z]/I_1$. To see this, sending y_{ij} and z_{ij} to the images of p_{ij} and q_{ij} in P/P^2, we have a natural homomorphism from $k[X, Y, Z]$ to $gr_P(k[X])$. Let I' be the kernel of this homomorphism. Then, since $k[X]_P[Y, Z]/(I')$ is isomorphic to $gr_{(P)}(k[X]_P)$, which is a regular domain of the same dimension as $k[X]_P$, I' must have a primary component of height r^2 $(=\dim k[X]_P[Y, Z] - \dim k[X]_P)$. But it can be easily checked that $I_1 \subseteq I'$. Hence I_1 is just the primary component of I' mentioned above, because I_1 is prime and $\operatorname{ht} I_1 = r^2$. Consequently, we have $I_1 = I'$, which shows that $gr_P(k[X]) \cong k[X, Y, Z]/I_1$.

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