J. Math. Kyoto Univ. (JMKYAZ)
21-3 (1981) 417-500

The characters of the discrete series for semisimple Lie groups

By

Takeshi HIRAI

(Received January 14, 1980)

Introduction

Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. For a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we denote by $H^{\mathfrak{h}}$ the corresponding Cartan subgroup of G. Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$ such that its toroidal part has the possible maximal dimension, and put $B=H^{\mathfrak{b}}$. Denote by $\mathfrak{b}_B^*$ the space of linear forms Λ of $\mathfrak{b}$ into $\boldsymbol{C}$ such that $B\ni\exp X\mapsto\exp\Lambda(X)\in\boldsymbol{C}$ $(X\in\mathfrak{b})$ defines a unitary character of B, and by $\mathfrak{b}_B^{*\prime}$ its subset consisting of regular elements. When every root of $(\mathfrak{g}_c, \mathfrak{b}_c)$ (or simply of $\mathfrak{b}$) is imaginary, we call $\mathfrak{b}$ compact. In that case, Harish-Chandra proved the existence and the uniqueness of a certain kind of invariant eigendistribution on G for $\Lambda\in\mathfrak{b}_B^{*\prime}$. When B is compact, it is characterized as a unique tempered invariant eigendistribution which coincides on $B\cap G'$ with a certain function, where G' denotes the set of all regular elements in G. We define the same kind of invariant eigendistribution π_Λ for $\Lambda\in\mathfrak{b}_B^{*\prime}$ evenwhen some roots of $\mathfrak{b}$ are not imaginary ($\pi_\Lambda=\Theta_\Lambda$ in the above case, and for the exact definition, see below).

The purpose of this paper is to give a global explicit formula of the invariant analytic function π'_Λ on G' corresponding canonically to π_Λ. We assume that G is acceptable in the sense of Harish-Chandra [2(b), § 18] for convenience. But this is not an essential restriction, and the essential assumption we made here is the connectedness of G. Our main results are Theorems 1 and 2 in § 5 which give the explicit formulas of the functions $\pi'_\Lambda(h)$ on $H^{\mathfrak{h}}\cap G'$ for every $\mathfrak{h}$.

When B is compact, G has the discrete series representations, and their characters are equal to Θ_Λ's except the known multiplicative sign ± 1. Thus we get the explicit character formula for these representations. Many researches have been made in this direction, for instance, Hecht [3], Martens [6], Midorikawa [7(a), (b)], Schmid [8(a), (b)] and Hirai [5(a), (b), (e)] (cf. also Arthur [10]). The first two authors treat essentially the holomorphic discrete series, and the next two authors treat some type (or types) of linear groups.

The method of the present paper goes along the same line as in the previous paper [5(e)]. Thus we apply the necessary and sufficient condition in [5(c)] for that a given invariant analytic function on G' defines canonically an invariant eigendistribution on G.

Let us define exactly the invariant eigendistribution π_Λ for $\Lambda \in \mathfrak{b}_B^{*\prime}$. Let $P(\mathfrak{h})$ be the set of positive roots of $\mathfrak{h}$ with respect to a lexicographic order, and ρ half the sum of all roots in $P(\mathfrak{h})$. We put for $h \in H^{\mathfrak{h}}$,

$$\Delta^{\mathfrak{h}}(h\,;\,P(\mathfrak{h}))=\xi_\rho(h)\Pi_{\gamma\in P(\mathfrak{h})}(1-\xi_\gamma(h)^{-1})\,,$$

where ξ_ρ and ξ_γ denote the character of $H^{\mathfrak{h}}$ canonically corresponding to ρ and γ (cf. [2(b), § 18]). We denote by $W(\mathfrak{h}_c)$ the Weyl group of $\mathfrak{g}_c$ acting on $\mathfrak{h}_c$, and by $W_G(\mathfrak{h})$ (resp. $W_G(H^{\mathfrak{h}})$) the group of transformations on $\mathfrak{h}$ (resp. on $H^{\mathfrak{h}}$) obtained as the restrictions of the inner automorphisms of $\mathfrak{g}$ (resp. of G) leaving $\mathfrak{h}$ (resp. $H^{\mathfrak{h}}$) invariant. Then $W_G(\mathfrak{h})$ is a subgroup of $W(\mathfrak{h}_c)$, and $W_G(\mathfrak{b})$ can be canonically identified with $W_G(B)$ because B is connected. We fix $P(\mathfrak{b})$ and put

$$\zeta_\Lambda(b)=\sum_{s\in W_G(\mathfrak{b})}\mathrm{sgn}(s)\xi_{s\Lambda}(b) \qquad (b\in B)\,,$$

where sgn(s) denotes the usual sign of $s \in W_G(\mathfrak{b}) \subset W(\mathfrak{b}_c)$.

We define π_Λ as a unique invariant eigendistribution on G with the following properties:

1) $\pi'_\Lambda(b)=\Delta^{\mathfrak{b}}(b\,;\,P(\mathfrak{b}))^{-1}\zeta_\Lambda(b) \quad (b\in B\cap G')$,
2) $\Delta^{\mathfrak{h}}(h\,;\,P(\mathfrak{h}))\pi'_\Lambda(h)$ is bounded on $H^{\mathfrak{h}}\cap G'$ for every $\mathfrak{h}$.

Note that the existence and the uniqueness of such a distribution are guaranteed by the discussions in [5(d), §§ 8-9]. In the case where the center Z_G of G is finite, Harish-Chandra defined the *temperedeness* of a distribution on G and also gave a criterion for an invariant eigendistribution on G to be tempered. When its infinitesimal character is regular, this condition reduces to the same thing as 2) above (for all these, see [2(d)]). In analogy of this fact, we call in this paper the distribution π_Λ "tempered" evenwhen Z_G is no longer finite, for the sake of simplicity. Thus the result of this paper is, in short, the presentation of global formulas of all "tempered" invariant eigendistributions on G of the possible highest height (cf. [5(d)]) with regular infinitesimal characters. In such distributions, the characters of the discrete series representations are the most important ones.

Now let us add one word about the background of the present work. Midorikawa's formula in [7(c)] of discrete series characters for some types of simple groups (of class I in the classification of the present paper) was very impressive and encouraging for the author of the present paper when he first knew it on the occasion of a meeting on harmonic analysis in Japan, 1975, Summer. Even though his formula gives only a sum of sgn$(w)\Theta_{w\Lambda}$ over $w \in W(\mathfrak{b}_c)$ (in his case, $w\Lambda$ belongs to $\mathfrak{b}_B^*$ for any $\Lambda \in \mathfrak{b}_B^*$ and $w \in W(\mathfrak{b}_c)$), it makes the author possible to imagine how the general formulas must be. In fact he draws up the present formulas in Theorems 1 and 2 from the above Midorikawa's one and the formulas for $Sp(n, \boldsymbol{R})$ in [5(e)] and for $SO_0(p, q)$ (not published) explicitly calculated.

We call a simple root system *of class* I if it is of type A_1, D_{2n}, E_7, E_8 or G_2, *of class* II if it is of type B_n $(n\geqq 2)$, C_n $(n\geqq 3)$ or F_4, and *of class* III otherwise. A root system is called of class I-III if any of its simple component is of class I or III. Let A be a connected component of $H^{\mathfrak{h}}$. We denote by $\Sigma_R(\mathfrak{h})$ (resp. $\Sigma_R(A)$) the root system consisting of real roots of $\mathfrak{h}$ (resp. those such that $\xi_\alpha(h)>0$ for $h\in A$). Then every simple component of $\Sigma_R(A)$ is of class I or II if $\mathfrak{g}$ has a compact Cartan subalgebra and especially if G has a compact Cartan subgroup. The formula of the function $\tilde{\kappa}^{\mathfrak{h}}=\Delta^{\mathfrak{h}}\pi'_\Lambda$ on A is given by means of $\Sigma_R(A)$, and hence varies on $H^{\mathfrak{h}}$ according to its connected component A. It is simple when $\Sigma_R(A)$ is of class I-III, but becomes complicated when such is not the case.

Put $\Sigma=\Sigma_R(A)$, $P=\Sigma_R(A)\cap P(\mathfrak{h})$, and let $W(\Sigma)$ be the Weyl group of Σ, and $M(P)$ the set of all maximal orthogonal systems in P. Assume that $\mathfrak{g}$ has a compact Cartan subalgebra and that Σ is of class I. Then the formula is given as follows (§ 5, Theorem 1). Take the unique standard element F in $M(P)$ (§ 1.7), and let ν_F be the Cayley transformation of $\mathfrak{g}_c$ corresponding to F (§ 2.2). Then the Cartan subalgebra $\mathfrak{h}^F=\nu_F(\mathfrak{h}_c)\cap\mathfrak{g}$ is compact (§ 2.3). Put $A_U=\{h\in A\,;\,\xi_\alpha(h)=1\ (\alpha\in\Sigma)\}$ and $\mathfrak{h}_V=\sum_{\alpha\in\Sigma}\boldsymbol{R}H_\alpha$. Then $h\in A$ is decomposed uniquely as $h=h_U\exp X$ with $h_U\in A_U$, $X\in\mathfrak{h}_V$.

Theorem. *Put* $\mathfrak{b}=\mathfrak{h}^F$ *and* $P(\mathfrak{b})=\nu_F P(\mathfrak{h})$. *If a tempered invariant eigendistribution* π *is given on* B *as* $\Delta^{\mathfrak{b}}\pi'=\zeta_\Lambda$ *for* $\Lambda\in\mathfrak{b}_B^{*\prime}$, *then it is given on* $A\subset H^{\mathfrak{h}}$ *as follows*:

$$(\varepsilon_R^{\mathfrak{h}}\Delta^{\mathfrak{h}}\pi')(h)=(-1)^{\# F}\sum_{s\in W_G(\mathfrak{b})}\ \sum_{u\in W(\Sigma)/I(F)}\operatorname{sgn}(s)Y'(h\,;\,F,\,u,\,s\Lambda)\,,$$

where

$$\varepsilon_R^{\mathfrak{h}}(h)=\operatorname{sgn}\{\prod_{\alpha\in\Sigma_R(\mathfrak{h})}(1-\xi_\alpha(h)^{-1})\}\,,\quad I(F)=\{u\in W(\Sigma)\,;\,uF\subset F\cup -F\}\,,$$

$$Y'(h\,;\,F,\,u,\,\Lambda)=\operatorname{sgn}\{\prod_{\gamma\in F}(\Lambda,\,\nu_F\gamma)\}\xi_\Lambda(h_U)\prod_{\gamma\in F}\exp\{-|(u\gamma)(X)|\,|(\Lambda,\,\nu_F\gamma)|/|\gamma|^2\}\,,$$

and the second sum runs over any fixed complete system of representatives of $W(\Sigma)/I(F)$.

We can understand that this formula is given essentially by means of $M(P)$ because $W(\Sigma)/I(F)$ corresponds bijectively onto $M(P)$ in such a way that $uI(F)\mapsto(uF\cup -uF)\cap P$.

When Σ is no longer of class I, the formula of $\Delta^{\mathfrak{h}}\pi'$ on A is given essentially in the same but much more complicated manner (§ 5, Theorem 2). Note that in this case $M(P)$ contains different types of orthogonal systems and even more we are obliged to use the set $M^{or}(P)$ of certain ordered orthogonal systems in P (§ 1.6).

The contents of this paper are arranged as follows. In § 1, some properties of root systems are studied, especially those of orthogonal systems and strongly orthogonal systems of roots. In § 2, we study the relations between Cartan subgroups and also between their connected components. In § 3, we define certain fundamental functions Y, Z, Y' and Z' on $A \subset H^{\mathfrak{h}}$ by means of $\Sigma_R(A)$, and study their properties using results in §§ 1-2. Section 4 is devoted to recall the fundamental results in [2(d)] and [5(c)] and prepare some notations. In § 5, the explicit formulas for $\tilde{\kappa}^{\mathfrak{h}} = \Delta^{\mathfrak{h}} \pi'$ are given on every $A \subset H^{\mathfrak{h}}$, by Theorem 1 when $\Sigma_R(A)$ is of class I-III, and by Theorem 2 in general. The definitions and the notations necessary to understand these formulas are given in §§ 1.2, 1.5, 1.7, 2.1, 2.2, 2.5, 3.1, 4.1 and 4.3.

Our proof of these theorems is reduced to prove that the given system of functions $\tilde{\kappa}^{\mathfrak{h}}$ satisfies the conditions (a), (b), (c) and (d) in § 4. In § 6, we outline this proof, and show that the conditions (a), (b) and (d) are satisfied. The condition (c) connects the function $\tilde{\kappa}^{\mathfrak{h}} | A$ with a neighbouring one $\tilde{\kappa}^{\mathfrak{h}^\alpha} | A^\alpha$ with $\alpha \in \Sigma_R(A)$, where A^α denotes the connected component of $H^{\mathfrak{h}^\alpha}$ containing $\{h \in A\,;\ \xi_\alpha(h) = 1\}$. The proof of this condition is given in §§ 7-9. We treat in § 8 the case where $\Sigma_R(A)$ is of class I-III and in § 9 the general case.

The formula of $\tilde{\kappa}^{\mathfrak{h}}$ on $A \subset H^{\mathfrak{h}}$ in § 5 are valid for all regular Λ at the same time, and is the most reduced ones when we consider $\pi'_\Lambda(h)$ as a function of two variable $\Lambda \in \mathfrak{b}_B^{*\prime}$ and $h \in A \cap G'$ in the sense that there exists no cancellation between the summands in the formula. However fix a $\Lambda \in \mathfrak{b}_B^{*\prime}$ and consider $\pi'_\Lambda(h)$ as a function of a one variable h. When $\Sigma_R(A)$ is of class I-III, the formula for it in Theorem 1 remains also to be reduced, but when $\Sigma_R(A)$ contains simple components of class II, the formula in Theorem 2 can contain the terms cancelling out each other. In fact, as is known, the reduced expression of $\pi'_\Lambda(h)$ in this sense depends on the Weyl chamber of $\mathfrak{b}_B^{*\prime}$ containing Λ, or more exactly the $W_G(\mathfrak{b})$-orbit $\mathcal{O}$ of the Weyl chambers. Therefore in this point of view, we may have $\sharp(W(\mathfrak{b}_c)/W_G(\mathfrak{b}))$ different reduced expressions of $\pi'_\Lambda(h)$ according to the orbit $\mathcal{O} \ni \Lambda$. However the differences are very minor when $\Sigma_R(A)$ is of class I-III.

In § 6 of [5(e)], all these reduced but complicated expressions are given for $Sp(3, \boldsymbol{R})$, and in § 9 of that paper, it is shown for $Sp(n, \boldsymbol{R})$ how the overwhelming complexities come out when we tried to give the reduced expression for every orbit $\mathcal{O}$ separately ($\sharp(W(\mathfrak{b}_c)/W_G(\mathfrak{b})) = 2^n$ in this case). Thus the author realized that it is really difficult to write down the reduced expression for each $\mathcal{O}$ in a reasonably simple form. In this way, he was lead to study a formula valid for all Λ expressed essentially by the terminologies of root systems and various Weyl groups, even if it contains some cancellation when Λ is fixed. (Recall the multiplicity formula of weight of Kostant or that of K-multiplicity of Blattner. They contain always negative terms.) In other point of view, such a formula has some adventages, for exemple, when we consider the important sum $\sum \operatorname{sgn}(w) \pi_{w\Lambda}$ over $\{w \in W(\mathfrak{b}_c)\,;\ w\Lambda \in \mathfrak{b}_B^{*\prime}\}$.

We know the simple character formula for the holomorphic discrete series (this corresponds to a special $\mathcal{O}$). When G is simple and has such series, $\Sigma_R(A)$ contains simple components of class II for some A if and only if G is locally isomorphic to $Sp(n, \boldsymbol{R})$. For $G = Sp(n, \boldsymbol{R})$, we have shown in [5(e), § 9] that this simple formula can be deduced easily from the general formula valid for all Λ, which is essentially the same thing for G as that in Theorem 2 in the present paper.

Afterwords we added Appendix on this point.

TABLE OF CONTENTS

§1. Elementary facts about root systems

For the later uses, we collect in this section some elementary facts about root systems, especially on the orthogonal systems of roots.

1.1. Let Σ be a root system. We put for $\alpha\in\Sigma$,

$$\Sigma^{\alpha}=\{\gamma\in\Sigma\,;\ \gamma\perp\alpha\}. \tag{1.1}$$

If Σ is simple, the type of Σ^{α} depends only on whether α is a short root or a long root. The correspondence between the type of Σ and that of Σ^{α} is given in [5(d), Remark 4.1]. Since we will quote it frequently, let us transcribe it here.

Σ	A_n	B_n $(n\geqq 2)$	C_n $(n\geqq 2)$	D_n $(n\geqq 3)$	E_6	E_7	E_8	F_4	G_2
Σ^{α}	A_{n-2}	$B_{n-2}+A_1^{(l)}$	C_{n-1}	$D_{n-2}+A_1$	A_5	D_6	E_7	C_3	$A_1^{(s)}$
		B_{n-1}	$C_{n-2}+A_1^{(s)}$					B_3	$A_1^{(l)}$

Table 1.1.

(In this table, the notations A_{-1}, A_0 etc. stand for $\emptyset$; $A_1^{(l)}$, $A_1^{(s)}$ denote the root systems of type A_1 consisting of long roots or short roots respectively; $B_1=A_1^{(s)}$, $C_1=A_1^{(l)}$, $D_1=\emptyset$, $D_2=A_1+A_1$. For B_n, C_n, F_4 and G_2, the upper column for Σ^{α}

is the case where α is long and the lower one is the case where α is short.)

1.2. *Orthogonal systems.* We denote by $W(\Sigma)$ the Weyl group of a root system Σ, by P the set of positive roots in Σ with respect to an order $\boldsymbol{P}$ in Σ and by Π the set of simple roots. Here we understand that Σ is a subset of a Euclidean space V, and $\boldsymbol{P}$ in Σ is induced from an order in V which makes V an ordered vector space. Note that if α and β in Σ satisfy $\alpha=\beta+\gamma_1+\gamma_2+\cdots+\gamma_p$ with $\gamma_i\in P$, then we have $\alpha>\beta$ with respect to $\boldsymbol{P}$. However the order $\boldsymbol{P}$ in Σ is not uniquely determined by the set P in general. The reflexion corresponding to a root α is denoted by s_α. The operation of $w\in W(\Sigma)$ on $\alpha\in\Sigma$ is denoted by $w\alpha$. For a set F and $w\in W(\Sigma)$, we define wF by

$$wF=\{w\alpha\,;\ \alpha\in F\}.$$

A set of roots $F=\{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ is called an orthogonal system if $\alpha_i\perp\alpha_j$ (orthogonal) for $i\neq j$. It is called a strongly orthogonal system if any two roots in F are strongly orthogonal to each other. Here, by definition, a root α is strongly orthogonal to another root $\beta\neq\pm\alpha$ if $\alpha\pm\beta\notin\Sigma$. In that case we get $\alpha\perp\beta$. Denote by $M(\Sigma)$ and $M(P)$ the sets of all maximal orthogonal systems in Σ and P respectively. We study here the properties of $M(\Sigma)$ and $M(P)$.

We devide the simple root systems into the following three classes:

Class I : A_1, D_{2n} $(n\geqq 2)$, E_7, E_8, G_2;
Class II : B_n $(n\geqq 2)$, C_n $(n\geqq 3)$, F_4;
Class III : A_n $(n\geqq 2)$, D_{2n+1} $(n\geqq 1)$, E_6.

A root system Σ is called of class I-III if all the simple components of it are of class I or III.

Let us first list up Σ, P, Π and $M(P)$ for every simple root system as explicitly as is necessary in the following. Here we denote by $e_1, e_2, \cdots, e_i, e_{i+1}, \cdots$ an orthonormal system of vectors in a Euclidean space and use the lexicographic order with respect to this system.

TYPE A_n. $P=\{e_i-e_j\ (1\leqq i<j\leqq n+1)\}$.

Put $k=[(n+1)/2]$, then every F in $M(\Sigma)$ is conjugate under $W(\Sigma)$ to the following:

$$F^0=\{e_1-e_{n+1},\ e_2-e_n,\ \cdots,\ e_k-e_{n-k+2}\}. \tag{1.2}$$

TYPE B_n. $P=\{e_i\pm e_j\ (1\leqq i<j\leqq n),\ e_i\ (1\leqq i\leqq n)\}$.

$$\Pi=\{e_1-e_2,\ e_2-e_3,\ \cdots,\ e_{n-1}-e_n,\ e_n\}. \tag{1.3}$$

Every F in $M(\Sigma)$ is conjugate under $W(\Sigma)$ to one of the following: for $0\leqq r\leqq[n/2]$,

$$F^r=\{e_{2i-1}\pm e_{2i}\ (1\leqq i\leqq r),\ e_j\ (2r+1\leqq j\leqq n)\}. \tag{1.4}$$

TYPE C_n. $P=\{2e_i\ (1\leqq i\leqq n),\ e_i\pm e_j\ (1\leqq i<j\leqq n)\}$.

$$\Pi=\{e_1-e_2,\ e_2-e_3,\ \cdots,\ e_{n-1}-e_n,\ 2e_n\}. \tag{1.5}$$

Every F in $M(\Sigma)$ is conjugate under $W(\Sigma)$ to one of the following: for $0 \leqq r \leqq [n/2]$,

(1.6) $$F^r = \{2e_j\ (1 \leqq j \leqq n-2r),\ e_{n-2r+2i-1} \pm e_{n-2r+2i}\ (1 \leqq i \leqq r)\}.$$

TYPE D_N. $P = \{e_i \pm e_j\ (1 \leqq i < j \leqq N)\}$.

(1.7) $$\Pi = \{e_1 - e_2,\ e_2 - e_3,\ \cdots,\ e_{N-1} - e_N,\ e_{N-1} + e_N\}.$$

Then every F in $M(\Sigma)$ is conjugate under $W(\Sigma)$ to the following:

(1.8) $$F^0 = \{e_{2i-1} \pm e_{2i}\ (1 \leqq i \leqq n)\}, \quad (n = [N/2]).$$

TYPE E_6.

$$\Sigma = \{\pm e_i \pm e_j\ (1 \leqq i < j \leqq 5),$$
$$\pm 2^{-1}(e_6 + e_7 - e_8 + \sum_{1 \leqq i \leqq 5} (-1)^{\nu_i} e_i) \text{ with } \sum_{1 \leqq i \leqq 5} \nu_i \text{ odd}\},$$

$$\Pi = \{2^{-1}(e_1 - \sum_{2 \leqq i \leqq 4} e_i \pm (e_5 + e_6 + e_7 - e_8)),\ e_i - e_{i+1}\ (2 \leqq i \leqq 4),\ e_4 + e_5\}.$$

Every $F \in M(\Sigma)$ is conjugate under $W(\Sigma)$ to the following:

$$F^0 = \{e_1 \pm e_2,\ e_3 \pm e_4\}.$$

TYPE E_7.

$$\Sigma = \{\pm e_i \pm e_j\ (1 \leqq i < j \leqq 6),\ \pm(e_7 - e_8),$$
$$\pm 2^{-1}(e_7 - e_8 + \sum_{1 \leqq i \leqq 6} (-1)^{\nu_i} e_i) \text{ with } \sum_{1 \leqq i \leqq 6} \nu_i \text{ odd}\},$$

$$\Pi = \{2^{-1}(e_1 - e_2 - \cdots - e_7 + e_8),\ e_i - e_{i+1}\ (2 \leqq i \leqq 5),\ e_5 + e_6,\ e_7 - e_8\}.$$

Every $F \in M(\Sigma)$ is conjugate under $W(\Sigma)$ to the following:

(1.9) $$F^0 = \{e_1 \pm e_2,\ e_3 \pm e_4,\ e_5 \pm e_6,\ e_7 - e_8\}.$$

TYPE E_8.

$$\Sigma = \{\pm e_i \pm e_j\ (1 \leqq i \leqq j \leqq 8),\ 2^{-1}(\sum_{1 \leqq i \leqq 8} (-1)^{\nu_i} e_i) \text{ with } \sum_{1 \leqq i \leqq 8} \nu_i \text{ even}\},$$

$$\Pi = \{2^{-1}(e_1 - e_2 - \cdots - e_7 + e_8),\ e_i - e_{i+1}\ (2 \leqq i \leqq 7),\ e_7 + e_8\}.$$

Every $F \in M(\Sigma)$ is conjugate under $W(\Sigma)$ to the following:

(1.10) $$F^0 = \{e_1 \pm e_2,\ e_3 \pm e_4,\ e_5 \pm e_6,\ e_7 \pm e_8\}.$$

TYPE F_4.

(1.11) $$P = \{e_i \pm e_j\ (1 \leqq i < j \leqq 4),\ e_j\ (1 \leqq j \leqq 4),\ 2^{-1}(e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

We denote the four simple roots as follows:

(1.12) $$\alpha = 2^{-1}(e_1 - e_2 - e_3 - e_4),\ \beta = e_2 - e_3,\ \gamma = e_3 - e_4,\ \delta = e_4.$$

Every $F \in M(\Sigma)$ is conjugate under $W(\Sigma)$ to one of the following:

(1.13) $F^0=\{e_1, e_2, e_3, e_4\}$, $F^1=\{e_1\pm e_2, e_3, e_4\}$, $F^2=\{e_1\pm e_2, e_3\pm e_4\}$.

TYPE G_2. Let $\Pi=\{\alpha, \beta\}$, where β is longer than α. Then

$$P=\{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}.$$

The set $M(P)$ consists of the following three elements

(1.14) $F^0=\{\alpha, 3\alpha+2\beta\}$, $F'=\{2\alpha+\beta, \beta\}$, $F''=\{\alpha+\beta, 3\alpha+\beta\}$.

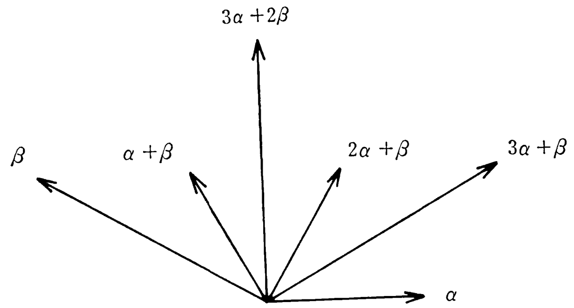

Figure 1.1. Type G_2

The above assertions on the conjugacy of orthogonal systems can be proved by applying the result in Table 1.1.

1.3. Now let us prove an elementary lemma on the orthogonal systems. For a subset F of Σ, denote by Σ_F the subset of Σ consisting of all elements which can be expressed as linear combinations of the elements in F, that is,

(1.15) $$\Sigma_F=\Sigma\cap(\textstyle\sum_{\alpha\in F}\boldsymbol{R}\alpha).$$

Lemma 1.1. *Let Σ be a simple root system. If Σ is of class* I *or* II, *then $\Sigma=\Sigma_F$ for any maximal orthogonal system F in Σ. In particular, the number of elements in F is equal to the rank of Σ. If Σ is of class* I *or* III, *any orthogonal system is strongly orthogonal. If Σ is of class* II, *for any orthogonal system F in Σ, take a strongly orthogonal system F' in Σ_F with maximal number of elements, then $\Sigma_{F'}=\Sigma_F$ and F' is also strongly orthogonal in Σ. Let Σ be of class* III, *then $\Sigma\neq\Sigma_F$ for any orthogonal system F in Σ, and the simple components of Σ_F are of class* I.

Proof. The assertions can be easily proved by using Table 1.1 and the above list of maximal orthogonal systems. Note that, for Σ of class II, a maximal *strongly orthogonal* system is not necessarily a maximal *orthogonal* system, and so is it if and only if its number of elements is maximal, that is, equal to the rank of Σ. Q. E. D.

Corollary. *Let Σ be a root system and F an orthogonal system in Σ. Then Σ_F does not contain simple components of class* III.

1.4. *Root system of class* I *or* III.

For $w \in W(\Sigma)$, we denote by $\mathrm{sgn}(w)$ the usual sign of w. Here we wish to prove the following lemma.

Lemma 1.2. *Let Σ be a root system of class* I-III. *For any two elements $F, F' \in M(P)$, there exists $w \in W(\Sigma)$ such that $wF = F'$. If an element $w \in W(\Sigma)$ satisfies $wF = F$ for some $F \in M(P)$, then* $\mathrm{sgn}(w)=1$.

Proof. It is sufficient to prove the lemma when Σ is simple. The first assertion is proved in § 1.2 and therefore it rests only to prove the second one. When Σ is of type A_1 or G_2, we see easily that $wF=F$ means $w=1$. Therefore the assertion is true. When Σ is of type A_n, we may take as F the special one F^0 in § 1.2, and in this case we see $\mathrm{sgn}(w)=1$ by an explicit calculation.

For the case of types D_n and E_n, we prove the assertion by induction on the rank of Σ. (Note that $D_3 = A_3$.) We reduce the proof for Σ to that for Σ^α with some $\alpha \in \Sigma$. This is possible because of the result on the type of Σ^α in Table 1.1. First consider the case of type D_4. Then Σ is given in § 1.2 and we take as F the following system:

$$F^0 = \{e_1 \pm e_2,\ e_3 \pm e_4\}. \tag{1.16}$$

For this case, we can prove easily that if $w \in W(\Sigma)$ satisfies $wF=F$, then $\mathrm{sgn}(w)=1$, and that for any two elements α_1, α_2 in F, there exists a $w \in W(\Sigma)$ such that $w\alpha_1 = \alpha_2$, $wF=F$.

Now consider the general case. We apply the following lemma.

Lemma 1.3. *Let Σ be of type D_N ($N \geqq 4$), E_6, E_7 or E_8 and let $F \in M(\Sigma)$. Then for any two elements $\alpha_1, \alpha_2 \in F$, there exists a subsets F' of F containing α_1, α_2 such that $\Sigma_{F'}$ is of type D_4.*

For a moment, we take Lemma 1.3 for granted. Assume that there exists an $\alpha \in F$ such that $w\alpha = \alpha$. Then by Proposition 1 in [1, Chap. V, § 3, p. 75] we see that $w \in W(\Sigma^\alpha)$. Therefore the situation is reduced to the case of Σ^α and $F - \{\alpha\} \in M(\Sigma^\alpha)$. Because the signs of w in $W(\Sigma)$ and in $W(\Sigma^\alpha)$ coincide. Thus the assertion is proved by the induction hypothesis. (Note that if Σ is of type E_6, Σ^α is of type A_5.)

Now assume that $\alpha \neq w\alpha$ for any $\alpha \in F$. Then Σ is of type D_N or E_n, and by Lemma 1.3, for a fixed $\alpha \in F$, there exists a subset F' of F containing α, $w\alpha$ such that $\Sigma_{F'}$ is of type D_4. Then as is proved for D_4, there exists a $w' \in W(\Sigma_{F'})$ such that $w'F' = F'$ and $\alpha = w'w\alpha$. Since $w'\gamma = \gamma$ for any $\gamma \in \Sigma$ such that $\gamma \perp \alpha$, $\gamma \perp w\alpha$, we get $w'F = F$. Put $w'' = w'w$, then $w''\alpha = \alpha$, $w''F = F$, and $\mathrm{sgn}(w'') = \mathrm{sgn}(w)$ because $\mathrm{sgn}(w')=1$. Therefore the situation is reduced to the above case already discussed. Thus the lemma is now proved modulo Lemma 1.3. Q. E. D.

Proof of Lemma 1.3. Let Σ be of class I or III, then as is seen in § 1.2, every $F\in M(\Sigma)$ is conjugate to the fixed $F^0\in M(\Sigma)$ under $W(\Sigma)$. If Σ is of type D_N, E_6, E_7 or E_8, the assertion in the lemma can be easily seen from the explicit forms of F^0 given in (1.8), (1.9) or (1.10) respectively. Q. E. D.

1.5. *Root systems of class* II.

Let Σ be a simple root system of class II. Every Σ contains long roots and short roots. Along with the maximal orthogonal systems in P, we consider ordered system E of roots with the following properties (B1), (B2) and (B3). Consider an order $\boldsymbol{P}$ in Σ corresponding to the set P.

(B1) For $E=(\alpha_1, \alpha_2, \cdots, \alpha_n)$, the underlying set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, which is denoted by E^*, is a maximal orthogonal system in P, i.e., $E^*\in M(P)$.

(B2) In E, long roots are placed before short roots. Let $\alpha_1, \alpha_2, \cdots, \alpha_l$ be long roots and $\alpha_{l+1}, \alpha_{l+2}, \cdots, \alpha_n$ short roots, and put $m=[l/2]$, then

$$(1.17)\qquad \begin{cases} \alpha_{2i-1}>\alpha_{2i}\ (1\leqq i\leqq m),\ \alpha_1>\alpha_3>\cdots>\alpha_{2m-1}; \\ \alpha_{l+1}>\alpha_{l+2}>\cdots>\alpha_{n-1}>\alpha_n . \end{cases}$$

(B3) For $1\leqq i\leqq m$, $2^{-1}(\alpha_{2i-1}\pm\alpha_{2i})$ are (short) roots in P. (If $2^{-1}(\alpha_{2i-1}-\alpha_{2i})$ is a root, then so is $2^{-1}(\alpha_{2i-1}+\alpha_{2i})$, because $\alpha_{2i-1}\perp\alpha_{2i}$.)

The set of all such ordered systems E in P is denoted by $M^{or}(\boldsymbol{P})$. Note that the correspondence $E\mapsto F=E^*$ from $M^{or}(\boldsymbol{P})$ to $M(P)$ is surjective but not 1-1 when Σ is of type C_n $(n\geqq 3)$ or F_4 (cf. § 1.2).

Definition 1.1. An element $E\in M^{or}(\boldsymbol{P})$ is called *standard* if it satisfies the following. Let E be given as in (1.17) and put $\Sigma_j=\{\gamma\in\Sigma\,;\ \gamma\perp\alpha_r\ (1\leqq r\leqq j-1)\}$ for $j\geqq 1$, $\Sigma_0=\Sigma$. For $1\leqq i\leqq m$, α_{2i-1} (or α_{2i} resp.) is the highest long root in Σ_{2i-1} (or Σ_{2i} such that $2^{-1}(\alpha_{2i-1}+\alpha_{2i})$ are roots in P resp.), and when l is odd, α_l is the highest long root in Σ_l, and for $l+1\leqq j\leqq n$, α_j is the highest short root in Σ_j. An element $F\in M(P)$ is called *standard* with respect to P if $F=E^*$ for some standard element $E\in M^{or}(\boldsymbol{P})$. We define the *type* of $F\in M(P)$ or $E\in M^{or}(\boldsymbol{P})$ as the number l of long roots in it.

To exclude useless complications, we choose as $\boldsymbol{P}$ the canonical order in Σ corresponding to P defined below, and in that case $M^{or}(\boldsymbol{P})$ is denoted simply as $M^{or}(P)$.

Let Σ be of type B_n. Then all positive short roots are mutually orthogonal. The arrangement of them such that

$$(1.18)\qquad (\gamma_1, \gamma_2, \cdots, \gamma_n),\quad \gamma_1>\gamma_2>\cdots>\gamma_n\ (\text{for } \boldsymbol{P})$$

does not depend on the choice of the order $\boldsymbol{P}$ corresponding to P because for any two such roots γ, γ', we have $\gamma-\gamma'\in\Sigma=P\cup-P$ and hence $\gamma>\gamma'$ or $\gamma<\gamma'$ is determined by the set P only. Let Σ be of type C_n. Then all positive long roots are mutually orthogonal, and the arrangement of them such that

$$(1.18')\qquad (\alpha_1, \alpha_2, \cdots, \alpha_n),\quad \alpha_1>\alpha_2>\cdots>\alpha_n\ (\text{for } \boldsymbol{P})$$

does not depent on the choice of $\boldsymbol{P}$ because for any two of such roots α, α', we have $2^{-1}(\alpha-\alpha')\in\Sigma$. Let Σ be of type F_4. The highest long root α_1 in P is uniquely determined by P only as is seen from the theory of highest weights of the finite-dimensional representations. Then Σ^{α_1} is of type C_3, and as is said above the long roots in $P\cap\Sigma^{\alpha_1}$ is uniquely arranged as $\alpha_2>\alpha_3>\alpha_4$ by P only. Thus the ordered set

$$(1.18'')\qquad (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

is uniquely determined by P.

Let us call an order $\boldsymbol{P}$ corresponding to P *canonical* when it is defined as follows: for Σ of type B_n or C_n, $\boldsymbol{P}$ is the lexicographic order with respect to $\gamma_1, \gamma_2, \cdots, \gamma_n$ in (1.18) or to $\alpha_1, \alpha_2, \cdots, \alpha_n$ in (1.18') respectively, and for Σ of type F_4, $\boldsymbol{P}$ is the lexicographic order with respect to $\gamma_1=2^{-1}(\alpha_1+\alpha_2)$, $\gamma_2=2^{-1}(\alpha_1-\alpha_2)$, $\gamma_3=2^{-1}(\alpha_3+\alpha_4)$, $\gamma_4=2^{-1}(\alpha_3-\alpha_4)$, where α_i's are given in (1.18''). When we realize Σ as in § 1.2, the canonical order $\boldsymbol{P}$ is nothing but the special lexicographic order given there.

Let $\boldsymbol{P}$ be canonical. For $F\in M(P)$, let $\alpha_1, \alpha_2, \cdots, \alpha_l$ be its long roots and $\alpha_{l+1}, \alpha_{l+2}, \cdots, \alpha_n$ its short roots numbered as

$$(1.19)\qquad \alpha_1>\alpha_2>\cdots>\alpha_l\,;\ \alpha_{l+1}>\alpha_{l+2}>\cdots>\alpha_n\,.$$

We put

$$(1.20)\qquad \tilde{F}=(\alpha_1, \alpha_2, \cdots, \alpha_l, \alpha_{l+1}, \alpha_{l+2}, \cdots, \alpha_n)\,,$$

then it always belongs to $M^{or}(\boldsymbol{P})$ (cf. § 1.2). The set of all $\tilde{F}$ is denoted by $\tilde{M}(P)$: $\tilde{M}(P)=\{\tilde{F}\,;\ F\in M(P)\}\subset M^{or}(\boldsymbol{P})$. Note that every standard element $E\in M^{or}(\boldsymbol{P})$ is given as $E=\tilde{F}$ with $F\in M(P)$. Moreover, for every simple Σ, all the standard elements in $M(P)$ are given by the elements F^0 and F^{γ}'s in the list in § 1.2.

Here we must remark the following. Assume that Σ is not of type F_4. Let $\alpha\in\Sigma$ and $\boldsymbol{P}$ be canonical. Then its restriction on the simple component of class II of Σ^{α} (if exists) is again canonical for the set of positive roots $P^{\alpha}=\Sigma^{\alpha}\cap P$ (see §§ 1.1-1.2). When Σ is of type F_4, we are not sure about this fact, but we have for Σ with rank $\Sigma\leqq 3$ the following.

Lemma 1.5. *Let Σ be of type B_2, B_3 or C_3. Let $\boldsymbol{P}$ and $\boldsymbol{P}'$ be two orders in Σ both corresponding to P. Then $M^{or}(\boldsymbol{P})$ and $M^{or}(\boldsymbol{P}')$ coincide with cach other. More exactly, if $E\in M^{or}(\boldsymbol{P})$, $E'\in M^{or}(\boldsymbol{P}')$ satisfy $E^*=(E')^*$, then $E=E'$, and if E is standard in $M^{or}(\boldsymbol{P})$, then so is it in $M^{or}(\boldsymbol{P}')$ and vice versa.*

Proof. It is sufficient to remark the following. Firstly, let α, β be two roots in Σ such that $\alpha-\beta$ or $2^{-1}(\alpha-\beta)$ is in Σ, then $\alpha>\beta$ or $\alpha<\beta$ is determined by P only. Secondly, the highest root in P is long and uniquely determined by P. Q. E. D.

By this lemma, we may restrict ourselves to use only the canonical order in

Σ corresponding to P. This is always done in the sequel unless the contrary is explicitly noted. Thus $M^{or}(\boldsymbol{P})$ is denoted simply by $M^{or}(P)$.

For $E=(\alpha_1, \alpha_2, \cdots, \alpha_n)\in M^{or}(P)$ given by (1.17), we put

$$(1.21)\qquad \begin{aligned} P(E) &= \{\alpha_1, \alpha_2, \cdots, \alpha_n, 2^{-1}(\alpha_{2i-1}\pm\alpha_{2i})\ (1\leqq i\leqq m)\},\\ F_0(E) &= \{\alpha_1, \alpha_2, \cdots, \alpha_l, \alpha_{l+2i-1}\pm\alpha_{l+2i}\ (1\leqq i\leqq [(n-l)/2]),\\ &\qquad \text{and } \alpha_n \text{ if } n-l \text{ is odd}\}. \end{aligned}$$

Then $F_0(E)$ belongs to $M(P)$ since the order is canonical, moreover is a strongly orthogonal system of roots which is called *associated* to E. We make $u\in W(\Sigma)$ operate on E by

$$(1.22)\qquad uE=(u\alpha_1, u\alpha_2, \cdots, u\alpha_n).$$

Note that uE does not necessarily belong to $M^{or}(P)$. Put

$$(1.23)\qquad \begin{aligned} W(E\,;P) &= \{u\in W(\Sigma)\,;\ uE\in M^{or}(P)\},\\ V(E) &= \{v\in W(E\,;P)\,;\ (vE)^*=E^*\}, \end{aligned}$$

and for $F\in M(P)$,

$$(1.24)\qquad U(\tilde{F})=\{u\in W(\Sigma)\,;\ u\tilde{F}\in \tilde{M}(P)\}.$$

Let us study $M^{or}(P)$ according to the type of Σ. We realize Σ as in § 1.2.

TYPE B_n. Let $F^r\in M(P)$ be given in (1.4), then the corresponding element $\tilde{F}^r\in\tilde{M}(P)$ are standard and the sets of roots $P(\tilde{F}^r)$ and $F_0(\tilde{F}^r)$ are given as follows:

$$(1.25)\qquad \begin{aligned} \tilde{F}^r=(&e_1+e_2, e_1-e_2, e_3+e_4, e_3-e_4, \cdots,\\ &e_{2r-1}+e_{2r}, e_{2r-1}-e_{2r}, e_{2r+1}, e_{2r+2}, \cdots, e_n), \end{aligned}$$

$$(1.26)\qquad P(\tilde{F}^r)=F^r\cup F^0,\quad F_0(\tilde{F}^r)=F^k \text{ with } k=[n/2].$$

Moreover $V(\tilde{F}^r)=\{1\}$, and therefore $M^{or}(P)=\tilde{M}(P)$ in this case, and the correspondence $E\mapsto E^*$ of $M^{or}(P)$ onto $M(P)$ is bijective. Every element w in $W(\Sigma)$ can be expressed as

$$(1.27)\qquad we_i=\varepsilon_i e_{\sigma(i)}\quad (1\leqq i\leqq n),$$

where $\varepsilon_i=\pm 1$ and $\sigma=(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is an element of the n-th symmetric group $\mathfrak{S}_n$. Put $\varepsilon=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ and express w by (ε, σ). Let $\varepsilon^0=(1, 1, \cdots, 1)$, then

$$(1.28)\qquad U(\tilde{F}^r)=\{w=(\varepsilon^0, \sigma)\,;\ \sigma \text{ satisfies } (1.28')\},$$

where

$$(1.28')\qquad \begin{cases} \sigma(2i-1)<\sigma(2i)\ (1\leqq i\leqq r),\quad \sigma(1)<\sigma(3)<\cdots<\sigma(2r-1)\,;\\ \sigma(2r+1)<\sigma(2r+2)<\cdots<\sigma(n). \end{cases}$$

TYPE C_n. The element $\tilde{F}^r\in\tilde{M}(P)$ corresponding to $F^r\in M(P)$ in (1.6) is standard and given by

$$\tilde{F}^r=(2e_1,\ 2e_2,\ \cdots,\ 2e_{n-2r},\ e_{n-2r+1}+e_{n-2r+2},\ e_{n-2r+1}-e_{n-2r+2},\ e_{n-2r+3}+e_{n-2r+4},\ e_{n-2r+3}-e_{n-2r+4},\ \cdots,\ e_{n-1}+e_n,\ e_{n-1}-e_n). \tag{1.29}$$

Put $k=[n/2]$, then

$$P(\tilde{F}^r)=F^r\cup F^k,\quad F_0(\tilde{F}^r)=F^0. \tag{1.30}$$

Every element $w\in W(\Sigma)$ can be expressed as in (1.27) and so it is denoted by (ε, σ) as above. Then

$$V(\tilde{F}^r)=\{w=(\varepsilon^0, \sigma);\ \sigma \text{ satisfies } (1.31')\}, \tag{1.31}$$

where $\varepsilon^0=(1, 1, \cdots, 1)$ and

$$\begin{cases} \sigma(2i-1)<\sigma(2i)\ (1\leqq i\leqq k-r), & \sigma(1)<\sigma(3)<\cdots<\sigma(2k-2r-1); \\ \sigma(j)=j\ (n-2r+1\leqq j\leqq n). \end{cases} \tag{1.31'}$$

The group $U(\tilde{F}^r)$ can be given similarly.

TYPE F_4. For F^0, F^1, $F^2\in M(P)$ in (1.13), the corresponding elements $\tilde{F}^0$, $\tilde{F}^1$, $\tilde{F}^2\in\tilde{M}(P)$ are standard and can be written down easily, and further

$$P(\tilde{F}^r)=F^r\cup F^0,\quad F_0(\tilde{F}^r)=F^2\quad (r=0,\ 1,\ 2). \tag{1.32}$$

Moreover, denoting the simple roots as in (1.12), we get

$$V(\tilde{F}^0)=V(\tilde{F}^1)=\{1\},\quad V(\tilde{F}^2)=\{1,\ s_\alpha,\ s_\delta s_\alpha\}. \tag{1.33}$$

Furthermore

$$\begin{aligned} &U(\tilde{F}^0)=\{1,\ s_\alpha,\ s_\delta s_\alpha\},\quad U(\tilde{F}^2)=\{1,\ s_\beta,\ s_\gamma s_\beta\}. \\ &U(\tilde{F}^1) \text{ contains 18 elements.} \end{aligned} \tag{1.34}$$

The elements in $M(P)$ conjugate under $W(\Sigma)$ to F^r ($r=0$, 1 or 2) are listed up below. The corresponding elements in $\tilde{M}(P)$ can be written down easily.

For F^0: F^0, $F^0_{+1}=s_\alpha F^0$, $F^0_{-1}=s_\delta s_\alpha F^0$,

where $F^0_\varepsilon=\{2^{-1}(e_1+\varepsilon_2e_2+\varepsilon_3e_3+\varepsilon_4e_4);\ \varepsilon_2\varepsilon_4\varepsilon_4=\varepsilon\}$ for $\varepsilon=\pm1$.

For F^1: $\{e_i\pm e_j,\ e_k,\ e_l\}$,

where $i<j$, $k<l$ and $\{i, j, k, l\}=\{1, 2, 3, 4\}$, and

$$\{e_1+\varepsilon e_i,\ e_j+\varepsilon' e_k,\ 2^{-1}(e_1-\varepsilon e_i\pm(e_j-\varepsilon' e_k))\},$$

where ε, $\varepsilon'=\pm1$, $j<k$, $\{i, j, k\}=\{2, 3, 4\}$.

For F^2: $\{e_1\pm e_i,\ e_j\pm e_k\}$,

where $j<k$, $\{i, j, k\}=\{2, 3, 4\}$.

Note that for any type of Σ, the strongly orthogonal system $F_0(E)$ associated to a standard element $E\in M^{or}(P)$ is always equal to the unique standard element in $M(P)$ which is strongly orthogonal.

1.6. Let Σ be a simple root system of class II.

Lemma 1.6. *Let* $\{\alpha, \beta\}$ *and* $\{\alpha', \beta'\}$ *be two pairs of long roots (short roots resp.) in* Σ *such that* $2^{-1}(\alpha\pm\beta)$ *and* $2^{-1}(\alpha'\pm\beta')$ ($\alpha\pm\beta$ *and* $\alpha'\pm\beta'$ *resp.) are roots in* Σ. *Then* $\alpha\perp\beta$ *and there exists a* $w\in W(\Sigma)$ *such that* $w\alpha=\alpha'$, $w\beta=\beta'$.

Proof. We see from the explicit form of Σ in §1.2 that $\alpha\perp\beta$. On the other hand, there exists a $w'\in W(\Sigma)$ such that $w'\alpha=\alpha'$. Then $w'\beta$ and β' are two roots in $\Sigma^{\alpha'}$ with the same length. The root system $\Sigma^{\alpha'}$ is simple except when Σ is of type B_n and α' is long, or Σ is of type C_n and α' is short (cf. Table 1.1). If $\Sigma^{\alpha'}$ is simple, there exists $w''\in W(\Sigma^{\alpha'})$ such that $w''(w'\alpha)=\alpha'$. Put $w=w''w'$, then $w\alpha=\alpha'$, $w\beta=\beta'$. If $\Sigma^{\alpha'}$ is not simple, the roots β and β' are characterized in Σ up to their signs by the condition that $2^{-1}(\alpha\pm\beta)$ and $2^{-1}(\alpha'\pm\beta')$ ($\alpha\pm\beta$ and $\alpha'\pm\beta'$ resp.) are again roots in Σ. Therefore we get $w'\beta=\varepsilon\beta'$ with $\varepsilon=1$ or -1 in this case. Since $\alpha'\perp\beta'$, it is enough to put $w=w'$ or $s_{\beta'}w'$ according as $\varepsilon=1$ or -1. Q. E. D.

Lemma 1.7. (a) *For any two elements* E, E' *in* $M^{or}(P)$ *of the same type, there exists uniquely a* $w\in W(E;P)$ *such that* $wE=E'$. (b) *For any* $F\in M(P)$, $W(\tilde{F};P)$ *is a direct product of* $U(\tilde{F})$ *and* $V(\tilde{F})$ *in the following sense: for every* $w\in W(\tilde{F};P)$, *there exist uniquely* $u\in U(\tilde{F})$, $v\in V(\tilde{F})$ *such that* $w=uv$, *and conversely any element of this form belongs to* $W(\tilde{F};P)$.

Proof. Let $\{\alpha, \beta\}$ be as in Lemma 1.6, then $\Sigma'=\{\gamma\in\Sigma;\gamma\perp\alpha,\beta\}$ is again a simple root system of class II or of type A_1. Therefore the assertion (a) follows from Lemma 1.6. For the assertion (b), note that $u\in W(\Sigma)$ belongs to $U(\tilde{F})$ if and only if u does not change the order relations between the long roots in $\tilde{F}$ and also between the short roots in $\tilde{F}$. Then it is not difficult to see that the assertion (b) holds. Q. E. D.

Let l be the type of an $E\in M^{or}(P)$ and $M^{or}(P, l)$ be the subset of $M^{or}(P)$ consisting of elements of type l. Then it follows from Lemma 1.7(a) that there exists a natural bijective correspondence:

$$W(E;P)\ni u\mapsto uE\in M^{or}(P, l). \tag{1.35}$$

On the other hand, consider a collection D of m ordered pairs (α_i, β_i) $(1\leqq i\leqq m)$ of long roots in P, and at most one long root $\gamma\in P$, and some number of short roots in P such that

(B′1) the underlying set F of D belongs to $M(P)$, and

(B′2) $2^{-1}(\alpha_i\pm\beta_i)$ are roots in P for $1\leqq i\leqq m$.

Let $M'(P)$ be the set of all such D's. We define the type of D as the number of long roots in it. Note that under the condition (B′2), we get always $\alpha_i>\beta_i$.

There exists a natural 1-1 correspondence between $M^{or}(P)$ and $M'(P)$ as follows: for an $E\in M^{or}(P)$ given in (1.17), we make correspond $D\in M'(P)$ consisting of long root pairs $(\alpha_{2i-1}, \alpha_{2i})$ $(1\leqq i\leqq m)$, and a long root α_l if l is odd, and short roots $\alpha_{l+1}, \alpha_{l+2}, \cdots, \alpha_n$.

Now we make operate $W(\Sigma)$ naturally on $D\in M'(P)$. Then for any $D, D' \in M'(P)$ of the same type, there exists a $u\in W(\Sigma)$ such that $uD=D'$ by Lemma 1.6. For a $D\in M'(P)$, we define $I(D)$ as a subgroup of $W(\Sigma)$ consisting of u such that (1) let $F\in M(P)$ be the underlying set of D, then $uF\subset F\cup -F$, and (2) for $1\leqq i\leqq m$, $\{u\alpha_i, u\beta_i\}\subset\{\pm\alpha_{i'}, \pm\beta_{i'}\}$ for some i'. Let l be the type of D and $M'(P, l)$ be the subset of $M'(P)$ consisting of elements of type l. Then we get a natural bijective correspondence:

$$(1.36) \qquad M'(P, l)\ni D'=uD\mapsto uI(D)\in W(\Sigma)/I(D).$$

Through the natural correspondence between $M^{or}(P, l)$ and $M'(P, l)$, the bijections (1.35) and (1.36) give us $W(E; P)$ as a complete system of representatives of the coset space $W(\Sigma)/I(D)$, where $E\in M^{or}(P)$ correspnds to D.

1.7. Let Σ be a simple root system of class I or III. Since it has no importance to introduce an order in Σ corresponding to P in this case, we introduce the following formal convention in accordance with the case of class II.

CONVENTION 1.1. We put simply as $M^{or}(P)=\tilde{M}(P)=M(P)$, $P(E)=F_0(E)=E^*=F$, where E denotes $F\in M(P)$ itself viewed formally as an element of $M^{or}(P)$. The type l of any element in $M^{or}(P)$ is by definition $l=0$. An element $F\in M(P)$ is called *standard* if it can be obtained as follows: (1) pick up the highest positive root α_1 of Σ (determined by P only), (2) for every simple component of type A_1 of Σ^{α_1}, pick up the unique positive root in it, and (3) for another simple component of Σ^{α_1}, repeat the picking up in (1) and (2), and so on. The set $W(F; P)$ denotes an arbitrary complete system of representatives of $W(\Sigma)/I(F)$ consisting of u's such that $uF\in M(P)$, where

$$(1.37) \qquad I(F)=\{u\in W(\Sigma);\ uF\subset F\cup -F\}.$$

Remark 1.3. Taking into account Lemma 1.2, we see that there exists a natural bijection: $W(F; P)\ni u\mapsto uF\in M(P)$. In the case of class I or III, we do not fix a special choice of a system of representatives of $W(\Sigma)/I(F)$ contrary to the case of class II (cf. (1.35), (1.36)).

In the sequel, we apply frequently Convention 1.1 when the discussions are parallel for all classes of root systems. In this sense, we shall use the notations $M^{or}(P)$, $\tilde{M}(P)$, $\tilde{F}$, $P(E)$, $F_0(E)$ etc. for any root system Σ under the following definition.

Definition 1.2. Let $\Sigma_1, \Sigma_2, \cdots, \Sigma_p$ be all the simple components of Σ, and put $P_i=P\cap\Sigma_i$. Then $M^{or}(P)$ is defined as the set of all ordered sets $E=(E_1, E_2, \cdots, E_p)$ with $E_i\in M^{or}(P_i)$ for $1\leqq i\leqq p$. We define

$$P(E)=P(E_1)\cup P(E_2)\cup\cdots\cup P(E_p)\,,$$
(1.38)
$$F_0(E)=F_0(E_1)\cup F_0(E_2)\cup\cdots\cup F_0(E_p)\,.$$

We call $F_0(E)$ the strongly orthogonal system *associated* to E. Let l_i be the type of E_i, then we call $(l_1, l_2, \cdots, l_p)$ the *type* of E. When every E_i is standard, E is called *standard*. The set $W(E\,;P)$ is defined as the product of $W(E_i\,;P_i)$ over $1\leqq i\leqq p$, where $W(E_i\,;P_i)=\{u\in W(\Sigma_i)\,;\, uE_i\in M^{or}(P_i)\}$ if Σ_i is of class II, and $W(E_i\,;P_i)$ is a complete system of representatives of $W(\Sigma_i)/I(F_i)$ with $F_i=E_i^*$ such that $uF_i\in M(P_i)$ otherwise.

Note that for all standard elements E of $M^{or}(P)$ the associated strongly orthogonal systems $F_0(E)$ coincide with each other. This has an important significance in the sequel (see § 5.2).

For the later use, we also introduce the following definition.

Definition 1.3. For $E\in M^{or}(P)$, a sign $\varepsilon(E)=\pm1$ is defined as follows. When Σ is simple, let $E=(\alpha_1, \alpha_2, \cdots, \alpha_l, \alpha_{l+1}, \cdots, \alpha_n)$ be as in (1.17). By Convention 1.1, we put $l=0$ if Σ is of class I or III. Put $m=[l/2]$ and

$$\varepsilon(E)=(-1)^n(-1)^m=(-1)^n(-1)^{[l/2]}\,. \tag{1.39}$$

When Σ is not simple, express E as in Definition 1.2 and put

$$\varepsilon(E)=\varepsilon(E_1)\cdot\varepsilon(E_2)\cdots\varepsilon(E_p)\,. \tag{1.39'}$$

1.8. Let Σ be a simple root system.

Lemma 1.8. *Assume that $\gamma, \gamma'\in\Sigma$ are orthogonal but not strongly orthogonal to each other. Then Σ is one of the types B_n, C_n and F_4, and γ, γ' are short roots in Σ. Moreover put $F'=\{\gamma, \gamma'\}$, then $\Sigma_{F'}=\{\pm\gamma, \pm\gamma', \pm\gamma\pm\gamma'\}$ and is of type B_2.*

More explicitly we can list up the possible pairs $F'=\{\gamma, \gamma'\}$ of positive roots as follows. (1) In the case of B_n, $\{\gamma, \gamma'\}$ is any orthogonal pair of positive short roots: $\{\gamma, \gamma'\}=\{e_i, e_j\}$ for some $i<j$, and is conjugate under $W(\Sigma)$ to $\{e_1, e_2\}$. (2) In the case of C_n, $\{\gamma, \gamma'\}=\{e_i\pm e_j\}$ for some $i<j$, and is conjugate to $\{e_1\pm e_2\}$. (3) In the case of F_4, $\{\gamma, \gamma'\}$ is any orthogonal pair of positive short roots and is conjugate to $\{e_1, e_2\}$.

§ 2. Structure of Cartan subgroups

Let G be a connected real semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $H^{\mathfrak{h}}$ the Cartan subgroup of G corresponding to $\mathfrak{h}$. In this section we study the structure of every $H^{\mathfrak{h}}$ and the relations between them.

2.1. For $g\in G$, we denote by τ_g the inner automorphism of G corresponding to $g:\tau_g x=gxg^{-1}$ $(x\in G)$. Let H be a subgroup of G and C a subset of G (resp. of $\mathfrak{g}$). Then we denote by $N_H(C)$ and $Z_H(C)$ the subgroups of H consisting of elements h such that $\tau_h(C)\subset C$ and $\tau_h(x)=x$ $(x\in C)$ (resp. $\mathrm{Ad}(h)C\subset C$ and $\mathrm{Ad}(h)X=X$ $(X\in C)$) respectively. Then by definition, $H^{\mathfrak{h}}=Z_G(\mathfrak{h})$. Put

$$W_G(C)=N_G(C)/Z_G(C). \tag{2.1}$$

Let θ be a Cartan involution of $\mathfrak{g}$, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$ and denote by exp the exponential map of $\mathfrak{g}$ into G. Then the map $(k, X)\mapsto k\exp X$ of $K\times\mathfrak{p}$ into G is an analytic diffeomorphism onto G. Moreover θ can be lifted up to an automorphism of G given by $k\exp X\mapsto k\exp(-X)$ $(k\in K, X\in\mathfrak{p})$. Assume that $\theta\mathfrak{h}=\mathfrak{h}$. Put $\mathfrak{h}_{\mathfrak{k}}=\mathfrak{h}\cap\mathfrak{k}$, $\mathfrak{h}_{\mathfrak{p}}=\mathfrak{h}\cap\mathfrak{p}$, then

$$\mathfrak{h}=\mathfrak{h}_{\mathfrak{k}}+\mathfrak{h}_{\mathfrak{p}} \qquad \text{(direct)}. \tag{2.2}$$

Moreover put $H^{\mathfrak{h}}_K=H^{\mathfrak{h}}\cap K$, $H^{\mathfrak{h}}_{\mathfrak{p}}=\exp(\mathfrak{h}\cap\mathfrak{p})$, then

$$H^{\mathfrak{h}}=H^{\mathfrak{h}}_K H^{\mathfrak{h}}_{\mathfrak{p}} \qquad \text{(direct)}. \tag{2.3}$$

In this case $W_G(\mathfrak{h})$ and $W_G(H^{\mathfrak{h}})$ are canonically isomorphic to $W_K(\mathfrak{h})$ and $W_K(H^{\mathfrak{h}})$ respectively (see [2(b), § 16]).

2.2. Denote by $\Sigma(\mathfrak{h})$ the set of roots of $(\mathfrak{g}_c, \mathfrak{h}_c)$ or simply of $\mathfrak{h}$. Let $P(\mathfrak{h})$ be the set of positive roots in $\Sigma(\mathfrak{h})$ with respect to an order in $\Sigma(\mathfrak{h})$. A root α of $\mathfrak{h}$ is called *real* or *imaginary* if $\alpha(\mathfrak{h})\subset\boldsymbol{R}$ or $\alpha(\mathfrak{h})\subset\sqrt{-1}\,\boldsymbol{R}$. We denote by $\Sigma_R(\mathfrak{h})$ the set of real roots of $\mathfrak{h}$ and put

$$P_R(\mathfrak{h})=\Sigma_R(\mathfrak{h})\cap P(\mathfrak{h}). \tag{2.4}$$

For a root α, we choose root vectors $X_{\pm\alpha}$ from the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$ in such a way that $H_\alpha=[X_\alpha, X_{-\alpha}]$ holds, where $H_\alpha\in\mathfrak{h}_c$ is the element corresponding to α under the Killing form of $\mathfrak{g}_c$. We put

$$H'_\alpha=2|\alpha|^{-2}H_\alpha, \quad X'_{\pm\alpha}=\sqrt{2}\,|\alpha|^{-1}X_{\pm\alpha}, \tag{2.5}$$

where $|\alpha|$ denotes the length of α with respect to the Killing form. An imaginary root α is called *compact* or *singular* if $\mathfrak{g}\cap(\boldsymbol{C}H_\alpha+\boldsymbol{C}X_\alpha+\boldsymbol{C}X_{-\alpha})$ is isomorphic to $\mathfrak{su}(2)$ or $\mathfrak{sl}(2, \boldsymbol{R})$. If α is real, we can choose $X_{\pm\alpha}$ from $\mathfrak{g}$. So done, put

$$\nu_\alpha=\exp\left\{-\sqrt{-1}\,\frac{\pi}{4}\mathrm{ad}(X'_\alpha+X'_{-\alpha})\right\} \tag{2.6}$$

and $\mathfrak{h}^\alpha=\nu_\alpha(\mathfrak{h}_c)\cap\mathfrak{g}$. Then $\mathfrak{h}^\alpha$ is another Cartan subalgebra of $\mathfrak{g}$ not conjugate to $\mathfrak{h}$ under any automorphism of $\mathfrak{g}$. Let σ_α be the hyperplane of $\mathfrak{h}$ defined by $\alpha(X)=0$. Then

$$\mathfrak{h}=\sigma_\alpha+\boldsymbol{R}H'_\alpha, \quad \mathfrak{h}^\alpha=\sigma_\alpha+\boldsymbol{R}(X'_\alpha-X'_{-\alpha}). \tag{2.7}$$

The root $\beta=\nu_\alpha\alpha=\alpha\circ(\nu_\alpha^{-1}|\mathfrak{h}^\alpha_c)$ is a singular imaginary root of $\mathfrak{h}^\alpha$ and $H'_\beta=$

$\sqrt{-1}(X'_\alpha - X'_{-\alpha})$. We know the following fact [5(d), Lem. 4.5].

Lemma 2.1. *The set* $\Sigma_R(\mathfrak{h}^\alpha)$ *of real roots of* $\mathfrak{h}^\alpha$ *is given by*

$$\Sigma_R(\mathfrak{h}^\alpha) = \{\nu_\alpha\gamma\,;\ \gamma \in \Sigma_R(\mathfrak{h}),\ \gamma \perp \alpha\}. \tag{2.8}$$

Put $\Sigma = \Sigma_R(\mathfrak{h})$, then $\Sigma_R(\mathfrak{h}^\alpha) = \nu_\alpha \Sigma^\alpha$, where Σ^α is defined in (1.1). Let $\gamma \in \Sigma^\alpha$, then

$$\nu_\alpha\gamma \mid \sigma_\alpha = \gamma \mid \sigma_\alpha\,, \quad (\nu_\alpha\gamma)(H'_\beta) = \gamma(H'_\alpha) = 0\,,$$

and $H_{\nu_\alpha\gamma} = \nu_\alpha H_\gamma = H_\gamma$. Moreover we get the following.

Lemma 2.2. Let $\gamma \perp \alpha$. *If* α *and* γ *are strongly orthogonal to each other, then* $\nu_\alpha X_{\pm\gamma} = X_{\pm\gamma}$. *Otherwise,*

$$\nu_\alpha X_{\pm\gamma} = -\frac{\sqrt{-1}}{2}\,\mathrm{ad}(X'_\alpha + X'_{-\alpha}) X_{\pm\gamma}\,. \tag{2.9}$$

Proof. If α and γ are strongly orthogonal to each other, we get $[X_{\pm\alpha}, X_{\pm\gamma}] = 0$ and therefore $\nu_\alpha X_{\pm\gamma} = X_{\pm\gamma}$. Assume it is not the case, then by Lemma 1.8 we know that $\Sigma' = \Sigma \cap (\boldsymbol{R}\alpha + \boldsymbol{R}\gamma)$ is of type B_2 and α, γ are short roots in Σ'. By an explicit calculation on the Lie algebra $\mathfrak{sp}(2, \boldsymbol{R})$, we get without difficulty that

$$\mathrm{ad}(X'_\alpha)\mathrm{ad}(X'_{-\alpha}) X_{\pm\gamma} = \mathrm{ad}(X'_{-\alpha})\mathrm{ad}(X'_\alpha) X_{\pm\gamma} = 2X_{\pm\gamma}\,, \tag{2.10}$$

and therefore $\mathrm{ad}(X'_\alpha + X'_{-\alpha})^2 X_{\pm\gamma} = 4X_{\pm\gamma}$. From this we get the above equality (2.9). Q. E. D.

We know the following [2(a), Lem. 46, p. 255]: *Assume that* $\theta\mathfrak{h} = \mathfrak{h}$. *Then for any* $\alpha \in \Sigma_R(\mathfrak{h})$, *the root vectors* $X_{\pm\alpha}$ *in* $\mathfrak{g}$ *can be normalized in such a way that* $\tilde{\theta} X_\alpha = -X_{-\alpha}$, *where* $\tilde{\theta}$ *denotes the conjugation of* $\mathfrak{g}_c$ *with respect to the compact real form* $\mathfrak{k} + \sqrt{-1}\,\mathfrak{p}$. In that case, $\theta\mathfrak{h}^\alpha = \mathfrak{h}^\alpha$. But in the sequel, we do not demand the normalization $\tilde{\theta} X_\alpha = -X_{-\alpha}$ unless the contrary is explicitly noted.

Let $F = \{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ be a strongly orthogonal system of real roots of $\mathfrak{h}$. Then $\nu_{\alpha_1}, \nu_{\alpha_2}, \cdots, \nu_{\alpha_s}$ commute with each other. The transformation ν_F is defined by the product of them as

$$\nu_F = \nu_{\alpha_1}\nu_{\alpha_2} \cdots \nu_{\alpha_s}\,. \tag{2.11}$$

and the Cartan subalgebra $\mathfrak{h}^F$ is defined as $\mathfrak{h}^F = \nu_F(\mathfrak{h}_c) \cap \mathfrak{g}$. Then putting $\sigma_F = \bigcap_{\alpha \in F} \sigma_\alpha$, we get

$$\mathfrak{h} = \sigma_F + \sum_{\alpha \in F} \boldsymbol{R} H'_\alpha\,, \quad \mathfrak{h}^F = \sigma_F + \sum_{\alpha \in F} \boldsymbol{R}(X'_\alpha - X'_{-\alpha})\,. \tag{2.12}$$

2.3. *Connected components of a Cartan subgroup.* For a root α of $\mathfrak{h}$, we define a character ξ_α of $H^{\mathfrak{h}}$ by

$$\mathrm{Ad}(h) X_\alpha = \xi_\alpha(h) X_\alpha \quad (h \in H^{\mathfrak{h}})\,. \tag{2.13}$$

Then we know that if α is real, ξ_α takes only real values 1 or -1 on $H^{\mathfrak{h}}_K$.

Let A be a connected component of $H^{\mathfrak{h}}$. Assuming $\theta\mathfrak{h}=\mathfrak{h}$, we get the following direct product decomposition of A:

(2.14) $$A=A_K\exp(\mathfrak{h}_{\mathfrak{p}}), \quad \text{where } A_K=A\cap K.$$

Define a subset $\Sigma_R(A)$ of $\Sigma_R(\mathfrak{h})$ by

(2.15) $$\Sigma_R(A)=\{\alpha\in\Sigma_R(\mathfrak{h})\,;\ \xi_\alpha(h)>0 \text{ for } h\in A\},$$

and put $P_R(A)=P(\mathfrak{h})\cap\Sigma_R(A)$. Note that the condition $\xi_\alpha(A)>0$ is equivalent to $\xi_\alpha(A_K)=\{1\}$.

Let D be a subset of $H^{\mathfrak{h}}$ and $\mathfrak{d}$ a subset of $\Sigma(\mathfrak{h})$. Then we put

(2.16) $$D(\mathfrak{d})=\{h\in D\,;\ \xi_\alpha(h)=1\ (\alpha\in\mathfrak{d})\}.$$

Lemma 2.3. *Let $\alpha\in\Sigma_R(A)$ and put $\beta=\nu_\alpha\alpha$, $\mathfrak{a}=\mathfrak{h}^\alpha$. Then $A(\{\alpha\})\subset H^{\mathfrak{a}}(\{\beta\})$. Moreover let C be the connected component of $H^{\mathfrak{a}}$ containing $A(\{\alpha\})$. Then*

$$\Sigma_R(C)=\{\nu_\alpha\gamma\,;\ \gamma\in\Sigma_R(A),\ \gamma\perp\alpha\}.$$

Proof. Let $h\in A(\{\alpha\})$, then $\mathrm{Ad}(h)X_{\pm\alpha}=\xi_{\pm\alpha}(h)X_{\pm\alpha}=X_{\pm\alpha}$. Since $\mathfrak{h}^\alpha=\mathfrak{a}$ is given by (2.7), we get $\mathrm{Ad}(h)X=X$ for any $X\in\mathfrak{a}$. Hence $h\in H^{\mathfrak{a}}$. On the other hand, a root vector for β is given by $X'_\beta=\nu_\alpha X'_\alpha=2^{-1}(\sqrt{-1}H'_\alpha+X'_\alpha+X'_{-\alpha})$. Since $\mathrm{Ad}(h)X'_\beta=\xi_\beta(h)X'_\beta$ by definition, we see that $\xi_\beta(h)=1$. This proves that $A(\{\alpha\})\subset H^{\mathfrak{a}}(\{\beta\})$.

The second assertion can be proved by using Lemma 2.1 and (2.7) (cf. [5(d), Lem. 4.5]). Q. E. D.

Corollary. *Let F be a strongly orthogonal system of roots in $\Sigma_R(A)$ and put $\mathfrak{a}=\mathfrak{h}^F$. Then $A(F)\subset H^{\mathfrak{a}}(\nu_F F)$.*

Proof. Note that if $\alpha, \alpha'\in\Sigma_R(A)$ are strongly orthogonal to each other, then so are also in $\Sigma(\mathfrak{h})$. Then the corollary follows immediately from the lemma. Q. E. D.

Now let $M(\Sigma_R(A))$ be as in § 1.2. We wish to prove the following fact.

Proposition 2.4. *Let $F\in M(\Sigma_R(A))$ be strongly orthogonal. Then the dimension of the vector part of $\mathfrak{h}^F$ takes the minimum possible value, or equivalently, $\Sigma_R(\mathfrak{h}^F)=\emptyset$.*

To prove the proposition, we need a lemma.

Lemma 2.5. *Assume that $\Sigma_R(\mathfrak{h})$ is isomorphic to a multiple of the root system of type A_1. Then for any connected component A of $H^{\mathfrak{h}}$, $\Sigma_R(A)=\Sigma_R(\mathfrak{h})$.*

Proof. Take a Cartan involution θ of $\mathfrak{g}$ such that $\theta\mathfrak{h}=\mathfrak{h}$. Put $D=Z_G(\mathfrak{h}_{\mathfrak{k}})$, then $D\supset H^{\mathfrak{h}}$, and by [2(b), Cor. 3 of Lem. 26, p. 481] we see that

$$D=Z_K(\mathfrak{h}_t)\exp\mathfrak{z}_{\mathfrak{p}}(\mathfrak{h}_t) \qquad \text{(direct)},$$

where $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{h}_t)$ denotes the centralizer of $\mathfrak{h}_t$ in $\mathfrak{p}$. Note that $T=\exp(\mathfrak{h}_t)$ is a total subgroup in the sense of [4, p. 247] and that $Z_K(\mathfrak{h}_t)=Z_K(T)$. On the other hand, we see that $Z_K(T)$ is connected by [Cor. 2.8, loc. cit.]. Hence D is connected. Let $\mathfrak{d}$ be the subalgebra of $\mathfrak{g}$ corresponding to D, then $\mathfrak{d}=\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}_t)$. Let $\mathfrak{g}_c=\mathfrak{h}_c+\sum_{\alpha\in\Sigma(\mathfrak{h})}\mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}_c$. Then we get

$$\mathfrak{d}=\mathfrak{h}+\sum_{\alpha\in\Sigma_R(\mathfrak{h})}\mathfrak{g}_\alpha\cap\mathfrak{g}=\mathfrak{h}'+\sum_{\alpha\in P_R(\mathfrak{h})}\mathfrak{g}^{(\alpha)},$$

where

$$\mathfrak{h}'=\{X\in\mathfrak{h}\,;\ \alpha(X)=0\ (\alpha\in\Sigma_R(\mathfrak{h}))\},\quad \mathfrak{g}^{(\alpha)}=\boldsymbol{R}H_\alpha+\boldsymbol{R}X_\alpha+\boldsymbol{R}X_{-\alpha}\cong\mathfrak{sl}(2,\boldsymbol{R}).$$

Note that $\mathfrak{h}=\mathfrak{h}'+\sum_{\alpha\in P_R(\mathfrak{h})}\boldsymbol{R}H_\alpha$. Since D is connected, we see that $\mathrm{Ad}_{\mathfrak{d}}(D)$ is canonically isomorphic to a multiple of $SL(2,\boldsymbol{R})/\{\pm1\}$. Then from the result on $SL(2,\boldsymbol{R})$, we see that for every $\alpha\in\Sigma_R(\mathfrak{h})$, $\xi_\alpha(h)>0$ $(h\in H^{\mathfrak{h}})$. This proves our assertion. Q. E. D.

Proof of Proposition 2.4. By Corollary of Lemma 2.3, we see that $A(F)\subset H^{\mathfrak{a}}$ with $\mathfrak{a}=\mathfrak{h}^F$. Let C be the connected component of $H^{\mathfrak{a}}$ containing $A(F)$. Let us prove that $\Sigma_R(C)=\emptyset$. In fact, let $\gamma\in\Sigma_R(C)$, then by Corollary of Lemma 2.3, there exists $\gamma'\in\Sigma_R(A)$ such that $\gamma=\nu_F\gamma'$, $\gamma'\perp\alpha$ for all α in F. This contradicts the maximality of F as an orthogonal system. Now, since $\Sigma_R(C)=\emptyset$, for any two roots α, α' in $\Sigma_R(\mathfrak{a})$, $\alpha\pm\alpha'$ are no longer roots. This means that $\Sigma_R(\mathfrak{a})$ is isomorphic to a multiple of the root system of type A_1. On the other hand, in this case, $\Sigma_R(\mathfrak{a})=\Sigma_R(C)$ by Lemma 2.5. Hence $\Sigma_R(\mathfrak{a})=\emptyset$. Thus the proposition is now proved. Q. E. D.

As a corollary of this proposition, we give the following significant result. We call a Cartan subalgebra of $\mathfrak{g}$ compact if its vector part is trivial.

Corollary 1. *Assume that $\mathfrak{g}$ has a compact Cartan subalgebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and A a connected component of $H^{\mathfrak{h}}$. Then the root system $\Sigma_R(\mathfrak{h})$ and $\Sigma_R(A)$ contain no simple component of class* III.

Proof. Let $\mathfrak{h}'$ be the vector part of $\mathfrak{h}$. We take an $F\in M(\Sigma_R(A))$ strongly orthogonal. Then the vector part of $\mathfrak{h}^F$ is equal to $\mathfrak{h}''=\{X\in\mathfrak{h}'\,;\ \alpha(X)=0\ (\alpha\in F)\}$ (cf. (2.12)). By Proposition 2.4, it follows from the assumption on $\mathfrak{g}$ that $\mathfrak{h}''=\{0\}$. This means that $\mathfrak{h}'$ is spanned over $\boldsymbol{R}$ by H_α $(\alpha\in F)$, and therefore that $\Sigma=\Sigma_F$ for $\Sigma=\Sigma_R(A)$. Then by Corollary of Lemma 1.1, we get the assertion of the corollary. Q. E. D.

Corollary 2. *Let A be a connected component of $H^{\mathfrak{h}}$. Then $M(\Sigma_R(A))\subset M(\Sigma_R(\mathfrak{h}))$.*

Proof. The number of elements in every $F\in M(\Sigma_R(A))$ is constant. By

Proposition 2.4, this must be equal to that for $M(\Sigma_R(\mathfrak{h}))$, because the Cartan subalgebra with minimal vector parts are mutually conjugate under G and hence the dimensions of these parts are equal to each other. Thus we have $M(\Sigma_R(A))\subset M(\Sigma_R(\mathfrak{h}))$. Q. E. D.

2.4. Let $F\in M(\Sigma_R(A))$ be strongly orthogonal, then by Corollary of Lemma 2.3, we see that $A(F)\subset H^{\mathfrak{a}}(\nu_F F)\subset H^{\mathfrak{a}}$, where $\mathfrak{a}=\mathfrak{h}^F$. Here we wish to prove the following more exact fact.

Proposition 2.6. *Let $F\in M(\Sigma_R(\mathfrak{h}))$ be strongly orthogonal, and assume that $\theta\mathfrak{h}=\mathfrak{h}$ and $\tilde{\theta}X_\alpha=-X_{-\alpha}$ $(\alpha\in F)$. Then*

$$W_G(H^{\mathfrak{h}})H_K^{\mathfrak{b}}(\nu_F F)=H_K^{\mathfrak{h}},\ \textit{where}\ \mathfrak{b}=\mathfrak{h}^F.$$

Proof. Put $B=H^{\mathfrak{b}}$, $B_K=H_K^{\mathfrak{b}}$ and $J=\nu_F F$. By Proposition 2.4, we know that the dimension of the toroidal part of B takes the maximal possible value. Therefore B is connected and so is B_K too. Let us first prove that $B_K(J)\subset H_K^{\mathfrak{h}}$. For $\alpha\in F$, put $\beta=\nu_F\alpha$ and $X_{\pm\beta}=\nu_F X_{\pm\alpha}$, then

$$(2.17)\qquad X'_\beta+X'_{-\beta}=X'_\alpha+X'_{-\alpha}\,,\quad X'_\beta-X'_{-\beta}=\sqrt{-1}\,H'_\alpha\,,\quad H'_\beta=\sqrt{-1}\,(X'_\alpha-X'_{-\alpha})\,.$$

Let $b\in B_K(J)$. Then for any $\beta\in J$, $\mathrm{Ad}(b)X_{\pm\beta}=\xi_{\pm\beta}(b)X_{\pm\beta}=X_{\pm\beta}$. Therefore by (2.12), we see that $\mathrm{Ad}(b)|\mathfrak{h}$ is the identity. This means that $b\in H^{\mathfrak{h}}$ and hence $B_K(J)\subset H_K^{\mathfrak{h}}$.

Now let us prove $W_G(H^{\mathfrak{h}})B_K(J)=H_K^{\mathfrak{h}}$. As is remarked in § 1.2, every element ω in $W_G(H^{\mathfrak{h}})$ has a representative in $N_K(\mathfrak{h})$. Therefore we get $W_G(H^{\mathfrak{h}})B_K(J)\subset H_K^{\mathfrak{h}}$. To prove the converse inclusion, take an arbitrary connected component A of $H^{\mathfrak{h}}$, then as is remarked above, $A_K\subset A(F')\subset H_K^{\mathfrak{b}'}(\nu_{F'}F')$ with $\mathfrak{b}'=\mathfrak{h}^{F'}$ for any strongly orthogonal $F'\in M(\Sigma_R(A))$. On the other hand, by Corollary 2 of Proposition 2.4, we have $M(\Sigma_R(A))\subset M(\Sigma_R(\mathfrak{h}))$. Therefore F' is conjugate to F under $W(\Sigma_R(\mathfrak{h}))$ (see § 1.2).

Let us study the relation between $H_K^{\mathfrak{b}'}(\nu_{F'}F')$ and $B_K(J)$. Note that the reflexion s_α corresponding to $\alpha\in\Sigma_R(\mathfrak{h})$ is realized by an element $g_\alpha\in G$ as $\mathrm{Ad}(g_\alpha)|\mathfrak{h}_c$, where

$$(2.18)\qquad g_\alpha=\exp 2^{-1}\pi(X'_\alpha-X'_{-\alpha})\,.$$

Therefore we can find a product g of g_α's such that $wF'=F$ with $w=\mathrm{Ad}(g)|\mathfrak{h}_c$. We see that $\mathrm{Ad}(g)\mathfrak{h}=\mathfrak{h}$, because $\mathrm{Ad}(g_\alpha)|\mathfrak{h}_c=s_\alpha$ and $H_\alpha\in\mathfrak{h}$. Since $g\in N_G(\mathfrak{h})=N_G(H^{\mathfrak{h}})$, g induces an element ω in $W_G(H^{\mathfrak{h}})$ as the inner automorphism by g. Moreover $\bar{\omega}=w$ under the canonical homomorphism $\omega\to\bar{\omega}$ of $W_G(H^{\mathfrak{h}})$ onto $W_G(\mathfrak{h})$.

Let $\{X_\alpha\}$ be the root vectors used to define ν_F and $\nu_{F'}$. Then there exists a non-zero constant c_α for every $\alpha\in F'$ such that

$$\mathrm{Ad}(g)X_\alpha=c_\alpha X_{w\alpha}\,,\quad \mathrm{Ad}(g)X_{-\alpha}=c_\alpha^{-1}X_{-w\alpha}\qquad(\alpha\in F')\,.$$

Put $c_\alpha=\varepsilon_\alpha\exp(t_\alpha|\alpha|^2)$ with $\varepsilon_\alpha=\pm1$, $t_\alpha\in\boldsymbol{R}$, and $g_0=g\cdot\prod_{\alpha\in F'}\exp(t_\alpha H_\alpha)$. Then ω is also realized by g_0 and $w=\mathrm{Ad}(g_0)|\mathfrak{h}_c$. Moreover $\mathrm{Ad}(g_0)X_{\pm\alpha}=\varepsilon_\alpha X_{\pm w\alpha}$ $(\alpha\in F')$, and hence by (2.12), we get $\mathrm{Ad}(g_0)\mathfrak{b}'=\mathfrak{b}$, $g_0H^{\mathfrak{b}'}g_0^{-1}=H^{\mathfrak{b}}=B$. Note that for $\beta=\nu_{F'}\alpha$ with $\alpha\in F'$, $H'_\beta=\sqrt{-1}(X'_\alpha-X'_{-\alpha})$. Then putting $\tau=\mathrm{Ad}(g_0)|\mathfrak{b}'_c$, we get $\tau H'_\beta=\varepsilon_\alpha\sqrt{-1}(X'_{w\alpha}-X'_{-w\alpha})=H'_{\varepsilon_\alpha\beta'}$, where $\beta'=\nu_F w\alpha$. Hence $\tau\beta=\varepsilon_\alpha\beta'$ and

$$\xi_\beta(h)=\xi_{\tau\beta}(g_0hg_0^{-1})=\xi_{\varepsilon_\alpha\beta'}(g_0hg_0^{-1}) \qquad (h\in H^{\mathfrak{b}'}).$$

This means that $g_0H^{\mathfrak{b}'}(\nu_{F'}F')g_0^{-1}=H^{\mathfrak{b}}(\nu_FF)=B(J)$. Since $A_K\subset H^{\mathfrak{b}'}(\nu_{F'}F')$ and ω leaves $H_K^{\mathfrak{h}}$ invariant, we get $A_K\subset B(J)\cap K=B_K(J)$, and therefore $A_K\subset W_G(H^{\mathfrak{h}})\cdot B_K(J)$. Hence $H_K^{\mathfrak{h}}\subset W_G(H^{\mathfrak{h}})B_K(J)$.

The proof of the proposition is now complete. Q. E. D.

2.5. Here we give a proposition (cf. [5(d), § 7.4, Lem. 7.9]) which will be applied latter. Let A be a connected component of $H^{\mathfrak{h}}$ as before. Then A is a direct product of A_K and $\exp(\mathfrak{h}_{\mathfrak{p}})$. We will utilize another decomposition of A in the sequel. Let $\mathfrak{h}_V$ be the subspace of $\mathfrak{h}_{\mathfrak{p}}$ spanned over $\boldsymbol{R}$ by H_α $(\alpha\in\Sigma_R(A))$, and put $\mathfrak{h}_U=\mathfrak{h}_{\mathfrak{k}}+\mathfrak{h}_V^{\perp}$, where $\mathfrak{h}_V^{\perp}$ denotes the orthogonal complement of $\mathfrak{h}_V$ in $\mathfrak{h}_{\mathfrak{p}}$ with respect to the Killing form of $\mathfrak{g}_c$. We put

$$A_V=\exp\mathfrak{h}_V\,,\quad A_U=A_K\exp\mathfrak{h}_U=A_K\exp\mathfrak{h}_V^{\perp}\,. \tag{2.19}$$

Then we have the direct product decomposition

$$A=A_UA_V\,. \tag{2.20}$$

The subsets A_U and A_V have intrinsic meanings independent of θ such that $\theta\mathfrak{h}=\mathfrak{h}$. In fact, $A_U=A(\Sigma_R(A))=\{h\in A\,;\,\xi_\alpha(h)=1\ (\alpha\in\Sigma_R(A))\}$. We see that for any $\alpha\in\Sigma_R(A)$, A_U commutes with g_α in (2.18), that is,

$$\omega_\alpha h=g_\alpha hg_\alpha^{-1}=h \qquad (h\in A_U)\,. \tag{2.21}$$

Remark 2.1. The subspaces $\mathfrak{h}_V$ and $\mathfrak{h}_U$ of $\mathfrak{h}$ may depend on the connected component A in question. However this is not the case when $\mathfrak{g}$ has a compact Cartan subalgebra. In fact, in this case, $\mathfrak{h}_V$ and $\mathfrak{h}_U$ coincide with the vector part and the toroidal part of $\mathfrak{h}$ respectively because of Corollary 1 of Proposition 2.4. Moreover $A_V=\exp\mathfrak{h}_{\mathfrak{p}}$, $A_U=A_K$.

Proposition 2.7. *Let F be a strongly orthogonal system of roots in $\Sigma_R(A)$. For $\omega\in W_G(A)$, let $w=\bar{\omega}$ be the element in $W_G(\mathfrak{h})$ induced canonically from ω. Assume $wF\subset F\cup -F$. Then there exists an element $g\in G$ with the following properties:* (i) *g leaves A invariant and commutes with ν_F on $\mathfrak{g}_c$.* (ii) *There exist $n_\alpha=0$ or 1 for $\alpha\in F$ such that*

$$ghg^{-1}=\omega(\prod_{\alpha\in F}\omega_\alpha^{n_\alpha})h \qquad (h\in A)\,,$$

in particular, $ghg^{-1}=\omega h$ $(h\in A_U)$, where $\omega_\alpha h=g_\alpha hg_\alpha^{-1}$ with g_α in (2.18), *and that if $\alpha,\alpha'\in F$ satisfy that both $2^{-1}(\alpha'\pm\alpha)$ are again roots in $\Sigma_R(A)$, then $(-1)^{n_\alpha}(-1)^{n_{\alpha'}}=1$.*

Moreover g leaves also $\mathfrak{h}$ and $\mathfrak{h}^F$ invariant. Put $w'=\mathrm{Ad}(g)|\mathfrak{h}_c$, $v=\mathrm{Ad}(g)|\mathfrak{h}_c^F$, then $w'\in W_G(\mathfrak{h})$, $v\in W_G(\mathfrak{h}^F)$ and

(2.22) $$\nu_F(w'X)=\nu(\nu_F X) \quad (X\in\mathfrak{h}_c), \qquad w'=w(\prod_{\alpha\in F} s_\alpha^{n_\alpha}).$$

The element w' *can be written also as* $w'=(\prod_{\alpha\in F}s_\alpha^{m_\alpha})w$, *where* $m_\alpha=0$ *or* 1 *and* m_α*'s have the same property as* n_α*'s.*

Proof. Take an element $g_\omega\in G$ such that $g_\omega h g_\omega^{-1}=\omega h$ $(h\in A)$. For $\alpha\in\Sigma_R(A)$, we get

$$\mathrm{Ad}(g_\omega)X_\alpha=c_\alpha X_{w\alpha}, \quad \mathrm{Ad}(g_\omega)X_{-\alpha}=c_\alpha^{-1}X_{-w\alpha},$$

where c_α is a non-zero real constant. Put $c_\alpha=\varepsilon_\alpha\exp(t_\alpha|\alpha|^2)$ with $\varepsilon_\alpha=\pm1$, $t_\alpha\in\boldsymbol{R}$, and

(2.23) $$g=g_\omega\{\prod_{\alpha\in F} g_\alpha^{n_\alpha}\exp(-\varepsilon_\alpha t_\alpha H_\alpha)\}.$$

where $n_\alpha=(1-\varepsilon_\alpha)/2$. It follows from (2.21) that g leaves A invariant and we get the expression of ghg^{-1} in (ii). Since F is strongly orthogonal, we see that for any two different α, α' in F, $\mathrm{Ad}(g_\alpha)X_{\pm\alpha'}=X_{\pm\alpha'}$ and g_α, $g_{\alpha'}$ commute with each other. Note that $\mathrm{Ad}(g_\alpha)X_{\pm\alpha}=-X_{\mp\alpha}$, then we get for $\alpha\in F$, $\mathrm{Ad}(g)X_{\varepsilon\alpha}=X_{\varepsilon\varepsilon_\alpha w\alpha}$ $(\varepsilon=\pm1)$. Therefore we get from (2.6) that

$$\mathrm{Ad}(g)\circ\nu_\alpha\circ\mathrm{Ad}(g)^{-1}=\exp\left\{-\sqrt{-1}\,\frac{\pi}{4}\,\mathrm{ad}(\mathrm{Ad}(g)X'_\alpha+\mathrm{Ad}(g)X'_{-\alpha})\right\}$$

$$=\exp\left\{-\sqrt{-1}\,\frac{\pi}{4}\,\mathrm{ad}(X'_{w\alpha}+X'_{-w\alpha})\right\}=\nu_{w\alpha}.$$

Thus using (2.11), we get $\mathrm{Ad}(g)\circ\nu_F\circ\mathrm{Ad}(g)^{-1}=\nu_F$ on $\mathfrak{g}_c$.

For the equality in (ii), put $\gamma=2^{-1}(\alpha'-\alpha)$, $\gamma'=2^{-1}(\alpha'+\alpha)$. Then $c_\alpha=c_{\gamma'}/c_\gamma$, $c_{\alpha'}=c_{\gamma'}c_\gamma$ and hence $(-1)^{n_\alpha}(-1)^{n_{\alpha'}}=\mathrm{sgn}(c_\alpha c_{\alpha'})=\mathrm{sgn}(c_{\gamma'})^2=1$. For another expression of w', note that $w'=w(\prod_{\alpha\in F}s_\alpha^{n_\alpha})=(\prod_{\alpha\in F}s_{w\alpha}^{n_\alpha})w$. Then it is sufficient to put $m_\alpha=n_\gamma$ with $\gamma=w^{-1}\alpha$ or $-w^{-1}\alpha$ such that $\gamma\in F$, because $wF\subset F\cup -F$.

The rest of the lemma is easy to prove. Q. E. D.

§ 3. Fundamental functions on a Cartan subgroup

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and A a connected component of $H^\mathfrak{h}$. In this section, we define some fundamental functions on every A. Here we need the direct product decomposition of A given in § 2.5. Let $\mathfrak{h}_V$ and $\mathfrak{h}_U=\mathfrak{h}_\mathfrak{k}+\mathfrak{h}_V^\perp$ be as in § 2.5, and put $A_V=\exp\mathfrak{h}_V$, $A_U=\{h\in A\,;\,\xi_\alpha(h)=1\ (\alpha\in\Sigma_R(A)\}$. Then $A_U=A_K\exp\mathfrak{h}_V^\perp$ and

(3.1) $$A=A_U A_V \qquad \text{(direct)}.$$

By definition, the root system $\Sigma_R(A)$ consists of real roots α of $\mathfrak{h}$ for which $\xi_\alpha(h)>0$ for $h\in A$. We imbed $W(\Sigma_R(A))$ into $W_G(A)$ in such a way that the reflexion s_α with respect to $\alpha\in\Sigma_R(A)$ corresponds to $\omega_\alpha|A$, that is,

(3.2) $$s_\alpha h=\omega_\alpha h=g_\alpha h g_\alpha^{-1} \qquad (h\in A),$$

where g_α is given by (2.18). Then the subgroup $W(\Sigma_R(A))$ of $W_G(A)$ acts on A_U trivially because $\xi_\alpha(h)=1$ for $\alpha\in\Sigma_R(A)$ and $h\in A_U$ and hence $g_\alpha h g_\alpha^{-1}=h$. Therefore expressing $h\in A$ as $h=h_U\exp X$ with $h_U\in A_U$, $X\in\mathfrak{h}_V$, we get

$$uh=h_U\exp uX \qquad (u\in W(\Sigma_R(A))). \tag{3.3}$$

We fix the set $P(\mathfrak{h})$ of positive roots in $\Sigma_R(\mathfrak{h})$, and put $P_R(A)=\Sigma_R(A)\cap P(\mathfrak{h})$.

Let F be an orthogonal system of roots in $\Sigma_R(\mathfrak{h})$, we define a map p_F of $\mathfrak{h}$ into itself by

$$p_F X=X-\sum_{\alpha\in F}|\alpha|^{-2}\alpha(X)H_\alpha \qquad (X\in\mathfrak{h}).$$

Then it is the projection onto the subspace $\mathfrak{h}(F)=\{X\in\mathfrak{h}\,;\,\alpha(X)=0\ (\alpha\in F)\}$ along $\sum_{\alpha\in F}\boldsymbol{R}H_\alpha$. For any $u\in W(\Sigma_R(\mathfrak{h}))$, we have $p_F(u^{-1}X)=u^{-1}p_{uF}(X)$. In particular, if $u\in W(\Sigma_F)$ with $\Sigma=\Sigma_R(\mathfrak{h})$, then $p_F(u^{-1}X)=p_F(X)$. When $F\subset\Sigma_R(A)$, we get $p_F(\mathfrak{h}_V)\subset\mathfrak{h}_V$, and hence $\mathfrak{h}_V=p_F(\mathfrak{h}_V)+\sum_{\alpha\in F}\boldsymbol{R}H_\alpha$, because by definition $\mathfrak{h}_V$ is spanned by H_α $(\alpha\in\Sigma_R(A))$.

3.1. Let $E\in M^{or}(P_R(A))$ and $F=E^*\in M(P_R(A))$, and $F_0=F_0(E)$ be the strongly orthogonal system in $M(P_R(A))$ associated to E given by (1.21), (1.38). Then $p_{F_0}=p_F$, and by Corollary of Lemma 2.4, $A(F_0)\subset H^{\mathfrak{b}}(\nu_{F_0}F_0)\subset H^{\mathfrak{b}}$ with $\mathfrak{b}=\mathfrak{h}^{F_0}$. The space $\mathfrak{h}\cap\mathfrak{b}=\sigma_{F_0}$ in (2.12) is given as $\sigma_{F_0}=\mathfrak{h}_U+p_{F_0}\mathfrak{h}_V$. Therefore we get $A(F_0)=A_U\exp(p_{F_0}\mathfrak{h}_V)$, $\nu_{F_0}X=X$ for $X\in\mathfrak{h}_U+p_{F_0}\mathfrak{h}_V$ and

$$\mathfrak{b}=\mathfrak{h}_U+p_{F_0}\mathfrak{h}_V+\sqrt{-1}\,\nu_{F_0}\Big(\sum_{\alpha\in F_0}\boldsymbol{R}H_\alpha\Big). \tag{3.4}$$

Put $B=H^{\mathfrak{b}}$, $B_U=H^{\mathfrak{b}}_U$, then they are connected because $\mathfrak{b}$ has no real root. Denote by $\mathfrak{b}_c^*$ the dual space of $\mathfrak{b}_c$ over $\boldsymbol{C}$, and by $\mathfrak{b}_B^*$ its additive subgroup consisting of $\Lambda\in\mathfrak{b}_c^*$ such that

$$\xi_\Lambda(\exp X)=e^{\Lambda(X)} \qquad (X\in\mathfrak{b}) \tag{3.5}$$

defines a unitary character of B. (This can be expressed as $\mathfrak{b}_B^*=\log B^*$, where B^* is the group of all unitary characters of B.) We denote by $\mathfrak{b}_B^{*\prime}$ the subset of $\mathfrak{b}_B^*$ consisting of regular elements. Let $\Lambda\in\mathfrak{b}_B^*$, then $\Lambda(\mathfrak{b})\subset\sqrt{-1}\boldsymbol{R}$ and therefore $\Lambda(\mathfrak{h}_U+p_{F_0}\mathfrak{h}_V)\subset\sqrt{-1}\boldsymbol{R}$, $\Lambda(\nu_{F_0}H_\alpha)=(\Lambda,\nu_{F_0}\alpha)\in\boldsymbol{R}$ for any $\alpha\in\Sigma_{F_0}$ $(\supset F)$, where $(\cdot,\cdot)$ denotes the inner product induced from the Killing form of $\mathfrak{g}_c$. Therefore we can define for $\Lambda\in\mathfrak{b}_B^{*\prime}$,

$$\mathrm{sgn}_{P(E)}(\Lambda)=\mathrm{sgn}\Big\{\prod_{\gamma\in P(E)}(\Lambda,\nu_{F_0}\alpha)\Big\}, \tag{3.6}$$

where $P(E)$ is given by (1.21), (1.38), and $F_0=F_0(E)$.

For every $\Lambda\in\mathfrak{b}_B^{*\prime}$, we define certain fundamental functions Y and Z on A as follows. Express $h\in A$ as $h=h_U\exp X$ with $h_U\in A_U$, $X\in\mathfrak{h}_V$. For $E\in M^{or}(P_R(A))$ and $u\in W(\Sigma_R(A))$, put

$$\begin{aligned} Y(h\,;E,u,\Lambda)=&\,\mathrm{sgn}_{P(E)}(\Lambda)\xi_\Lambda(h_U)\\ &\times\exp\Lambda(p_{F_0}(u^{-1}X))\cdot\prod_{\alpha\in F}\exp\{-\alpha(u^{-1}X)|(\Lambda,\nu_{F_0}\alpha)|/|\alpha|^2\}, \end{aligned} \tag{3.7}$$

where $F=E^*$, $F_0=F_0(E)$. From (3.3), we see that

$$Y(h\,;\,E,\,u,\,\Lambda)=Y(u^{-1}h\,;\,E,\,1,\,\Lambda)\,, \tag{3.8}$$

where 1 denotes the unit element in $W(\Sigma_R(A))$. Moreover put

$$Z(h\,;\,E,\,\Lambda,\,P_R(A))=\sum_s \operatorname{sgn}(s)\sum_u Y(h\,;\,E,\,u,\,s\Lambda)\,, \tag{3.9}$$

where the sum runs over all s and u such that

$$s\in W_G(\mathfrak{h}^{F_0})\,,\quad u\in W(E\,;\,P_R(A))\,. \tag{3.9$'$}$$

Here $W(E\,;\,P_R(A))$ is given by Definition 1.2 as follows. Let $\Sigma_1, \Sigma_2, \cdots, \Sigma_p$ be all the simple components of $\Sigma=\Sigma_R(A)$, and $P_i=\Sigma_i\cap P$ with $P=P_R(A)$, and let E_i $(1\leqq i\leqq p)$ be the part of E in Σ_i. Then $W(E\,;\,P)$ is the product of $W(E_i\,;\,P_i)$, where $W(E_i\,;\,P_i)=\{u\in W(\Sigma_i)\,;\,uE_i\in M^{or}(P_i)\}$ if Σ_i is of class II, otherwise $W(\Sigma_i\,;\,P_i)$ is an arbitrary complete system of representatives of $W(\Sigma_i)/I(F_i)$ with $F_i=E_i^*$ consisting of elements u such that $uF_i\in M(P_i)$.

We will prove in § 3.3 that the definition of the function Z does not depend on the choice of $W(E_i\,;\,P_i)$'s for Σ_i's of class I or III. Note that since $\mathfrak{h}^{F_0}$ has no real root, $W_G(\mathfrak{h}^{F_0})$ is generated by the reflexions corresponding to compact roots of $\mathfrak{h}^{F_0}$ [2(d), § 16, p. 277].

Remark 3.1. When $\mathfrak{g}$ has a compact Cartan subalgebra, we have $p_{F_0}\mathfrak{h}_V=\{0\}$. Hence $\mathfrak{b}=\mathfrak{h}_U+\sqrt{-1}\,\nu_{F_0}\mathfrak{h}_V$ in (3.4), and the factor $\exp\Lambda(p_{F_0}(u^{-1}X))$ in (3.7) disappears.

3.2. Let us prove here an important property of the function Y. Let $E\in M^{or}(P_R(A))$. Assume that $w\in W(\Sigma_R(A))$ satisfies that $wF=F$, $wF_0=F_0\cup -F_0$ for $F=E^*$, $F_0=F_0(E)$. Then by Proposition 2.7 there exist $v\in W_G(\mathfrak{h}^{F_0})$, $g\in G$ such that

$$\begin{aligned}&ghg^{-1}=h\ (h\in A_U)\,,\quad \mathrm{Ad}(g)\,|\,\mathfrak{h}_c^{F_0}=v\,,\\ &v(\nu_{F_0}X)=\nu_{F_0}(w'X)\ (X\in\mathfrak{h}_c)\ \text{with}\ w'=(\prod_{\alpha\in F_0}s_\alpha^{m_\alpha})w\,,\end{aligned} \tag{3.10}$$

where $m_\alpha=0$ or 1 for $\alpha\in F_0$, and $(-1)^{m_\alpha}(-1)^{m_{\alpha'}}=1$ for any two roots α, α' in F_0 such that $2^{-1}(\alpha\pm\alpha')\in\Sigma_R(A)$.

Lemma 3.1. *Assume that* $E\in M^{or}(P_R(A))$ *and* $w\in W(\Sigma_R(A))$ *satisfy* $wF=F$, $wF_0\subset F_0\cup -F_0$ *for* $F=E^*$, $F_0=F_0(E)$. *Take* $v\in W_G(\mathfrak{h}^{F_0})$ *satisfying* (3.10). *Then for any* $u\in W(\Sigma_R(A))$,

$$\operatorname{sgn}_{P(E)}(\Lambda)Y(h\,;\,E,\,uw,\,\Lambda)=\operatorname{sgn}(w)\operatorname{sgn}(v)\operatorname{sgn}_{w^{-1}P(E)}(\Lambda)Y(h\,;\,E,\,u,\,v\Lambda)\,, \tag{3.11}$$

where

$$\operatorname{sgn}_{w^{-1}P(E)}(\Lambda)=\operatorname{sgn}\{\prod_{\gamma\in P(E)}(\Lambda,\,\nu_{F_0}w^{-1}\gamma)\}\,. \tag{3.12}$$

Proof. Note that

$$\mathrm{sgn}(w)=(\prod_{\alpha\in F_0}(-1)^{m_\alpha})\cdot\mathrm{sgn}(w')=(\prod_{\alpha\in F_0}(-1)^{m_\alpha})\cdot\mathrm{sgn}(v).$$

It follows from $wF_0\subset F_0\cup -F_0$ that

$$\mathrm{sgn}_{P(E)}(v\Lambda)=\mathrm{sgn}(w)\mathrm{sgn}(v)\mathrm{sgn}_{w^{-1}P(E)}(\Lambda). \tag{3.13}$$

In fact, the left hand side is equal to

$$(*)\qquad \begin{aligned}\mathrm{sgn}\{\prod_{\gamma\in P(E)}(v\Lambda,\nu_{F_0}\gamma\}&=\mathrm{sgn}\{\prod_{\gamma\in P(E)}(\Lambda,\nu_{F_0}w'^{-1}\gamma)\}\\ &=\mathrm{sgn}\{\prod_{\gamma\in P(E)}(\Lambda',(\prod_{\alpha\in F_0}s_\alpha^{m_\alpha})\gamma)\},\end{aligned}$$

where $\Lambda'=w\nu_{F_0}^{-1}\Lambda$. Therefore it is sufficient to prove that the last term is equal to

$$(**)\qquad (\prod_{\alpha\in F_0}(-1)^{m_\alpha})\cdot\mathrm{sgn}\{\prod_{\gamma\in P(E)}(\Lambda',\gamma)\}.$$

To do so, we must take into account of the difference between $F_0=F_0(E)$ and $P(E)$. Put $\Sigma=\Sigma_R(A)$, and let $\Sigma=\Sigma_1\cup\Sigma_2\cup\cdots\cup\Sigma_p$ and $E=E_1\cup E_2\cup\cdots\cup E_p$ be the decompositions as above. If Σ_j is of class I or III, then $F_0(E_j)=P(E_j)=E_j^*$ and the product of $s_\alpha^{m_\alpha}$ over $F_0(E_j)$ in (*) comes out of $\mathrm{sgn}\{\cdot\}$ as the product of $(-1)^{m_\alpha}$ over $F_0(E_j)$. Assume Σ_j be of class II and express E_j as $E_j=(\alpha_1,\alpha_2,\cdots,\alpha_l,\alpha_{l+1},\cdots,\alpha_n)$ with the properties (B1), (B2) and (B3) in § 1.5. Then $P(E_j)$ and $F_0(E_j)$ are given by (1.21). Therefore the difference between them consists of two kinds: (1) $P(E_j)$ contains α_{2i-1}, α_{2i}, $2^{-1}(\alpha_{2i-1}\pm\alpha_{2i})$, and $F_0(E_j)$ contains α_{2i-1}, α_{2i} $(1\leqq i\leqq m)$; (2) $P(E_j)$ contains α_{l+2i-1}, α_{l+2i}, and $F_0(E_j)$ contains $\alpha_{l+2i-1}\pm\alpha_{l+2i}$ $(1\leqq i\leqq[(n-l)/2])$. Put $(\alpha',\alpha)=(\alpha_{2i-1},\alpha_{2i})$ in (1), and $(\alpha',\alpha)=(\alpha_{l+2i-1}+\alpha_{l+2i},\ \alpha_{l+2i-1}-\alpha_{l+2i})$ in (2). Moreover put $P'=\{\alpha',\alpha,\gamma'=2^{-1}(\alpha'+\alpha),\gamma=2^{-1}(\alpha'-\alpha)\}$, then $\Sigma'=P'\cup -P'$ is a root system of type B_2. Corresponding to the case (1) or (2), we get the following respectively:

$$\mathrm{sgn}\{\prod_{\delta\in P'}(\Lambda',s_\alpha^{m_\alpha}s_{\alpha'}^{m_{\alpha'}}\delta)\}=(-1)^{m_\alpha}(-1)^{m_{\alpha'}}\mathrm{sgn}(\prod_{\delta\in P'}(\Lambda',\delta)\},\quad\text{or}$$

$$\mathrm{sgn}\{\prod_{\delta=\gamma,\gamma'}(\Lambda',s_\alpha^{m_\alpha}s_{\alpha'}^{m_{\alpha'}}\delta)\}=\mathrm{sgn}\{\prod_{\delta=\gamma,\gamma'}(\Lambda',\delta)\}.$$

Thus we have seen that the product of $s_\alpha^{m_\alpha}$ over $\alpha\in F_0(E_j)$ in (*) comes out of $\mathrm{sgn}\{\cdot\}$ as the product of $(-1)^{m_\alpha}$ over $\alpha\in F_0(E_j)\cap E_j^*$.

On the other hand, $F_0(E_j)-E_j^*$ is a union of pairs $\{\alpha,\alpha'\}$ such that $2^{-1}(\alpha'\pm\alpha)\in\Sigma_R(A)$. Since $(-1)^{m_\alpha}(-1)^{m_{\alpha'}}=1$ for every such pair, we get $\prod_{\alpha\in F_0(E_j)\cap E_j^*}(-1)^{m_\alpha}=\prod_{\alpha\in F_0(E_j)}(-1)^{m_\alpha}$. Thus we see that the terms in (*) and (**) are equal to each other, and get the equality (3.13).

Now it follows from (3.10), $|w\alpha|=|\alpha|$ and $wF=F$ that

$$\begin{aligned}\sum_{\alpha\in F}\alpha((uw)^{-1}X)|(\Lambda,\nu_{F_0}\alpha)|/|\alpha|^2&=\sum_{\alpha\in F}(w\alpha)(u^{-1}X)|(v\Lambda,\nu_{F_0}w\alpha)|/|w\alpha|^2\\ &=\sum_{\alpha\in F}\alpha(u^{-1}X)|(v\Lambda,\nu_{F_0}\alpha)|/|\alpha|^2.\end{aligned}\tag{3.14}$$

Finally consider the factor $\xi_\Lambda(h_U)$. Since $A_U \subset H^{\mathfrak{b}'}$ with $\mathfrak{b}'=\mathfrak{h}^{F_0}$, we can take from $\mathfrak{h}^{F_0}$ an element X_U such that $h_U=\exp X_U$. Recall that $W(\Sigma_R(A))$ acts on A_U trivially, then we get by (3.10),

$$\xi_\Lambda(h_U)=\xi_\Lambda(g^{-1}h_U g)=\exp(\Lambda,\ \mathrm{Ad}(g^{-1})X_U) \tag{3.15}$$
$$=\exp(\Lambda,\ v^{-1}X_U)=\exp(v\Lambda,\ X_U)=\xi_{v\Lambda}(h_U)\,.$$

For the factor $\exp \Lambda(p_{F_0}(u^{-1}X))$, we have

$$p_{F_0}((uw)^{-1}X)=\nu_{F_0}p_{F_0}(w^{-1}u^{-1}X)=\nu_{F_0}w^{-1}p_{wF_0}(u^{-1}X) \tag{3.15'}$$
$$=\nu_{F_0}w'^{-1}p_{F_0}(u^{-1}X)=v^{-1}\nu_{F_0}p_{F_0}(u^{-1}X)=v^{-1}p_{F_0}(u^{-1}X)\,.$$

Then the equality in the lemma follows from (3.13)-(3.15′). Q.E.D.

Corollary. *Let E, w be as in Lemma* 3.1. *Then for any $u\in W(\Sigma_R(A))$ and $h\in A$,*

$$\sum_s \mathrm{sgn}(s)\mathrm{sgn}_{P(E)}(s\Lambda)\mathrm{sgn}_{w^{-1}P(E)}(s\Lambda)Y(h\,;\,E,\ uw,\ s\Lambda) \tag{3.16}$$
$$=\mathrm{sgn}(w)\cdot\sum_s \mathrm{sgn}(s)Y(h\,;\,E,\ u,\ s\Lambda)\,,$$

where s runs over $W_G(\mathfrak{h}^{F_0})$ with $F_0=F_0(E)$.

This is an immediate consequence of the lemma.

Remark 3.2. Let $V(E)$ be the subset in $W(E\,;P_R(A))$ defined in (1.23). Then every element w in $V(E)$ satisfies the assumption in Lemma 3.1 because of the condition (B2) and Lemma 1.8.

3.3. Let $E\in M^{or}(P_R(A))$. Let Σ_i be a simple component of $\Sigma_R(A)$ of class I or III, and P_i, E_i and $F_i=E_i^*$ be as in § 3.1. We prove here that the definition of the function $Z(h\,;\,E,\ \Lambda,\ P_R(A))$ does not depend on the choice of $W(E_i\,;\,P_i)$. We define a subgroup $K(F_i)$ of $I(F_i)$ by

$$K(F_i)=\{w\in W(\Sigma_i)\,;\ wF_i=F_i\}\,. \tag{3.17}$$

Then applying Lemma 3.1 and its corollary to $w\in K(F_i)$, we get the following.

Lemma 3.2. *Let $E\in M^{or}(P_R(A))$. Let Σ_i be a simple component of class* I *or* III *of $\Sigma_R(A)$ and P_i, $K(F_i)$ be as above. For $w\in K(F_i)$, take $v\in W_G(\mathfrak{h}^{F_0})$ for which* (3.10) *holds. Then for any $u\in W(\Sigma_R(A))$ and $h\in A$,*

$$Y(h\,;\,E,\ uw,\ \Lambda)=\mathrm{sgn}(v)Y(h\,;\,E,\ u,\ v\Lambda)\,.$$

Moreover

$$\sum_s \mathrm{sgn}(s)Y(h\,;\,E,\ uw,\ s\Lambda)=\sum_s \mathrm{sgn}(s)Y(h\,;\,E,\ u,\ s\Lambda)\,, \tag{3.18}$$

where s runs over $W_G(\mathfrak{h}^{F_0})$. Thus the definition of the function $Z(h\,;\,E,\ \Lambda,\ P_R(A))$ does not depend on the choice of $W(E_i\,;\,P_i)$.

Proof. We know by Lemma 1.2 that $\mathrm{sgn}(w)=1$ for $w\in K(F_i)$. Moreover $w^{-1}P(E)=P(E)$ and hence $\mathrm{sgn}_{P(E)}(s\Lambda)=\mathrm{sgn}_{w^{-1}P(E)}(s\Lambda)$ for E. Therefore the equality (3.11) turns out to be the first one in the lemma. The rest of the lemma follows from it immediately. Q. E. D.

3.4. Here we assume that $\Sigma_R(A)$ is of class I or III. We define two new functions Y' and Z' on $A\subset H^{\mathfrak{h}}$ which do not differ essentially from Y and Z respectively.

Express $h\in A$ as $h=h_U \exp X$ with $h_U\in A_U$, $X\in\mathfrak{h}_V$. For $F\in M(P_R(A))$ we put $\mathrm{sgn}_F(\Lambda)=\mathrm{sgn}\{\prod_{\alpha\in F}(\Lambda, \nu_F\alpha)\}$, and

$$Y'(h\,;F,u,\Lambda)=\mathrm{sgn}_F(\Lambda)\xi_\Lambda(h_U)\times\exp\Lambda(p_F(u^{-1}X))\cdot\prod_{\alpha\in F}\exp\{-|\alpha(u^{-1}X)|\cdot|(\Lambda,\nu_F\alpha)|/|\alpha|^2\},\tag{3.19}$$

$$Z'(h\,;F,\Lambda,P_R(A))=\sum_s \mathrm{sgn}(s)\sum_u Y'(h\,;F,u,s\Lambda),\tag{3.20}$$

where the sum runs over all s in $W_G(\mathfrak{h}^F)$ and u in a complete system of representatives of $W(\Sigma_R(A))/I(F)$ with $I(F)$ in (1.37).

We must first prove that Z' is well-defined, or the sum does not depend on the choice of a system of representatives of $W(\Sigma_R(A))/I(F)$. Let $J(F)$ be the subgroup of $I(F)$ generated by the commuting family $\{s_\alpha\,;\alpha\in F\}$, then $I(F)$ is a semidirect product of $J(F)$ and $K(F)$. We see easily that for any $w\in J(F)$,

$$Y'(h\,;F,uw,\Lambda)=Y'(h,F,u,\Lambda).\tag{3.21}$$

Now let $w\in K(F)$. Then repeating the proofs of Lemmas 3.1 and 3.2 word for word, we get the following.

Lemma 3.3. *Assume that $\Sigma_R(A)$ is of class* I-III *and let $F\in M(P_R(A))$. For $w\in K(F)$, take $v\in W_G(\mathfrak{h}^F)$ for which* (3.10) *holds. Then for $u\in W(\Sigma_R(A))$,*

$$Y'(h\,;F,uw,\Lambda)=\mathrm{sgn}(v)Y'(h\,;F,u,v\Lambda).$$

From this lemma and (3.21), we see that for any $u\in W(\Sigma_R(A))$ and $w\in I(F)$,

$$\sum_s \mathrm{sgn}(s)Y'(h,F,uw,s\Lambda)=\sum_s \mathrm{sgn}(s)Y'(h,F,u,s\Lambda)\quad(s\in W_G(\mathfrak{h}^F)).\tag{3.22}$$

This proves that the definition (3.20) of Z' does not depend on the choice of a complete system of representatives of $W(\Sigma_R(A))/I(F)$. Hence Z' is well-defined.

Now let us study the relations between the functions Y, Z and those Y', Z'. Denote by $A^+(P)$ with $P=P_R(A)$ the open subset of A given by

$$A^+(P)=\{h\in A\,;\ \xi_\alpha(h)>1\ (\alpha\in P_R(A))\}.\tag{3.23}$$

Then $A\cap H'^{\mathfrak{h}}(R)=\{h\in A\,;\ \xi_\alpha(h)\neq 1\ (\alpha\in\Sigma_R(A))\}$ is the disjoint union of $uA^+(P)$ over $u\in W(\Sigma_R(A))$. Put

$$\mathfrak{h}_V^+=\{X\in\mathfrak{h}_V\,;\ \alpha(X)>0\ (\alpha\in P_R(A))\}.\tag{3.24}$$

Then every element h in $A^+(P)$ is uniquely expressed as

$$h=h_U \exp X \quad \text{with } h_U\in A_U\,,\ X\in\mathfrak{h}_V^+\,. \tag{3.25}$$

Lemma 3.4. *Assume that $\Sigma_R(A)$ is of class* I-III. *Let $P_R(A)$ be the set of positive roots. Then for $F\in M(P_R(A))$,*

$$Z'(h\,;F,\Lambda,P_R(A))=Z(h\,;F,\Lambda,P_R(A)) \qquad (h\in A^+(P))\,,$$

and if $u\in W(\Sigma_R(A))$ satisfies $uF\in M(P_R(A))$, then

$$Y'(h\,;F,u,\Lambda)=Y(h\,;F,u,\Lambda) \qquad (h\in A^+(P))\,.$$

Proof. Let us first prove the second equality. Assume $uF\in M(P_R(A))$. Then $u\alpha>0$ for $\alpha\in F$ and therefore $(u\alpha)(X)=\alpha(u^{-1}X)>0$ for $X\in\mathfrak{h}_V^+$. Hence by the expression (3.25) for $h\in A^+(P)$, we get the second equality. The first one follows from this immediately by taking a sum over a complete system of representatives of $W(\Sigma_R(A))/I(F)$ consisting of elements u such that $uF\in M(P_R(A))$. Q. E. D.

3.5. Let us now study the boundedness of the functions Y, Z, Y' and Z'. This property relates directly with the temperedness of the invariant eigendistributions which will be given later by means of these functions. These distributions cover the discrete series characters.

Lemma 3.5. *Let $E\in M^{or}(P_R(A))$ and $u\in W(E\,;P_R(A))$. Then for $h\in A^+(P)$,*

$$|Y(h\,;E,u,\Lambda)|\leqq 1\,,$$

$$|Z(h\,;E,\Lambda,P_R(A))|\leqq \sharp W_G(\mathfrak{h}^{F_0})\cdot\sharp W(E\,;P_R(A))\,,$$

where $F_0=F_0(E)$, and for a set C, $\sharp C$ denotes the number of elements in C.

Assume that $\Sigma_R(A)$ is of class I-III. *Let $F\in M(P_R(A))$ and $u\in W(\Sigma_R(A))$, then for any $h\in A$,*

$$|Y'(h\,;F,u,\Lambda)|\leqq 1\,,$$

$$|Z'(h\,;F,\Lambda,P_R(A))|\leqq \sharp W_G(\mathfrak{h}^{F})\cdot\sharp W(\Sigma_R(A))/\sharp I(F)\,.$$

Note that $\Lambda(p_{F_0}\mathfrak{h}_V)\subset\sqrt{-1}\,\boldsymbol{R}$, then the above inequalities are easy to prove.

3.6. Consider $W(\Sigma_R(A))$ and $W_G(A^+(P))$ as subgroups of $W_G(A)$ canonically. Then every element ω in $W_G(A)$ is expressed uniquely as

$$\omega=u\omega' \quad \text{with } u\in W(\Sigma_R(A)),\ \ \omega'\in W_G(A^+(P))\,, \tag{3.26}$$

and it operates on $h=h_U\exp X\in A$ with $h_U\in A_U$, $X\in\mathfrak{h}_V$ as

$$\omega h=(\omega' h_U)\exp(u\bar{\omega}'X)\,, \tag{3.27}$$

where $\bar{\omega}'$ denotes the element in $W_G(\mathfrak{h})$ induced canonically from ω'. (In our convention, $\bar{\omega}=u\bar{\omega}'$.)

Let us study how the fundamental functions behave under $W_G(A^+(P))$. Let

$\omega\in W_G(A^+(P))$, then $\bar{\omega}P_R(A)=P_R(A)$. From this, we see that $w=\bar{\omega}$ satisfies for any standard element E of $M^{or}(P_R(A))$ that $wF=F$, $wF_0=F_0$ for $F=E^*$, $F_0=F_0(E)$. In fact, let Σ be a simple component of $\Sigma_R(A)$ and put $P=\Sigma\cap P_R(A)$. Then the standard element of the given type of $M^{or}(P)$ is uniquely determined by P. Since $wP_R(A)=P_R(A)$ for $w=\bar{\omega}$, we have $wE=E$ (recall that $M^{or}(P)=M(P)$ by definition if Σ is of class I or III). Hence $wF=F$, $wF_0=F_0$.

Note 3.2. For the simple root system Σ of type A_n $(n\geqq 2)$, E_6 or D_N $(N\geqq 4)$, there exists at least one outer automorphism w_1 of Σ such that $w_1P=P$, which gives an automorphism of the Dynkin diagram of Σ. In particular, for D_N $(N\geqq 5)$ realized as in (1.7), we have

$$w_1(e_i-e_{i+1})=e_i-e_{i+1}\ (1\leqq i\leqq N-2)\,,\quad w_1(e_{N-1}\pm e_N)=e_{N-1}\mp e_N\,. \tag{3.28}$$

For D_4 realized as in (1.7), w_1 is an element of the group generated by the reflexions corresponding to the vectors

$$e_4,\ e_1-e_2-e_3-e_4\,,\quad e_1-e_2-e_3+e_4\,. \tag{3.29}$$

Note that for D_N $(N\geqq 4)$, $w_1F=F$ for any $F\in M(P)$.

Applying Proposition 2.7 to ω, $w=\bar{\omega}$ and $F_0=F_0(E)$, we see that there exist $g\in G$, $v\in W_G(\mathfrak{h}^{F_0})$ satisfying (3.10). Then we have the following.

Lemma 3.6. *Let $\omega\in W_G(A^+(P))$ and put $w=\bar{\omega}$. Let E be a standard element in $M^{or}(P_R(A))$. Then for any $u\in W(\Sigma_R(A))$,*

$$Y(\omega^{-1}h\,;E,\,u,\,\Lambda)=\mathrm{sgn}(w)\mathrm{sgn}(v)Y(h\,;E,\,wuw^{-1},\,v\Lambda)\,,$$

where $v\in W_G(\mathfrak{h}^{F_0})$ is given by (3.10), *and moreover*

$$Z(\omega^{-1}h\,;E,\,\Lambda,\,P_R(A))=\mathrm{sgn}(w)Z(h\,;E,\,\Lambda,\,P_R(A))\,.$$

Let $\Sigma_R(A)$ be of class I-III *and F a standard element in $M(P_R(A))$. Then*

$$Y'(\omega^{-1}h\,;F,\,u,\,\Lambda)=\mathrm{sgn}(w)\mathrm{sgn}(v)Y'(h\,;F,\,wuw^{-1},\,v\Lambda)\quad(u\in W(\Sigma_R(A)))\,,$$

$$Z'(\omega^{-1}h\,;F,\,\Lambda,\,P_R(A))=\mathrm{sgn}(w)Z'(h\,;F,\,\Lambda,\,P_R(A))\,.$$

Proof. Let $\omega\in W_G(A^+(P))$, then by (3.27), $\omega^{-1}h=\omega^{-1}h_U\exp\bar{\omega}^{-1}X$ for $h=h_U\exp X\in A$, where $\omega^{-1}h_U\in A_U$, $\bar{\omega}^{-1}X=w^{-1}X\in\mathfrak{h}_V$. First consider the factor $\mathrm{sgn}_{P(E)}(\Lambda)$ in Y. Then since $wF=F$, $wF_0=F_0$, we get as in (3.13),

$$\mathrm{sgn}_{P(E)}(v\Lambda)=\mathrm{sgn}(w)\mathrm{sgn}(v)\mathrm{sgn}_{w^{-1}P(E)}(\Lambda)$$

Moreover since $wE=E$, we get $w^{-1}P(E)=P(E)$ and

$$\mathrm{sgn}_{P(E)}(v\Lambda)=\mathrm{sgn}(w)\mathrm{sgn}(v)\mathrm{sgn}_{P(E)}(\Lambda)\,. \tag{3.30}$$

Next consider $\xi_\Lambda(\omega^{-1}h_U)$. Let X_U be an element in $\mathfrak{h}^{F_0}$ such that $\exp X_U=h_U$, then

$$(3.31)\qquad \xi_\Lambda(\omega^{-1}h_U)=\xi_\Lambda(g^{-1}h_U g)=\exp \Lambda(\mathrm{Ad}(g^{-1})X_U)$$
$$=\exp \Lambda(v^{-1}X_U)=\exp(v\Lambda)(X_U)=\xi_{v\Lambda}(h_U)\,.$$

For the factor $\exp \Lambda(p_{F_0}(u^{-1}\bar{\omega}^{-1}X))$, we have as for (3.15′),

$$(3.32)\qquad \Lambda(p_{F_0}(u^{-1}\bar{\omega}^{-1}X))=\Lambda(p_{F_0}(w^{-1}\cdot wu^{-1}w^{-1}X)=v\Lambda(p_{F_0}(wu^{-1}w^{-1}X))\,.$$

For the rest part of Y, note that for $\alpha\in F$,

$$\alpha(u^{-1}\bar{\omega}^{-1}X)=(wu\alpha)(X)\,,\quad |(\Lambda,\ \nu_{F_0}\alpha)|=|(v\Lambda,\ \nu_{F_0}w\alpha)|\,.$$

Therefore noting that $wF=F$, we get

$$(3.33)\qquad \sum_{\alpha\in F}\alpha(u^{-1}\bar{\omega}^{-1}X)|(\Lambda,\ \nu_{F_0}\alpha)|/|\alpha|^2=\sum_{\alpha\in F}(wu\alpha)(X)|(\Lambda,\ \nu_{F_0}w\alpha|/|w\alpha|^2$$
$$=\sum_{\alpha\in F}(wuw^{-1}\alpha)(X)|(v\Lambda,\ \nu_{F_0}\alpha)|/|\alpha|^2.$$

Thus we get the first equality in the lemma from (3.30)-(3.33).

Now let us deduce the second equality from the first. We get from the first equality that

$$Z(\omega^{-1}h\,;\,E,\ \Lambda,\ P_R(A))=\sum_u\ \sum_s \mathrm{sgn}(s)Y(\omega^{-1}h\,;\,E,\ u,\ s\Lambda)$$
$$=\mathrm{sgn}(w)\sum_u\ \sum_s \mathrm{sgn}(s)Y(h\,;\,E,\ wuw^{-1},\ s\Lambda)\,,$$

where u and s runs over $W(E\,;P_R(A))$ and $W_G(\mathfrak{h}^{F_0})$ respectively. Let Σ^1 (resp. Σ^2) be the root system consisting of all simple components of $\Sigma_R(A)$ of class I or III (resp. of class II), then $w\Sigma^i=\Sigma^i$ for $i=1, 2$. Put $P^i=\Sigma^i\cap P_R(A)$, and let $E^i\in M^{or}(P^i)$ be the standard elements canonically corresponding to E. Then $wF^1=F^1$ with $F^1=(E^1)^*$ and $wE^2=E^2$, because $wP^i=P^i$ $(i=1, 2)$. Every element $u\in W(E\,;P_R(A))$ is expressed uniquely as $u=u_1u_2$ with $u_i\in W(E^i\,;P^i)$. Noting that $u_i'=wu_iw^{-1}\in W(\Sigma^i)$ for $i=1, 2$, it is sufficient for us to see that when u_i runs over $W(E^i\,;P^i)$ $(i=1, 2)$, u_1' runs over a complete system of representatives of $W(\Sigma^1)/I(F^1)$ such that $u_1'F^1\in M(P^1)$, and u_2' runs over $W(E^2\,;P^2)$.

First consider u_1. Since $wP^1=P^1$ and $wu_1w^{-1}F^1=wu_1F^1$, we see that $wuw^{-1}F^1$ belongs to $M(P^1)$ and runs over $M(P^1)$ exactly once when u_1 runs over $W(E^1\,;P^1)$. This gives us the desired fact for u_1'. Now consider u_2. Let Σ be a simple component of Σ^2 and put $\Sigma'=w\Sigma$. Then the transformation $w|\Sigma$ of Σ onto Σ' preserves the order relations because $wP=P'$ with $P=P^2\cap\Sigma$, $P'=P^2\cap\Sigma'$. Let L and L' be the standard elements of $M^{or}(P)$ and $M^{or}(P')$ corresponding to E^2 respectively. Then $wL=L'$. Let v be $W(\Sigma)$-component of u_2, then $W(\Sigma')$-component of $u_2'=wu_2w^{-1}$ is $v'=wvw^{-1}$. On the other hand, Σ'-component of $u_2'E^2$ is equal to $v'L'=w(vL)$. Since $w|\Sigma$ preserves the order relations and $vL\in M^{or}(P)$, we get $w(vL)\in M^{or}(P')$. This proves that $u_2'E^2\in M^{or}(P^2)$ and hence $u_2'\in W(E^2\,;P^2)$. Moreover, as is easily seen, u_2' runs over $W(E^2\,;P^2)$ exactly once. Thus we get the second equality in the lemma from the first.

The assertion on Y' and Z' can be proved similarly. Q.E.D.

3.7. Assume that $\Sigma_R(A)$ is of class I-III. Let us study the symmetries of Y', Z' under $W(\Sigma_R(A))\subset W_G(A)$.

Lemma 3.7. *Assume that* $\Sigma_R(A)$ *is of class* I-III. *Then the functions* Y' *and* Z' *satisfy the following: for* $F\in M(P_R(A))$ *and* $w, u\in W(\Sigma_R(A))$,

$$Y'(wh; F, u, \Lambda)=Y'(h; F, w^{-1}u, \Lambda),$$

$$Z'(wh; F, \Lambda, P_R(A))=Z'(h; F, \Lambda, P_R(A)).$$

Proof. Express h as $h=h_U\exp X$ with $h_U\in A_U$, $X\in\mathfrak{h}_V$. Then by (3.3), $wh=h_U\exp wX$. Since $\alpha(u^{-1}wX)=\alpha((w^{-1}u)^{-1}X)$, we get the first equality. To deduce the second equality from the first, note that when u runs over a complete system of representatives of $W(\Sigma_R(A))/I(F)$, so does $w^{-1}u$. We know by (3.22) that the definition (3.20) of Z' does not depend on a complete system of representatives of $W(\Sigma_R(A))/I(F)$. Therefore we get the second equality from the first. Q.E.D.

3.8. When $\Sigma_R(A)$ is of class I-III, the function $Z(h; E, \Lambda, P_R(A))$ does not depend on the choice of the root vectors $X_{\pm\alpha}$ $(\alpha\in F=E^*)$ used to define the transformation ν_F (see § 6.1). If $\Sigma_R(A)$ contains a simple component of class II, this is not the case. However the dependence of the definition of the function Z on the choice of root vectors $X_{\pm\alpha}(\alpha\in F_0=F_0(E))$ is so little that we can cancell it out by putting the following simple regularity condition on the choice of these root vectors (see § 6.1).

CONDITION 3.1. In the definition (3.9) of the function $Z(h; E, \Lambda, P_R(A))$, we choose the root vectors $X_{\pm\alpha}(\alpha\in F_0=F_0(E))$ in such a way that there exists a system of root vectors $X_{\pm\gamma}(\gamma\in F'=F_0(E)\cup E^*)$ such that if $\gamma, \gamma'\in F'$ and $\gamma'\pm\gamma\in F'$, then $X_{\gamma'+\varepsilon\gamma}=\varepsilon^a[X_{\varepsilon\gamma}, X_{\gamma'}]$ for $\varepsilon=\pm1$, where $a=1$ or 0 according as the simple component of $\Sigma_R(A)$ containing γ, γ' is of type B_n with n odd or not.

In § 9, we need the following lemma. For $\delta\in\Sigma_R(A)$, define $\nu=\nu_\delta$ by means of root vectors $X_{\pm\delta}\in\mathfrak{g}$ such that $[X_\delta, X_{-\delta}]=H_\delta$. Then for $\eta\in\Sigma_R(A)^\delta$, we have $\nu\eta\in\Sigma_R(A')$ by Lemma 2.1, where A' is the connected component of $H^{\mathfrak{h}^\alpha}$ containing $\{h\in A; \xi_\alpha(h)=1\}$. Define a root vector $X_{\nu\eta}\in\mathfrak{g}$ for $\nu\eta$ from a root vector $X_\eta\in\mathfrak{g}$ for η as follows: $X_{\nu\eta}=\nu X_\eta=X_\eta$ if η and δ are strongly orthogonal, and $X_{\nu\eta}=\varepsilon\sqrt{-1}\,\nu X_\eta$ otherwise, where $\varepsilon=\pm1$ such that $\varepsilon\eta>0$ (cf. Lem. 2.2).

Lemma 3.8. *Let* $\gamma, \gamma'\in\Sigma_R(A)$ *be such that* $\gamma'\perp\gamma$, $\gamma'\pm\gamma\in\Sigma_R(A)$, *and* $\delta\in\Sigma_R(A)$ *orthogonal to* γ, γ'. *Then for* $a=0, 1$, *the relation* $X_{\gamma'+\varepsilon\gamma}=\varepsilon^a[X_{\varepsilon\gamma}, X_{\gamma'}]$ $(\varepsilon=\pm1)$ *gives* $X_{\nu\gamma'+\varepsilon\nu\gamma}=\varepsilon^a[X_{\varepsilon\nu\gamma}, X_{\nu\gamma'}]$ *or* $X_{\nu\gamma'+\varepsilon\nu\gamma}=\varepsilon^{a+1}[-X_{\varepsilon\nu\gamma}, X_{\nu\gamma'}]$ *according as* γ *is strongly orthogonal to* δ *or not. In the latter case,* γ, γ' *and* δ *are three short roots in a simple component of type* B_n $(n\geqq3)$ *or* F_4 *of* $\Sigma_R(A)$.

Proof. It is sufficient to note that γ and δ are not strongly orthogonal if and only if so are not γ' and δ. In fact, in this case, the simple component of $\Sigma_R(A)$ containing γ, γ' and δ must be of type B_n ($n\geqq 3$) or F_4. Q. E. D.

Note 3.3. Note that if a root system Σ is of type B_n ($n\geqq 3$) or F_4, then for a short root $\delta\in\Sigma$, the root system Σ^δ is of type B_{n-1} or B_3 respectively. To explain the significance of this lemma, let us take the case where $\Sigma_R(A)$ is simple and of type B_n with $n=2k+1$ odd. Then for any standard element $E\in M^{or}(P_R(A))$, $F_0(E)=\{e_{2i-1}\pm e_{2i}\ (1\leqq i\leqq k),\ e_n\}$ in the realization in (1.3) of $P_R(A)$. Put $\delta=e_n$, $\nu=\nu_\delta$ and $e'_i=\nu e_i\in\Sigma_R(A')$ for $1\leqq i\leqq n-1$. Let E' be the system obtained from νE by removing out the last element $\nu\delta$. Then E' is a standard element of $M^{or}(P_R(A'))$ for $P_R(A')=\nu(P_R(A)\cap\Sigma_R(A)^\delta)$, and $F_0(E')=\{e'_{2i-1}\pm e'_{2i}\ (1\leqq i\leqq k)\}$. Put $F_0=F_0(E)$, $F'_0=F_0(E')$, then we have $\nu_{F_0}=\nu_{F'_0}\nu_\delta$, where ν_{F_0} is defined by means of a system of root vectors $S=\{X_\eta\,;\ \eta\in F_0\cup -F_0\}$ and $\nu_{F'_0}$ by $S'=\{X_{\nu\eta}\,;\ \eta\in F_0\cup -F_0,\ \neq\pm\delta\}$ canonically defined from S. We see from Lemma 3.8 that if the original system S of root vectors satisfies Condition 3.1 for E, then so does for E' the new system S'. (For the general case of $\Sigma_R(A)$, cf. § 9.6.)

For the significance of Condition 3.1, see also Lemmas 7.5 and A1.

§ 4. Recapturation of fundamental theorems

Let G be a connected real semisimple Lie group and G' its subset consisting of regular elements. Let us recall here a theorem in [5(c), § 4] giving a necessary and sufficient condition for that an invariant analytic function on G' defines canonically an invariant eigendistribution on G. This theorem is the principal tool to calculate the discrete series characters. We also recall a characterization of the discrete series characters as a special kind of tempered invariant eigendistributions on G [2(d), §§ 40-41].

4.1. As is well-known, the character of an irreducible quasi-simple representation of G on a Hilbert space is an invariant eigendistribution on G. On the other hand, an invariant eigendistribution π on G coincides essentially with an (invariant) analytic function on G' which will be denoted by π':

$$\pi(f)=\int_G f(g)\pi'(g)dg \qquad (f\in C_0^\infty(G)), \tag{4.1}$$

where dg is a fixed Haar measure on G and $C_0^\infty(G)$ is the space of all C^∞-functions on G with compact supports.

Let $H^{\mathfrak{h}}$ be a Cartan subgroup of G corresponding to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Assume that G is acceptable [2(b), § 18]. Then the character ξ_ρ of $H^{\mathfrak{h}}$ is canonically well-defined, where ρ denotes half the sum of roots in $P(\mathfrak{h})$. Put for $h\in H^{\mathfrak{h}}$,

$$(4.2)\qquad \Delta^{\mathfrak{h}}(h)=\xi_{\rho}(h)\prod_{\alpha\in P(\mathfrak{h})}(1-\xi_{\alpha}(h)^{-1}),\qquad \Delta'^{\mathfrak{h}}_{R}(h)=\prod_{\alpha\in P_R(\mathfrak{h})}(1-\xi_{\alpha}(h)^{-1}).$$

Let $H'^{\mathfrak{h}}$, $H'^{\mathfrak{h}}(R)$ be the subsets of $H^{\mathfrak{h}}$ defined by $\Delta^{\mathfrak{h}}(h)\neq 0$, $\Delta'^{\mathfrak{h}}_{R}(h)\neq 0$ respectively and put

$$(4.3)\qquad \varepsilon^{\mathfrak{h}}_{R}(h)=\mathrm{sgn}(\Delta'^{\mathfrak{h}}_{R}(h))\qquad (h\in H'^{\mathfrak{h}}(R)).$$

For an invariant eigendistribution π on G, put

$$(4.4)\qquad \tilde{\kappa}^{\mathfrak{h}}(h)=\Delta^{\mathfrak{h}}(h)\pi'(h)\qquad (h\in H'^{\mathfrak{h}}),\qquad \kappa^{\mathfrak{h}}(h)=\varepsilon^{\mathfrak{h}}_{R}(h)\Delta^{\mathfrak{h}}(h)\pi'(h)=\varepsilon^{\mathfrak{h}}_{R}(h)\tilde{\kappa}^{\mathfrak{h}}(h)\qquad (h\in H'^{\mathfrak{h}}).$$

Then it is known that $\tilde{\kappa}^{\mathfrak{h}}$ and $\kappa^{\mathfrak{h}}$ can be extended analytically from $H'^{\mathfrak{h}}$ onto $H'^{\mathfrak{h}}(R)$ [2(b), § 19]. Note that the definitions of $\tilde{\kappa}^{\mathfrak{h}}$ and $\kappa^{\mathfrak{h}}$ depend on the set $P(\mathfrak{h})$ of positive roots of $\mathfrak{h}$. In the case when this is crucial in the discussions, we denote $\Delta^{\mathfrak{h}}(h)$, $\tilde{\kappa}^{\mathfrak{h}}(h)$ and $\kappa^{\mathfrak{h}}(h)$ respectively as

$$\Delta^{\mathfrak{h}}(h\,;\,P(\mathfrak{h})),\quad \tilde{\kappa}^{\mathfrak{h}}(h\,;\,P(\mathfrak{h})),\quad \kappa^{\mathfrak{h}}(h\,;\,P(\mathfrak{h})).$$

Let $\mathfrak{h}'$ be a Cartan subalgebra conjugate to $\mathfrak{h}$ under G. Choose a $g\in G$ such that $\mathrm{Ad}(g)\mathfrak{h}=\mathfrak{h}'$. Then there exists a $w\in W(\mathfrak{h}'_c)=W(\Sigma(\mathfrak{h}'))$ such that

$$\mathrm{Ad}(g)P(\mathfrak{h})=wP(\mathfrak{h}').$$

In this case we get

$$(4.5)\qquad \tilde{\kappa}^{\mathfrak{h}}(h)=\mathrm{sgn}(w)\tilde{\kappa}^{\mathfrak{h}'}(ghg^{-1})\qquad (h\in H'^{\mathfrak{h}}).$$

For an element $\omega\in W_G(H^{\mathfrak{h}})$, denote by $\bar{\omega}$ the corresponding element in $W_G(\mathfrak{h})\subset W(\mathfrak{h}_c)$ and put $\varepsilon(\omega)=\mathrm{sgn}(\bar{\omega})$. Then $\Delta^{\mathfrak{h}}(\omega h)=\varepsilon(\omega)\Delta^{\mathfrak{h}}(h)$ and hence

$$(4.6)\qquad \tilde{\kappa}^{\mathfrak{h}}(\omega h)=\varepsilon(\omega)\tilde{\kappa}^{\mathfrak{h}}(h)\qquad (h\in H'^{\mathfrak{h}}(R)).$$

Moreover, define a sign $\varepsilon(\omega, h)=\pm 1$ for $\omega\in W_G(H^{\mathfrak{h}})$ and $h\in H^{\mathfrak{h}}$ as in [5(c), § 2.1, (2.2)], then it is locally constant in h and it holds that

$$(4.7)\qquad \kappa^{\mathfrak{h}}(\omega h)=\varepsilon(\omega, h)\kappa^{\mathfrak{h}}(h)\qquad (h\in H'^{\mathfrak{h}}(R)).$$

Take a connected component A of $H^{\mathfrak{h}}$. Then

$$(4.8)\qquad A\cap H'^{\mathfrak{h}}(R)=\{h\in A\,;\,\xi_{\alpha}(h)\neq 1\ (\alpha\in P_R(A))\},$$

and this set is a disjoint union of $uA^{+}(P)$ over all $u\in W(\Sigma_R(A))\subset W_G(A)$, where $A^{+}(P)$ is given by (3.23). Note that for $h\in A^{+}(P)$, $\varepsilon^{\mathfrak{h}}_{R}(h)=1$ and so

$$\tilde{\kappa}^{\mathfrak{h}}(h)=\kappa^{\mathfrak{h}}(h)\qquad (h\in A^{+}(P)),$$

and that we can define $\varepsilon(\omega)=\mathrm{sgn}(\bar{\omega})$ for $\omega\in W_G(A^{+}(P))$ as above, and then

$$(4.9)\qquad \tilde{\kappa}^{\mathfrak{h}}(\omega h)=\varepsilon(\omega)\tilde{\kappa}^{\mathfrak{h}}(h)\qquad (h\in A^{+}(P)).$$

The reason why we define two kinds of functions $\tilde{\kappa}^{\mathfrak{h}}$ and $\kappa^{\mathfrak{h}}$ is the following. They are both natural from the various points of view, for exemple, $\kappa^{\mathfrak{h}}$ can be extended to a continuous function on the whole $H^{\mathfrak{h}}$ [5(c), § 2.2, Th. 1]. Concerning the discrete series characters in the present question, the situation is as follows. When $\Sigma_R(A)$ is class I, $\kappa^{\mathfrak{h}}$ for such a character π can be expressed on the whole A by a single formula using the functions $Z'(h, F, \Lambda, P_R(A))$ with $F\in M(P_R(A))$. On the contrary, when $\Sigma_R(A)$ is not of class I, such is not the case and we give $\tilde{\kappa}^{\mathfrak{h}}$ on $A^+(P)$ by a formula using the functions $Z(h\,;E, \Lambda, P_R(A))$ with $E\in M^{or}(P_R(A))$, and for other connected component $uA^+(P)$ with $u\in W(\Sigma_R(A))$ of $A\cap H'^{\mathfrak{h}}(R)$, $\tilde{\kappa}^{\mathfrak{h}}$ on it is determined from $\tilde{\kappa}^{\mathfrak{h}}|A^+(P)$ by means of (4.6).

4.2. Let us give the necessary and sufficient condition mentioned at the beginning of this section.

Denote by $\mathrm{Car}(\mathfrak{g})$ ($\mathrm{Car}(G)$ resp.) the set of conjugate classes under G of Cartan subalgebras of $\mathfrak{g}$ (Cartan subgroups of G resp.). We fix a complete system of representatives Ω of $\mathrm{Car}(\mathfrak{g})$. Then $H^{\mathfrak{h}}$ ($\mathfrak{h}\in\Omega$) gives a complete system of representatives of $\mathrm{Car}(G)$. Assume that the set of positive roots $P(\mathfrak{h})$ is fixed for every $\mathfrak{h}\in\Omega$. Denote by $\mathrm{Car}^0(G)$ the set of all conjugate classes under G of connected components of Cartan subgroups of G, and let Ω^0 be a complete system of representatives of $\mathrm{Car}^0(G)$ such that every element A in Ω^0 is a connected component of $H^{\mathfrak{h}}$ for some $\mathfrak{h}\in\Omega$.

Assume that we are given a system of functions $\zeta(h\,; A^+(P), P(\mathfrak{h}))$ on $A^+(P)$ for every $A\in\Omega^0$ with $H^{\mathfrak{h}}\supset A$ and that they satisfy the following condition.

(a) *Symmetry condition. For any* $\omega\in W_G(A^+(P))$ *and* $h\in A^+(P)$,

$$\zeta(\omega h\,; A^+(P), P(\mathfrak{h}))=\varepsilon(\omega)\zeta(h\,; A^+(P), P(\mathfrak{h}))\,. \tag{4.10}$$

Then, extending this system of functions, we can define uniquely for any Cartan subalgebra $\mathfrak{h}'$ and any $P(\mathfrak{h}')$ a function $\tilde{\kappa}^{\mathfrak{h}'}(h)=\tilde{\kappa}^{\mathfrak{h}'}(h\,; P(\mathfrak{h}'))$ on $H'^{\mathfrak{h}'}(R)$ in such a way that they altogether satisfy the conditions (4.5) and (4.6).

Let $S(\mathfrak{h}_c)$ be the symmetric algebra of $\mathfrak{h}_c$ and $I(\mathfrak{h}_c)$ its subalgebra of elements invariant under $W(\mathfrak{h}_c)$. Let $\mathfrak{h}'$ be another Cartan subalgebra. Then we say that a homomorphism $\chi^{\mathfrak{h}}$ of $I(\mathfrak{h}_c)$ into $\boldsymbol{C}$ is consistent with a homomorphism $\chi^{\mathfrak{h}'}$ of $I(\mathfrak{h}'_c)$ into $\boldsymbol{C}$ if $\chi^{\mathfrak{h}}(D)=\chi^{\mathfrak{h}'}(\nu D)$ $(D\in I(\mathfrak{h}_c))$ for an inner automorphism ν of $\mathfrak{g}_c$ such that $\nu\mathfrak{h}'_c=\mathfrak{h}_c$.

We consider the following condition on the system of functions $\zeta(h\,; A^+(P), P(\mathfrak{h}))$ $(A\in\Omega^0)$.

(b) *Differential equations. There exists a system of mutally consistent homomorphisms* $\chi^{\mathfrak{h}}$ *of* $I(\mathfrak{h}_c)$ *for* $\mathfrak{h}\in\Omega$ *such that for* $A\in\Omega^0$ *with* $H^{\mathfrak{h}}\supset A$,

$$D\zeta(h\,; A^+(P), P(\mathfrak{h}))=\chi^{\mathfrak{h}}(D)\zeta(h\,; A^+(P), P(\mathfrak{h}))\qquad (D\in I(\mathfrak{h}_c))\,. \tag{4.11}$$

To state the necessary and sufficient condition, we need one more condition on ζ's. For $\gamma\in\Sigma(\mathfrak{h})$, put

$$\begin{aligned} &\Xi_\gamma=\{h\in H^{\mathfrak{h}}\,; \xi_\gamma(h)=1\}\,,\\ &\Xi'_\gamma(R)=\{h\in\Xi_\gamma\,; \xi_\delta(h)\neq 1 \quad \text{for any} \quad \delta\in\Sigma_R(\mathfrak{h}),\ \neq\pm\gamma\}\,. \end{aligned} \tag{4.12}$$

Let $A\in\Omega^0$ with $H^{\mathfrak{h}}\supset A$. Let $\Sigma_R(A)$, $P_R(A)=\Sigma_R(A)\cap P(\mathfrak{h})$ be as in § 2.3. Denote by $\Pi_R(A)$ the set of simple roots of $\Sigma_R(A)$ with respect to $P_R(A)$. Then any wall of $A^+(P)$ in A is given as $\Xi_\alpha\cap\mathrm{Cl}(A^+(P))$ for some $\alpha\in\Pi_R(A)$, where $\mathrm{Cl}(E)$ denotes the closure of a set E. For an $\alpha\in\Pi_R(A)$, put $\mathfrak{h}^\alpha=\nu_\alpha(\mathfrak{h}_c)\cap\mathfrak{g}$ and let A^α be the connected component of $H^{\mathfrak{h}^\alpha}$ containing the wall $\Xi_\alpha\cap\mathrm{Cl}(A^+(P))$. We know that $\beta=\nu_\alpha\alpha$ is a singular imaginary root of $\mathfrak{h}^\alpha$ and it follows from (2.7) that

$$\Xi_\beta=\Xi_\alpha=H^{\mathfrak{h}}\cap H^{\mathfrak{h}^\alpha}\,,\quad \Xi_\beta\cap A^\alpha\supset\Xi_\alpha\cap A\,.$$

Note that $\Xi'_\alpha(R)\subset H'^{\mathfrak{h}}(R)$ and that, under the condition (a), the function $\tilde{\kappa}^{\mathfrak{h}}$ coming from some ζ by the conditions (4.5) and (4.6) is analytic on $H'^{\mathfrak{h}}(R)$, and moreover that, under the condition (b), the function $\tilde{\kappa}^{\mathfrak{h}}|A^+(P)=\zeta(\cdot\,;A^+(P),\,P(\mathfrak{h}))$ has finite limit values everywhere on the walls of $A^+(P)$. Thus the following equation has meaning: let $P(\mathfrak{h}^\alpha)=\nu_\alpha P(\mathfrak{h})$, then for $h\in\Xi'_\alpha(R)\cap\mathrm{Cl}(A^+(P))$,

$$\text{(4.13)}\qquad \frac{d}{dt}\tilde{\kappa}^{\mathfrak{h}}(h\exp(tH_\alpha)\,;P(\mathfrak{h}))|_{t=+0}=\frac{1}{\sqrt{-1}}\frac{d}{dt}\tilde{\kappa}^{\mathfrak{h}^\alpha}(h\exp(t\sqrt{-1}H_\beta\,;P(\mathfrak{h}^\alpha))|_{t=0}\,,$$

where the left hand side denotes the limit values for $t\to+0$.

Let us rewrite this equation by means of ζ's. There exist unique $\mathfrak{h}'\in\Omega$ and $C\in\Omega^0$ such that $C\subset H^{\mathfrak{h}'}$, $\mathrm{Ad}(g)\mathfrak{h}^\alpha=\mathfrak{h}'$ and $gA^\alpha g^{-1}=C$ for some $g\in G$. Note that we fix $P(\mathfrak{h})$, $P(\mathfrak{h}')$ for $\mathfrak{h},\mathfrak{h}'\in\Omega$ apriori. Since $W(\Sigma_R(C))\subset W_G(C)$, we may assume that $g(A^\alpha)^+(P_R(A^\alpha))g^{-1}=C^+(P_R(C))$, or equivalently, $\mathrm{Ad}(g)P_R(A^\alpha)=P_R(C)$. (Note that $\Sigma_R(C)=(\mathrm{Ad}(g)\circ\nu_\alpha)\Sigma_R(A)^\alpha$, where $\Sigma_R(A)^\alpha=\{\gamma\in\Sigma_R(A)\,;\gamma\perp\alpha\}$.) There exists $w\in W(\mathfrak{h}'_c)$ such that $\mathrm{Ad}(g)P(\mathfrak{h}^\alpha)=wP(\mathfrak{h}')$. Then under the condition (a), we get for $h\in A^{\alpha+}(P)=(A^\alpha)^+(P_R(A^\alpha))$,

$$\begin{aligned}\tilde{\kappa}^{\mathfrak{h}^\alpha}(h\,;P(\mathfrak{h}^\alpha))&=\mathrm{sgn}(w)\tilde{\kappa}^{\mathfrak{h}'}(ghg^{-1}\,;P(\mathfrak{h}'))\\&=\mathrm{sgn}(w)\zeta(ghg^{-1}\,;C^+(P_R(C)),\,P(\mathfrak{h}'))\,.\end{aligned}$$

Thus, rewriting the equation (4.13), we have the following condition on ζ's.

(c) *Boundary condition. For $A\in\Omega^0$ with $H^{\mathfrak{h}}\supset A$ and $\alpha\in\Pi_R(A)$, take $\mathfrak{h}'\in\Omega$, $C\in\Omega^0$ and $g\in G$ such that* $\mathrm{Ad}(g)\mathfrak{h}^\alpha=\mathfrak{h}'$ *and C is the connected component of $H^{\mathfrak{h}'}$ containing $g(\Xi_\alpha\cap A)g^{-1}$ and moreover that $\nu_\alpha P_R(A)=\mathrm{Ad}(g)^{-1}P_R(C)$. Let w be an element in $W(\mathfrak{h}'_c)$ such that* $(\mathrm{Ad}(g)\circ\nu_\alpha)P(\mathfrak{h})=wP(\mathfrak{h}')$. *Then for $h\in\Xi'_\alpha(R)\cap\mathrm{Cl}(A^+(P))$,*

$$\text{(4.14)}\qquad H_\alpha\zeta(h\,;A^+(P),\,P(\mathfrak{h}))=\mathrm{sgn}(w)H_\beta\zeta(h\,;C^+(P),\,P(\mathfrak{h}'))\,,$$

where the left hand side denotes the limit value at h.

The following theorem is a version of Theorem 3 in [5(c), § 4.3].

Theorem A. *Let Ω and Ω^0 be complete systems of representatives of* $\mathrm{Car}(\mathfrak{g})$ *and* $\mathrm{Car}^0(G)$ *respectively such that for any $A\in\Omega^0$, $H^{\mathfrak{h}}\supset A$ for some $\mathfrak{h}\in\Omega$. For an invariant eigendistribution π on G, put for $A\in\Omega^0$,*

$$\text{(4.15)}\qquad \zeta(h\,;A^+(P),\,P(\mathfrak{h}))=\Delta^{\mathfrak{h}}(h\,;P(\mathfrak{h}))\pi'(h)\qquad(h\in A^+(P)\cap G')\,.$$

where $\mathfrak{h}\in\Omega$ *is such that* $A\subset H^{\mathfrak{h}}$. *Then* ζ *can be extended to an anlytic function on* $A^{+}(P)$, *and the system of functions* $\zeta(\cdot\,;A^{+}(P),P(\mathfrak{h}))$ $(A\in\Omega^{0})$ *satisfies the conditions* (a), (b) *and* (c) *above.*

Conversely assume that a system of analytic functions $\zeta(\cdot\,;A^{+}(P),P(\mathfrak{h}))$ *on* $A^{+}(P)$ *is given for* $A\in\Omega^{0}$ *in such a manner that the conditions* (a), (b) *and* (c) *hold. Then it defines uniquely an invariant analytic function* π' *on* G' *by* (4.15), *and an invariant eigendistribution* π *on* G *by* (4.1).

To deduce this theorem from Theorem 3 in [5(c), § 4.3], it is enough to apply Lemma 7.9 in [5(c), § 7.4].

4.3. When G has a finite center, Harish-Chandra defined the temperedness of a distribution on G in [2(d)]. He also gave a criterion for an invariant eigendistribution π on G to be tempered by means of the functions $\tilde{\kappa}^{\mathfrak{h}}$ (or $\kappa^{\mathfrak{h}}$) corresponding to it [2(d), § 19]. For the present purpose it is sufficient for us to quote it partially as follows.

Theorem B. *Assume that the center of* G *is finite. Then the following condition* (d) *is sufficient for an invariant eigendistribution* π *on* G *to be tempered. Moreover it is necessary if the infinitesimal character of* π *is regular.*

(d) *Boundedness. The functions* $\tilde{\kappa}^{\mathfrak{h}}$ *(or* $\kappa^{\mathfrak{h}}$*) corresponding to* π *is bounded on* $H'^{\mathfrak{h}}(R)$ *for every Cartan subalgebra* $\mathfrak{h}$.

In this paper, we use the following conventional terminology: evenwhen G has an infinite center, an invariant eigendistribution π on G satisfying the condition (d) is called *tempered.*

In [5(d), §§ 8–9], we have studied the existence and the uniqueness of tempered invariant eigendistributions in general for G with finite center. The method and the discussions there can also be applied to G with infinite center. In particular, we get the following theorem which serves very conveniently to calculate explicitly tempered invariant eigendistributions, especially the discrete series characters. To state it, we need a certain order in $\mathrm{Car}^{0}(G)$. Let A be a connected component of $H^{\mathfrak{h}}$ and denote by $[A]$ its conjugate class in $\mathrm{Car}^{0}(G)$. Take $\alpha\in\Sigma_{R}(A)$ and let A^{α} be the connected component of $H^{\mathfrak{h}^{\alpha}}$ containing $A\cap\Xi_{\alpha}$. Then we put $[A]\prec[A^{\alpha}]$. Extending this relation transitively, we get an order $\prec$ in $\mathrm{Car}^{0}(G)$ which is naturally related with the condition (c) in § 4.2. In the proof of the existence theorem, Theorem 11 in [5(d), § 8], we gave a method of constructing tempered invariant eigendistributions on G inductively according to the order $\prec$ in $\mathrm{Car}^{0}(G)$. Depending on this inductive process, we proved the uniqueness theorem, Theorem 12 in [5(d), § 9]. In the way of the proof of this theorem, we get the following result.

Theorem C. *Let* G *be a connected semisimple Lie group not necessarily center finite, and* π *an invariant eigendistribution on it. Put* $\tilde{\kappa}^{\mathfrak{h}}=\Delta^{\mathfrak{h}}\pi'$ *on* $H'^{\mathfrak{h}}$ *for* $\mathfrak{h}\in\Omega$, *and put for* $A\in\Omega^{0}$, $\zeta(\cdot\,;A^{+}(P),P(\mathfrak{h}))=\tilde{\kappa}^{\mathfrak{h}}|A^{+}(P)$, *where* $H^{\mathfrak{h}}\supset A$. *Then*

$\tilde{\kappa}^{\mathfrak{h}}|A$ is uniquely determined by $\zeta(\cdot\,;A^+(P),P(\mathfrak{h}))$. Assume that π is tempered and its infinitesimal character is regular. Then for any $A\in\Omega^0$, the function $\zeta(\cdot\,;A^+(P),P(\mathfrak{h}))$ is uniquely determined by the family of functions $\zeta(\cdot\,;C^+(P),P(\mathfrak{h}'))$ with $H^{\mathfrak{h}'}\supset C$ such that $[C]\succ[A]$ under the conditions (a), (b), (c) *and* (d) *all restricted on $\Omega^0(A)=\{C\in\Omega^0\,;[C]\succ[A]$ or $=[A]\}$.*

4.4. Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$ such that the dimension of its vector part takes the possible minimal value. Then $B=H^{\mathfrak{b}}$ is connected. Let $\mathfrak{b}_B^*$ and $\mathfrak{b}_B^{*\prime}$ be as in § 3.1. For $\Lambda\in\mathfrak{b}_B^*$, define a function ζ_Λ on B by

$$\zeta_\Lambda(b)=\sum_{w\in W_G(\mathfrak{b})}\mathrm{sgn}(w)\xi_{w\Lambda}(b)\,. \tag{4.16}$$

Note that in this case $W_G(\mathfrak{b})$ is canonically isomorphic to $W_G(B)$, and identifying them with each other, we get $\xi_{w\Lambda}(b)=\xi_\Lambda(w^{-1}b)$. Moreover note that, since $\Sigma_R(B)=\emptyset$, we get $M(P_R(B))=\emptyset$, $M^{or}(P_R(B))=\emptyset$ and hence putting $E=\emptyset$, we get from (3.9) and (3.20) that

$$\zeta_\Lambda(b)=Z(b\,;E,\Lambda,P_R(B))=Z'(b\,;E,\Lambda,P_R(B))\,. \tag{4.17}$$

(In this case, the symbol $P_R(B)$ has no meaning in the definition of Z and Z'.)

Assume now that B is compact. Then G has the discrete series representations and their characters are characterized as a special class of tempered invariant eigendistributions on G as follows ([2(c), Part II] and [2(d), Part III]).

Theorem D. *Assume that G has a compact Cartan subgroup B. For $\Lambda\in\mathfrak{b}_B^*$, there exists at least one tempered invariant eigendistribution π on G such that the corresponding function $\tilde{\kappa}^{\mathfrak{b}}=\Delta^{\mathfrak{b}}\pi'$ on $B\cap G'$ is equal to ζ_Λ in* (4.16). *In particular, if Λ is regular, π is unique and equal to the distribution Θ_Λ in* [2(c), § 24]. *The distribution*

$$(-1)^q\varepsilon(\Lambda)\Theta_\Lambda$$

is the character of a representation of G in the discrete series. Here

$$\varepsilon(\Lambda)=\mathrm{sgn}\{\prod_{\gamma\in P(\mathfrak{b})}(\Lambda,\gamma)\}\,,\qquad q=2^{-1}(\dim G-\dim K)\,, \tag{4.18}$$

where K is a maximal compact subgroup of G. Moreover any such character is given in this form.

§ 5. Formulas for the discrete series characters

Fix a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that its vector part has the possible minimal dimension, and put $B=H^{\mathfrak{b}}$. We fix once for all the set $P(\mathfrak{b})$ of positive roots in $\Sigma(\mathfrak{b})$. In this section, we give explicitly a tempered invariant eigendistribution π_Λ such that $\tilde{\kappa}^{\mathfrak{b}}=\zeta_\Lambda$, for any $\Lambda\in\mathfrak{b}_B^{*\prime}$. More exactly, for any fixed set of positive roots $P(\mathfrak{h})$ in $\Sigma(\mathfrak{h})$, we give the function $\tilde{\kappa}^{\mathfrak{h}}$ on $H'^{\mathfrak{h}}(R)$ corresponding to π_Λ for any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Note that since Λ is taken to be regular, π_Λ is uniquely determined by Λ. When B is compact, π_Λ is equal to the distribution Θ_Λ in § 4.4. Therefore the discrete series character $(-1)^q\varepsilon(\Lambda)\Theta_\Lambda$ is written down quite explicitly on every $H'^{\mathfrak{h}}$. The proof of these formulas will be given in the subsequent sections.

5.1. Let $\mathfrak{h}$ be a Cartan subalgebra and A a connected component of $H^{\mathfrak{h}}$. The explicit form on A of the function $\tilde{\kappa}^{\mathfrak{h}}$ essentially depends on the type of the root system $\Sigma_R(A)$ and is simple when $\Sigma_R(A)$ is of class I-III. Let us first treat this case.

Assume that $\Sigma_R(A)$ is of class I-III. Take an element $F\in M(\Sigma_R(A))$, then by Lemma 1.2 and Proposition 2.4, $\mathfrak{h}^F$ is conjugate to $\mathfrak{b}$ under G. Let g_0 be an element in G such that $\mathrm{Ad}(g_0)\mathfrak{b}=\mathfrak{h}^F$, and denote $\Lambda\circ\mathrm{Ad}(g_0)^{-1}|\mathfrak{h}_c^F$ by $\mathrm{Ad}(g_0)\Lambda$, then $\mathrm{Ad}(g_0)\Lambda\in(\mathfrak{h}_c^F)^*$. (Note that since $\Sigma_R(A)$ is of class I-III, $M^{or}(P_R(A))=\tilde{M}(P_R(A))=M(P_R(A))$ formally, by definition in § 1.7.)

Theorem 1. *Let $\Lambda\in\mathfrak{b}_B^{*\prime}$. Let π'_Λ be the invariant analytic function on G' corresponding to the invariant eigendistribution π_Λ. Let A be a connected component of a Cartan subgroup $H^{\mathfrak{h}}$. When $\Sigma_R(A)$ is of class* I-III, *the function $\pi'_\Lambda|A\cap G'$ is given as follows. Let F be the unique standard element in $M(P_R(A))$ with respect to $P_R(A)=\Sigma_R(A)\cap P(\mathfrak{h})$. Then there exist an element $g_0\in G$ and a $w_0\in W(\mathfrak{b}_c)$ such that*

$$\mathrm{Ad}(g_0)\mathfrak{b}=\mathfrak{h}^F\,,\quad \mathrm{Ad}(g_0)w_0P(\mathfrak{b})=\nu_FP(\mathfrak{h})\,. \tag{5.1}$$

Then for $h\in A\cap H'^{\mathfrak{h}}(R)$,

$$\begin{aligned}\kappa^{\mathfrak{h}}(h\,;P(\mathfrak{h}))&\equiv(\varepsilon_R^{\mathfrak{h}}\Delta^{\mathfrak{h}})(h\,;P(\mathfrak{h}))\pi'_\Lambda(h)\\&=\mathrm{sgn}(w_0)\varepsilon(F)Z'(h\,;F,\mathrm{Ad}(g_0)\Lambda,P_R(A))\,,\end{aligned} \tag{5.2}$$

where the function Z' on A is given by (3.20), *and the sign $\varepsilon(F)$ in Definition* 1.3 *is given by*

$$\varepsilon(F)=(-1)^{\#F}\,. \tag{5.3}$$

Remark 5.1. Midorikawa gave in [7] an analogous formula for the sum of the discrete series characters with the same infinitesimal character. But he excluded the type GI for the simple factors of G (for the definition of type GI, see [9, p. 429]), and assumed that G is linear. Thus he treated the sum of $\mathrm{sgn}(w)\Theta_{w\Lambda}$ over $w\in W(\mathfrak{b}_c)$. Note that the formula in Theorem 1 is a reduced one, that is, there occurs no cancellation between terms appearing in it.

5.2. Now let us treat the general case. Let A be a connected component of $H^{\mathfrak{h}}$. For any simple component Σ of $\Sigma_R(A)$ of class II, we consider the canonical order corresponding to $P=\Sigma\cap P_R(A)$. Let $\{E_1, E_2, \cdots, E_p\}$ be the set of all standard elements in $M^{or}(P_R(A))$. Then as is seen in § 1.5 and § 1.7, the associated strongly orthogonal systems $F_0(E_i)$ coincide with each other (denote it by F_0). There exist $g_0\in G$ and $w_0\in W(\mathfrak{b}_c)$ such that

(5.4) $$\mathrm{Ad}(g_0)\mathfrak{b}=\mathfrak{h}^{F_0}, \quad \mathrm{Ad}(g_0)w_0P(\mathfrak{b})=\nu_{F_0}P(\mathfrak{h}).$$

Here we put the following regularity condition on the choice of the root vectors $X_{\pm\alpha}(\alpha\in F_0)$ used to define ν_{F_0} (cf. Lemmas 6.2 and 7.5).

CONDITION 5.1. Put $F'=\bigcup_{1\leq i\leq r}(E_i)^*\supset F_0$. The root vectors $X_{\pm\alpha}$ $(\alpha\in F_0)$ are so chosen that there exists a system of root vectors $X_{\pm\gamma}$ $(\gamma\in F')$ such that if $\gamma, \gamma'\in F'$ and $\gamma'\pm\gamma\in F'$, then $X_{\gamma'+\varepsilon\gamma}=\varepsilon^a[X_{\varepsilon\gamma}, X_{\gamma'}]$ for $\varepsilon=\pm 1$, where $a=1$ or 0 according as the simple component of $\Sigma_R(A)$ containing γ, γ' is of type B_n with n odd or not.

Let $A^+(P)$ with $P=P_R(A)=\Sigma_R(A)\cap P(\mathfrak{h})$ be a connected component of $A\cap H'^{\mathfrak{h}}(R)$ given by

(5.5) $$A^+(P)=\{h\in A\,;\ \xi_\alpha(h)>1\ (\alpha\in P_R(A))\}.$$

Then $A\cap H'^{\mathfrak{h}}(R)$ is a disjoint union of $uA^+(P)$ over $u\in W(\Sigma_R(A))$, where $W(\Sigma_R(A))$ is imbedded canonically into $W_G(A)$. We give the function $\tilde{\kappa}^{\mathfrak{h}}$ corresponding to π_Λ on every $uA^+(P)$ separately as follows.

Theorem 2. *Let $\Lambda\in\mathfrak{b}_B^*$ be regular and π_Λ the tempered invariant eigendistribution on G such that $\tilde{\kappa}^{\mathfrak{b}}=\Delta^{\mathfrak{b}}\pi'_\Lambda=\zeta_\Lambda$. Put $\tilde{\kappa}^{\mathfrak{h}}(h\,;P(\mathfrak{h}))=\Delta^{\mathfrak{h}}(h\,;P(\mathfrak{h}))\pi'_\Lambda(h)$ for $h\in H'^{\mathfrak{h}}$. Then for any connected component A of $H^{\mathfrak{h}}$, $\tilde{\kappa}^{\mathfrak{h}}$ is given on it as follows. Let $E_i\in M^{or}(P_R(A))$ $(1\leqq i\leqq r)$ be all the standard elements in $M^{or}(P_R(A))$ and F_0 the strongly orthogonal system associated to them. Let $g_0\in G$ and $w_0\in W(\mathfrak{b}_c)$ be as in* (5.4). *Then for $h\in A^+(P)$,*

(5.6) $$\tilde{\kappa}^{\mathfrak{h}}(h\,;P(\mathfrak{h}))=\mathrm{sgn}(w_0)\sum_{1\leq i\leq r}\varepsilon(E_i)Z(h\,;E_i, \mathrm{Ad}(g_0)\Lambda, P_R(A)),$$

where the function Z is given by (3.9)-(3.9′) *and the sign $\varepsilon(E_i)$ is given by* (1.39)-(1.39′). *Moreover for $u\in W(\Sigma_R(A))$, $h\in A^+(P)$,*

(5.7) $$\tilde{\kappa}^{\mathfrak{h}}(uh\,;P(\mathfrak{h}))=\mathrm{sgn}(u)\tilde{\kappa}^{\mathfrak{h}}(h\,;P(\mathfrak{h})).$$

When the Cartan subgroup B is compact, the discrete series character $(-1)^q\varepsilon(\Lambda)\Theta_\Lambda$ is given on $A^+(P)$ by

(5.8) $$(-1)^q\,\mathrm{sgn}(w_0)\varepsilon(\Lambda)\sum_{1\leq i\leq r}\frac{\varepsilon(E_i)Z(h\,;E_i, \mathrm{Ad}(g_0)\Lambda, P_R(A))}{\Delta^{\mathfrak{h}}(h\,;P(\mathfrak{h}))}$$

where q and $\varepsilon(\Lambda)$ are given in (4.18).

It follows from Lemma 3.4 that when $\Sigma_R(A)$ is of class I-III, the formula (5.2) in Theorem 1 and those (5.6)-(5.7) in Theorem 2 concide with each other.

Remark 5.2. The character formula for $Sp(n, \boldsymbol{R})$ given in [5(e)] is essentially the same as that given above. Let $M_0^{or}(P_R(A))$ be the subset of $M^{or}(P_R(A))$ consisting of E such that $E^*\in M(P_R(A))$ is strongly orthogonal. Note that $M_0^{or}(P_R(A))=W(\tilde{F}_0\,;P_R(A))\tilde{F}_0$, where $W(\tilde{F}_0\,;P_R(A))$ is the subset of $W(\Sigma_R(A))$ defined in (1.23). In [5(e)], we use essentially $M_0^{or}(P_R(A))$ or equivalently

$W(\tilde{F}_0; P_R(A))$ to give a formula for $\tilde{\kappa}^{\mathfrak{h}}$ on A, and the roll of $M^{or}(P_R(A))$ is implicit there. (Determine $W(\tilde{F}_0; P_R(A))$ as a subset of a symmetric group as in (1.28)-(1.28′) and compair it with the formula in [5(e), §7, Th. 5].)

Remark 5.3. Assume that $\Sigma_R(A)$ is of class II. For an $F \in M(P_R(A))$, the set $W(\tilde{F}; P_R(A))$ is expressed as a product of $U(\tilde{F})$ in (1.24) and $V(\tilde{F})$ in (1.23) by Lemma 1.7. Note that the subset $U(\tilde{F})\tilde{F}$ in $\tilde{M}(P_R(A))$ can be canonically identified with the subset of $M(P_R(A))$ consisting of elements conjugate to F under $W(\Sigma_R(A))$. These facts mean that anyhow the formula (5.6) for $\tilde{\kappa}^{\mathfrak{h}}$ can be rewritten by means of $M(P_R(A))$ not using $M^{or}(P_R(A))$ explicitly. By Remark 3.1, for any $w \in V(\tilde{F})$, F and w satisfy the condition in Lemma 3.1 and hence the equality (3.11) holds. We ask in what extent this equality serves to simplify our formula (5.6) for $\tilde{\kappa}^{\mathfrak{h}}$.

Remark 5.4. When $\Sigma_R(A)$ contains simple component of class II, the general formula (5.6) for $\tilde{\kappa}^{\mathfrak{h}}$ can be reduced to more simple forms in some special cases, for exemple, for the holomorphic discrete series (cf. [3] and [6]) for the groups locally isomorphic to $Sp(n, \boldsymbol{R})$. For $Sp(n, \boldsymbol{R})$, we have shown in [5(e)] that this reduction of the general formula can be carried out easily. See also Appendix of this paper. However, as is remarked in Introduction, the formula (5.6) is the most reduced one when we consider the function $\tilde{\kappa}^{\mathfrak{h}}|A$ as a function of two variables $h \in A \cap G'$ and $\Lambda \in \mathfrak{b}_B^{*\prime}$.

§6. Outline of the proof of main theorems

To prove Theorems 1 and 2, we apply Theorems A and B. To prove Theorem 1 only, it is a short passage to apply Theorem C. But, because of Lemma 3.4, Theorem 2 contain Theorem 1 essentially, and therefore we give here a proof for Theorem 2 only.

6.1. First of all, we must prove that the function $\tilde{\kappa}^{\mathfrak{h}}$ in Theorem 2 is well-defined. Next we study the relation between $\tilde{\kappa}^{\mathfrak{h}}$ and $\tilde{\kappa}^{\mathfrak{h}'}$ when $\mathfrak{h}$ and $\mathfrak{h}'$ are conjugate to each other under G. Thus we shall prove in this subsection the following.

(1) For any connected component A of $H^{\mathfrak{h}}$, the function $\tilde{\kappa}^{\mathfrak{h}}|A^+(P)$ defined by (5.6) does not depend on the choice of root vectors $X_{\pm\alpha}$ for $\alpha \in F_0$ $(=F_0(E_i)$ for any $i)$ satisfying Condition 5.1. (These root vectors are used to define the automorphism ν_{F_0} and the Cartan subalgebra $\mathfrak{h}^{F_0} = \mathfrak{g} \cap \nu_{F_0}\mathfrak{h}_c$.)

(2) Let $\mathfrak{h}$ and $\mathfrak{h}'$ be two Cartan subalgebras of $\mathfrak{g}$ conjugate to each other under G (we admit the case $\mathfrak{h} = \mathfrak{h}'$). Let $P(\mathfrak{h})$ and $P'(\mathfrak{h}')$ be the sets of positive roots of $\mathfrak{h}$ and $\mathfrak{h}'$ respectively, and choose a $g \in G$ such that $\mathrm{Ad}(g)\mathfrak{h} = \mathfrak{h}'$. For $h \in H'^{\mathfrak{h}}$, put $h^g = ghg^{-1}$, then

$$(6.1) \qquad \Delta^{\mathfrak{h}}(h; P(\mathfrak{h}))^{-1}\tilde{\kappa}^{\mathfrak{h}}(h; P(\mathfrak{h})) = \Delta^{\mathfrak{h}'}(h^g; P'(\mathfrak{h}'))^{-1}\tilde{\kappa}^{\mathfrak{h}'}(h^g; P'(\mathfrak{h}')).$$

Let us analyse (1) more in detail. Let $\{X^*_{\pm\alpha}\}$ be another choice of root vectors for $\alpha\in F_0$ such that $X^*_{\pm\alpha}\in\mathfrak{g}$, $[X^*_\alpha, X^*_{-\alpha}]=H_\alpha$, and satisfying Condition 5.1. Using these root vectors, we define an automorphism $\nu^*_{F_0}$ analogously as ν_{F_0}, and put $\mathfrak{h}^{F_0,*}=\mathfrak{g}\cap\nu^*_{F_0}\mathfrak{h}_c$. We can choose $g^*_0\in G$ and $w^*_0\in W(\mathfrak{b}_c)$ in such a way that

$$\mathrm{Ad}(g^*_0)\mathfrak{b}=\mathfrak{h}^{F_0,*},\quad \mathrm{Ad}(g^*_0)w^*_0P(\mathfrak{b})=\nu^*_{F_0}P(\mathfrak{h}). \tag{6.2}$$

For a while, we denote the function $Z(h\,;E_i, \mathrm{Ad}(g^*_0)\Lambda, P_R(A))$ on A by $Z(h\,;E_i, \mathrm{Ad}(g^*_0)\Lambda, P_R(A)\,;\nu^*_{F_0})$ indicating explicitly the automorphism $\nu^*_{F_0}$ under use. Then it is sufficient to prove the following equality:

$$\mathrm{sgn}(w^*_0)\sum_{1\leq i\leq r}\varepsilon(E_i)Z(h\,;E_i, \mathrm{Ad}(g^*_0)\Lambda, P_R(A)\,;\nu^*_{F_0}) \tag{6.3}$$

$$=\mathrm{sgn}(w_0)\sum_{1\leq i\leq r}\varepsilon(E_i)Z(h\,;E_i, \mathrm{Ad}(g_0)\Lambda, P_R(A)\,;\nu_{F_0}).$$

Therefore it is also sufficient to prove that for $1\leqq i\leqq r$,

$$\mathrm{sgn}(w^*_0)Z(h\,;E_i, \mathrm{Ad}(g^*_0)\Lambda, P_R(A)\,;\nu^*_0) \tag{6.4}$$

$$=\mathrm{sgn}(w_0)Z(h\,;E_i, \mathrm{Ad}(g_0)\Lambda, P_R(A)\,;\nu_{F_0}).$$

This equality will be proved in more general setting in the proof of the assertion (2) (Lemma 6.2).

Now let us analyse (2). Let $X_{\pm\alpha}$ for $\alpha\in F_0$, ν_{F_0}, $g_0\in G$ and $w\in W(\mathfrak{b}_c)$ be those elements used to define $\tilde{\kappa}^{\mathfrak{h}}(h\,;P(\mathfrak{h}))$ by (5.6). Take a $g\in G$ such that $\mathrm{Ad}(g)\mathfrak{h}=\mathfrak{h}'$. We transform the above things by g as follows:

$$\begin{aligned}&\mathfrak{h}^g=\mathrm{Ad}(g)\mathfrak{h}\ (=\mathfrak{h}'),\ P^g(\mathfrak{h}^g)=\mathrm{Ad}(g)P(\mathfrak{h}),\ A^g=gAg^{-1},\ P^g_R(A^g)=\mathrm{Ad}(g)P_R(A),\\ &\alpha^g=\mathrm{Ad}(g)\alpha\in\Sigma_R(\mathfrak{h}^g),\ \ X^g_{\pm\alpha^g}=\mathrm{Ad}(g)X_{\pm\alpha}\ (\alpha\in F_0),\\ &E^g_i=\mathrm{Ad}(g)E_i,\ \ \nu^g_{F^g}=\mathrm{Ad}(g)\circ\nu_{F_0}\circ\mathrm{Ad}(g)^{-1}.\end{aligned} \tag{6.5}$$

Note that the canonical order in a simple component Σ of class II of $\Sigma_R(A)$ corresponding to $P=\Sigma\cap P_R(A)$ is transformed by g to that in $\Sigma^g=\mathrm{Ad}(g)\Sigma$ corresponding to $P^g=\mathrm{Ad}(g)P$. We see that (i) E^g_i's are standard with respect to $P^g_R(A^g)$ and $F^g_0=\mathrm{Ad}(g)F_0$ is associated to them, (ii) $\nu_{F^g_0}$ is defined canonically by $X^g_{\pm\alpha^g}(\alpha^g\in F^g_0)$, and (iii) $(\mathfrak{h}^g)^{F^g_0}\ (=\mathfrak{g}\cap\nu_{F^g_0}\mathfrak{h}^g_c)=\mathrm{Ad}(g)\mathfrak{h}^{F_0}$. Therefore we get

$$\mathrm{Ad}(gg_0)\mathfrak{b}=(\mathfrak{h}^g)^{F^g_0},\quad \mathrm{Ad}(gg_0)w_0P(\mathfrak{b})=\nu_{F^g_0}P^g(\mathfrak{h}^g). \tag{6.6}$$

Thus we see that the general case is reduced to the case where $\mathfrak{h}'=\mathfrak{h}$ and $P'(\mathfrak{h}')\neq P(\mathfrak{h})$ by the following lemma.

Lemma 6.1. *Assume that* (5.4) *holds. For a* $g\in G$, *put* $\mathfrak{h}^g=\mathrm{Ad}(g)\mathfrak{h}$, $P^g(\mathfrak{h}^g)=\mathrm{Ad}(g)P(\mathfrak{h})$ *etc. as in* (6.5), *then for* $h\in H^{\mathfrak{h}}$,

$$\Delta^{\mathfrak{h}}(h\,;P(\mathfrak{h}))=\Delta^{\mathfrak{h}^g}(h^g\,;P^g(\mathfrak{h}^g)),$$

and for $h\in A\subset H^{\mathfrak{h}}$,

$$Z(h\,;E_i, \mathrm{Ad}(g_0)\Lambda, P_R(A)\,;\nu_{F_0})=Z(h^g\,;E^g_i, \mathrm{Ad}(gg_0)\Lambda, P^g_R(A^g)\,;\nu_{F^g_0}). \tag{6.7}$$

Proof. The first equality is easy to prove. Let us prove the second one. Note that in the decomposition of A^g analogous as in (3.1), we get $(A^g)_U = gA_U g^{-1} = (A_U)^g$, $H_V^{\mathfrak{h}^g} = (H_V^{\mathfrak{h}})^g$. Then the second equality follows directly from the definitions (3.7) and (3.9)-(3.9′) of the functions Y and Z. Q.E.D.

Thus the proof of the assertion (2) is reduced to the case where $g=e$, hence $\mathfrak{h}'=\mathfrak{h}$. In this case, the equality (6.1) takes the following form:

$$\Delta^{\mathfrak{h}}(h\,;P(\mathfrak{h}))^{-1}\tilde{\kappa}^{\mathfrak{h}}(h\,;P(\mathfrak{h}))=\Delta^{\mathfrak{h}}(h\,;P'(\mathfrak{h}))^{-1}\tilde{\kappa}^{\mathfrak{h}}(h\,;P'(\mathfrak{h}))\,. \tag{6.8}$$

Note that there exist uniquely a $w\in W(\mathfrak{h}_c)$ and a $u_R\in W(\Sigma_R(A))$ such that

$$P'(\mathfrak{h})=wP(\mathfrak{h})\,,\quad P'_R(A)=u_R P_R(A)\,, \tag{6.9}$$

where $P'_R(A)=\Sigma_R(A)\cap P'(\mathfrak{h})$. Then $A^+(P'_R(A))=u_R A^+(P_R(A))$. Therefore taking into account (5.7), we see that the above equality is equivalent to the following: for $h\in A^+(P_R(A))$,

$$\operatorname{sgn}(w_0)\sum_{1\leq i\leq r}\varepsilon(E_i)Z(h\,;E_i,\operatorname{Ad}(g_0)A,\,P_R(A)\,;\nu_{F_0}) \tag{6.8′}$$

$$=\operatorname{sgn}(w)\operatorname{sgn}(u_R)\operatorname{sgn}(w'_0)\sum_{1\leq i\leq r}\varepsilon(E'_i)Z(u_R h\,;E'_i,\operatorname{Ad}(g'_0)A,\,P'_R(A)\,;\nu_{F'_0})$$

Here in the left hand side, $P_R(A)$, E_i, ν_{F_0} etc. are as in (5.4), (5.6), and in the right hand side, E'_i's are standard elements in $M^{or}(P'_R(A))$, $F'_0=F_0(E'_i)$ for any i, $\nu_{F'_0}$ is the canonical automorphism defined by a system of root vectors $X^*_{\pm\alpha}$ ($\alpha\in F'_0$) satisfying Condition 5.1 for E'_i's, and we assume the following, analogous as (5.4):

$$\operatorname{Ad}(g'_0)\mathfrak{b}=\mathfrak{h}^{F'_0}\ (\equiv\mathfrak{g}\cap\nu_{F'_0}\mathfrak{h}_c)\,,\quad \operatorname{Ad}(g'_0)w'_0P(\mathfrak{b})=\nu_{F'_0}P'(\mathfrak{h})\,. \tag{6.10}$$

We arrange the suffices for E_i, E'_i in such a way that E_i and E'_i are of the same type for every i. Then $\varepsilon(E_i)=\varepsilon(E'_i)$ by (1.39)-(1.39′), and since $P'_R(A)=u_R P_R(A)$, we have $E'_i=u_R E_i$. Recall that for every simple component Σ of class II (or of class I or III resp.) of $\Sigma_R(A)$, the corresponding part of E'_i is an ordered system (or a set resp.) belonging to $M^{or}(P)$ (or to $M(P)$ resp.) with $P=\Sigma\cap P_R(A)$. Therefore to get (6.8′), it is sufficient to prove that for $1\leqq i\leqq r$,

$$\operatorname{sgn}(w_0)Z(h\,;E_i,\operatorname{Ad}(g_0)A,\,P_R(A)\,;\nu_{F_0}) \tag{6.8″}$$

$$=\operatorname{sgn}(w)\operatorname{sgn}(u_R)\operatorname{sgn}(w'_0)Z(u_R h\,;u_R E_i,\operatorname{Ad}(g'_0)A,\,P'_R(A)\,;\nu'_{u_R F_0})\,.$$

We know that there exists a certain product g of g_α's in (2.18) for some $\alpha\in\Sigma_R(A)$ such that $\operatorname{Ad}(g)|\Sigma_R(A)=u_R$. Then $\operatorname{Ad}(g)\mathfrak{h}=\mathfrak{h}$, $h^g=u_R h$, $\operatorname{Ad}(g)E_i=u_R E_i$ and $\operatorname{Ad}(g)^{-1}P'(\mathfrak{h})=u_R^{-1}P'(\mathfrak{h})$. Thus, applying Lemma 6.1 to the right hand side of (6.8″), we can reduce (6.8″) to the case $u_R=1$. Therefore finally it is sufficient for us to prove the following.

Lemma 6.2. *Let A be a connected component of $H^{\mathfrak{h}}$, and let $P(\mathfrak{h})$, $P'(\mathfrak{h})$ be two sets of positive roots $\mathfrak{h}$ such that $P(\mathfrak{h})\cap\Sigma_R(A)=P'(\mathfrak{h})\cap\Sigma_R(A)$ $(=P_R(A))$. Let $\{E_i\}$ be all the standard elements in $M^{or}(P_R(A))$ and $F_0=F_0(E_i)$ be the strongly* *orthogonal system associated to them. Let* $\{X_{\pm\alpha}\}_{\alpha\in F_0}$ *and* $\{X^*_{\pm\alpha}\}_{\alpha\in F_0}$ *be two systems of root vectors both satisfying Condition* 5.1. *Define* ν_{F_0} *and* ν'_{F_0} *by using* $\{X_{\pm\alpha}\}$ *and* $\{X^*_{\pm\alpha}\}$ *respectively. Then for* $h\in A$,

$$\text{sgn}(w_0)Z(h\,;E_i,\,\text{Ad}(g_0)\Lambda,\,P_R(A)\,;\nu_{F_0}) = \text{sgn}(w)\text{sgn}(w_0')Z(h\,;E_i,\,\text{Ad}(g_0')\Lambda,\,P_R(A)\,;\nu'_{F_0}) \tag{6.11}$$

where $P'(\mathfrak{h})=wP(\mathfrak{h})$ *and*

$$\text{Ad}(g_0)w_0P(\mathfrak{b})=\nu_{F_0}P(\mathfrak{h}),\quad \text{Ad}(g_0')w_0'P(\mathfrak{b})=\nu'_{F_0}P'(\mathfrak{h}). \tag{6.12}$$

Note that the equality in the lemma contains (6.4) as a special case where $w=1$.

Proof. Put $F'=\bigcup_{1\leq i\leq r}(E_i)^*$. Then by assumption, there exist systems of root vectors $\{X_{\pm\delta}\}_{\delta\in F'}$ and $\{X^*_{\pm\delta}\}_{\delta\in F'}$ such that if $\delta,\,\delta'\in F'$ and $\delta'\pm\delta\in F'$, then for $a=0$ or 1,

$$[X_{\varepsilon\delta},\,X_{\delta'}]=\varepsilon^a X_{\delta'+\varepsilon\delta},\quad [X^*_{\varepsilon\delta},\,X^*_{\delta'}]=\varepsilon^a X^*_{\delta'+\varepsilon\delta}\ (\varepsilon=\pm1). \tag{*}$$

Let E_r be the unique system among E_i's for which the number of short roots in it is maximal for any simple component of class II, and put $F_r=(E_r)^*$. Since we utilize the canonical orders, we see that if $\delta,\,\delta'\in F'$ and $\delta'\pm\delta\in F'$, then $\delta,\,\delta'\in F_r$, and that every root in $F'-F_r$ is expressed as $\delta'+\delta$ or $\delta'-\delta$ with some $\delta,\,\delta'\in F_r$ (see § 1.5). We see in § 2.4 that there exists a $g\in A_V$ such that $\text{Ad}(g)X_{\pm\delta}=\varepsilon_\delta X^*_{\pm\alpha}$ $(\delta\in F_r)$ with $\varepsilon_\delta=\pm1$. Then it follows from (*) and what was said above that there exist $\varepsilon_\gamma=\pm1$ for $\gamma\in F'$ such that $\text{Ad}(g)X_{\pm\gamma}=\varepsilon_\gamma X^*_{\pm\gamma}$. Note that if $\alpha,\,\alpha'\in F_0$ and $2^{-1}(\alpha\pm\alpha')\in F'$, then by (*), $\varepsilon_\alpha=\varepsilon_{\alpha'}=\varepsilon_\delta\varepsilon_{\delta'}$ with $\delta'=2^{-1}(\alpha+\alpha')$, $\delta=2^{-1}(\alpha-\alpha')\in F_r$ and hence $\varepsilon_\alpha\varepsilon_{\alpha'}=1$. Since g commutes with every element in A and induces the identity mapping on $\mathfrak{h}$, we see by Lemma 6.1 that we may assume from the beginning that $X_{\pm\alpha}=\varepsilon_\alpha X^*_{\pm\alpha}$ for $\alpha\in F_0$. Now put $n_\alpha=(1-\varepsilon_\alpha)/2$ and let w' be the product of $s_\alpha^{n_\alpha}$ over $\alpha\in F_0$. Then $w'\in J(F_0)$, the group generated by the commuting family $\{s_\alpha\,;\,\alpha\in F_0\}$, and

$$\nu'_{F_0}|\mathfrak{h}_c=\nu_{F_0}\circ w'|\mathfrak{h}_c. \tag{6.13}$$

This means in particular that $\mathfrak{g}\cap\nu_{F_0}\mathfrak{h}_c=\mathfrak{g}\cap\nu'_{F_0}\mathfrak{h}_c=\mathfrak{h}^{F_0}$ (put). Then $\text{Ad}(g_0)\mathfrak{b}=\text{Ad}(g_0')\mathfrak{b}=\mathfrak{h}^{F_0}$ by assumption. Hence $s'=\text{Ad}(g_0')\text{Ad}(g_0)^{-1}|\mathfrak{h}^{F_0}$ belongs to $W_G(\mathfrak{h}^{F_0})$. Then it follows from $P'(\mathfrak{h})=wP(\mathfrak{h})$ and (6.12) that

$$\nu_{F_0}^{-1}\circ\text{Ad}(g_0)\circ w_0|\mathfrak{b}_c=w^{-1}\circ(\nu'_{F_0})^{-1}\circ\text{Ad}(g_0')\circ w_0'|\mathfrak{b}_c.$$

Therefore we get by (6.13)

$$\nu_{F_0}^{-1}\circ\text{Ad}(g_0)\circ w_0|\mathfrak{b}_c=w^{-1}w'^{-1}\circ\nu_{F_0}^{-1}\circ s'\circ\text{Ad}(g_0)\circ w_0'|\mathfrak{b}_c,$$

and hence

$$\text{sgn}(w_0)=\text{sgn}(w)\text{sgn}(w')\text{sgn}(s')\text{sgn}(w_0'). \tag{6.14}$$

Then inserting (6.14) into (6.11), it takes the following form under the condition that $X^*_{\pm\alpha}=\varepsilon_\alpha X_{\pm\alpha}$ $(\alpha\in F_0)$:

$$(6.11') \qquad Z(h\,;\,E_i,\ \mathrm{Ad}(g_0)\Lambda,\ P_R(A)\,;\,\nu_{F_0})$$
$$=\mathrm{sgn}(w')\mathrm{sgn}(s')Z(h\,;\,E_i,\ \mathrm{Ad}(g_0')\Lambda,\ P_R(A)\,;\,\nu'_{F_0}),$$

where $w'\in J(F_0)$ is given by (6.13).

We see from (3.9)-(3.9′) that to prove this equality, it is sufficient to obtain

$$(6.15) \qquad Y(h\,;\,E_i,\ u,\ ss'\,\mathrm{Ad}(g_0)\Lambda\,;\,\nu_{F_0})=\mathrm{sgn}(w')Y(h\,;\,E_i,\ u,\ s\cdot\mathrm{Ad}(g_0')\Lambda\,;\,\nu'_{F_0})$$

for any $s\in W_G(\mathfrak{h}^{F_0})$ and $u\in W(E_i\,;\,P_R(A))$, where $W(E_i\,;\,P_R(A))$ is given in Definition 1.2. The function $Y(h\,;\,E,\ u,\ \Lambda)$ in (3.7) is denoted here by $Y(h\,;\,E,\ u,\ \Lambda\,;\,\nu_{F_0})$ indicating explicitly the automorphism ν_{F_0} in use. Put $\Lambda_1=ss'\,\mathrm{Ad}(g_0)\Lambda$, then $s\cdot\mathrm{Ad}(g_0')\Lambda=\Lambda_1$. Therefore the above equality becomes

$$(6.15') \qquad Y(h\,;\,E_i,\ u,\ \Lambda_1\,;\,\nu_{F_0})=\mathrm{sgn}(w')Y(h\,;\,E_i,\ u,\ \Lambda_1\,;\,\nu'_{F_0}),$$

where $\nu'_{F_0}=\nu_{F_0}\circ w'$ on $\mathfrak{h}_c$ with $w'\in J(F_0)$.

To prove this equality, first note that $\mathrm{sgn}(w')=\prod_{\alpha\in F_0}\varepsilon_\alpha$. We obtain just as in the proof of Lemma 3.1 that

$$\mathrm{sgn}\{\prod_{\gamma\in P(E_i)}(\Lambda_1,\ \nu_{F_0}\gamma)\}=(\prod_{\alpha\in F_0\cap F_i}\varepsilon_\alpha)\mathrm{sgn}\{\prod_{\gamma\in P(E_i)}(\Lambda_1,\ \nu'_{F_0}\gamma)\},$$

where $F_i=(E_i)^*$. By Condition 5.1, we have $\varepsilon_\alpha\varepsilon_{\alpha'}=1$ if $2^{-1}(\alpha\pm\alpha')\in F'$, whence $\prod_{\alpha\in F_0\cap F_i}\varepsilon_\alpha=\prod_{\alpha\in F_0}\varepsilon_\alpha=\mathrm{sgn}(w')$. Thus the proof of Lemma 6.2 is now complete.

Q. E. D.

The assertions (1) and (2) at the beginning of this subsection are now completely proved.

6.2. Now we fix a complete system Ω (resp. Ω^0) of representatives of $\mathrm{Car}(G)$ (resp. $\mathrm{Car}^0(G)$) in such a way that every $A\in\Omega^0$ is contained in $H^{\mathfrak{h}}$ for some $\mathfrak{h}\in\Omega$. We put for any $A\in\Omega^0$,

$$(6.16) \qquad \zeta(h\,;\,A^+(P)\,;\,P(\mathfrak{h}))=\tilde{\kappa}^{\mathfrak{h}}(h\,;\,P(\mathfrak{h})) \qquad (h\in A^+(P)).$$

To prove Theorem 2, it is sufficient by Theorems A and B to prove the following: for the system of functions $\zeta(\cdot\,;\,A^+(P))=\zeta(\cdot\,;\,A^+(P)\,;\,P(\mathfrak{h}))$, there hold the conditions (a), (b), (c) and (d) in Theorems A and B. In this section, we prove (a), (b) and (d). The "boundary condition" (c) will be studied in §§ 7-9.

Let us first prove the condition (a).

Lemma 6.3. *Let A be an arbitrary connected component of $H^{\mathfrak{h}}$. Then for any $\omega\in W_G(A^+(P))$,*

$$(6.17) \qquad \tilde{\kappa}^{\mathfrak{h}}(\omega h\,;\,P(\mathfrak{h}))=\varepsilon(\omega)\tilde{\kappa}^{\mathfrak{h}}(h\,;\,P(\mathfrak{h})) \qquad (h\in A^+(P)),$$

where $\varepsilon(\omega)=\mathrm{sgn}(\bar{\omega})$ is as in § 4.1, *and $A^+(P)$ with $P=P_R(A)=\Sigma_R(A)\cap P(\mathfrak{h})$ is given in* (5.5). *In particular, the system of functions $\zeta(\cdot\,;\,A^+(P))=\zeta(\cdot\,;\,A^+(P)\,;\,P(\mathfrak{h}))$*

given above satisfies the condition (a).

Proof. The relation (6.17) follows from Lemma 3.6. This is also contained in the assertion (2) in § 6.1. Q. E. D.

For the condition (d), we have the following.

Lemma 6.4. *For any Cartan subgroup* $H^{\mathfrak{h}}$, *the function* $\tilde{\kappa}^{\mathfrak{h}}$ *given by* (5.6)-(5.7) *is uniformaly bounded on it. This means that the system of functions* $\zeta(\cdot\,;A^+(P))=\zeta(\cdot\,;A^+(P)\,;P(\mathfrak{h}))$ *satisfies the condition* (d).

Proof. The boundedness of the functioned $\tilde{\kappa}^{\mathfrak{h}}$ follows from Lemma 3.5. Q. E. D.

Next we study the condition (b). For any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, take an inner automorphism ν of $\mathfrak{g}_c$ such that $\nu\mathfrak{b}_c=\mathfrak{h}_c$ and let $\chi_\Lambda^{\mathfrak{h}}$ be the homomorphism of $I(\mathfrak{h}_c)$ into $\boldsymbol{C}$ given canonically by $\nu\Lambda=\Lambda\circ\nu^{-1}|\mathfrak{h}_c$. Then $\chi_\Lambda^{\mathfrak{h}}$'s are mutually consistent in the sense in § 4.2.

Lemma 6.5. *For any Cartan subgroup* $H^{\mathfrak{h}}$ *of* G, *the function* $\tilde{\kappa}^{\mathfrak{h}}$ *satisfies on* $H'^{\mathfrak{h}}(R)$ *the following differential equations*:

$$D\tilde{\kappa}^{\mathfrak{h}}(h\,;P(\mathfrak{h}))=\chi_\Lambda^{\mathfrak{h}}(D)\tilde{\kappa}^{\mathfrak{h}}(h\,;P(\mathfrak{h})) \qquad (D\in I(\mathfrak{h}_c))\,. \tag{6.18}$$

In particular, the system of functions $\zeta(\cdot\,;A^+(P))$ *satisfies the condition* (b).

Proof. Let A be a connected component of $H^{\mathfrak{h}}$. Then the function $\tilde{\kappa}^{\mathfrak{h}}|A^+(P)$ in (5.6) is a linear combination of the functions $Z(\cdot\,;E,\Lambda',P_R(A))$, and the latters are linear combinations of the functions $Y(\cdot\,;E,u,s\Lambda')$ in (3.7), where $u\in W(\Sigma_R(A))$, $s\in W_G(\mathfrak{h}^{F_0})$ with $F_0=F_0(E)$ and $\Lambda'=\mathrm{Ad}(g_0)\Lambda$ with $g_0\in G$ such that $\mathrm{Ad}(g_0)\mathfrak{h}^{F_0}=\mathfrak{b}$. Therefore it is sufficient to prove that for any $D\in I(\mathfrak{h}_c)$,

$$DY(h\,;E,u,s\Lambda')=\chi_\Lambda^{\mathfrak{h}}(D)Y(h\,;E,u,s\Lambda')\,. \tag{6.19}$$

Since $\nu=\nu_{F_0}^{-1}\mathrm{Ad}(g_0)^{-1}$ is an inner automorphism of $\mathfrak{g}_c$ such that $\nu\mathfrak{b}_c=\mathfrak{h}_c$, the linear form $\lambda=\nu_{F_0}^{-1}s\Lambda'$ on $\mathfrak{h}_c$ induces the homomorphism $\chi_\Lambda^{\mathfrak{h}}$ of $I(\mathfrak{h}_c)$. Let us rewrite $Y(h\,;E,u,s\Lambda')$ as follows. Express $X\in\mathfrak{h}_c$ as $X=X_U+X_V$ with $X_U\in(\mathfrak{h}_U)_c$, $X_V\in(\mathfrak{h}_V)_c$, and define a linear form λ' on $\mathfrak{h}_c$ by

$$\lambda'(X)=(s\Lambda')(X_U+p_{F_0}(u^{-1}X_V))-\sum_{\alpha\in F}\alpha(u^{-1}X_V)|(s\Lambda',\nu_{F_0}\alpha)|/|\alpha|^2\,,$$

where $F=E^*$, $F_0=F_0(E)$ and the projection $p_{F_0}=p_F$ is extended from $\mathfrak{h}$ to $\mathfrak{h}_c$ naturally. Then, since $u|\mathfrak{h}_U=1$, we get

$$\lambda'=u'\lambda \quad \text{with} \quad u'=u(\prod_{\alpha\in F}s_\alpha^{n_\alpha})\in W(\Sigma_R(A))\,,$$

where $n_\alpha=1$ or 0 according as $(s\Lambda',\nu_{F_0}\alpha)>0$ or <0. Fix $h_0\in A$ and let $h=h_0\exp X$ with $X\in\mathfrak{h}$. Then $Y(h\,;E,u,s\Lambda')$, as a function in X, is a constant multiple of $\xi_{\lambda'}(\exp X)=\exp\lambda'(X)=\exp(u'\lambda)(X)$. Since λ and $u'\lambda$ induce the same homomorphism $\chi_{\Lambda}^{\mathfrak{h}}$ of $I(\mathfrak{h}_c)$ into $\boldsymbol{C}$, we get

$$D\xi_{\lambda'}(\exp X)=\chi_{\Lambda}^{\mathfrak{h}}(D)\xi_{\lambda'}(\exp X),$$

and hence (6.19). This proves the lemma. Q. E. D.

§7. Lemmas to prove the boundary condition (c)

We have proved that the system of functions $\zeta(\cdot\,; A^+(P))$ satisfies the conditions (a), (b) and (d) in Theorems A and B. In order to prove the boundary condition (c), we prepare here some lemmas.

7.1. Note that the system of functions $\tilde{\kappa}^{\mathfrak{h}}(\cdot\,; P(\mathfrak{h}))$ in Theorem 2 satisfies the assertion (2) in § 6.1. Then we see that we can choose Ω, Ω^0, $P(\mathfrak{h})$ for $\mathfrak{h}\in\Omega$, and $P_R(A)$ for $A\in\Omega^0$ arbitrarily. Therefore we will choose if necessary these things as convenient as possible to prove the condition (c). Note that on the simple components of class II of $\Sigma_R(A)$, we consider the canonical order corresponding to $P_R(A)$.

Take an $A\in\Omega^0$ and an $\alpha\in\Pi_R(A)$ with $\mathfrak{h}\in\Omega$ such that $A\subset H^{\mathfrak{h}}$, where $\Pi_R(A)$ denotes the set of simple roots in $\Sigma_R(A)$ with respect to $P_R(A)=\Sigma_R(A)\cap P(\mathfrak{h})$. Put $\mathfrak{h}'=\mathfrak{h}^{\alpha}\equiv\mathfrak{g}\cap\nu_{\alpha}\mathfrak{h}_c$ and let A' be the connected component of $H^{\mathfrak{h}'}$ containing $A_{\alpha}=\{h\in A\,; \xi_{\alpha}(h)=1\}$. Then as in Lemma 2.1,

$$\Sigma_R(\mathfrak{h}')=\{\nu_{\alpha}\gamma\,; \gamma\in\Sigma_R(\mathfrak{h}),\ \gamma\perp\alpha\},\tag{7.1}$$

$$\Sigma_R(A')=\{\nu_{\alpha}\gamma\,; \gamma\in\Sigma_R(A),\ \gamma\perp\alpha\}.\tag{7.2}$$

We choose the set $P(\mathfrak{h}')$ of positive roots of $\mathfrak{h}'$ in such a way that $P(\mathfrak{h}')=\nu_{\alpha}P(\mathfrak{h})$. Then the function $\tilde{\kappa}^{\mathfrak{h}}(h\,; P(\mathfrak{h}))|A^+(P)$ is given by (5.6), and the one $\tilde{\kappa}^{\mathfrak{h}'}(h'\,; P(\mathfrak{h}'))|A'^+(P')$ is given by the analogous formula. Put $\beta=\nu_{\alpha}\alpha$ and

$$M_{\alpha}=\{h\in\mathrm{Cl}(A^+(P))\cap A_{\alpha}\,; \xi_{\gamma}(h)\neq 1 \text{ for } \gamma\in\Sigma_R(A),\ \neq\pm\alpha\}.\tag{7.3}$$

Then the condition (c) is equivalent to the following.

Lemma 7.1. *Let the notations be as above. Then for* $h_0\in M_{\alpha}$,

$$H_{\alpha}\tilde{\kappa}^{\mathfrak{h}}(h_0\,; P(\mathfrak{h}))=H_{\beta}\tilde{\kappa}^{\mathfrak{h}'}(h_0\,; P(\mathfrak{h}')),\tag{7.4}$$

where

$$H_{\alpha}\tilde{\kappa}^{\mathfrak{h}}(h_0\,; P(\mathfrak{h}))=\lim_{t\to+0}\frac{d}{dt}\tilde{\kappa}^{\mathfrak{h}}(h_0\exp(tH_{\alpha})\,; P(\mathfrak{h})),$$

$$H_{\beta}\tilde{\kappa}^{\mathfrak{h}'}(h_0\,; P(\mathfrak{h}'))=\lim_{t\to 0}\frac{1}{\sqrt{-1}}\frac{d}{dt}\tilde{\kappa}^{\mathfrak{h}'}(h_0\exp(t\sqrt{-1}H_{\beta})\,; P(\mathfrak{h}')).$$

Note that for $h_0\in M_{\alpha}$, $h_0\exp(tH_{\alpha})$ belongs to $A^+(P)$ if $t>0$ is sufficiently small, and $h_0\exp(t\sqrt{-1}H_{\beta})$ belongs to $A'(P')$ for real t.

To prove Lemma 7.1, we separate the cases according to the class of the simple component $\Sigma(\alpha)$ of $\Sigma_R(A)$ containing α. The case of class I or III will be studied in § 8 and that of class II (type B_n, C_n and F_4) in § 9.

7.2. Hereafter in this section, we denote by Λ or Λ' a regular element in $\mathfrak{d}_D^*$ with $\mathfrak{d}=\mathfrak{h}^{F_0}$, $D=H^{\mathfrak{d}}$, $F_0=F_0(E)$ for $E\in M^{or}(P_R(A))$. To prove Lemma 7.1 for the case of class I or III, we apply the following elementary lemma.

Lemma 7.2. *Let $E\in M^{or}(P_R(A))$ and $\alpha\in\Sigma_R(A)$. For $u\in W(\Sigma_R(A))$, put*

$$J(h)=Y(h\,;E,u,\Lambda')+Y(h\,;E,s_\alpha u,\Lambda') \qquad (h\in A).$$

Then the function J is even under the transformation $h\rightarrow s_\alpha h$ on A, and hence $H_\alpha J(h)=0$ on $A_\alpha=\{h\in A\,;\,\xi_\alpha(h)=1\}$.

Proof. By (3.8), we get

$$Y(h\,;E,u,\Lambda')=Y(u^{-1}h\,;E,1,\Lambda'). \tag{7.5}$$

The assertion follows from this immediately. Q. E. D.

7.3. For the case of class II, we apply also the following lemma.

Lemma 7.3. *Let $E\in M^{or}(P_R(A))$ and $\alpha\in\Sigma_R(A)$. For $u\in W(\Sigma_R(A))$, put*

$$J(h)=\sum_s \mathrm{sgn}(s)Y(h\,;E,u,s\Lambda) \qquad (h\in A),$$

where s runs over $W_G(\mathfrak{h}^{F_0})$ with $F_0=F_0(E)$. Assume that $w=u^{-1}s_\alpha u$ satisfies that $wF=F$, $wF_0\subset F_0\cup -F_0$ with $F=E^*$ and that for any $s\in W_G(\mathfrak{h}^{F_0})$,

$$\mathrm{sgn}_{P(E)}(s\Lambda)\mathrm{sgn}_{w^{-1}P(E)}(s\Lambda)=-1, \tag{7.6}$$

where $\mathrm{sgn}_{P(E)}(\Lambda)$ is defined in (3.6). Then the function J is even under $h\rightarrow s_\alpha h$ on A, and hence $H_\alpha J(h)=0$ on A_α.

Proof. By assumption, w satisfies the condition in Lemma 3.1. Therefore we get the equality (3.16) in its corollary. Note that in this case, $\mathrm{sgn}(w)=-1$, $uw=s_\alpha u$, then using (7.6), we get from (3.16),

$$\sum_s \mathrm{sgn}(s)Y(h\,;E,s_\alpha u,s\Lambda)=\sum_s \mathrm{sgn}(s)Y(h\,;E,u,s\Lambda).$$

This means by (7.5) that $J(s_\alpha h)=J(h)$. Hence the assertion of the lemma.

Q. E. D.

When $\Sigma(\alpha)$ is of type B_n or C_n, we also apply the following lemma.

Lemma 7.4. *Let $E\in M^{or}(P_R(A))$ and $\alpha\in\Sigma_R(A)$. For $u\in W(\Sigma_R(A))$, assume that there exist $u'\in W(E\,;P_R(A))$ and $w\in W(\Sigma_R(A))$ such that $s_\alpha u=u'w$ and that the following conditions hold: $wF=F$, $wF_0\subset F_0\cup -F_0$ with $F=E^*$, $F_0=F_0(E)$, and for any $s\in W_G(\mathfrak{h}^{F_0})$,*

$$\mathrm{sgn}_{P(E)}(s\Lambda)\mathrm{sgn}_{w^{-1}P(E)}(s\Lambda)=\mathrm{sgn}(w). \tag{7.7}$$

Put for $h\in A$,

$$J(h)=\sum_{s}\operatorname{sgn}(s)\{Y(h\,;E,u,sA)+Y(h\,;E,u',sA)\},$$

where s runs over $W(\mathfrak{h}^{F_0})$. Then $J(s_\alpha h)=J(h)$ for $h\in A$ and hence $H_\alpha J(h)=0$ for $h\in A_\alpha$.

Proof. By the assumption on w, we can apply Lemma 3.1 to it and then get the equality (3.16) in its corollary. By (7.7), this equality turns out to

$$\sum_{s}\operatorname{sgn}(s)Y(h\,;E,u'w,sA)=\sum_{s}\operatorname{sgn}(s)Y(h\,;E,u',sA).$$

Since $u'w=s_\alpha u$, this gives us $J(s_\alpha h)=J(h)$ by (7.5), and hence $H_\alpha J(h)=0$ on A_α.
Q. E. D.

7.4. *Case of type B_2.* An essential part of our proof of the condition (c) for the class II is reduced to the case of type $B_2\cong C_2$. This has been already seen in [5(e)] for $Sp(n,\boldsymbol{R})$ (cf. also [5(b)]). Here we give three lemmas essentially for the type B_2.

Let A be a connected component of a Cartan subgroup $H^{\mathfrak{h}}$ of G, and F a strongly orthogonal system in $P_R(A)$. Assume that there exist in F two roots α,α' such that $\gamma=2^{-1}(\alpha'-\alpha)$, $\gamma'=2^{-1}(\alpha'+\alpha)$ are again roots in $P_R(A)$. Then, as is seen in § 1.2, the simple component Σ of $\Sigma_R(A)$ containing α,α' is of class II, and $\Sigma_{\{\alpha,\alpha'\}}=\{\pm\alpha,\pm\alpha',\pm\gamma,\pm\gamma'\}$ is of type B_2. Moreover all roots in $F-\{\alpha,\alpha'\}$, except at most one short root in $F\cap\Sigma$, are strongly orthogonal to γ and γ' (cf. § 1.5 and Lemma 1.8). For a system of root vectors $X_{\pm\delta}\in\mathfrak{g}$ ($\delta\in F$) such that $[X_\delta,X_{-\delta}]=H_\delta$, we define ν_F as in § 2.2. Choosing root vectors $X_{\pm\eta}$ ($\eta=\gamma,\gamma'$), we define ν_γ and put $F'=\nu_\gamma(F-\{\alpha,\alpha'\})\cup\{\nu_\gamma\gamma'\}$. Assume that any root in $F-\{\alpha,\alpha'\}$ is strongly orthogonal to γ, then we can give the root vectors for $F'\cup-F'$ as follows: $\nu_\gamma X_{\pm\delta}=X_{\pm\delta}$ for $\delta\in F-\{\alpha,\alpha'\}$ because $[X_{\pm\gamma},X_{\pm\delta}]=0$, and $\sqrt{-1}\,\varepsilon\nu_\gamma X_{\varepsilon\gamma'}=2^{-1}\varepsilon\cdot\operatorname{ad}(X'_\gamma+X'_{-\gamma})X_{\varepsilon\gamma'}$ for $\varepsilon\nu_\gamma\gamma'$ ($\varepsilon=\pm1$) because of Lemma 2.2.

Lemma 7.5. *Let F, α, α', γ and γ' be as above. Assume that any root in $F-\{\alpha,\alpha'\}$ is strongly orthogonal to γ,γ'. Suppose that X_α, $X_{\alpha'}$ are so chosen that for some root vectors $Y_{\pm\gamma}$, $Y_{\gamma'}$ for $\pm\gamma$ and γ', $X_{\gamma'\pm\gamma}=[Y_{\pm\gamma},Y_{\gamma'}]$ ($\alpha=\gamma'-\gamma$, $\alpha'=\gamma'+\gamma$). Then there exist root vectors $X_{\pm\delta}$ ($\delta=\gamma,\gamma'$) and a $g\in G(\alpha,\alpha')$ such that $\operatorname{Ad}(g)\nu_F|\mathfrak{h}_c=\nu_{F'}\nu_\gamma|\mathfrak{h}_c$, where $G(\alpha,\alpha')$ denotes the analytic subgroup corresponding to the subalgebra generated by $X_{\pm\delta}\in\mathfrak{g}$ ($\delta\in\Sigma_{\{\alpha,\alpha'\}}$) which is locally isomorphic to $Sp(2,\boldsymbol{R})$.*

Proof. It follows from $g\in G(\alpha,\alpha')$ that $\operatorname{Ad}(g)X_{\pm\delta}=X_{\pm\delta}$ for $\delta\in F-\{\alpha,\alpha'\}$, whence $\operatorname{Ad}(g)$ and ν_δ commute with each other. Therefore it is sufficient for us to treat the case where $F=\{\alpha,\alpha'\}$. Moreover it is sufficient to prove the assertion for a special choice of root vectors $X_{\pm\alpha}$, $X_{\pm\alpha'}$. In fact, let $X^*_{\pm\alpha}$, $X^*_{\pm\alpha'}$ be other root vectors such that $X^*_{\gamma'\pm\gamma}=[Y^*_{\pm\gamma},Y^*_{\gamma'}]$ for some root vectors $Y^*_{\pm\gamma}$,

$Y^*_{\gamma'}$ for $\pm\gamma$, γ'. Then there exists $h=\exp(aH_\alpha+bH_{\alpha'})$ with $a, b\in \boldsymbol{R}$ such that $\mathrm{Ad}(h)Y^*_{\pm\delta}=\varepsilon_\delta Y_{\pm\delta}$ with $\varepsilon_\delta=\pm 1$ for $\delta=\gamma, \gamma'$. Therefore $\mathrm{Ad}(h)X^*_{\pm\gamma',\pm\gamma}=\varepsilon X_{\pm\gamma',\pm\gamma}$ with $\varepsilon=\varepsilon_\gamma\varepsilon_{\gamma'}$. Denote by ν^*_F the transformation analogous as ν_F, defined by $X^*_{\pm\alpha}$, $X^*_{\pm\alpha'}$. Then we get $\mathrm{Ad}(h)\nu^*_F|\mathfrak{h}_c=\nu_F(s_{\gamma'}s_\gamma)^p|\mathfrak{h}_c$ with $p=(1-\varepsilon)/2$. Assume that there holds $\mathrm{Ad}(g)\nu_F|\mathfrak{h}_c=\nu_{F'}\nu_\gamma|\mathfrak{h}_c$, then we get $\mathrm{Ad}(gh)\nu^*_F|\mathfrak{h}_c=\nu_{F'}\nu_\gamma \mathrm{Ad}(g_1)|\mathfrak{h}_c$, where $g_1=(g_{\gamma'}g_\gamma)^p$ with g_γ in (2.18). Take the root vectors $\mathrm{Ad}(g_1)^{-1}X_{\pm\gamma}$, $\mathrm{Ad}(g_1)^{-1}X_{\pm\gamma'}$ for $\pm\mathrm{Ad}(g_1)^{-1}\delta=\pm(-1)^p\delta$ $(\delta=\gamma, \gamma')$. Then we see from (2.6) that $\nu^*_{F'}=\mathrm{Ad}(g_1)^{-1}\nu_{F'}\mathrm{Ad}(g_1)$, $\nu^*_\gamma=\mathrm{Ad}(g_1)^{-1}\nu_\gamma\mathrm{Ad}(g_1)$ are defined by means of these root vectors. Thus we get $\mathrm{Ad}(g^*)\nu^*_F|\mathfrak{h}_c=\nu^*_{F'}\nu^*_\gamma|\mathfrak{h}_c$ with $g^*=g_1^{-1}gh$.

Now let us prove $\mathrm{Ad}(g)\nu_F|\mathfrak{h}_c=\nu_{F'}\nu_\gamma|\mathfrak{h}_c$ holds for a special choice of $X_{\pm\alpha}$, $X_{\pm\alpha'}$. Since $G(\alpha, \alpha')$ is locally isomorphic to $Sp(2, \boldsymbol{R})$, it is sufficient to prove this for $Sp(2, \boldsymbol{R})$, the group of real matrices x of degree 4 satisfying

$$xJ^tx=J, \quad \text{where } J=\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \quad \text{with} \quad E=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Denote by $d(a_1, a_2, a_3, a_4)$ the diagonal matrice with diagonal elements a_1, a_2, a_3, a_4. We take as $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{sp}(2, \boldsymbol{R})$ consisting of elements of the form $X=d(-t_1, -t_2, t_1, t_2)$ $(t_1, t_2\in \boldsymbol{R})$. The roots α and α' are given by $\alpha(X)=2t_1$, $\alpha'(X)=2t_2$. We take the root vectors as follows $(X'_\delta=\sqrt{2}\,|\delta|^{-1}X_\delta)$:

$$X'_\alpha=\begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}, \quad X'_{\alpha'}=\begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix}, \quad X'_\gamma=\begin{pmatrix} -{}^tV & 0 \\ 0 & V \end{pmatrix}, \quad X'_{\gamma'}=\begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix},$$

where

$$T=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad U=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad V=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad W=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We put $X'_{-\delta}={}^t(X'_\delta)$. This choice of root vectors satisfies the condition in the lemma, and by a simple calculation, we get $\mathrm{Ad}(g)\nu_F=\nu_{F'}\nu_\gamma$ with $g=\frac{1}{\sqrt{2}}\begin{pmatrix} E & -W \\ W & E \end{pmatrix}$ in $Sp(2, \boldsymbol{R})$. Now the proof of the lemma is complete. Q.E.D.

To reduce the proof of the condition (c) in the general case to the case of type B_2, we need also the following lemma.

Lemma 7.6. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Assume that there exist two singular imaginary roots δ, δ' in $\Sigma(\mathfrak{h})$ such that $\eta=2^{-1}(\delta'-\delta)$ and $\eta'=2^{-1}(\delta'+\delta)$ are roots in $\Sigma(\mathfrak{h})$. Let $\mathfrak{g}_{2,c}$ be the subalgebra of $\mathfrak{g}_c$ generated by $X_{\pm\gamma}$ $(\gamma=\delta, \delta', \eta, \eta')$, then $\mathfrak{g}_2=\mathfrak{g}\cap\mathfrak{g}_{2,c}$ is isomorphic to $\mathfrak{sp}(2, \boldsymbol{R})$. The group $W_G(\mathfrak{h})$ contain an element σ such that*

$$\sigma\delta=\varepsilon\delta', \quad \sigma\delta'=\varepsilon\delta \qquad (\varepsilon=1 \text{ or } -1)$$

and $\sigma X=X$ for any $X\in\mathfrak{h}$ such that $\delta(X)=\delta'(X)=0$.

Proof. Since δ, δ' are singular imaginary, we see that one of η, η' is compact and the other is singular imaginary, the reflexion corresponding to this compact root belongs to $W_G(\mathfrak{h})$, and is an element looked for. Q. E. D.

Note 7.1. Let λ and λ' be two imaginary roots of a Cartan subalgebra $\mathfrak{h}$. Assume that $\lambda+\lambda'$ is also a root of $\mathfrak{h}$. Then $\lambda+\lambda'$ is singular if and only if one of λ and λ' is compact and the other is singular.

Now let G be locally isomorphic to $Sp(2, \boldsymbol{R})$, and assume for $A\subset H^{\mathfrak{h}}$ that $\Sigma_R(A)$ is of type B_2. Let

$$P_R(A)=\{\alpha,\ \alpha',\ \gamma,\ \gamma'\}. \tag{7.8}$$

where $\gamma=2^{-1}(\alpha'-\alpha)$, $\gamma'=2^{-1}(\alpha'+\alpha)$. Then $\Pi_R(A)=\{\alpha, \gamma\}$, $M(P_R(A))=\{F, F'\}$, where

$$F=\{\alpha,\ \alpha'\},\quad F'=\{\gamma,\ \gamma'\}. \tag{7.9}$$

In this case, $M^{or}(P_R(A))=\tilde{M}(P_R(A))$ and

$$W(\tilde{F}\,;\ P_R(A))=W(\tilde{F}'\,;\ P_R(A))=\{1\}, \tag{7.10}$$

$$P(\tilde{F})=\{\alpha,\ \alpha',\ \gamma,\ \gamma'\},\quad P(\tilde{F}')=\{\gamma,\ \gamma'\}\ ;\quad F_0(\tilde{F})=F_0(\tilde{F}')=F. \tag{7.11}$$

Let $\mathfrak{h}'=\mathfrak{h}^{\alpha}$ and $A'\subset H^{\mathfrak{h}'}$ be as in §7.1. Then $\Sigma_R(A')=\{\pm\nu_\alpha\alpha'\}$ and $H_{\nu_\alpha\alpha'}=\nu_\alpha H_{\alpha'}=H_{\alpha'}$. Put $P(\mathfrak{h}')=\nu_\alpha P(\mathfrak{h})$, then $P_R(A')=\{\nu_\alpha\alpha'\}$ and $M(P_R(A'))=F''$, where $F''=\{\nu_\alpha\alpha'\}$. Note that $\varepsilon(\tilde{F})=-1$, $\varepsilon(\tilde{F}')=1$, $\varepsilon(F'')=-1$.

Put $\mathfrak{b}=\mathfrak{h}^F$, then $W_G(\mathfrak{b})$ is generated by the reflexion corresponding to the compact roots of $\mathfrak{b}$. Put $\delta=\nu_F\alpha$, $\delta'=\nu_F\alpha'$, then they are singular imaginary roots of $\mathfrak{b}$. Since exactly one of $\nu_F\gamma=2^{-1}(\delta'-\delta)$ and $\nu_F\gamma'=2^{-1}(\delta'+\delta)$ is compact, there exists only one non-trivial element σ in $W_G(\mathfrak{b})$ and

$$\sigma\delta=\varepsilon\delta',\quad \sigma\delta'=\varepsilon\delta\qquad (\varepsilon=1 \text{ or } -1). \tag{7.12}$$

Put $P(\mathfrak{b})=\nu_F P(\mathfrak{h})$ and $P(\mathfrak{h}')=\nu_\alpha P(\mathfrak{h})$. Then the formula in §5.2 gives us

$$\tilde{\kappa}^{\mathfrak{h}}(h\,;\ P(\mathfrak{h}))=-Z_0(h)+Z_1(h)\qquad (h\in A^+(P)),$$

$$\tilde{\kappa}^{\mathfrak{h}'}(h'\,;\ P(\mathfrak{h}'))=-Z_2(h')\qquad (h'\in A'^+(P)),$$

where for $\Lambda\in\mathfrak{b}_B^{*\prime}$ with $B=H^{\mathfrak{b}}$,

$$Z_0(h)=\sum_s \operatorname{sgn}(s)Y(h\,;\ \tilde{F},\ 1,\ s\Lambda)\qquad (h\in A),$$

$$Z_1(h)=\sum_s \operatorname{sgn}(s)Y(h\,;\ \tilde{F}',\ 1,\ s\Lambda)\qquad (h\in A),$$

$$Z_2(h')=\sum_s \operatorname{sgn}(s)Y(h\,;\ F'',\ 1,\ s\Lambda)\qquad (h'\in A'),$$

where s runs over $\{1, \sigma\}$. Therefore the following lemma asserts that the condition (c) is satisfied in the case of type B_2.

Lemma 7.7 (cf. [5(e), Lem. 8.3]). *Put $\beta=\nu_\alpha\alpha$. Then for $h_0\in M_\alpha$ in* (7.3),

$$H_\alpha(-Z_0+Z_1)(h_0)=-H_\beta Z_2(h_0).$$

Proof. Since A is connected, A_K consists of single element h_K. Put $h\in A$, $h'\in A'$ as $h=h_K\exp X$, $h'=h_K\exp X'$ with $X\in\mathfrak{h}$, $X'\in\mathfrak{h}'$ such that

$$X=t_1H_\alpha+t_2H_{\alpha'}=(-t_1+t_2)H_\gamma+(t_1+t_2)H_{\gamma'},$$

$$X'=t_1\sqrt{-1}\,H_\beta+t_2H_{\nu_\alpha\alpha'}=t_1\sqrt{-1}\,H_\beta+t_2H_{\alpha'},$$

and $(\Lambda,\delta)=2m_1$, $(\Lambda,\delta')=2m_2$. Then $\Lambda\to\sigma\Lambda$ is given by $(2m_1,2m_2)\to(\varepsilon 2m_1,\varepsilon 2m_2)$. Since $\xi_\gamma(h_K)=\xi_{\gamma'}(h_K)=1$, h_K belongs to the center of G and hence $\xi_\Lambda(h_K)=\xi_{\sigma\Lambda}(h_K)=c$ (*put*). Hence we get

$$Z_0(h)=c\cdot\mathrm{sgn}\{m_1m_2(m_2^2-m_1^2)\}\{\exp(-2|m_1|t_1-2|m_2|t_2)+\exp(-2|m_2|t_1-2|m_1|t_2)\},$$

$$Z_1(h)=2c\cdot\mathrm{sgn}(m_2^2-m_1^2)\exp\{-|-m_1+m_2|(-t_1+t_2)-|m_1+m_2|(t_1+t_2)\},$$

$$Z_2(h')=c\cdot\mathrm{sgn}(m_2)\exp(2m_1\sqrt{-1}t_1-2|m_2|t_2)-c\cdot\mathrm{sgn}(\varepsilon m_1)\exp(\varepsilon 2m_2\sqrt{-1}t_1-2|m_1|t_2).$$

On the other hand, denote by $(\partial/\partial t_1)_0$ the differentiation at $t_1=0$. Then for $h_0\in M_\alpha$,

$$H_\alpha(-Z_0+Z_1)(h_0)=(\partial/\partial t_1)_0(-Z_0+Z_1)(h_0),\quad H_\beta Z_2(h_0)=(\sqrt{-1})^{-1}(\partial/\partial t_1)_0Z_2(h_0),$$

Therefore we get

$$H_\alpha Z_0(h_0)=2c\cdot\mathrm{sgn}\{m_1m_2(m_2^2-m_1^2)\}\{-|m_1|\exp(-2|m_2|t_2)-|m_2|\exp(-2|m_1|t_2)\},$$

$$H_\alpha Z_1(h_0)$$
$$=2c\cdot\mathrm{sgn}(m_2^2-m_1^2)(|-m_1+m_2|-|m_1+m_2|)\exp\{-(|-m_1+m_2|+|m_1+m_2|)t_2\},$$

$$H_\beta Z_2(h_0)=2c\cdot\mathrm{sgn}(m_1m_2)\{|m_1|\exp(-2|m_2|t_2)-|m_2|\exp(-2|m_1|t_2)\}.$$

Note that

$$|-m_1+m_2|-|m_1+m_2|=-2\cdot\mathrm{sgn}(m_1m_2)\cdot\min(|m_1|,|m_2|),$$

$$|-m_1+m_2|+|m_1+m_2|=2\cdot\max(|m_1|,|m_2|).$$

Then the equality in the lemma follows. Q.E.D.

§8. Proof of Lemma 7.1 (Case of class I or III)

In this section, we study the case where the simple component $\Sigma(\alpha)$ of $\Sigma_R(A)$ containing α is of class I or III. The case of class II (type B_n, C_n or F_4) will be studied in the next section. The case of type C_n has been studied in the previous paper [5(e)]. However we will give in the next section a new general proof for the case of class II. This proof follows essentially the idea in the previous paper [5(e)].

8.1. Let us first give a property of an orthogonal system of roots. Let Σ be a root system of class I or III, P the set of positive roots in Σ and Π the set of simple roots.

Lemma 8.1. *Assume that Σ is of class* I *or* III. *Let $\alpha\in\Pi$. Then for $F\in M(P)$, there exist only the following two possibilities*: (i) $\alpha\in F$, (ii) $s_\alpha F\in M(P)$ *and* $s_\alpha F\neq F$.

Proof. Assume that $\alpha\notin F$. Then we get $s_\alpha F\in M(P)$, because for any $\gamma\in P$, $\neq\alpha$, we have $s_\alpha\gamma\in P$. On the other hand, there exists at least one $\gamma\in F$ not orthogonal to α. We see from the result in §1.2 that the angle between α and γ is an integral multiple of $\pi/6$. Hence $s_\alpha\gamma$ is not orthogonal to γ and therefore $s_\alpha\gamma\notin F$, and $s_\alpha F\neq F$. Q.E.D.

Put $\Sigma^\alpha=\{\gamma\in\Sigma\,;\,\gamma\perp\alpha\}$ and $P^\alpha=P\cap\Sigma^\alpha$. Let $M(P\,;\alpha)$ be the subset of $M(P)$ consisting of F such that $F\ni\alpha$. For $F\in M(P\,;\alpha)$, put $F^\alpha=F-\{\alpha\}$. Then we get easily the following.

Lemma 8.2. *The map $F\mapsto F^\alpha$ gives a bijective correspondence between $M(P\,;\alpha)$ and $M(P^\alpha)$.*

Let F_0 be the standard element in $M(P)$ and F_1 that in $M(P^\alpha)$. Then $F_1\cup\{\alpha\}$ is an element of $M(P)$, and there exists a $u_0\in W(\Sigma)$ such that $u_0F_0=F_1\cup\{\alpha\}$.

Lemma 8.3. *Let F_0, $\alpha\in\Pi$ and F_1 be as above. Then it follows from $u_0F_0=F_1\cup\{\alpha\}$ that* $\mathrm{sgn}(u_0)=1$.

Proof. We may assume that Σ is simple. Note that it is sufficient for us to prove the assertion for any fixed P. In fact, let $P'\neq P$ be the set of positive roots with respect to another order in Σ and $\alpha'\in\Pi'$ a simple root in P'. Then there exists a unique $w\in W(\Sigma)$ such that $wP'=P$ and therefore $\alpha=w\alpha'\in\Pi$. Thus $F'_0=w^{-1}F_0$ and $F'_1=w^{-1}F_1$ are standard in $M(P')$ and $M(P'^{\alpha'})$ respectively. Hence $u_0F_0=F_1\cup\{\alpha\}$ means that $u'_0F'_0=F_1\cup\{\alpha'\}$ with $u'_0=w^{-1}u_0w$.

In general, when $\alpha\in F_0$, we get $F_0=F_1\cup\{\alpha\}$, and therefore the assertion follows from Lemma 1.2. Moreover note that when F_0 and F_1 contains a root γ in common, the proof of the assertion can be reduced from the root system Σ to the subsystem Σ^γ.

The assertion is easy to prove for the type A_n. For the type G_2, the assertion is seen to hold from Figure 1.1. Let us study the type D_N ($N\geqq4$). We apply the realization of Σ, P and Π in §1.2. Then

$$\Pi=\{e_1-e_2,\ e_2-e_3,\ \cdots,\ e_{N-1}-e_N,\ e_{N-1}+e_N\},$$

$$F_0=\{e_1\pm e_2,\ e_3\pm e_4,\ \cdots,\ e_{2n-1}\pm e_{2n}\}\qquad(n=[N/2]).$$

Assume that $\alpha\in\Pi$ is not in F_0, then $\alpha=e_{2i}-e_{2i+1}$ for some i. Moreover F_1 is the set obtained from F_0 by removing $e_{2i-1}\pm e_{2i}$, $e_{2i+1}\pm e_{2i+2}$ out and adding $e_{2i-1}\pm e_{2i+2}$, $e_{2i}+e_{2i+1}$ instead. Therefore the situation is reduced to the case of type D_4, and $u_0=s_\alpha s_\gamma$ with $\gamma=e_{2i}-e_{2i+2}$ is an element such that $u_0F_0=F_1\cup\{\alpha\}$ and that $\mathrm{sgn}(u_0)=1$. Thus the assertion holds by Lemma 1.2.

Now consider the case of type E_6. In the realization in §1.2,

$$\Pi=\{2^{-1}(e_1-\sum_{2\leq i\leq 4} e_i\pm(e_5+e_6+e_7-e_8)),\ e_i-e_{i+1}\ (2\leqq i\leqq 4),\ e_4+e_5\},$$

$$F_0=\{e_1\pm e_2,\ e_3\pm e_4\}.$$

Let α be a root in Π not belonging to F_0. Then $F_0\cap F_1$ contain $\gamma=e_1+e_2$, the highest root in P, except when $\alpha=e_2-e_3$. Therefore the proof is reduced to the case of Σ^{γ}, which is of type A_5 by Table 1.1. For $\alpha=e_2-e_3$, we have $F_1=\{e_1\pm e_4,\ e_2+e_3\}$, and the assertion is seen to hold.

Next consider the case of type E_7. In the realization in § 1.2,

$$\Pi=\{2^{-1}(e_1-e_2-\cdots-e_7+e_8),\ e_i-e_{i+1}\ (2\leqq i\leqq 5),\ e_5+e_6,\ e_7-e_8\},$$

$$F_0=\{e_1\pm e_2,\ e_3\pm e_4,\ e_5\pm e_6,\ e_7-e_8\}.$$

Let α be a root in Π not belonging to F_0. Then $F_0\cap F_1$ contains $\gamma=e_1+e_2$, the highest root in P, except when $\alpha=e_2-e_3$. Therefore the proof is reduced to the case of Σ^{γ} which is of type D_6, and the assertion holds for it as is seen above. For $\alpha=e_2-e_3$, we have

$$F_1=\{e_1\pm e_4,\ e_2+e_3,\ e_5\pm e_6,\ e_7-e_8\},$$

Hence $F_0\cap F_1\ni\gamma=e_7-e_8$ and therefore the assertion holds in this case too. Finally consider the case of type E_8. Using the explicit forms of Π and F_0 in § 1.2, we see as for type E_7 that for $\alpha\in\Pi$, either F_0 contains α itself or $F_0\cap F_1$ contains a root γ in common. Moreover Σ^{γ} is of type E_7. Hence the assertion holds in this case too. Thus the proof of the lemma is now complete. Q. E. D.

8.2. To prove Lemma 7.1 in the case where $\Sigma(\alpha)$ is of class I or III, let us first assume that $\Sigma_R(A)$ itself is of class I-III. Let $F\in M(P_R(A))$ be standard. Take a $g_0\in G$ and a $w_0\in W(\mathfrak{b}_c)$ such that

$$\mathrm{Ad}(g_0)\mathfrak{b}=\mathfrak{h}^F,\quad \mathrm{Ad}(g_0)w_0P(\mathfrak{b})=\nu_F P(\mathfrak{h}).\tag{8.1}$$

Then by (5.6), we have for $h\in A^+(P)=A^+(P_R(A))$,

$$\tilde{\kappa}^{\mathfrak{h}}(h\,;\,P(\mathfrak{h}))=\varepsilon(F)\mathrm{sgn}(w_0)Z(h\,;\,F,\ \mathrm{Ad}(g_0)\Lambda,\ P_R(A)).\tag{8.2}$$

Moreover by (3.9) and Lemma 3.2,

$$Z(h\,;\,F,\ \mathrm{Ad}(g_0)\Lambda,\ P_R(A))=\sum_{s\in W_G(\mathfrak{h}^F)}\mathrm{sgn}(s)\sum_{u\in U}Y(h\,;\,F,\ u,\ s\cdot\mathrm{Ad}(g_0)\Lambda),\tag{8.3}$$

where U denotes a complete system of representatives of $W(\Sigma_R(A))/I(F)$ such that $uF\in M(P_R(A))$.

Let $\mathfrak{h}'=\mathfrak{h}^{\alpha}$ and $A'\subset H^{\mathfrak{h}'}$ be as in § 7.1, and put $P(\mathfrak{h}')=\nu_{\alpha}P(\mathfrak{h})$. Put $P_R(A')=\Sigma_R(A')\cap P(\mathfrak{h}')$ and let $F'\in M(P_R(A'))$ be standard. Take $g_0'\in G$ and $w_0'\in W(\mathfrak{b}_c)$ such that

$$\mathrm{Ad}(g_0')\mathfrak{b}=\mathfrak{h}'^{F'},\quad \mathrm{Ad}(g_0')w_0'P(\mathfrak{b})=\nu_{F'}P(\mathfrak{h}').\tag{8.1'}$$

Then the function $\tilde{\kappa}^{\mathfrak{h}'}(\cdot\,;\,P(\mathfrak{h}'))$ is given as follows: for $h'\in A'^+(P)=A'^+(P_R(A'))$,

$$\tilde{\kappa}^{\mathfrak{h}'}(h'\,;\,P(\mathfrak{h}'))=\varepsilon(F')\mathrm{sgn}(w_0')Z(h'\,;\,F',\ \mathrm{Ad}(g_0')\Lambda,\ P_R(A')),\tag{8.2'}$$

where

$$(8.3') \quad Z(h'; F', \mathrm{Ad}(g_0')\Lambda, P_R(A')) = \sum_{s' \in W_G(\mathfrak{h}^{\prime F'})} \mathrm{sgn}(s') \sum_{u' \in U'} Y(h'; F', u', s'\mathrm{Ad}(g_0')\Lambda).$$

Here U' denotes a complete system of representatives of $W(\Sigma_R(A'))/I(F')$ such that $u'F' \in M(P_R(A'))$.

Note that $\varepsilon(F') = -\varepsilon(F)$. Then we see that to prove Lemma 7.1, it is sufficient to show the following: for $h_0 \in M_\alpha \subset A \cap A'$,

$$(8.4) \quad \mathrm{sgn}(w_0) H_\alpha Z(h_0; F, \mathrm{Ad}(g_0)\Lambda, P_R(A)) = -\mathrm{sgn}(w_0') H_\beta Z(h_0; F', \mathrm{Ad}(g_0')\Lambda, P_R(A)).$$

Let us prove this equality. Put $\Sigma = \Sigma_R(A)$, $P = P_R(A)$ for brevity. Then $\Sigma_R(A') = \nu_\alpha \Sigma^\alpha$, $P_R(A') = \nu_\alpha P^\alpha$ with $P^\alpha = P \cap \Sigma^\alpha$. First we reduce the left hand side of (8.4) by applying Lemma 7.2. In the expression (8.3), we devide $U = \{u\}$ into two subsets U_1 and U_2 according as $uF \ni \alpha$ (i.e., $uF \in M(P; \alpha)$) or not, and consider the sum Z_i over $s \in W_G(\mathfrak{h}^F)$ and $u \in U_i$ for $i = 1, 2$: $Z = Z_1 + Z_2$.

First consider the sum Z_2. Since $uF \not\ni \alpha$, we get $s_\alpha uF \in M(P)$, $\neq uF$ by Lemma 8.1. Therefore we may assume by Lemma 3.2 that $s_\alpha u$ belongs to U and hence to U_2 together with u. The next lemma follows from Lemma 7.2 directly.

Lemma 8.4. *For u, $s_\alpha u \in U_2$, put for every $s \in W_G(\mathfrak{h}^F)$ and $\Lambda_1 = \mathrm{Ad}(g_0)\Lambda$,*

$$J(h) = Y(h; F, u, s\Lambda_1) + Y(h; F, s_\alpha u, s\Lambda_1) \qquad (h \in A).$$

Then $J(s_\alpha h) = J(h)$ on A and hence $H_\alpha J(h_0) = 0$ on A_α.

By this lemma, we get $H_\alpha Z_2(h_0) = 0$ on A_α, and therefore for $h_0 \in A_\alpha$,

$$(8.5) \quad H_\alpha Z(h_0; F, \mathrm{Ad}(g_0)\Lambda, P_R(A)) = \sum_s \mathrm{sgn}(s) \sum_{u \in U_1} H_\alpha Y(h_0; F, u, s \cdot \mathrm{Ad}(g_0)\Lambda).$$

On the other hand, we get from Lemma 8.2 a natural bijective correspondence between U_1 and U' in (8.3′) as follows: for any $u' \in U'$, since $u'F' \in M(P')$, we see that $\nu_\alpha^{-1} u'F' \cup \{\alpha\}$ belongs to $M(P; \alpha)$, and therefore there exists a unique $u \in U_1$ such that

$$(8.6) \quad uF = \nu_\alpha^{-1} u'F' \cup \{\alpha\}.$$

Let $u_0 \in U_1$ be such that $u_0 F = \nu_\alpha^{-1} F' \cup \{\alpha\}$. Then $\mathrm{sgn}(u_0) = 1$ by Lemma 8.3. We may assume that any $u \in U_1$ has the form $u = (\nu_\alpha^{-1} u' \nu_\alpha) u_0$, where $u' \in U'$ is given by (8.6), because $\nu_\alpha^{-1} u' \nu_\alpha \in W(\Sigma^\alpha)$.

Thus the proof of Lemma 7.1 is reduced to compair $H_\alpha Y(h_0; F, u, s \cdot \mathrm{Ad}(g_0)\Lambda)$ with $H_\beta Y(h_0; F', u', s'\mathrm{Ad}(g_0')\Lambda)$ for $u = (\nu_\alpha^{-1} u' \nu_\alpha) u_0$ and $h_0 \in M_\alpha$.

8.3. We can choose an $a_0 \in G$ such that $\mathrm{Ad}(a_0)|\mathfrak{h} = u_0$ and $a_0 h a_0^{-1} = h$ for $h \in A_U$. Then under an appropriate choice of root vectors, we get

$$(8.7) \quad \nu_{F'} \nu_\alpha = \mathrm{Ad}(a_0) \circ \nu_F \circ \mathrm{Ad}(a_0)^{-1}.$$

In fact, let $X_{\pm\gamma}$ ($\gamma\in F$) be root vectors used to define ν_F. Put $F''=\nu_\alpha^{-1}F'$, then $F''\cup\{\alpha\}=u_0F$. For $\gamma\in F$, put $X^*_{\pm u_0\gamma}=\mathrm{Ad}(a_0)X_{\pm\gamma}$ and define ν_{u_0F} by means of these root vectors. Then $\nu_{u_0F}=\nu_{F''}\nu_\alpha=\nu_\alpha\nu_{F''}=\mathrm{Ad}(a_0)\circ\nu_F\circ\mathrm{Ad}(a_0)^{-1}$. For $\gamma'\in F'$, put $X_{\pm\gamma'}=\nu_\alpha X^*_{\pm\gamma''}=X^*_{\pm\gamma''}$ with $\gamma''=\nu_\alpha^{-1}\gamma'\in F''$. Then we get $\nu_{F'}=\nu_\alpha\nu_{F''}\nu_\alpha^{-1}=\nu_{F''}$, and therefore the equality (8.7).

Assume (8.7) holds. Then $\mathfrak{h}'^{F'}=\mathrm{Ad}(a_0)\mathfrak{h}^F$. Put $g_0''=a_0g_0$, then applying $\mathrm{Ad}(a_0)$ on the both sides of the equalities in (8.1), we get

$$\mathrm{Ad}(g_0'')\mathfrak{b}=\mathrm{Ad}(a_0)\mathfrak{h}^F=\mathfrak{h}'^{F'}\,,$$

$$\mathrm{Ad}(g_0'')w_0P(\mathfrak{b})=\mathrm{Ad}(a_0)\nu_FP(\mathfrak{h})=\nu_{F'}\nu_\alpha u_0P(\mathfrak{h})=\nu_{F'}vP(\mathfrak{h}')\,,$$

where $v=\nu_\alpha u_0\nu_\alpha^{-1}\in W(\mathfrak{h}_c')$. By the result in § 6.1, we may take g_0'' as g_0' in (8.3'). Then, since $\mathrm{sgn}(v)=\mathrm{sgn}(u_0)=1$ by Lemma 8.3, we get for w_0' in (8.4),

$$\mathrm{sgn}(w_0')=\mathrm{sgn}(w_0)\mathrm{sgn}(v)=\mathrm{sgn}(w_0)\,. \tag{8.8}$$

For $s'\in W_G(\mathfrak{h}'^{F'})$ and $u'\in U'$, put

$$s=\mathrm{Ad}(a_0)^{-1}\circ s'\circ\mathrm{Ad}(a_0)\,|\,\mathfrak{h}^F\,,\quad u=(\nu_\alpha^{-1}u'\nu_\alpha)u_0\,, \tag{8.9}$$

then $s\in W_G(\mathfrak{h}^F)$, $u\in U_1$, and (8.6) holds. In view of (8.8), it is sufficient for us to prove the following.

Lemma 8.5. *Let $g_0'=a_0g_0$, and let s, s' and u, u' be as in* (8.9). *Then for $h_0\in M_\alpha$,*

$$H_\alpha Y(h_0\,;\,F,\,u,\,s\cdot\mathrm{Ad}(g_0)\Lambda)=-H_\beta Y(h_0\,;\,F',\,u',\,s'\mathrm{Ad}(g_0')\Lambda)\,.$$

Proof. Note that the subspace $\mathfrak{h}_V'$ of $\mathfrak{h}'$ is given by $\{X\in\mathfrak{h}_V\,;\,\alpha(X)=0\}$. Then we see that $h_0\in M_\alpha$ can be expressed as $h_0=h_U\exp X'$ with $h_U\in A_U$, $X'\in\mathfrak{h}_V'$, where $\gamma(X')>0$ for any $\gamma\in P^\alpha$. Put

$$h=h_0\exp(tH_\alpha)=h_U\exp(X'+tH_\alpha)\,,$$

$$h'=h_0\exp(t\sqrt{-1}\,H_\beta)=h_U\exp(t\sqrt{-1}\,H_\beta)\exp X',$$

then $h\in A^+(P)$ for sufficiently small $t>0$, and $h'\in A'^+(P)$ for t real. For brevity, put $\Lambda_1=s\cdot\mathrm{Ad}(g_0)\Lambda$, $\Lambda_2=s'\mathrm{Ad}(g_0')\Lambda$, then we get from (3.7) that

$$Y(h\,;\,F,\,u,\,\Lambda_1)=\mathrm{sgn}\{\prod_{\gamma\in F}(\Lambda_1,\,\nu_F\gamma)\}\xi_{\Lambda_1}(h_U)$$
$$\times\exp\Lambda_1(p_F(u^{-1}X'))\cdot\prod_{\gamma\in F}\exp\{-(u\gamma)(X'+tH_\alpha)|(\Lambda_1,\,\nu_F\gamma)|/|\gamma|^2\}, \tag{8.10}$$

$$Y(h'\,;\,F',\,u',\,\Lambda_2)=\mathrm{sgn}\{\prod_{\gamma'\in F'}(\Lambda_2,\,\nu_{F'}\gamma')\}\xi_{\Lambda_2}(h_U\exp(t\sqrt{-1}\,H_\beta))$$
$$\times\exp\Lambda_2(p_{F'}(u'^{-1}X'))\cdot\prod_{\gamma'\in F'}\exp\{-(u'\gamma')(X')|(\Lambda_2,\,\nu_{F'}\gamma')|/|\gamma'|^2\}. \tag{8.10'}$$

For $\gamma'\in F'$, choose $\gamma\in F$ such that $\nu_\alpha^{-1}\gamma'=u_0\gamma$. Then $|\gamma'|=|\gamma|$, and by (8.9) $\nu_\alpha^{-1}u'\gamma'=u\gamma$, and therefore $(u'\gamma')(X')=(u\gamma)(X')$ for $X'\in\mathfrak{h}_V'$. By (8.9) and $g_0'=a_0g_0$, we get

$$\Lambda_2=\mathrm{Ad}(a_0)\cdot s\cdot\mathrm{Ad}(g_0)\Lambda=\mathrm{Ad}(a_0)\Lambda_1\,.$$

Therefore by $\gamma'=\nu_\alpha u_0\gamma$ and (8.7),

$$(\Lambda_2,\ \nu_{F'}\gamma')=(\mathrm{Ad}(a_0)\Lambda_1,\ \mathrm{Ad}(a_0)\nu_F\gamma)=(\Lambda_1,\ \nu_F\gamma)\,.$$

Moreover, since $\nu_\alpha^{-1}u'\nu_\alpha=uu_0^{-1}\in W(\Sigma^\alpha)$, we get $u'^{-1}X'=\nu_\alpha(u_0u^{-1}X')=u_0u^{-1}X'$, and hence $p_{F'}(u'^{-1}X')=u_0p_F(u^{-1}X')$, and finally $\Lambda_2(p_{F'}(u'^{-1}X'))=\Lambda_1(p_F(u^{-1}X'))$. Since $a_0h_Ua_0^{-1}=h_U$, we get $\xi_{\Lambda_1}(h_U)=\xi_{\Lambda_2}(h_U)$. Thus we get from (8.10) and (8.10′) that

$$H_\alpha Y(h_0\,;\,F,\,u,\,\Lambda_1)=-(\Lambda_1,\,\nu_Fu^{-1}\alpha)\cdot Y(h_0\,;\,F',\,u',\,\Lambda_2)\,,$$

$$H_\beta Y(h_0\,;\,F',\,u',\,\Lambda_2)=(\Lambda_2,\,\nu_{F'}\beta)\cdot Y(h_0\,;\,F',\,u',\,\Lambda_2)\,.$$

Since $\nu_\alpha^{-1}u'\nu_\alpha\in W(\Sigma^\alpha)$, we see from (8.9) that $u^{-1}\alpha=u_0^{-1}\alpha$, and therefore by (8.7),

$$(\Lambda_1,\,\nu_Fu^{-1}\alpha)=(\Lambda_1,\,\nu_Fu_0^{-1}\alpha)=(\Lambda_1\,,\,\mathrm{Ad}(a_0)^{-1}\nu_{F'}\nu_\alpha\alpha)=(\Lambda_2,\,\nu_{F'}\beta)\,.$$

Hence we get finally

$$H_\alpha Y(h_0\,;\,F,\,u,\,\Lambda_1)=-H_\beta Y(h_0\,;\,F',\,u',\,\Lambda_2)\,.$$

This completes the proof of the lemma. Q. E. D.

Thus Lemma 7.1 is proved when $\Sigma_R(A)$ itself is of class I-III.

8.4. Consider the general case where $\Sigma(\alpha)$ is of class I or III, but $\Sigma_R(A)$ is not necessarily of classe I-III. Note that the above calculations to prove Lemma 7.1 can be carried out essentially only on $\Sigma(\alpha)$. Therefore putting $\Sigma=\Sigma(\alpha)$, the calculation for the general case is quite similar as that given above. Thus the proof of Lemma 7.1 is now complete in the case where $\Sigma(\alpha)$ is of class I or III.

§ 9. Proof of Lemma 7.1 (Case of class II)

In order to prove the boundary condition (c), we continue to prove Lemma 7.1. In this section, we treat the case where $\Sigma(\alpha)\subset\Sigma_R(A)$ is of class II. (For type C_n, cf. [5(e), § 8].)

9.1. First let us study some properties of ordered system of mutually orthogonal roots in a simple root system of class II, i. e., of type B_n, C_n or F_4. Let P be the set of positive roots in Σ, and Π the set of simple roots in P. We consider the canonical order in Σ corresponding to P.

Lemma 9.1. *Let $F\in M(P)$ and $\alpha\in\Pi$. Then there exist only the following three posibilities*: (i) $F\ni\alpha$; (ii) $s_\alpha F=F$, *in this case* $F\not\ni\alpha$; (iii) $s_\alpha F\in M(P),\neq F$, *in this case* $F\not\ni\alpha$. *When Σ is of type B_n and α is a shoot root, or Σ is of type C_n and α is a long root, the case* (iii) *does not exist.*

Proof. The first assertion is easy to prove. For the second assertion, realize Σ, P and Π as in § 1.2, then $\alpha=e_n$ for B_n, and $\alpha=2e_n$ for C_n. Then we see easily that the assertion holds in these cases. Q. E. D.

Let $\alpha\in\Pi$ and $E\in M^{or}(P)$. Then the relation of E with α is devided into three cases (i), (ii) and (iii) according as $F=E^*\in M(P)$ is in case (i), (ii) or (iii) for α. Further we devide the cases (ii) and (iii) of E into (iia), (iib) and into (iiia), (iiib) respectively according as $s_\alpha E\in M^{or}(P)$ or not.

Lemma 9.2. *Let $F\in M(P)$ and $\alpha\in\Pi$. In order that F is in case* (ii) *for α, it is necessary and sufficient that there exists a $\gamma_0\in F$ such that $\gamma_0\not\perp\alpha$ and $s_\alpha\gamma_0\in F$. In this case, all the roots in F other than γ_0, $s_\alpha\gamma_0$ are orthogonal to α.*

Proof. The sufficiency of the condition is clear. Let us prove the necessity. Since $F\not\ni\alpha$, we can find a $\gamma_0\in F$ such that $\gamma_0\not\perp\alpha$. Then $F=s_\alpha F$ contains $s_\alpha\gamma_0$. Therefore any element in F other than γ_0, $s_\alpha\gamma_0$ is orthogonal to $\gamma_0-s_\alpha\gamma_0\neq 0$, hence to α. Q. E. D.

9.2. Let us study the case (ii) more in detail. To prove Lemma 7.1, we will apply Lemma 7.2 to the case (iia) of E, and Lemmas 7.3 and 7.4 to the case (iib). To this end and also for another purpose, we list up the possible pairs (α, E) in case (iib). First note the following. Let $E\in M^{or}(P)$ be

$$E=(\alpha_1, \alpha_2, \cdots, \alpha_l, \alpha_{l+1}, \cdots, \alpha_n), \tag{9.1}$$

where α_i $(1\leqq i\leqq l)$ are long roots, and α_j $(l+1\leqq j\leqq n)$ are short roots. Then by definition, it satisfies the conditions (B1), (B2) and (B3). In particular, $2^{-1}(\alpha_{2i-1}+\alpha_{2i})$ are again roots in Σ for $1\leqq i\leqq m$, where $m=[l/2]$. We see that wE with $w\in W(\Sigma)$ belongs to $M^{or}(P)$ if and only if the following condition analogous as (1.17) holds:

$$\begin{aligned} &w\alpha_{2i-1}>w\alpha_{2i}\ (1\leqq i\leqq m), \quad w\alpha_1>w\alpha_3>\cdots>w\alpha_{2m-3}>w\alpha_{2m-1}: \\ &w\alpha_{l+1}>w\alpha_{l+2}>\cdots>w\alpha_{n-1}>w\alpha_n . \end{aligned} \tag{9.2}$$

By Lemma 9.2, we know that there exist in $F=E^*$ exactly two roots γ_0, $\gamma_0'=s_\alpha\gamma_0$ which are not orthogonal to α and are permuted with each other under s_α.

Realize Σ, P and Π as in § 1.2. Then the canonical order in Σ is the order given there. Let $\gamma_0>\gamma_0'=s_\alpha\gamma_0$.

Case of B_n. (1B) For $\alpha=e_k-e_{k+1}$ (long root), $(\gamma_0, \gamma_0')=(e_k, e_{k+1})$ (short roots), and in E there exists some $i>l$ such that

$$\alpha_i=e_k, \quad \alpha_{i+1}=e_{k+1}. \tag{9.3}$$

(2B) For $\alpha=e_n$ (short root), $(\gamma_0, \gamma_0')=(e_j+e_n, e_j-e_n)$ (long roots) for some $j<n$, and in E there exists some i $(1\leqq i\leqq m)$ such that

$$\alpha_{2i-1}=e_j+e_n, \quad \alpha_{2i}=e_j-e_n. \tag{9.4}$$

Case of C_n. (1C) For $\alpha=2e_n$, $(\gamma_0, \gamma_0')=(e_j+e_n, e_j-e_n)$ for some $j<n$, and

in E there exists some $i>l$ such that

(9.5) $$\alpha_i=e_j+e_n\,,\quad \alpha_{i+1}=e_j-e_n\,.$$

(2C) For $\alpha=e_k-e_{k+1}$, $(\gamma_0, \gamma_0')=(2e_k, 2e_{k+1})$, and in E, we have one of the following:

(9.6) $$\alpha_{2i-1}=2e_k\,,\quad \alpha_{2i}=2e_{k+1}\quad \text{for some}\quad i\ (1\leqq i\leqq m)\,,$$

(9.6′) $$\alpha_{2i-1}=2e_k\,,\quad \alpha_{2j-1}=2e_{k+1}\quad \text{for some}\quad 1\leqq i<j\leqq m\,.$$

CASE OF F_4. (1F) For $\alpha=e_k-e_{k+1}$ with $k=2$ or 3, we have the following possibilities:

(9.7) $$(\gamma_0, \gamma_0')=(e_k, e_{k+1})\,,$$

(9.7ε) $$(\gamma_0, \gamma_0')=(2^{-1}(e_1+\varepsilon e_i+e_k-e_{k+1}),\ 2^{-1}(e_1+\varepsilon e_i-e_k+e_{k+1}))\quad \text{with}\quad \varepsilon=\pm 1\,,$$

where $\{i, k, k+1\}=\{2, 3, 4\}$. Let l be the type of $E\in M^{or}(P)$, then the elements E in case (iib) are given as follows:

($l=0$) $E=\tilde{F}^0$ for (γ_0, γ_0') in (9.7),

$E=\tilde{F}^0_{-\varepsilon}$ for (γ_0, γ_0') in (9.7ε),

where F^0, $F^0_{-\varepsilon}$ are given in § 1.5 for type F_4;

($l=2$) $E=(e_1+e_i, e_1-e_i, \gamma_0, \gamma_0')$ for (9.7),

$E=(e_1-\varepsilon e_i, e_k+e_{k+1}, \gamma_0, \gamma_0')$ for (9.7ε).

(2F) For $\alpha=2^{-1}(e_1-e_2-e_3-e_4)$, we have $(\gamma_0, \gamma_0')=(e_1-e_i, e_j+e_k)$, where $j<k$, $\{i, j, k\}=\{2, 3, 4\}$. The elements $E\in M^{or}(P)$ in case (iib) are given as follows:

($l=2$) $E=(\gamma_0, \gamma_0', 2^{-1}(e_1+e_i+e_j-e_k), 2^{-1}(e_1+e_i-e_j+e_k))$;

($l=4$) $E=(e_1+e_i, e_j-e_k, \gamma_0, \gamma_0')$.

(3F) For $\alpha=e_4$, we have $(\gamma_0, \gamma_0')=(e_i+e_4, e_i-e_4)$ for some i ($1\leqq i\leqq 3$). The elements $E\in M^{or}(P)$ in case (iib) are given as follows: let $j<k$ and $\{i, j, k\}=\{1, 2, 3\}$, then

($l=2$) $E=(\gamma_0, \gamma_0', e_j, e_k)$ for any i;

($l=4$) $E=(\gamma_0, \gamma_0', e_j+e_k, e_j-e_k)$ for $i=1$ ($j=2$, $k=3$),

(9.8) $E=(\gamma_0, \eta, \gamma_0', \eta')$ with $\{\eta, \eta'\}=\{e_j\pm e_k\}$ for $i=1$,

$E=(e_j+e_k, e_j-e_k, \gamma_0, \gamma_0')$ for $i=2, 3$ ($j=1$).

Summarizing these results, we get the following.

Lemma 9.3. *Let P be the set of positive roots in Σ and Π the set of simple roots in P. Let $\alpha\in\Pi$, and $E\in M^{or}(P)$ be in case* (iib) *for α. Then there hold the following.*

(1) *Let* $\gamma_0, \gamma_0'=s_\alpha\gamma_0$ *be two roots in* E *not orthogonal to* α. *Then they are long or short according as* α *is short or long.*

(2) *Assume* α *is short. Then* γ_0, γ_0' *are contained in* E *in two different manners: let* $\gamma_0>\gamma_0'$, *then either*

(iib1) *there exists an* $i\leqq m=[l/2]$ *such that* $(\alpha_{2i-1}, \alpha_{2i})=(\gamma_0, \gamma_0')$, *or*

(iib2) *for type* C_n $(n\geqq 4)$ *or* F_4, $\alpha_{2i-1}=\gamma_0$, $\alpha_{2j-1}=\gamma_0'$ *for some* $1\leqq i<j\leqq m$. *In the latter case, put* $\alpha'=2^{-1}(\alpha_{2i}-\alpha_{2j})$, *then* $\alpha'\in\Sigma$ *and* $s_{\alpha'}E$ *also belongs to* $M^{or}(P)$.

(3) *Assume* α *is long. Let* $\gamma_0>\gamma_0'$, *then* $(\alpha_j, \alpha_{j+1})=(\gamma_0, \gamma_0')$ *for some* $j>l$. *Put*

$$(\delta_0, \delta_0')=(\gamma_0+\gamma_0', \gamma_0-\gamma_0'), \tag{9.9}$$

then δ_0, δ_0' *are long roots in* Σ *and* $\delta_0>\delta_0'$, $\delta_0'=\alpha$.

Note that the case of short α corresponds to (2B), (2C), (2F) and (3F), and the case (iib2) in (2) corresponds to (9.6′) in (2C) and (9.8) in (3F). The case of long α corresponds to (1B), (1C) and (1F).

Let $\alpha\in\Pi$ be a long root. For every $E\in M^{or}(P)$ in case (iib) for α, we make associate an element $\iota_\alpha E\in M^{or}(P)$ in case (i) as follows: take out from E the two roots γ_0, γ_0' and then insert the ordered pair of roots (δ_0, δ_0') in an appropriate place so as to get an element in $M^{or}(P)$. Then $\iota_\alpha E$ exists uniquely. We call the pair $\{E, \iota_\alpha E\}$ an α-pair. (Let D and D' be the elements in $M'(P)$ in § 1.6 corresponding naturally to E and $\iota_\alpha E$ respectively. Then $D'=(D-\{\gamma_0, \gamma_0'\})\cup\{\delta_0, \delta_0')\}$.)

Let $\alpha\in\Pi$ be long. Let $E\in M^{or}(P)$ be in case (i) for α. We say that E is in case (i2) if the type l of E is odd and $\alpha_l=\alpha$, otherwise we say that E is in case (i1).

Lemma 9.4. *Let* $\alpha\in\Pi$ *be long. If* Σ *is of type* B_n, C_{2d} *or* F_4, *there does not exist the case* (i2) *for* α. *If* Σ *is of type* C_{2d+1}, *the type* l *of an* $E\in M^{or}(P)$ *is always odd, and there exists the case* (i2) *for* α. *In general, every element in* $M^{or}(P)$ *in case* (i1) *for* α *makes an* α*-pair with an element in* $M^{or}(P)$ *in case* (iib) *for* α. *In this way, the elements in case* (i1) *and those in case* (iib) *correspond bijectively.*

Proof. The first assertion is easy to prove. Let us prove the second one. Let $\delta\in P$ be a long root orthogonal to α such that $2^{-1}(\delta\pm\alpha)$ are roots in Σ. Then we have $\delta>\alpha$. In fact, express δ as a linear combination of simple roots, then α must be contained in it with positive integral coefficient. Hence we get $\delta>\alpha$.

Let $E\in M^{or}(P)$ be in case (i1) for α. Then there exists in (9.1) some i $(1\leqq i\leqq m)$ such that $\{\alpha_{2i-1}, \alpha_{2i}\}$ contains α. Since $2^{-1}(\alpha_{2i-1}\pm\alpha_{2i})\in P$ and α is simple, we get $\alpha_{2i}=\alpha$. Put $\gamma_0=2^{-1}(\alpha_{2i-1}+\alpha_{2i})$, $\gamma_0'=2^{-1}(\alpha_{2i-1}-\alpha_{2i})$, then $\gamma_0>\gamma_0'$, $\gamma_0\perp\gamma_0'$. Take out from E the two roots $\alpha_{2i-1}, \alpha_{2i}$, and then insert (γ_0, γ_0') in an appropriate place, then we get an element E' in $M^{or}(P)$ in case (iib), because $s_\alpha(\gamma_0, \gamma_0')=(\gamma_0', \gamma_0)$. Thus we get finally $\iota_\alpha E'=E$.

Now the bijectiveness of the correspondence ι_α is clear. Q.E.D.

9.3. Now let us study the case (iii) for $E\in M^{or}(P)$. To prove Lemma 7.1, we apply Lemma 7.2 to the case (iiia) and Lemma 7.4 to the case (iiib).

Lemma 9.5. *Assume* $E\in M^{or}(P)$ *is in case* (iiib) *for* $\alpha\in\Pi$. *Then there exist exactly* 4 *roots in* $F=E^*$ *which are not orthogonal to* α. *Moreover there exists a unique non-trivial element* $v\in W(\Sigma)$, *depending on* E *such that* $vE\in M^{or}(P)$ *and* $v\alpha=\alpha$, $v\gamma=\gamma$ *for any* $\gamma\in F$ *orthogonal to* α. *Then* $\mathrm{sgn}(v)=-1$, *and* $E'=vE$ *is in case* (iiib) *for* α *too.*

Proof. Assume that in E in (9.1), $\alpha_i>\alpha_j$ and $\alpha_i-\alpha_j$ is a multiple of a positive root δ ($\alpha_i-\alpha_j=\delta$ or 2δ). Then we get $s_\alpha\alpha_i>s_\alpha\alpha_j$ in case (iii). In fact, since $\alpha_i\perp\alpha_j$, we see that $\alpha_i+\alpha_j$ is also a multiple of a positive root $\delta'\perp\delta$. If $s_\alpha\alpha_i<s_\alpha\alpha_j$, δ must be equal to α because α is simple, and hence $\delta'\perp\alpha$. Therefore $s_\alpha\delta'=\delta'$ and $s_\alpha\alpha_i=\alpha_j$. This means by Lemma 9.2 that E is in case (ii) for α.

Apply the condition (9.2) for $wE\in M^{or}(P)$ to $w=s_\alpha$. Then for an E in case (iiib), it follows from $s_\alpha E\notin M^{or}(P)$ that there exist α_i, α_j in E such that the relation $\alpha_i>\alpha_j$ appears in (1.17) and $s_\alpha\alpha_i<s_\alpha\alpha_j$. In this case, as is seen above $\alpha_i-\alpha_j$ is not a multiple of a root. This does not occur for F_4.

The proofs for B_n and C_n are given separately using the realization of Σ and P in §1.2. First consider the case of B_n. Then by Lemma 9.2, we see that $\alpha=e_k-e_{k+1}$ for some k. Then by the result in §1.2, we see that in the case (iii), $F=E^*$ contains one of the following sets of roots:

(1) $e_k+e_i,\ e_{k+1}\ (k+1<i)$;	(1′) $e_{k+1}+e_i,\ e_k\ (k+1<i)$;
(2) $e_i\pm e_k,\ e_{k+1}\ (k>i)$;	(2′) $e_i\pm e_{k+1},\ e_k\ (k+1>i)$;
(3) $e_i\pm e_k,\ e_j\pm e_{k+1}\ (i,\ j<k)$;	(4) $e_i\pm e_k,\ e_{k+1}\pm e_j\ (i<k,\ k+1<j)$;
(5) $e_k\pm e_i,\ e_j\pm e_{k+1}\ (j<k,\ k+1<i)$;	(6) $e_k\pm e_i,\ e_{k+1}\pm e_j\ (k+1<i,\ j)$.

As is easily seen, in the cases (1)-(5), we get $s_\alpha E\in M^{or}(P)$ and so E is in case (iiia). In the case (6), E is in case (iiib). Put $v=s_{\alpha'}$ with $\alpha'=e_i-e_j$, then $vE\in M^{or}(P)$ and the assertion of the lemma holds.

Consider the case of C_n. By Lemma 9.2, we see that $\alpha=e_k-e_{k+1}$ for some k. Exactly as for B_n, we get the possibilities as (1)-(6). (In (1)-(2′), e_k and e_{k+1} must be replaced by $2e_k$ and $2e_{k+1}$ respectively.) The proof of the assertion can be carried out similarly. The proof of the lemma is now complete. Q.E.D.

9.4. In the sequel, we will need the following lemma. Let $L,\ L'\in M^{or}(P_R(A))$ be such that $u_0L=L'$ for some $u_0\in W(\Sigma_R(A))$. We can choose an $a_0\in G$ such that

$$\mathrm{Ad}(a_0)|\mathfrak{h}_c=u_0,\quad a_0ha_0^{-1}=h\qquad(h\in A_U)\,. \tag{9.10}$$

Then choosing root vectors for $F_0\cup-F_0$ with $F_0=F_0(L)$ and for $F_0'\cup-F_0'$ with $F_0'=F_0(L')$ appropriately, we have

$$\mathrm{Ad}(a_0)\circ\nu_{F_0}\circ\mathrm{Ad}(a_0)^{-1}=\nu_{F_0'}\,. \tag{9.11}$$

Put $\mathfrak{h}^{F_0}=\mathfrak{g}\cap\nu_{F_0}\mathfrak{h}_c$, $\mathfrak{h}^{F'_0}=\mathfrak{g}\cap\nu_{F'_0}\mathfrak{h}_c$, then $\mathrm{Ad}(a_0)\mathfrak{h}^{F_0}=\mathfrak{h}^{F'_0}$. Let $g_0\in G$ be such that

$$\mathrm{Ad}(g_0)\mathfrak{b}=\mathfrak{h}^{F_0}, \tag{9.12}$$

then $\mathrm{Ad}(a_0g_0)\mathfrak{b}=\mathfrak{h}^{F'_0}$. For $\lambda\in\mathfrak{b}_B^{*\prime}$, put $\mu=\mathrm{Ad}(g_0)\lambda$, $\mu'=\mathrm{Ad}(a_0)\mu=\mathrm{Ad}(a_0g_0)\lambda$, and define $Y(h;L,u,\mu)$, $Y(h;L',u,\mu')$ on A for $u\in W(\Sigma_R(A))$ by means of ν_{F_0} and $\nu_{F'_0}$ respectively (cf. (3.7)). Then we have the following lemma.

Lemma 9.6. *Let $L, L'\in M^{or}(P_R(A))$ be such that $u_0L=L'$ for some $u_0\in W(\Sigma_R(A))$. Then under* (9.10)-(9.12), *there holds that for any $u\in W(\Sigma_R(A))$,*

$$Y(h;L,u,\mu)=Y(h;L',uu_0^{-1},\mu').$$

Proof. This equality follows from the definition (3.7) of the function Y directly. In fact, it is sufficient to note that

$$(\mu,\nu_{F_0}\gamma)=(\mathrm{Ad}(a_0)\mu,\ \mathrm{Ad}(a_0)\nu_{F_0}\gamma)=(\mu',\nu_{F'_0}u_0\gamma)\qquad(\gamma\in\Sigma_R(A)),$$

$$\xi_\mu(h_U)=\xi_\mu(a_0h_Ua_0^{-1})=\xi_{\mu'}(h_U)\qquad(h_U\in A_U),$$

$$p_{F_0}(u^{-1}X)=u_0^{-1}p_{F'_0}(u_0u^{-1}X)\qquad(X\in\mathfrak{h}_V).$$

Q. E. D.

9.5. Let us now return to the proof of Lemma 7.1. Let A be a connected component of $H^{\mathfrak{h}}$, and let $\alpha\in\Pi_R(A)$. Then we must calculate $H_\alpha\tilde{\kappa}^{\mathfrak{h}}(h_0;P(\mathfrak{h}))$ for $h_0\in M_\alpha$, where the function $\tilde{\kappa}^{\mathfrak{h}}$ is given on $A^+(P)\subset A$ by (5.6). This function is a linear combination of $Z(h;E_i,\mathrm{Ad}(g_0)\Lambda,P_R(A))$, where $P_R(A)=\Sigma_R(A)\cap P(\mathfrak{h})$. In turn, the latter function is a linear combination of $Y(h;E_i,u,s\cdot\mathrm{Ad}(g_0)\Lambda)$ with $u\in W(E_i;P_R(A))\subset W(\Sigma_R(A))$, $s\in W_G(\mathfrak{h}^{F_0})$ with $F_0=F_0(E_i)$ (for $W(E_i;P_R(A))$, see Definition 1.2). Therefore it is our main task to calculate $H_\alpha Y(h_0;E_i,u,s\cdot\mathrm{Ad}(g_0)\Lambda)$ for $h_0\in M_\alpha$. Put $\Sigma=\Sigma(\alpha)$, and let P and Π be as usual the set of positive roots and that of simple roots respectively. Let $E\in M^{or}(P)$, $u\in W(\Sigma)$ and $s\in W_G(\mathfrak{h}^{F_0})$. Let $g\in G$ be an element such that $\mathrm{Ad}(g)\mathfrak{b}=\mathfrak{h}^{F_0}$ and put $\Lambda_1=\mathrm{Ad}(g)\Lambda$. For simplicity, we write in this subsection the function $Y(h;\cdot,\cdot,s\Lambda_1)$ on A as $Y(h;E,u,s\Lambda_1)$, indicating only E, u for the simple component Σ of $\Sigma_R(A)$ (fixing corresponding things for other components of $\Sigma_R(A)$).

First of all, we get the following result by applying Lemma 7.2.

Lemma 9.7. *Let $\alpha\in\Pi$ and $E\in M^{or}(P)$. For $u\in W(\Sigma)$, assume that $uE\in M^{or}(P)$, and uE is in case* (iia) *or* (iiia) *for α, that is, $s_\alpha uE\in M^{or}(P)$. Put for any $s\in W_G(\mathfrak{h}^{F_0})$,*

$$J(h)=Y(h;E,u,s\Lambda_1)+Y(h;E,s_\alpha u,s\Lambda_1)\qquad(h\in A),$$

where $F_0=F_0(E)$. Then $J(s_\alpha h)=J(h)$ on A and hence $H_\alpha J(h_0)=0$ on A_α.

Proof. First note that the case (iia) and (iiia) cover exactly the case where $s_\alpha uE\in M^{or}(P)$. The assertion of the lemma follows directly from Lemma 7.2.

Q. E. D.

Next consider the case (iib). We apply Lemmas 7.3 and 7.4 to this case. Let $\alpha\in\Pi$ be short, then we have two cases (iib1) and (iib2) in Lemma 9.3.

Lemma 9.8. *Let $\alpha\in\Pi$ be short. For $E\in M^{or}(P)$ and $u\in W(\Sigma)$, assume that $uE\in M^{or}(P)$ and is in case* (iib1) *for α. Put*

$$J(h)=\sum_{s\in W_G(\mathfrak{h}^{F_0})}\operatorname{sgn}(s)Y(h\,;\,E,\,u,\,s\Lambda_1)\qquad(h\in A)\,,$$

where $F_0=F_0(E)$. Then $J(s_\alpha h)=J(h)$ on A and hence $H_\alpha J(h_0)=0$ on A_α.

Proof. Express E as in (9.1), and put $E'=uE=(\alpha_1',\alpha_2',\cdots,\alpha_l',\alpha_{l+1}',\cdots,\alpha_n')$. Then $uE=E'$ means that $u\alpha_j=\alpha_j'$ for $1\leqq j\leqq n$. Since E' is in case (iib1) in Lemma 9.3, wet get $(\alpha_{2i-1}',\alpha_{2i}')=(\gamma_0,\gamma_0')$ for some $i\leqq m$ (cf. Lemma 9.3). Put $w=u^{-1}s_\alpha u$, then $wF=F$ with $F=E^*$, $wF_0=F_0$ and moreover

$$w\alpha_{2i-1}=\alpha_{2i},\ w\alpha_{2i}=\alpha_{2i-1}\,;\ w\alpha_j=\alpha_j\ \text{for}\ j\neq 2i-1,\,2i\,.$$

Therefore putting $D=P(E)\cap w^{-1}P(E)$ and $\gamma=2^{-1}(\alpha_{2i-1}-\alpha_{2i})$, we get from the definition of $P(E)$ in (1.21) that

$$P(E)=D\cup\{\gamma\}\,,\quad w^{-1}P(E)=D\cup\{-\gamma\}\,.$$

Hence for any $s\in W_G(\mathfrak{h}^{F_0})$,

$$\operatorname{sgn}_{w^{-1}P(E)}(s\Lambda_1)=-\operatorname{sgn}_{P(E)}(s\Lambda_1)\,.$$

These facts mean that the assumption in Lemma 7.3 is satisfied for u, E and α. Thus the assertion follows from Lemma 7.3. Q. E. D.

Now consider the case (iib2) for a short $\alpha\in\Pi$.

Lemma 9.9. *Let $\alpha\in\Pi$ be short. For $E\in M^{or}(P)$ and $u\in W(\Sigma)$, assume that $E'=uE\in M^{or}(P)$ and is in case* (iib2) *for α. Then Σ is of type C_n $(n\geqq 4)$ or F_4, and in $E'=(\alpha_1',\alpha_2',\cdots,\alpha_n')$, we have $\gamma_0=\alpha_{2i-1}'$, $\gamma_0'=\alpha_{2j-1}'$ for some $1\leqq i<j\leqq m$. Put $\alpha'=2^{-1}(\alpha_{2i}'-\alpha_{2j}')$, then $\alpha'\in\Sigma$ and $s_{\alpha'}E'=s_{\alpha'}uE\in M^{or}(P)$. Put*

$$J(h)=\sum_{s\in W_G(\mathfrak{h}^{F_0})}\operatorname{sgn}(s)\{Y(h\,;\,E,\,u,\,s\Lambda_1)+Y(h\,;\,E,\,s_{\alpha'}u,\,s\Lambda_1)\}\,,$$

where $F_0=F_0(E)$. Then $J(s_\alpha h)=J(h)$ on A and hence $H_\alpha J(h_0)=0$ on A_α.

Proof. Put $u'=s_{\alpha'}u$ and $w=u'^{-1}s_\alpha u$. Then $s_\alpha u=u'w$, $wF=F$ with $F=E^*$, $wF_0=F_0$. Moreover $\operatorname{sgn}(w)=1$ and $w^{-1}P(E)=P(E)$. Therefore the assumption in Lemma 7.4 is satisfied. Thus the lemma follows from Lemma 7.4. Q. E. D.

Note that when $\alpha\in\Pi$ is long, every element $E'\in M^{or}(P)$ in case (iib) forms an α-pair with exactly one element $\iota_\alpha E'\in M^{or}(P)$ in case (i1).

Let us now consider the case (iiib). We apply Lemma 7.4 and obtain the following.

Lemma 9.10. *Let $\alpha\in\Pi$ and $E\in M^{or}(P)$. For $u\in W(\Sigma)$, assume that $E'=uE\in M^{or}(P)$ and is in case* (iiib) *for α. Let $v\in W(\Sigma)$ be the unique non-trivial element such that $E''=vE'\in M^{or}(P)$ and $v\alpha=\alpha$, $v\gamma=\gamma$ for any $\gamma\in E'^*$ orthogonal* *to* α. *Put* $u'=vu$, *then* $E''=u'E$. *Moreover* E'' *is in case* (iiib). *Put for* $h\in A$,

$$J(h)=\sum_{s\in W_G(\mathfrak{h}^{F_0})}\mathrm{sgn}(s)\{Y(h;E,u,s\Lambda_1)+Y(h;E,u',s\Lambda_1)\},$$

where $F_0=F_0(E)$. *Then* $J(s_\alpha h)=J(h)$ *on* A *and hence* $H_\alpha J(h_0)=0$ *on* A_α.

Proof. We apply Lemma 7.4. Put $w=u'^{-1}s_\alpha u$, then $s_\alpha u=u'w$, $wF=F$ with $F=E^*$, $wF_0=F_0$, and $\mathrm{sgn}(w)=1$ because $\mathrm{sgn}(v)=-1$ by Lemma 9.5. Moreover we get $w^{-1}P(E)=P(E)$. In fact, Σ is of type B_n or C_n in this case, and we see in the proof of Lemma 9.5 that $\alpha=e_k-e_{k+1}$ for some k, and the four elements in $F'=E'^*$ not orthogonal to α are given as $e_k\pm e_i$, $e_{k+1}\pm e_j$ for some $i, j<k$. Then $w=u^{-1}s_{\alpha'}s_\alpha u$ with $\alpha'=e_i-e_j$. Since $uP(E)=P(uE)$, this gives us $w^{-1}P(E)=P(E)$. Therefore the equality (7.7) in Lemma 7.4 holds. Thus the assertion of the lemma follows from Lemma 7.4. Q.E.D.

We see from Lemmas 9.7-9.10 that to calculate $H_\alpha\tilde{\kappa}^{\mathfrak{h}}(h_0;P(\mathfrak{h}))$, it is sufficient to pick up such summands $Y(h;E_i,u,s\cdot\mathrm{Ad}(g_0)\Lambda)$ with $u\in W(E_i;P_R(A))$, $s\in W_G(\mathfrak{h}^{F_0})$ that satisfy the following conditions. Put $\Sigma=\Sigma(\alpha)$.

(1) When $\alpha\in\Pi$ is short, the Σ-part of uE_i is in case (i) for α, i.e., uE_i contains α.

(2) When $\alpha\in\Pi$ is long, the Σ-part of uE_i is in case (i) or in case (iib) for α. (The latter corresponds to (1B), (1C) and (1F) in the list in §9.2.)

9.6. *The case of short* α. We assume $\Sigma\equiv\Sigma(\alpha)=\Sigma_R(A)$. Let us prove Lemma 7.1 when $\alpha\in\Pi$ is short. Put $\mathfrak{h}'=\mathfrak{h}^\alpha$, $P(\mathfrak{h}')=\nu_\alpha P(\mathfrak{h})$, and let $A'\subset H^{\mathfrak{h}'}$ be as in §7.1. Our aim is to prove the equality

$$H_\alpha\tilde{\kappa}^{\mathfrak{h}}(h_0;P(\mathfrak{h}))=H_\beta\tilde{\kappa}^{\mathfrak{h}'}(h_0;P(\mathfrak{h}'))\qquad(h_0\in M_\alpha),\tag{9.13}$$

where the left hand side denotes the limit value as $h\in A^+(P)$ tends to $h_0\in M_\alpha$.

Let us realize Σ and P as in §1.2. Since α is short, we see from the result in §9.4 that

$$\begin{aligned}H_\alpha\tilde{\kappa}^{\mathfrak{h}}(h_0;P(\mathfrak{h}))\\ =\mathrm{sgn}(w_0)\sum_{1\leq j\leq r}\varepsilon(E_j)\sum_{u\in U_{j1}}\sum_{s\in W_0}\mathrm{sgn}(s)H_\alpha Y(h_0;E_j,u,s\cdot\mathrm{Ad}(g_0)\Lambda),\end{aligned}\tag{9.14}$$

where w_0, E_j, g_0 are as in (5.4), $W_0=W_G(\mathfrak{h}^{F_0})$ with $F_0=F_0(E_j)$ for any j, and

$$U_{j1}=\{u\in W(\Sigma);\ uE_j\in M^{or}(P)\text{ and in case (i) for }\alpha,\text{ i.e., }uE_j\ni\alpha\}.\tag{9.15}$$

Recall that $\Sigma_R(A')=\nu_\alpha\Sigma_R(A)^\alpha$, $P_R(A')=\Sigma_R(A')\cap P(\mathfrak{h}')=\nu_\alpha P_R(A)^\alpha$. We put $\Sigma'=\nu_\alpha\Sigma^\alpha$, $P'=\nu_\alpha P^\alpha$. Note that the canonical order in Σ corresponding to P induces in Σ^α the canonical one corresponding to P^α, except perhaps for Σ of type F_4, and in this exceptional case, we can apply Lemma 1.5 to Σ^α because it is of type B_3 or C_3. Let E'_j $(1\leqq j\leqq r')$ be a complete system of standard elements in $M^{or}(P')$, then $F_0(E'_j)$ coincide mutually for all j (denote it by F'_0). Take $w'_0\in W(\mathfrak{h}_c)$ and $g'_0\in G$ in such a way that

(9.16) $$\mathrm{Ad}(g'_0)\mathfrak{b}=\mathfrak{h}'^{F'_0},\quad \mathrm{Ad}(g'_0)w'_0P(\mathfrak{b})=\nu_{F'_0}P(\mathfrak{h}')\,.$$

Then by the formula for $\tilde{\kappa}^{\mathfrak{h}'}$ analogous as (5.6), we get for $h_0\in M_\alpha$,

(9.17) $$H_\beta\tilde{\kappa}^{\mathfrak{h}'}(h_0\,;\,P(\mathfrak{h}'))$$
$$=\mathrm{sgn}(w'_0)\sum_{1\leq j\leq r'}\varepsilon(E'_j)\sum_{u'\in U'_j}\ \sum_{s'\in W'_0}\mathrm{sgn}(s')H_\beta Y(h_0\,;\,E'_j,\,u',\,s'\mathrm{Ad}(g'_0)\Lambda)\,,$$

where $W'_0=W_G(\mathfrak{h}'^{F'_0})$ and

(9.18) $$U'_j=W(E'_j\,;\,P')=\{u'\in W(\Sigma')\,;\,u'E'_j\in M^{or}(P')\}\,.$$

We see from (9.14), (9.17) that our task is to find (1) a natural correspondence between $\{E_j\,;\,U_{j1}\neq\emptyset\}$ and $\{E'_j\}$, and (2) if E_j corresponds to E'_j, a natural correspondence from U_{j1} onto U'_j, and finally (3) the relation between $H_\alpha Y(h_0\,;\,E_j,\,u,\,s\cdot\mathrm{Ad}(g_0)\Lambda)$ and $H_\beta Y(h_0\,;\,E'_j,\,u',\,s'\mathrm{Ad}(g'_0)\Lambda)$.

We discuss separately according to the type of Σ.

CASE OF B_n. Let Σ be of type B_n. Then Σ' is of type B_{n-1}. We have in P only one short simple root $\alpha=e_n$, the lowest root in P. Let l_j be the type of E_j. We see that $U_{j1}\neq\emptyset$ if and only if $l_j<n$, and that if $U_{j1}\neq\emptyset$, E_j contains α as its last element. In that case, let E''_j be the system obtained from E_j by removing out α, then $E'_j=\nu_\alpha E''_j$ gives us a complete system of standard elements in $M^{or}(P')$. Hence in an appropriate numbering of E_j's, we have E'_j for $1\leqq j\leqq r'$ with $r'=r$ or $r-1$ according as n is odd or even.

Note that $E_j=(E''_j,\,\alpha)$ and $E'_j=\nu_\alpha E''_j$. Then taking an appropriate system of root vectors for $F'_0\cup-F'_0$ satisfying Condition 5.1, we have $\mathrm{Ad}(g_1)\nu_{F_0}|\mathfrak{h}_c=\nu_{F'_0}\nu_\alpha|\mathfrak{h}_c$ with an element $g_1\in G$. In fact, we may put $g_1=e$, when $F_0\ni\alpha$, or equivalently, when n is odd. Otherwise put $\alpha'=\alpha_{n-1}$ in E_j $(\alpha=\alpha_n)$, then $\alpha'\pm\alpha\in F_0$, and α, α', $\alpha'\pm\alpha$ are strongly orthogonal to any element in $F_0-\{\alpha'\pm\alpha\}$. Apply Lemma 7.5 to F_0, α and α', then under an appropriate choice of root vectors for $\pm\alpha$, $F'_0\cup-F'_0$, we have $\mathrm{Ad}(g_1)\nu_{F_0}|\mathfrak{h}_c=\nu_{F'_0}\nu_\alpha|\mathfrak{h}_c$ with an element $g_1\in G(\alpha,\,\alpha')$, where $G(\alpha,\,\alpha')$ is given in Lemma 7.5. Then Condition 5.1 follows from Lemma 3.8.

In that case, putting $P(\mathfrak{h}')=\nu_\alpha P(\mathfrak{h})$, we have

(*) $$\mathrm{Ad}(g_1)\mathfrak{h}^{F_0}=\mathfrak{h}'^{F'_0},\quad \mathrm{Ad}(g_1)\nu_{F_0}P(\mathfrak{h})=\nu_{F'_0}P(\mathfrak{h}')\,.$$

On the other hand, consider the following term in the expression of $\tilde{\kappa}^{\mathfrak{h}'}(h'\,;\,P(\mathfrak{h}'))$ in (9.17):

(9.19) $$\mathrm{sgn}(w'_0)\varepsilon(E'_j)\sum_{s'\in W'_0}\mathrm{sgn}(s')Y(h'\,;\,E'_j,\,u',\,s'\mathrm{Ad}(g'_0)\Lambda)\,.$$

Then we see easily that this term, as a whole, does not depend on the choice of g'_0, w'_0 in (9.16). This means that in (9.19) we may take g'_0, w'_0 in (9.16) depending on $(j,\,u')$. Take g'_0 as $g'_0=g_1g_0$, then we have $w'_0=w_0$. In fact, we get from (5.4) and (*) that

$$\mathrm{Ad}(g'_0)\mathfrak{b}=\mathfrak{h}'^{F'_0},\quad \mathrm{Ad}(g'_0)w_0P(\mathfrak{b})=\nu_{F'_0}P(\mathfrak{h}')\,.$$

Assume $U_{j1}\neq\emptyset$. For $u\in U_{j1}$, uE_j contains α as its last element, hence $u\alpha=\alpha$. Therefore $u\in W(\Sigma^{\alpha})$ and $uE_j=(uE_j'', \alpha)$. Put $u'=\nu_\alpha u\nu_\alpha^{-1}\in W(\Sigma')$, then $\nu_\alpha uE_j''=u'E_j'$. Moreover we see that $u\mapsto u'$ gives us a bijective correspondence from U_{j1} onto U_j'. Therefore we see from (9.14), (9.17) and $\varepsilon(E_j)=-\varepsilon(E_j')$ that to get the equality (9.13), it is sufficient to prove the following.

Lemma 9.11. *Let* $E_j=(E_j'', \alpha)$, $E_j'=\nu_\alpha E_j''$, *and put for* $u\in W(\Sigma^\alpha)$, $u'=\nu_\alpha u\nu_\alpha^{-1}\in W(\Sigma')$. *For* $s\in W_G(\mathfrak{h}^{F_0})$, *put* $s'=\mathrm{Ad}(g_1)\circ s\circ\mathrm{Ad}(g_1)^{-1}|\mathfrak{h}'^{F_0'}\in W_G(\mathfrak{h}'^{F_0'})$ *with* g_1 *in* (*), *and put* $g_0'=g_1g_0$. *Then, for* $h_0\in A_\alpha$,

$$H_\alpha Y(h_0\,;E_j,\,u,\,s\cdot\mathrm{Ad}(g_0)\Lambda)=-H_\beta Y(h_0\,;E_j',\,u',\,s'\mathrm{Ad}(g_0')\Lambda)\,.$$

Proof. The proof can be carried out analogously as for Lemma 8.5. In the proof of Lemma 8.5, it is sufficient to put $a_0=g_1$, $u_0=1$. Q. E. D.

Thus the proof of Lemma 7.1 is now complete when α is short, and $\Sigma(\alpha)=\Sigma_R(A)$ and is of type B_n.

CASE OF C_n. Let Σ be of type C_n. Then Σ^α is of type $C_{n-2}+A_1^{(s)}$ $(C_1=A_1^{(l)})$. We know that $\alpha=e_k-e_{k+1}$ for some $k<n$, and $A_1^{(s)}$-component of Σ^α is $\{\pm\alpha'\}$ with $\alpha'=e_k+e_{k+1}$. Let l_j be the type of E_j, then $U_{j1}\neq\emptyset$ if and only if $l_j<n$. For $u\in U_{j1}$, $L=uE_j$ contains α' together with α in such a way that for some $p>l_j$ depending on u,

$$(\alpha_p', \alpha_{p+1}')=(\alpha', \alpha)\,,$$

where we put $L=(\alpha_1', \alpha_2', \cdots, \alpha_n')$. We define for $L=uE_j$ an ordered system of roots in Σ^α by

$$\boldsymbol{j}_\alpha L=\boldsymbol{j}_\alpha(uE_j)=(\alpha'\,;\alpha_1', \alpha_2', \cdots, \alpha_{p-1}', \alpha_{p+2}', \cdots, \alpha_n')\,. \tag{9.20}$$

Let Σ_1 and Σ_2 be $A_1^{(s)}$-component and the other one of Σ^α. Then $\{\alpha'\}\in M(P_1)$ and $(\alpha_1', \alpha_2', \cdots, \alpha_{p-1}', \alpha_{p+1}', \cdots, \alpha_n')\in M^{or}(P_2)$, where $P_i=\Sigma_i\cap P$. We denote this fact symbolically as $\boldsymbol{j}_\alpha L\in M^{or}(P^\alpha)$ with $P^\alpha=\Sigma^\alpha\cap P$. Then we see that $uE_j\mapsto\boldsymbol{j}_\alpha(uE_j)$ gives us a bijective correspondence between $\{uE_j\,; u\in U_{j1}\}$ and $\{M\in M^{or}(P^\alpha)\,;\text{ type of } M=l_j\}$. Let E_j' be the standard element in $M^{or}(P')$ with $P'=\nu_\alpha P^\alpha$ of type $(0, l_j)$ with $l_j\leqq n-1$. Then for any $u\in U_{j1}$, there exists a unique $u'\in U_j'=W(E_j'\,;P')$ such that

$$\boldsymbol{j}_\alpha(uE_j)=u'E_j'\,, \tag{9.21}$$

and vice versa. Hence we see from (9.14) and (9.17) that our task is to study the relation between $H_\alpha Y(h_0\,;E_j,\,u,\,s\cdot\mathrm{Ad}(g_0)\Lambda)$ and $H_\beta Y(h_0\,;E_j',\,u',\,s'\mathrm{Ad}(g_0')\Lambda)$. The situation is quite similar as that for class I in § 8.3. (The relation (9.21) corresponds to (8.6).)

The following lemma corresponds to Lemma 8.3.

Lemma 9.12. *Let* $u_j\in U_{j1}$ *be such that* $\boldsymbol{j}_\alpha(u_jE_j)=E_j'$. *Then* $\mathrm{sgn}(u_j)=1$.

Proof. Put $l=l_j$, then we have

$E_j=(2e_1, 2e_2, \cdots, 2e_l, e_{l+1}+e_{l+2}, e_{l+1}-e_{l+2}, \cdots, e_{n-1}+e_n, e_{n-1}-e_n)$,

$\nu_\alpha^{-1}E_j'=(e_k+e_{k+1}; 2e_1', 2e_2', \cdots, 2e_l', e_{l+1}'+e_{l+2}', e_{l+1}'-e_{l+2}', \cdots, e_{n-1}'+e_n', e_{n-1}'-e_n')$,

where $e_p'=e_p$ if $p<k$, $e_p'=e_{p+2}$ if $p\geqq k$. When α is contained in E_j, we get $u_j=1$. Otherwise the situation is reduced to the case of C_3 or C_4. Checking case by case, we get $\mathrm{sgn}(u_j)=1$. Q.E.D.

Put $E_j^0=u_jE_j$ and $\nu_\alpha^{-1}E_j'=(\alpha'; \gamma_1, \gamma_2, \cdots, \gamma_{n-2})$. Then for some $q>l_j$, we have

$$E_j^0=(\gamma_1, \gamma_2, \cdots, \gamma_{q-1}, \alpha', \alpha, \gamma_q, \cdots, \gamma_{n-2}). \tag{9.22}$$

For $u'\in U_j'$, put $v=\nu_\alpha^{-1}u'\nu_\alpha\in W(\Sigma)$. Then $v\alpha=\alpha$, $v\alpha'=\alpha'$, hence $v\in W(\Sigma^{\alpha,\alpha'})$ with $\Sigma^{\alpha,\alpha'}=(\Sigma^\alpha)^{\alpha'}$ Moreover

$$\nu_\alpha^{-1}u'E_j'=(\alpha'; v\gamma_1, v\gamma_2, \cdots, v\gamma_{n-2}). \tag{9.23}$$

Let $u\in U_{j1}$ be the element corresponding to u' under (9.21). Then by (9.20),

$$uE_j=(v\gamma_1, v\gamma_2, \cdots, v\gamma_{p-1}, \alpha', \alpha, v\gamma_p, \cdots, v\gamma_{n-2}). \tag{9.24}$$

Therefore we see from (9.22), (9.24) that $u=vwu_j$, where w is given by

$$wE_j^0=(\gamma_1, \gamma_2, \cdots, \gamma_{p-1}, \alpha', \alpha, \gamma_p, \cdots, \gamma_{n-2}). \tag{9.25}$$

Let $a_j\in G$ be an element such that $\mathrm{Ad}(a_j)|\mathfrak{h}_c=u_j$ and $a_jha_j^{-1}=h$ for $h\in A_U$. Then,

$$\mathrm{Ad}(a_j)\circ\nu_{F_0}\circ\mathrm{Ad}(a_j)^{-1}=\nu_{F^0},$$

if the root vectors for $F_0\cup-F_0$, $F^0\cup-F^0$ are so chosen that $\mathrm{Ad}(a_j)X_{\pm\gamma}=X_{\pm u_j\gamma}$ $(\gamma\in F_0)$, where $F_0=F_0(E_j)$, $F^0=F_0(E_j^0)$.

Lemma 9.13. *Assume that ν_{F_0} and ν_{F^0} are normalized as above, and put $\Lambda_1=\mathrm{Ad}(g_0)\Lambda$, $\Lambda_1'=\mathrm{Ad}(a_j)\Lambda_1$, and $s_1'=\mathrm{Ad}(a_j)\cdot s\cdot\mathrm{Ad}(a_j)^{-1}|\mathfrak{h}^{F^0}\in W_G(\mathfrak{h}^{F^0})$, where $\mathfrak{h}^{F^0}=\mathfrak{g}\cap\nu_{F^0}\mathfrak{h}_c$. For $u\in U_{j1}$, let $u'\in U_j'$ be the element corresponding to u under* (9.21), *and put $v=\nu_\alpha^{-1}u'\nu_\alpha$. Then*

$$Y(h; E_j, u, s\Lambda_1)=Y(h; E_j^0, v, s_1'\Lambda_1').$$

Proof. We have by Lemma 9.6 that $Y(h; E_j, u, s\Lambda_1)=Y(h; E_j^0, uu_j^{-1}, s_1'\Lambda_1')$. Therefore, since $uu_j^{-1}=vw$, it is sufficient to prove that

$$Y(h; E_j^0, vw, s_1'\Lambda_1')=Y(h; E_j^0, v, s_1'\Lambda_1').$$

To do so, we apply Lemma 3.1. As is seen from (9.22), (9.25), E_j^0 and w satisfy the assumption in Lemma 3.1. Moreover we get $w^{-1}P(E_j^0)=P(E_j^0)$ and $\mathrm{sgn}(w)=1$. Therefore we get the desired equality from Corollary of Lemma 3.1. Q.E.D.

Now note that $\alpha'\pm\alpha\in F^0$ and that $\alpha, \alpha', \alpha'\pm\alpha$ are strongly orthogonal to any other roots in F^0. Apply Lemma 7.5 to F^0, α' and α, then for an appropriate choice of root vectors for $\pm\alpha$ and $F_0'\cup -F_0'$ with $F_0'=F_0(E_j')$, we have for some element g_1 in $G(\alpha, \alpha')$,

$$\mathrm{Ad}(g_1)\nu_{F^0}|\mathfrak{h}_c=\nu_{F_0'}\nu_\alpha|\mathfrak{h}_c\,, \tag{9.26}$$

and hence $\mathrm{Ad}(g_1)\mathfrak{h}^{F^0}=\mathfrak{h}'^{F_0'}$. Note that Condition 5.1 can be satisfied for (F^0, E_j^0) and (F_0', E_j'). If $\mathrm{Ad}(a_j)\circ\nu_{F_0}\circ\mathrm{Ad}(a_j)^{-1}=\nu_{F^0}$ holds, then putting $\mathfrak{h}'=\mathfrak{h}^\alpha$ as above, we get

$$\mathfrak{h}'^{F_0'}=\mathrm{Ad}(g_1a_j)\mathfrak{h}^{F_0}\,,\quad \nu_{F_0'}P(\mathfrak{h}')=\mathrm{Ad}(g_1a_j)\nu_{F_0}u_j^{-1}P(\mathfrak{h})\,. \tag{9.27}$$

According to the remark about the term (9.19), we put for (9.19) $g_0'=g_1a_jg_0$ with g_0 in (5.4). Then we get from (5.4) and (9.27) that

$$\mathrm{Ad}(g_0')\mathfrak{b}=\mathfrak{h}'^{F_0'},\quad \mathrm{Ad}(g_0')w_0P(\mathfrak{b})=\nu_{F_0'}(\nu_\alpha u_j^{-1}\nu_\alpha^{-1})P(\mathfrak{h}')\,.$$

Therefore w_0' in (9.16) satisfies that $\mathrm{sgn}(w_0')=\mathrm{sgn}(w_0)\mathrm{sgn}(u_j)$. Hence by Lemma 9.12,

$$\mathrm{sgn}(w_0')=\mathrm{sgn}(w_0)\,. \tag{9.28}$$

Note that $\varepsilon(E_j')=-\varepsilon(E_j)$. Then by Lemma 9.13 and (9.28), we see from (9.14), (9.17) that it is sufficient for us to prove the following.

Lemma 9.14. *Put* $g_0'=g_1a_jg_0$ *in* (9.16), *and* $\Lambda'=\mathrm{Ad}(g_0')\Lambda$, $\Lambda_1'=\mathrm{Ad}(a_jg_0)\Lambda$. *Put for* $u'\in U_j'$ *and* $s'\in W_G(\mathfrak{h}'^{F_0'})$, $v=\nu_\alpha^{-1}u'\nu_\alpha\in W(\Sigma)$ *and* $s_1'=\mathrm{Ad}(g_1)^{-1}\cdot s'\cdot\mathrm{Ad}(g_1)|\mathfrak{h}^{F^0}\in W_G(\mathfrak{h}^{F^0})$. *Then for any* u' *and* s',

$$H_\alpha Y(h_0\,;\,E_j^0,\,v,\,s_1'\Lambda_1')=-H_\beta Y(h_0\,;\,E_j',\,u',\,s'\Lambda')\qquad (h_0\in A_\alpha)\,.$$

Proof. Note that $v=\nu_\alpha^{-1}u'\nu_\alpha$ belongs to $W(\Sigma^{\alpha,\alpha'})$ or $v\alpha=\alpha$, $v\alpha'=\alpha'$. Then, taking into account (9.26) and (9.23)-(9.24), we can get the equality directly from the definition (3.7) of the function Y. Q. E. D.

Thus by Lemmas 9.13 and 9.14, the proof of Lemma 7.1 is now complete when α is short, $\Sigma(\alpha)=\Sigma_R(A)$ and is of type C_n.

CASE OF F_4. Let Σ be of type F_4, and $\alpha\in\Pi$ be short. Then Σ^α is of type B_3, and $\alpha=e_4$ or $2^{-1}(e_1-e_2-e_3-e_4)$. In any case, if $E\in M^{or}(P)$ is in case (i) for α, i.e., $E\ni\alpha$, then α is contained in E as its last element. Therefore the proof of the equality (9.13) can be carried out quite analogously as for type B_n. Thus the proof of Lemma 7.1 is now complete when α is short, and $\Sigma(\alpha)=\Sigma_R(A)$ is of type F_4.

9.7. *The case of long* α. We assume $\Sigma\equiv\Sigma(\alpha)=\Sigma_R(A)$. Let us prove Lemma 7.1 when $\alpha\in\Pi$ is long. Our aim is to prove the equality (9.13). Let us realize Σ and P as in § 1.2. Since α is long, we obtain from the result in § 9.5 that

$$\begin{aligned} &H_\alpha\tilde{\kappa}^{\mathfrak{h}}(h_0\,;\,P(\mathfrak{h}))\\ &=\mathrm{sgn}(w_0)\sum_{1\leq j\leq r}\sum_u\sum_{s\in W_0}\mathrm{sgn}(s)H_\alpha Y(h_0\,;\,E_j,\,u,\,s\cdot\mathrm{Ad}(g_0)\Lambda)\,, \end{aligned} \tag{9.29}$$

where w_0, E_j, g_0 are as in (5.4), $W_0=W_G(\mathfrak{h}^{F_0})$ with $F_0=F_0(E_j)$, and u runs over $U_{j1}\cup U_{j2}\cup U_{j3}$. Here U_{jp}'s are the subsets of $W(E_j\,;\,P)=\{u\,;\,uE_j\in M^{or}(P)\}$ given by

$$U_{j1}=\{u\,;\,uE_j \text{ is in case (i1) for } \alpha\},$$

$$U_{j2}=\{u\,;\,uE_j \text{ is in case (iib) for } \alpha\},$$

$$U_{j3}=\{u\,;\,uE_j \text{ is in case (i2) for } \alpha\}.$$

This expression of $H_\alpha\tilde{\kappa}^{\mathfrak{h}}(h_0\,;\,P(\mathfrak{h}))$ corresponds to (9.14) for a short α. On the other hand, the expression (9.17) of $H_\beta\tilde{\kappa}^{\mathfrak{h}'}(h_0\,;\,P(\mathfrak{h}'))$ holds in this case too.

We see in §9.3 that $U_{j3}=\emptyset$ except when Σ is of type C_{2d+1}. Moreover $U_{j1}\neq\emptyset$ (resp. $U_{j2}\neq\emptyset$) if and only if $l_j\geqq 2$ (resp. $n-l_j\geqq 2$), where l_j denotes the type of E_j. Note that l_j is always even for B_n and F_4, and $n-l_j$ is always even for C_n.

Let us normalize the numbering of E_j as $[(n-l_j)/2]=j$. Then we know by Lemma 9.4 that for any $u\in U_{j1}$, there exists a unique $\varphi_\alpha(u)\in U_{j+1,2}$ such that $\varphi_\alpha(u)E_{j+1}$ and uE_j form an α-pair:

$$\iota_\alpha(\varphi_\alpha(u)E_{j+1})=uE_j\,. \tag{9.30}$$

The map φ_α gives a bijection from U_{j1} onto $U_{j+1,2}$. Put for $u\in U_{j1}$,

$$uE_j=(\alpha_1',\,\alpha_2',\,\cdots,\,\alpha_n')\,. \tag{9.31}$$

Then by the definition of the case (i1),

$$(\alpha_{2p-1}',\,\alpha_{2p}')=(\alpha',\,\alpha) \tag{9.32}$$

for some p ($2p\leqq l_j$) with $\alpha'>\alpha$ such that $\alpha'\perp\alpha$, $2^{-1}(\alpha'\pm\alpha)\in P$. We call α' the counter-part of α in uE_j. Put

$$\gamma_0=2^{-1}(\alpha'-\alpha)\,,\quad \gamma_0'=2^{-1}(\alpha'+\alpha)\,. \tag{9.33}$$

Then $\Sigma_2=\{\pm\alpha,\,\pm\alpha',\,\pm\gamma_0,\,\pm\gamma_0'\}$ is a root system of type B_2. By the definition of ι_α and by Lemma 9.3(3), we see that for some $q\geqq l_{j+1}=l_j-2$,

$$\varphi_\alpha(u)E_{j+1}=(\alpha_1',\,\alpha_2',\,\cdots,\,\alpha_{2p-2}',\,\alpha_{2p+1}',\,\cdots,\,\alpha_q',\,\gamma_0',\,\gamma_0,\,\alpha_{q+1}',\,\cdots\,\alpha_n')\,. \tag{9.34}$$

We will apply Lemma 7.7 to these α-pairs with the help of Lemma 7.6. This gives us Lemmas 9.16 and 9.18 below. The assertion corresponding to Lemma 9.14 for a short α is Lemma 9.16 except when Σ is of type C_{2d+1}, and is Lemma 9.17 when Σ is of type C_{2d+1}.

To state these lemmas, we need some preparation. First remark the following.

(IB) If Σ is of type B_n, Σ^α is of type $B_{n-2}+A_1^{(l)}$ ($B_1=A_1^{(s)}$) and $\alpha=e_k-e_{k+1}$ for some $k<n$. The counter-part α' of α is unique and $\alpha'=e_k+e_{k+1}$. The $A_1^{(l)}$-component of Σ^α is $\{\pm\alpha'\}$.

(IC) If Σ is of type C_n, Σ^α is of type C_{n-1} and $\alpha=2e_n$, the lowest root in P, and as the counter-part of α, any long root $\alpha'=2e_k$ in P is possible ($k<n$).

(IF) If Σ is of type F_4, Σ^α is of type C_3 and $\alpha=e_2-e_3$ or e_3-e_4. Note that in any case, if α' is a long root in P such that $\alpha'\perp\alpha$, then $\alpha'>\alpha$, $2^{-1}(\alpha'\pm\alpha)\in\Sigma$. This means that any such α' is possible to become the counterpart of α.

As a cut-off of the root α, we define an operation $\boldsymbol{j}_\alpha$ as follows according to the type of Σ. Let uE_j for U_{j1} be as in (9.31)-(9.32). If Σ is of type B_n, put

(J1) $$\boldsymbol{j}_\alpha(uE_j)=(\alpha'\,;\,\alpha'_1,\,\alpha'_2,\,\cdots,\,\alpha'_{2p-2},\,\alpha'_{2p+1},\,\cdots,\,\alpha'_n)\,.$$

If Σ is of type C_{2d} or F_4, l_j is always even, and put

(J2) $$\boldsymbol{j}_\alpha(uE_j)=(\alpha'_1,\,\alpha'_2,\,\cdots,\,\alpha'_{2p-2},\,\alpha'_{2p+1},\,\cdots,\,\alpha'_{l_j},\,\alpha',^{\triangledown}\,\alpha'_{l_j+1},\,\cdots,\,\alpha'_n)\,.$$

where $^{\triangledown}$ indicates the boundary of long roots and short roots. If Σ is of type C_{2d+1}, l_j is always odd, and put

(J3) $$\boldsymbol{j}_\alpha(uE_j)=(\alpha'_1,\,\alpha'_2,\,\cdots,\,\alpha'_{2p-2},\,\alpha'_{2p+1},\,\cdots,\,\alpha'_{l_j-1},\,\alpha',\,\alpha'_{l_j},^{\triangledown}\,\alpha'_{l_j+1},\,\cdots,\,\alpha'_n)\,.$$

When Σ is of type C_{2d+1}, $U_{j3}\neq\emptyset$. Then by the definition of the case (i2), uE_j for $u\in U_{j3}$ has the following form:

(9.35) $$uE_j=(\alpha'_1,\,\alpha'_2,\,\cdots,\,\alpha'_{l_j-1},\,\alpha,^{\triangledown}\,\alpha'_{l_j+1},\,\cdots,\,\alpha'_n)\,,$$

that is, $\alpha'_{l_j}=\alpha$. We put in this case

(J4) $$\boldsymbol{j}_\alpha(uE_j)=(\alpha'_1,\,\alpha'_2,\,\cdots,\,\alpha'_{l_j-1},^{\triangledown}\,\alpha'_{l_j+1},\,\cdots,\,\alpha'_n)\,.$$

For $u\in U_{j1}$ in case (J3), we define $\hat{u}\in U_{j1}$ as a kind of exchange of α' and the last long root $\alpha''=\alpha'_{l_j}$ in uE_j (l_j is odd in this case): for uE_j in (9.31), put $\alpha''=\alpha'_{l_j}$ and

(9.36) $$\hat{u}E_j=(\alpha''_1,\,\alpha''_2,\,\cdots,\,\alpha''_{2q-1},\,\alpha''_{2q},\,\cdots,\,\alpha''_{l_j},^{\triangledown}\,\alpha'_{l_j+1},\,\cdots,\,\alpha'_n)\,,$$

where $\alpha''_{l_j}=\alpha'$, and $(\alpha''_{2q-1},\,\alpha''_{2q})=(\alpha'',\,\alpha)$ for some q ($2q<l_j$), and all the other roots are arranged in the same order as in uE_j (q is uniquely determined by the condition that $\hat{u}E_j\in M^{or}(P)$).

Let E'_j be the standard element in $M^{or}(P')$. We normalize the numbering of E'_j in such a way that the number of short roots in E_j and E'_j coincide with each other. Note that in the cases of (J1), (J2) and (J4), $\boldsymbol{j}_\alpha(uE_j)$ is an element of $M^{or}(P')$. But in case (J3), $\boldsymbol{j}_\alpha(uE_j)$ does not necessarily belong to $M^{or}(P')$ because α' and $\alpha''=\alpha'_{l_j}$ may be any pair of long roots in P^α.

In the cases of (J1) and (J2), we define for $u\in U_{j1}$ an element $\psi_\alpha(u)\in U'_j$ by

(Ψ) $$\nu_\alpha\boldsymbol{j}_\alpha(uE_j)=\psi_\alpha(u)E'_j\,.$$

In the case of (J4), we define for $u\in U_{j3}$ an element $\psi_\alpha(u)\in U'_j$ by the same equation (Ψ). Then we have the following lemma.

Lemma 9.15. *Let Σ be of type B_n, C_{2d} or F_4, then ψ_α gives a bijective correspondence between U_{j1} and U'_j. Let Σ be the type C_{2d+1}, then ψ_α gives a bijective correspondence between U_{j3} and U'_j.*

Proof. This can be proved case by case by applying the result in § 1.5.
Q. E. D.

Now we can state the lemmas mentioned before.

Lemma 9.16. *Let Σ be of type B_n, C_{2d} or F_4. For $u \in U_{j1}$, put $u(j)=u$, $u(j+1)=\varphi_\alpha(u) \in U_{j+1,2}$. Then for $h_0 \in M_\alpha$,*

$$\operatorname{sgn}(w_0) \sum_{i=j,j+1} \varepsilon(E_i) \sum_{s \in W_0} \operatorname{sgn}(s) H_\alpha Y(h_0\,;\, E_i,\, u(i),\, s\cdot \operatorname{Ad}(g_0)\Lambda) \tag{9.37}$$

$$=\operatorname{sgn}(w_0')\varepsilon(E_j') \sum_{s' \in W_0'} \operatorname{sgn}(s') H_\beta Y(h_0\,;\, E_j',\, \psi_\alpha(u),\, s' \operatorname{Ad}(g_0')\Lambda)\,,$$

where w_0, g_0, W_0 are as in (9.29), *and w_0', g_0', W_0' are as in* (9.17).

Lemma 9.17. *Let Σ be of type C_{2d+1}. Then for any $u \in U_{j3}$, it holds the following equality: for $h_0 \in M_\alpha$,*

$$\operatorname{sgn}(w_0)\varepsilon(E_j) \sum_{s \in W_0} \operatorname{sgn}(s) H_\alpha Y(h_0\,;\, E_j,\, u,\, s\cdot \operatorname{Ad}(g_0)\Lambda) \tag{9.38}$$

$$=\operatorname{sgn}(w_0')\varepsilon(E_j') \sum_{s' \in W_0'} \operatorname{sgn}(s') H_\beta Y(h_0\,;\, E_j',\, \psi_\alpha(u),\, s' \operatorname{Ad}(g_0)\Lambda)\,,$$

where w_0, g_0, W_0, w_0', g_0', W_0' are as in Lemma 9.16.

Moreover when Σ is of type C_{2d+1}, we have the following.

Lemma 9.18. *Let Σ be of type C_{2d+1}. For $u \in U_{j1}$, let $\hat{u} \in U_{j1}$ be given by* (9.36). *Put $u(j)=u$, $u(j+1)=\varphi_\alpha(u)$, $\hat{u}(j)=\hat{u}$, $\hat{u}(j+1)=\varphi_\alpha(\hat{u})$, then for $h_0 \in M_\alpha$,*

$$\sum_{i=j,j+1} \sum_{v=u(i),\hat{u}(i)} \sum_{s \in W_0} \operatorname{sgn}(s) H_\alpha Y(h_0\,;\, E_i,\, v,\, s\cdot \operatorname{Ad}(g_0)\Lambda)=0\,. \tag{9.39}$$

Assume Lemmas 9.16–9.18 for a moment, then we see that the equality (9.13) to be proved follows from them. In fact, when Σ is of type B_n, C_{2d} or F_4, it follows from Lemmas 9.15, 9.16, and (9.17), (9.29). When Σ is of type C_{2d+1}, it follows from Lemmas 9.15, 9.17, 9.18, and (9.17), (9.29).

9.8. We can prove Lemma 9.17 following Case of B_n in § 9.6 and using the analogous lemma as Lemma 9.11. We omit the details.

Thus it rests only to prove Lemmas 9.16 and 9.18. To do so, we apply Lemmas 7.6, 7.7. We need some preparation.

Let us introduce an operation $\boldsymbol{k}_\alpha$ on uE_j with $u \in U_{j1}$ as a cut-off of α and its counter-part α': for uE_j in (9.31)-(9.32),

$$\boldsymbol{k}_\alpha(uE_j)=(\alpha_1',\, \alpha_2',\, \cdots,\, \alpha_{2p-2}',\, \alpha_{2p+1}',\, \cdots,\, \alpha_n')\,. \tag{K}$$

Let α' be a long root in P such that $\alpha' \perp \alpha$, $2^{-1}(\alpha' \pm \alpha) \in P$. Let $U_{j1\alpha'}$ be the subset of U_{j1} consisting of u such that in uE_j, α' is the counter-part of α. Then U_{j1} is the disjoint union of $U_{j1\alpha'}$'s. (Note that if Σ is of type B_n and $\alpha=e_k-e_{k+1}$, then $U_{j1}=U_{j1\alpha'}$ with $\alpha'=e_k+e_{k+1}$.)

Fix such an α' from now on. Put $\Sigma^{\alpha\alpha'}=(\Sigma^\alpha)^{\alpha'}$ and $P^{\alpha\alpha'}=P \cap \Sigma^{\alpha\alpha'}$, then we see easily that $\boldsymbol{k}_\alpha(uE_j)$ is an element of $M^{or}(P^{\alpha\alpha'})$ of type l_j-2 for any $U_{j1\alpha'}$. Let

$$N_{\alpha'}=(\gamma_1, \gamma_2, \cdots, \gamma_{n-2}) \tag{9.40}$$

be the standard element of type l_j-2 of $M^{or}(P^{\alpha\alpha'})$, and define $\rho_{\alpha'}(u)\in W(\Sigma^{\alpha\alpha'})$ for $u\in U_{j1}$, by

$$\boldsymbol{k}_\alpha(uE_j)=\rho_{\alpha'}(u)N_{\alpha'}\,. \tag{P}$$

Then we see that $\rho_{\alpha'}(u)$ gives a bijective correspondence between $U_{j1\alpha'}$ and $W(N_{\alpha'}\,;\,P^{\alpha\alpha'})$.

Choose $x_{\alpha'}\in W(\Sigma)$ as

$$x_{\alpha'}E_j=(\gamma_1, \gamma_2, \cdots, \gamma_{l_j-2}, \alpha', \alpha, \gamma_{l_j-1}, \cdots, \gamma_{n-2})\,, \tag{9.41}$$

and put $K_{\alpha'}=x_{\alpha'}E_j$, $L_{\alpha'}=x_{\alpha'}E_{j+1}$. Let $b_{\alpha'}$ be an element in G such that $\mathrm{Ad}(b_{\alpha'})|\mathfrak{h}_c=x_{\alpha'}$, $b_{\alpha'}hb_{\alpha'}^{-1}=h$ for $h\in A_U$. Then choosing the root vectors for $F_{\alpha'}\cup -F_{\alpha'}$ with $F_{\alpha'}=F_0(K_{\alpha'})=F_0(L_{\alpha'})$ appropriately, we have

$$\mathrm{Ad}(b_{\alpha'})\circ\nu_{F_0}\circ\mathrm{Ad}(b_{\alpha'})^{-1}=\nu_{F_{\alpha'}}\,. \tag{9.42}$$

Hence $\mathfrak{h}^{F_{\alpha'}}=\mathrm{Ad}(b_{\alpha'})\mathfrak{h}^{F_0}$, where $\mathfrak{h}^{F_{\alpha'}}=\mathfrak{g}\cap\nu_{F_{\alpha'}}\mathfrak{h}_c$. Moreover by Lemma 9.6, we get for $u\in W(\Sigma)$,

$$\begin{aligned} &Y(h\,;\,E_j,\,u,\,\mu)=Y(h\,;\,K_{\alpha'},\,ux_{\alpha'}^{-1},\,\mathrm{Ad}(b_{\alpha'})\mu) \quad (h\in A)\,,\\ &Y(h\,;\,E_{j+1},\,u,\,\mu)=Y(h\,;\,L_{\alpha'},\,ux_{\alpha'}^{-1},\,\mathrm{Ad}(b_{\alpha'})\mu) \quad (h\in A)\,, \end{aligned} \tag{9.43}$$

where $\mu=\mathrm{Ad}(g_0)\lambda$ for $\lambda\in\mathfrak{b}_B^{*\prime}$, and g_0 is given in (5.4).

For $u\in U_{j1\alpha'}$, let $v=\rho_{\alpha'}(u)\in W(N_{\alpha'}\,;\,P^{\alpha\alpha'})$ be as in (P). Then by (9.31)-(9.34),

$$\left\{\begin{aligned} &(ux_{\alpha'}^{-1})K_{\alpha'}=uE_j=(v\gamma_1, v\gamma_2, \cdots, v\gamma_{2p-2}, \alpha', \alpha, v\gamma_{2p-1}, \cdots, v\gamma_{n-2})\,,\\ &(\varphi_\alpha(u)x_{\alpha'}^{-1})L_{\alpha'}=\varphi_\alpha(u)E_{j+1}=(v\gamma_1, v\gamma_2, \cdots, v\gamma_q, \gamma_0', \gamma_0, v\gamma_{q+1}, \cdots, v\gamma_{n-2})\,. \end{aligned}\right. \tag{9.44}$$

Put $g_{\alpha'}=b_{\alpha'}g_0$, and for $s\in W_G(\mathfrak{h}^{F_0})$,

$$s'=\mathrm{Ad}(b_{\alpha'})\cdot s\cdot\mathrm{Ad}(b_{\alpha'})^{-1}|\mathfrak{h}_c^{F_{\alpha'}}\,. \tag{9.45}$$

Then $s'\in W_G(\mathfrak{h}^{F_{\alpha'}})$, and for $\mu=\mathrm{Ad}(g_0)\lambda$,

$$\mathrm{Ad}(b_{\alpha'})s\mu=s'\mathrm{Ad}(b_{\alpha'}g_0)\lambda=s'\mathrm{Ad}(g_{\alpha'})\lambda\,. \tag{9.46}$$

Moreover put $M_0=(\alpha', \alpha)$, $M_1=(\gamma_0', \gamma_0)$, then $M^{or}(P_2)=\{M_0, M_1\}$, where $P_2=P\cap\Sigma_2$ with $\Sigma_2=\{\pm\alpha', \pm\alpha, \pm\gamma_0', \pm\gamma_0\}$, and $P(M_0)=P_2$, $P(M_1)=M_1^*$ by (1.21).

Let $h\in A$ be $h=h_U\exp X$ with $h_U\in A_U$, $X\in\mathfrak{h}_V$. Decompose X as $X=X_1+X_2$, where $X_1\in \boldsymbol{R}H_\alpha+\boldsymbol{R}H_{\alpha'}$ and $\alpha(X_2)=\alpha'(X_2)=0$, and put $h_1=\exp(X_1)$, $h_2=h_U\exp(X_2)$, then $h=h_1h_2$. Put for $\Lambda_1=s'\mathrm{Ad}(g_{\alpha'})\Lambda$,

$$\begin{aligned} Y(h\,;\,M_0,\,\Lambda_1)&=Y(h_1\,;\,M_0,\,\Lambda_1)\\ &=\mathrm{sgn}\{\prod_{\gamma\in P(M_0)}(\tau^{-1}\Lambda_1,\gamma)\}\prod_{\gamma\in M_0^*}\exp\{-\gamma(X_1)|(\tau^{-1}\Lambda_1,\gamma)|/|\gamma|^2\}, \end{aligned} \tag{9.47}$$

$$(9.48)\qquad Y(h\,;M_1,\Lambda_1)=Y(h_1\,;M_1,\Lambda_1)$$
$$=\mathrm{sgn}\{\prod_{\gamma\in P(M_1)}(\tau^{-1}\Lambda_1,\gamma)\}\prod_{\gamma\subset M_1^*}\exp\{-\gamma(X_1)|(\tau^{-1}\Lambda_1,\gamma)|/|\gamma|^2\},$$

$$(9.49)\qquad Y(h\,;N,v,\Lambda_1)=Y(h_2\,;N,v,\Lambda_1)$$
$$=\mathrm{sgn}\{\prod_{\gamma\in P(N)}(\tau^{-1}\Lambda_1,\gamma)\}\xi_{\Lambda_1}(h_U)\cdot\prod_{\gamma\in N*}\exp\{-(v\gamma)(X_2)|(\tau^{-1}\Lambda_1,\gamma)|/|\gamma|^2\},$$

where $N=N_{\alpha'}$, $\tau=\nu_{F_{\alpha'}}$.

Then we get from (9.43)-(9.46) the following lemma.

Lemma 9.19. *For $u\in U_{j1\alpha'}$, let $\rho_{\alpha'}(u)\in W(N_{\alpha'}\,;P^{\alpha\alpha'})$ be given by* (P). *For $s\in W_G(\mathfrak{h}^{F_0})$, put s' as in* (9.45). *Then for $h\in A$,*

$$Y(h\,;E_j,u,s\cdot\mathrm{Ad}(g_0)\Lambda)$$
$$=Y(h\,;M_0,s'\mathrm{Ad}(g_{\alpha'})\Lambda)\cdot Y(h\,;N,\rho_{\alpha'}(u),s'\mathrm{Ad}(g_{\alpha'})\Lambda),$$
$$Y(h\,;E_{j+1},\varphi_\alpha(u),s\cdot\mathrm{Ad}(g_0)\Lambda)$$
$$=Y(h\,;M_1,s'\mathrm{Ad}(g_{\alpha'})\Lambda)\cdot Y(h\,;N,\rho_{\alpha'}(u),s'\mathrm{Ad}(g_{\alpha'})\Lambda).$$

Put $\delta=\tau\alpha$, $\delta'=\tau\alpha'$, then they are singular imaginary roots in $\Sigma_1=\Sigma(\mathfrak{h}^{F_{\alpha'}})$ such that $2^{-1}(\delta\pm\delta')\in\Sigma_1$. Then applying Lemma 7.6, we see that there exists $\sigma'\in W_G(\mathfrak{h}^{F_{\alpha'}})$ such that

$$\sigma'\delta=\varepsilon\delta',\ \sigma'\delta'=\varepsilon\delta\ (\varepsilon=1\text{ or }-1)\,;\ \sigma'\eta=\eta\text{ for any }\eta\in\Sigma_1,\ \perp\delta,\delta'.$$

Note that for any $\gamma\in N$, $(\tau^{-1}s'\lambda',\gamma)=(\tau^{-1}\sigma's'\lambda',\gamma)$ for any $s'\in W_G(\mathfrak{h}^{F_{\alpha'}})$ and $\lambda'=\mathrm{Ad}(g_{\alpha'})\lambda$ with $\lambda\in\mathfrak{b}_B^{*\prime}$. Then we see that for any $v\in W(\Sigma^{\alpha\alpha'})$,

$$(9.50)\qquad Y(h\,;N,v,s'\lambda')=Y(h\,;N,v,\sigma's'\lambda').$$

On the other hand, applying Lemma 7.7 to Σ_2, we obtain the following.

Lemma 9.20. *Put $g_{\alpha'}=b_{\alpha'}g_0$, and let $s'\in W_G(\mathfrak{h}^{F_{\alpha'}})$. Then for $h_0=\exp X'$ with $X'=t'H_{\alpha'}$,*

$$(9.51)\qquad \sum_{q=0,1}\varepsilon(M_q)\sum_{y=s',\sigma's'}\mathrm{sgn}(y)H_\alpha Y(h_0\,;M_q,y\cdot\mathrm{Ad}(g_{\alpha'})\Lambda)$$
$$=-\sum_{y=s',\sigma's'}\mathrm{sgn}(y)H_\beta Y(h_0\,;\{\alpha'\},y\cdot\mathrm{Ad}(g_{\alpha'})\Lambda).$$

Here the function Y in the right hand side is given as follows: let h_1' be an element expressed as $h_1'=\exp(t\sqrt{-1}H_\beta)\cdot\exp X'$ with $X'=t'H_{\alpha'}$, then put for $\Lambda_1=s'\mathrm{Ad}(g_{\alpha'})\Lambda$,

$$(9.52)\qquad Y(h_1'\,;\{\alpha'\},\Lambda_1)$$
$$=\mathrm{sgn}\{(\tau^{-1}\Lambda_1,\alpha')\}\xi_{\Lambda_1}(\exp(t\sqrt{-1}H_\beta))\cdot\exp\{-\alpha'(X')|(\tau^{-1}\Lambda_1,\alpha')|/|\alpha'|^2\}.$$

Recall that any element $h_0\in M_\alpha$ is expressed as $h_0=h_U\exp X'$, where $h_U\in A_U$, $X'\in\mathfrak{h}_V'=\{X\in\mathfrak{h}_V\,;\alpha(X)=0\}$. This enables us to utilize Lemmas 9.19, 9.20 and (9.50) for our purpose. In fact, Lemma 9.16 follows from these results and the expressions of $Y(h'; E'_j, u', \Lambda')$ analogous as those in Lemma 9.19. On the other hand, Lemma 9.18 follows essentially from them by the help of Lemma 7.6.

9.9. *Proof of Lemma* 9.16. To finish the proof of Lemma 9.16, we need the following lemma analogous as Lemma 9.12.

Lemma 9.21. *Let* $u_j \in U_{j1}$ *be such that* $\nu_\alpha \boldsymbol{j}_\alpha(u_j E_j) = E'_j$. *Then* $\mathrm{sgn}(u_j)=1$.

Proof. This can be proved analogously as Lemma 9.12 using explicit forms of E_j and E'_j. Q. E. D.

Now let Σ be of type B_n, C_{2d} or F_4. Then $U_{j3}=0$ and l_j is always even. Let $Q_{\alpha'}$ be an ordered system of roots in Σ^α given by

$$
(9.53) \qquad \begin{aligned} \nu_\alpha^{-1} Q_{\alpha'} &= (\alpha'; \gamma_1, \gamma_2, \cdots, \gamma_{n-2}) && \text{for } B_n; \\ &= (\gamma_1, \gamma_2, \cdots, \gamma_{l_j-2}, \alpha'^{\nabla}, \gamma_{l_j-1}, \cdots, \gamma_{n-2}) && \text{for } C_{2d} \text{ or } F_4, \end{aligned}
$$

where γ_i's are in (9.40). Note that $\nu_\alpha^{-1} Q_{\alpha'}$ is obtained from $K_{\alpha'}$ by cutting off the root α, and that α is strongly orthogonal to any $\gamma \in \Sigma^\alpha$. Then we see that choosing root vectors for $G_{\alpha'} \cup -G_{\alpha'}$ with $G_{\alpha'} = F_0(Q_{\alpha'})$ appropriately, we have

$$
(9.54) \qquad \nu_{F_{\alpha'}} = \nu_{G_{\alpha'}} \nu_\alpha = \nu_\alpha \nu_{G_{\alpha'}}.
$$

Therefore we get $\mathfrak{h}^{F_{\alpha'}} = \mathfrak{h}'^{G_{\alpha'}}$ with $\mathfrak{h}' = \mathfrak{h}^\alpha$. Now let $z_{\alpha'} \in W(\Sigma')$ be an element such that $z_{\alpha'} E'_j = Q_{\alpha'}$. Take a $c_{\alpha'} \in G$ such that $\mathrm{Ad}(c_{\alpha'}) | \mathfrak{h}'_c = z_{\alpha'}$, $c_{\alpha'} h' c_{\alpha'}^{-1} = h'$ for $h' \in A'_U$. Since α is strongly orthogonal to any $\gamma \in \nu_\alpha^{-1} \Sigma' = \Sigma^\alpha$, we may assume in addition that $\mathrm{Ad}(c_{\alpha'})$ commutes with ν_α. Therefore choosing root vectors for $F'_0 \cup -F'_0$ with $F'_0 = F_0(E'_j)$ appropriately, we have

$$
(9.55) \qquad \mathrm{Ad}(c_{\alpha'}) \circ \nu_{F'_0} \circ \mathrm{Ad}(c_{\alpha'})^{-1} = \nu_{G_{\alpha'}}.
$$

Hence we get $\mathrm{Ad}(c_{\alpha'}) \mathfrak{h}'^{F'_0} = \mathfrak{h}'^{G_{\alpha'}}$.

Starting from ν_{F_0}, we normalize $\nu_{F_{\alpha'}}$ by (9.42), $\nu_{G_{\alpha'}}$ by (9.54), and finally $\nu_{F'_0}$ by (9.55). Then we have

$$
\mathrm{Ad}(b_{\alpha'}) \circ \nu_{F_0} \circ \mathrm{Ad}(b_{\alpha'})^{-1} = \mathrm{Ad}(c_{\alpha'}) \circ \nu_{F'_0} \nu_\alpha \circ \mathrm{Ad}(c_{\alpha'})^{-1}.
$$

Put $d_{\alpha'} = c_{\alpha'}^{-1} b_{\alpha'}$, $u_j = (\nu_\alpha^{-1} z_{\alpha'} \nu_\alpha)^{-1} x_{\alpha'}$. Then

$$
(9.56) \qquad \mathrm{Ad}(d_{\alpha'}) \circ \nu_{F_0} \circ \mathrm{Ad}(d_{\alpha'})^{-1} = \nu_{F'_0} \nu_\alpha, \quad \mathrm{Ad}(d_{\alpha'}) | \mathfrak{h}_c = u_j.
$$

Moreover it follows from (9.41) and (9.53) that

$$
(9.57) \qquad u_j \in U_{j1}, \quad \nu_\alpha \boldsymbol{j}_\alpha(u_j E_j) = E'_j.
$$

Hence u_j is independent of α', and so is $\mathrm{Ad}(d_{\alpha'}) | \mathfrak{h}_c = u_j$. Moreover the transformation $h \mapsto d_{\alpha'} h d_{\alpha'}^{-1}$ on A is independent of α', because $d_{\alpha'} h d_{\alpha'}^{-1} = h$ for $h \in A_U$. Using (9.55), we get from (5.4),

$$\text{(9.58)}\qquad \mathrm{Ad}(d_{\alpha'}g_0)\mathfrak{b}=\mathfrak{h}^{F'_0},\quad \mathrm{Ad}(d_{\alpha'}g_0)w_0P(\mathfrak{b})=\nu_{F'_0}(\nu_\alpha u_j\nu_\alpha^{-1})P(\mathfrak{h}'),$$

where $P(\mathfrak{h}')=\nu_\alpha P(\mathfrak{h})$. Hence we can take $d_{\alpha'}g_0$ as g'_0 in (9.16) for the term in (9.19). In that case, we get from (9.57)

$$\text{(9.59)}\qquad \mathrm{sgn}(w'_0)=\mathrm{sgn}(w_0)\cdot\mathrm{sgn}(u_j)=\mathrm{sgn}(w_0),$$

because $\mathrm{sgn}(u_j)=1$ by Lemma 9.21.

After this study on the relation between ν_{F_0} and $\nu_{F'_0}$, we will rewrite $Y(h';E'_j,u',\Lambda')$ analogously as for $Y(h;E_j,u,\cdot)$ and $Y(h;E_{j+1},u,\cdot)$ in Lemma 9.19. Firstly we have by Lemma 9.6 the following expression analogous as (9.43): for $u'\in W(\Sigma')$,

$$\text{(9.60)}\qquad Y(h';E'_j,u',\mu')=Y(h';Q_{\alpha'},u'z_{\alpha'}^{-1},\mathrm{Ad}(c_{\alpha'})\mu')\qquad (h'\in A'),$$

where $\mu'=\mathrm{Ad}(g'_0)\lambda$ with $\lambda\in b_B^{*\prime}$.

Note that $\mathrm{Ad}(c_{\alpha'})\mathrm{Ad}(g'_0)\lambda=\mathrm{Ad}(b_{\alpha'})\mathrm{Ad}(g_0)\lambda$. For $s\in W_0=W_G(\mathfrak{h}^{F_0})$, put $s'\in W_G(\mathfrak{h}^{F_{\alpha'}})=W_G(\mathfrak{h}^{G_{\alpha'}})$ as in (9.45) and

$$\text{(9.61)}\qquad s_2=\mathrm{Ad}(c_{\alpha'})^{-1}\cdot s'\cdot\mathrm{Ad}(c_{\alpha'})\,|\,\mathfrak{h}_c'^{F'_0}=\mathrm{Ad}(d_{\alpha'})\cdot s\cdot\mathrm{Ad}(d_{\alpha'})^{-1}\,|\,\mathfrak{h}_c'^{F'_0}.$$

Then $s_2\in W'_0=W_G(\mathfrak{h}'^{F'_0})$, and $\mathrm{Ad}(c_{\alpha'})\cdot s_2\cdot\mathrm{Ad}(g'_0)\Lambda=s'\cdot\mathrm{Ad}(g_{\alpha'})\Lambda$.

Put $U'_{j\alpha'}=\phi_\alpha(U_{j1\alpha'})$, then U'_j is a disjoint union of them. For $u'=\phi_\alpha(u)\in U'_{j\alpha'}$, put $v=\rho_{\alpha'}(u)\in W(N_{\alpha'};P^{\alpha\alpha'})$. Then we get from (Ψ) and (9.44) that

$$\text{(9.62)}\qquad \begin{aligned}\nu_\alpha^{-1}((u'z_{\alpha'}^{-1})Q_{\alpha'})=\nu_\alpha^{-1}(u'E'_j)&=(\alpha';v\gamma_1,v\gamma_2,\cdots,v\gamma_{n-2}) &&\text{for } B_n,\text{ or}\\ &=(v\gamma_1,v\gamma_2,\cdots,v\gamma_{l_j-2},\alpha'^{\nabla}_{,}v\gamma_{l_j-1},\cdots,v\gamma_{n-2}) &&\text{for } C_{2d}\text{ or } F_4.\end{aligned}$$

Express $h'\in A'$ as $h'=h'_1h'_2$, where $h'_1=\exp(t\sqrt{-1}H_\beta)\cdot\exp(t'H_{\alpha'})$, $h'_2=h_U\exp X'$ with $h_U\in A_U$, $X'\in\mathfrak{h}'_V$ such that $\alpha'(X')=0$. We put for $\Lambda_1=s'\mathrm{Ad}(g_{\alpha'})\Lambda$,

$$Y(h';\{\nu_\alpha\alpha'\},\Lambda_1)=Y(h_1;\{\alpha'\},\Lambda_1),\quad Y(h';\nu_\alpha N,u',\Lambda_1)=Y(h'_2;N,v,\Lambda_1),$$

where $N=N_{\alpha'}$, $u'=\phi_\alpha(u)\in U'_{j\alpha'}$, $v=\rho_{\alpha'}(u)$ for some $u\in U_{j1\alpha'}$, and the functions in the right hand sides are given by (9.52) and (9.49) respectively. Note that the decomposition $h'=h'_1h'_2$ is not necessarily unique on A'. However we need only to apply it for a fixed h_U.

Now we have the following lemma analogous as Lemma 9.19.

Lemma 9.22. *For* $s\in W_0=W_G(\mathfrak{h}^{F_0})$, *put* s' *as in* (9.45), *and* s_2 *as in* (9.61). *Then for* $u\in U_{j1\alpha'}$, $h'\in A'$,

$$\begin{aligned}&Y(h';E'_j,\phi_\alpha(u),s_2\mathrm{Ad}(g'_0)\Lambda)\\ &=Y(h';\{\nu_\alpha\alpha'\},s'\mathrm{Ad}(g_{\alpha'})\Lambda)\cdot Y(h';\nu_\alpha N,\rho_{\alpha'}(u),s'\mathrm{Ad}(g_{\alpha'})\Lambda).\end{aligned}$$

Thus the equality in Lemma 9.16 follows from Lemmas 9.19, 9.20, 9.22, (9.50) and (9.59), by summing up over α'. The proof of Lemma 9.16 is now complete.

9.10. *Proof of Lemma* 9.18. Let us prove Lemma 9.18. Put $\nu=\nu_{F_0}$, $\mathfrak{a}=\mathfrak{h}^{F_0}$, $l=l_j$ for simplicity. Recall that $l=l_j$ is always odd in this case. Fix an element $u\in U_{j1}$, and let $\gamma=\alpha'$, the counter-part of α in uE_j, and $\delta=\alpha'_l$, the last long root in uE_j (see (9.31)).

Let $N_{\alpha'}\in M^{or}(P^{\alpha\alpha'})$ be as in (9.40). Then $\boldsymbol{k}_\alpha(uE_j)=\rho_\gamma(u)N_\gamma$ for $u\in U_{j1\gamma}$. Since l is odd, the last long root in $\boldsymbol{k}_\alpha(uE_j)=\rho_\gamma(u)N_\gamma$ is δ. From the definition of $\hat{u}$, we see that $\hat{u}\in U_{j1}$, and in $\boldsymbol{k}_\alpha(\hat{u}E_j)=\rho_\delta(\hat{u})N_\delta$, the last long root is γ, and all the other elements coincide with the respective elements in $\rho_\gamma(u)N_\gamma$. Let $(\gamma_1', \gamma_2', \cdots, \gamma_{n-3}')$ be the common part, then

$$\rho_\gamma(u)N_\gamma=(\gamma_1', \gamma_2', \cdots, \gamma_{l-3}', \delta,^{\nabla} \gamma_{l-2}', \cdots, \gamma_{n-3}'),$$
(9.63)
$$\rho_\delta(u)N_\delta=(\gamma_1', \gamma_2', \cdots, \gamma_{l-3}', \gamma,^{\nabla} \gamma_{l-2}', \cdots, \gamma_{n-3}').$$

Now let us rewrite the last result in § 9.8. In (9.47)-(9.49), we see from (9.42), (9.46) that for $\eta\in\Sigma$, $\Lambda_1=s'\mathrm{Ad}(g_{\alpha'})\Lambda$,

$$(\tau^{-1}\Lambda_1, \eta)=(\nu^{-1}s\cdot\mathrm{Ad}(g_0)\Lambda, x_{\alpha'}^{-1}\eta),$$

where $\tau=\nu_{F_{\alpha'}}$. Let $E_j=(\alpha_1, \alpha_2, \cdots, \alpha_n)$, then by (9.40), (9.41), we get

$$x_{\alpha'}\alpha_{l-2}=\alpha',\ x_{\alpha'}\alpha_{l-1}=\alpha;\ x_{\alpha'}\alpha_p=\gamma_p\ (p\leqq l-3),\ x_{\alpha'}\alpha_{p+2}=\gamma_p\ (p\geqq l-2).$$

Since $\gamma=\alpha'$, $\delta=\alpha_l'=\gamma_{l-2}$, putting $\Lambda'=s\cdot\mathrm{Ad}(g_0)\Lambda$, we get

$$(\tau^{-1}\Lambda_1, \gamma_p)=(\Lambda', \nu\alpha_p)\ (p\leqq l-3),\quad (\tau^{-1}\Lambda_1, \gamma_p)=(\Lambda', \nu\alpha_{p+2})\ (p\geqq l-1);$$

$$(\tau^{-1}\Lambda_1, \gamma)=(\Lambda', \nu\alpha_{l-2}),\ (\tau^{-1}\Lambda_1, \delta)=(\Lambda', \nu\alpha_l),\ (\tau^{-1}\Lambda_1, \alpha)=(\Lambda', \nu\alpha_{l-1}).$$

Using these results, we obtain from Lemmas 9.19, 9.20 the following.

For $u\in U_{j1}$, put $u(j)=u$, $u(j+1)=\varphi_\alpha(u)$, then for $h_0\in M_\alpha$,

(9.64)
$$\sum_{i=j,j+1}\varepsilon(E_i)\sum_{s\in W_0}\mathrm{sgn}(s)H_\alpha Y(h_0;E_i, u(i), s\cdot\mathrm{Ad}(g_0)\Lambda)$$
$$=\varepsilon(E_j')\sum_{s\in W_0}\mathrm{sgn}(s)Y_j(h_0;u, s\cdot\mathrm{Ad}(g_0)\Lambda),$$

where the function Y_j in the right hand side is defined as follows. Let $h_0=h_U\exp X'$ with $h_U\in A_U$, $X'\in\mathfrak{h}_V'$, then for $\Lambda_1=s\cdot\mathrm{Ad}(g_0)\Lambda$,

(9.65)
$$Y_j(h_0;u, \Lambda_1)=U_j(h_0;u, \Lambda_1)\cdot V_j(h_0;u, \Lambda_1)$$

with

$$U_j(h_0;u, \Lambda_1)=\mathrm{sgn}\{(\Lambda_1, \nu\alpha_{l-2})(\Lambda_1, \nu\alpha_l)\}$$
$$\times\exp\{-\gamma(X')|(\Lambda_1, \nu\alpha_{l-2})|/|\alpha_{l-2}|^2\}\cdot\exp\{-\delta(X')|(\Lambda_1, \nu\alpha_l)|/|\alpha_l|^2\},$$
$$V_j(h_0;u, \Lambda_1)=\mathrm{sgn}\{(\Lambda_1, \nu\alpha_{l-1})\}\cdot\xi_{\Lambda_1}(h_U)$$
$$\times\mathrm{sgn}\{\prod_{\eta\in P(E)}(\Lambda_1, \nu\eta)\}\cdot\sum_{1\leq p\leq n-3}\exp\{-\gamma_p(X')|(\Lambda_1, \nu\beta_p)|/|\beta_p|^2\},$$

where $E=(\beta_1, \beta_2, \cdots, \beta_{n-3})=(\alpha_1, \alpha_2, \cdots, \alpha_{l-3},^{\nabla} \alpha_{l+1}, \cdots, \alpha_n)$.

It follows from what was seen above that $U_j(h_0;u, \Lambda_1)$ is obtained from $U_j(h_0;\hat{u}, \Lambda_1)$ by an exchange of the roots γ and δ, and

(9.66)
$$V_j(h_0;\hat{u}, \Lambda_1)=V_j(h_0;u, \Lambda_1).$$

On the other hand, put $\delta_1=\nu\alpha_{l-2}$, $\delta_2=\nu\alpha_l$. Then they are singular imaginary roots of $\mathfrak{a}$ such that $2^{-1}(\delta_1\pm\delta_2)\in\Sigma(\mathfrak{a})$. Hence we see by Lemma 7.6 that there exists a $\sigma\in W_0=W_G(\mathfrak{a})$ such that

$$\sigma\delta_1=\varepsilon\delta_2,\ \sigma\delta_2=\varepsilon\delta_1\ (\varepsilon=1 \text{ or } -1);\ \sigma\eta=\eta \text{ for } \eta\in\Sigma(\mathfrak{a}),\ \perp\delta_1,\ \delta_2.$$

Clearly we have $V_j(h_0\,;\,u,\,\sigma\Lambda_1)=V_j(h_0\,;\,u,\,\Lambda_1)=V_j(h_0\,;\,\hat{u},\,\Lambda_1)$. Moreover, since $|\alpha_{l-2}|=|\alpha_l|$, we get $U_j(h_0\,;\,u,\,\sigma\Lambda_1)=U_j(h_0\,;\,\hat{u},\,\Lambda_1)$. Hence

$$Y_j(h_0\,;\,u,\,\sigma\Lambda_1)=Y_j(h_0\,;\,\hat{u},\,\Lambda_1)\,.$$

This gives us for any $s\in W_0$,

$$\mathrm{sgn}(\sigma s)Y_j(h_0\,;\,u,\,\sigma s\cdot\mathrm{Ad}(g_0)\Lambda)+\mathrm{sgn}(s)Y_j(h_0\,;\,\hat{u},\,s\cdot\mathrm{Ad}(g_0)\Lambda)=0\,.$$

This equality together with (9.64) proves Lemma 9.18. Thus the proof of Lemma 7.1 is now complete when α is long and $\Sigma(\alpha)=\Sigma_R(A)$ is of class II (§§ 9.5-9.10).

9.11. Let us consider the general case where $\Sigma_R(A)$ is no longer simple. We see that the discussions to prove Lemma 7.1 can be carried out exclusively only on the simple component $\Sigma(\alpha)$ of $\Sigma_R(A)$ containing $\alpha\in\Pi$ in question. This fact comes out from the very definition of the functions Y and Z in § 3.1.

Now we have completed the proof of Lemma 7.1, and accordingly the proof of Theorem 2. Since Theorem 1 follows from Theorem 2 by means of the result in § 3.4, this is the end of our long proof (§§ 6-9). (How well everything works!)

Appendix. The case of the holomorphic discrete series for $Sp(n,\,\boldsymbol{R})$

1. As is remarked in Introduction, for the group $Sp(n,\,\boldsymbol{R})$, the character formula in Theorem 2 is an another expression of the formula given in [5(e), § 7]. Once this is admitted, the reduction of the general formula in Theorem 2 to the known one in the case of holomorphic discrete series [3, 6] is proved easily by using certain identities of the same kind as (A15) below in the symmetric groups [5(e), § 9]. This is remarked in Introduction. However, since the equivalence between the formula in Theorem 2 and that in [5(e)] is not so apparent, it has some worth to mention here how the reduction occurs for the holomorphic discrete series. It is a simple calculation given, in a certain sense, by going back to the formula in [5(e)] and following the proof there.

For the group $SO_0(2n,\,2n+1)$ and $\Lambda\in\mathfrak{b}_B^*$ in the Weyl chamber of Borel-de Siebenthal, the reduction of the same kind is given by S. Mikami [11]. Similarly as for $Sp(n,\,\boldsymbol{R})$, it is carried out using some identities like (A15) below in the symmetric groups. Note that for the group $SO_0(2n,\,2n)$, the formula in Theorem 1 contains no cancellation. In general, for $\Lambda\in\mathfrak{b}_B^*$ in the Weyl chamber of Borel-de Siebenthal, the reduced formula is given also by J. A. Vargas [12] by a completely different method.

2. Let $G=Sp(n, \boldsymbol{R})$ and A the connected component of the unit element of a split Cartan subgroup. Let $\mathfrak{h}$ be the corresponding Cartan subalgebra, then the root system $\Sigma_R(A)$ is identical with $\Sigma(\mathfrak{h})$ and of type C_n. We use the notations in (1.5), and introduce the co-ordinates $(t_1, t_2, \cdots, t_n)$ of $X\in\mathfrak{h}_c$ by $t_i=e_i(X)$ $(1\leqq i\leqq n)$. The standard systems in $M(P_R(A))$ are given by

$$F_k=\{2e_i\ (1\leqq i\leqq l),\ e_{l+2j-1}+e_{l+2j}\ (1\leqq j\leqq k)\},$$

where $l+2k=n$. Let $\mathfrak{S}_n$ be the symmetric group of order n and define its operation on i $(1\leqq i\leqq n)$ in such a way that $(s\sigma)(i)=\sigma(s(i))$ $(s, \sigma\in\mathfrak{S}_n)$. All the elements in $M^{or}(P_R(A))$ of type l are given as follows:

(A1) $$E=(2e_{\sigma(1)}, 2e_{\sigma(2)}, \cdots, 2e_{\sigma(l)}, e_{\sigma(l+1)}+e_{\sigma(l+2)}, e_{\sigma(l+1)}-e_{\sigma(l+2)}, \cdots, e_{\sigma(n-1)}+e_{\sigma(n)}, e_{\sigma(n-1)}-e_{\sigma(n)}),$$

where $\sigma\in\mathfrak{S}_n$ satisfies the condition

(A2) $$\begin{cases} \sigma(2i-1)<\sigma(2i)\ (1\leqq i\leqq m=[l/2]),\ \sigma(1)<\sigma(3)<\cdots<\sigma(2m-1); \\ \sigma(l+2i-1)<\sigma(l+2i)\ (1\leqq i\leqq k),\ \sigma(l+1)<\sigma(l+3)<\cdots<\sigma(l+2k-1) \\ \qquad =\sigma(n-1). \end{cases}$$

Put $N=[n/2]$, then $k+m=N$, $0\leqq k\leqq N$. Put $E_k=\tilde{F}_k$, then it is standard, and by (1.21) and (1.39), we have

(A3) $$P(E_k)=\{2e_i\ (1\leqq i\leqq l),\ e_{2j-1}\pm e_{2j}\ (1\leqq j\leqq m),\ e_{l+2i-1}\pm e_{l+2i}\ (1\leqq i\leqq k)\},$$

(A4) $$\varepsilon(E_k)=(-1)^{n-m}=(-1)^{n-N+k}.$$

3. Now put $\mathfrak{b}=\mathfrak{h}^{F_0}=\nu_{F_0}(\mathfrak{h}_c)\cap\mathfrak{g}$, and let B be the corresponding Cartan subgroup. Introduce the co-ordinates $(x_1, x_2, \cdots, x_n)$ of $X\in\mathfrak{b}_c$ by $x_i=e_i(\nu_{F_0}^{-1}X)$ $(1\leqq i\leqq n)$. Then the real subspace $\mathfrak{b}$ is characterized by $x_j\in\sqrt{-1}\boldsymbol{R}$ $(1\leqq j\leqq n)$, and the kernel of the mapping exp of $\mathfrak{b}$ onto B is the n-times product of $\sqrt{-1}2\pi\boldsymbol{Z}$. Therefore $\Lambda\in\mathfrak{b}_B^*$ is given by

$$\Lambda(X)=\sum_{1\leq j\leq n}\lambda_j x_j \qquad (X=(x_1, x_2, \cdots, x_n)\in\mathfrak{b}_c),$$

where $\lambda_j\in\boldsymbol{Z}$ for $1\leqq j\leqq n$ and X is denoted by its co-ordinates as $X=(x_1, x_2, \cdots, x_n)$. The group $W_G(\mathfrak{b})\cong W_G(B)$ is given as follows. Since the roots $\nu_{F_0}(2e_i)$ and $\nu_{F_0}(2e_{i+1})$ are both singular imaginary, exactly one of $\nu_{F_0}(e_i\pm e_{i+1})$ is compact. Assume that $\nu_{F_0}(e_i-\varepsilon'_i e_{i+1})$ is compact, where $\varepsilon'_i=1$ or -1. Put $\varepsilon_1=1$, $\varepsilon_i=\varepsilon'_1\varepsilon'_2\cdots\varepsilon'_{i-1}$ $(2\leqq i\leqq n)$, and introduce auxiliary co-ordinates for $X\in\mathfrak{b}$ and $\Lambda\in\mathfrak{b}_B^*$ as follows:

$$x'_i=\varepsilon_i x_i\ (1\leqq i\leqq n);\quad \lambda'_i=\varepsilon_i\lambda_i\ (1\leqq i\leqq n).$$

Then $\Lambda(X)=\lambda'_1x'_1+\lambda'_2x'_2+\cdots+\lambda'_nx'_n$ and $W_G(\mathfrak{b})$ consists of the transformations given by

(A5) $$X=(\varepsilon_1x'_1, \varepsilon_2x'_2, \cdots, \varepsilon_nx'_n)\mapsto sX=(\varepsilon_1x'_{s(1)}, \varepsilon_2x'_{s(2)}, \cdots, \varepsilon_nx'_{s(n)}),$$

where $s\in\mathfrak{S}_n$. Then for $\Lambda=(\varepsilon_1\lambda'_1, \varepsilon_2\lambda'_2, \cdots, \varepsilon_n\lambda'_n)$, we have $s\Lambda=(\varepsilon_1\lambda'_{s(1)}, \varepsilon_2\lambda'_{s(2)}, \cdots, \varepsilon_n\lambda'_{s(n)})$. The sign of this element in $W_G(\mathfrak{b})$ coincides with the usual sign $\mathrm{sgn}(s)$ of s in $\mathfrak{S}_n$. Thus the function $\tilde{\kappa}^{\mathfrak{b}}_\Lambda$ on B corresponding to Θ_Λ is given by

(A6) $$\tilde{\kappa}^{\mathfrak{b}}_\Lambda(\exp X)=\sum_{s\in\mathfrak{S}_n}\mathrm{sgn}(s)\exp\Big(\sum_{1\leq j\leq n}\lambda'_{s(j)}x'_j\Big).$$

Put $\lambda'_i=\nu_i p_i$, that is, $\lambda_i=\varepsilon_i\nu_i p_i$ with $\nu_i=\pm1$, $p_i>0$. We put on Λ the following condition:

(A7) $$p_1>p_2>\cdots>p_n>0\,,\quad \nu_1=\nu_2=\cdots=\nu_n=\nu\ (put)\,.$$

Then $\varepsilon(\Lambda)=\nu^n\varepsilon_1\varepsilon_2\cdots\varepsilon_n$. Assume that the root vectors used to define ν_{F_0} satisfy Condition 5.1. Then we assert that $\varepsilon_{n-2j-1}=\varepsilon_{n-2j}$ $(0\leqq j\leqq N-1)$ and so under the condition (A7), we have $\varepsilon(\Lambda)=\nu^n$, and for $\exp X\in A$ with $X=(t_1, t_2, \cdots, t_n)\in\mathfrak{h}$ such that $t_1>t_2>\cdots>t_n>0$,

(A8) $$\tilde{\kappa}^{\mathfrak{h}}_\Lambda(\exp X)=(-\nu)^n\sum_{s\in\mathfrak{S}_n}\mathrm{sgn}(s)\exp\{-\sum_{1\leq j\leq n}p_{s(j)}t_j\}\,.$$

Note that, with respect to the canonical order, simple roots are given by e_i-e_{i+1} $(1\leqq i\leqq n-1)$, $2e_n$.

4. Let us first prove that under Condition 5.1, we have $\varepsilon'_{n-2j-1}=1$, that is, $\nu_{F_0}(e_{n-2j-1}-e_{n-2j})$ is compact for $0\leqq j\leqq N-1$. Once this is so, we have $\varepsilon_{n-2j-1}=\varepsilon_{n-2j}$ for $0\leqq j\leqq N-1$, and $\varepsilon_1\varepsilon_2\cdots\varepsilon_l=1$ for $l+2k=n$. Let $X_{\pm2e_i}$ $(1\leqq i\leqq n)$ be the root vectors used to define ν_{F_0}. Then Condition 5.1 says the following: let $\alpha'=2e_{n-2j-1}$, $\alpha=2e_{n-2j}$ $(0\leqq j\leqq N-1)$, and $\gamma'=2^{-1}(\alpha'+\alpha)$, $\gamma=2^{-1}(\alpha'-\alpha)$, then there exist root vectors $Y_{\gamma'}$ and $Y_{\pm\gamma}$ for γ' and $\pm\gamma$ such that $X_{\alpha'}=[Y_\gamma, Y_{\gamma'}]$, $X_\alpha=[Y_{-\gamma}, Y_{\gamma'}]$. Thus we come to the situation in Lemma 7.5.

Lemma A1. *Let $\mathfrak{g}$, $\mathfrak{h}$, F, α, α', γ and γ' be as in Lemma 7.5. Let ν_F be the automorphism $\mathfrak{g}_c$ defined by the system of root vectors $\{X_{\pm\delta}\in\mathfrak{g}\,;\,\delta\in F\}$ satisfying $[X_\delta, X_{-\delta}]=H_\delta$. Assume that every root in $F-\{\alpha,\alpha'\}$ is strongly orthogonal to γ, γ' and that $X_{\alpha'}=[Y_\gamma, Y_{\gamma'}]$, $X_\alpha=[Y_{-\gamma}, Y_{\gamma'}]$ for some root vectors $Y_{\pm\gamma}$, $Y_{\gamma'}\in\mathfrak{g}$ for $\pm\gamma$ and γ'. Then the root $\nu_F\gamma=2^{-1}\nu_F(\alpha'-\alpha)$ of $\mathfrak{h}^F$ is compact, and the other $\nu_F\gamma'=2^{-1}\nu_F(\alpha'+\alpha)$ is singular.*

Proof. Let $\mathfrak{g}(\alpha,\alpha')$ be the subalgebra of $\mathfrak{g}$ generated by $Y_{\pm\gamma}$, $Y_{\pm\gamma'}$, where $Y_{-\gamma'}\in\mathfrak{g}$ is also a root vector for $-\gamma'$, then $\mathfrak{g}(\alpha,\alpha')\cong\mathfrak{sp}(2,\boldsymbol{R})$. Since $[X_{\pm\delta}, X]=0$ for $\delta\in F-\{\alpha,\alpha'\}$, $X\in\mathfrak{g}(\alpha,\alpha')$, it is sufficient to prove the lemma in the case where $\mathfrak{g}=\mathfrak{g}(\alpha,\alpha')$. Thus we may take $\mathfrak{g}=\mathfrak{sp}(2,\boldsymbol{R})$. We first prove the assertion for a special choice of $F=\{\alpha,\alpha'\}$ and $X_{\pm\alpha}$, $X_{\pm\alpha'}$ given in the proof of Lemma 7.5. Let T, U be as there. Note that $\nu_\alpha=\exp\{-\sqrt{-1}(\pi/4)\mathrm{ad}(X'_\alpha+X'_{-\alpha})\}$ is the inner automorphism of $\mathfrak{g}_c$ given by the element $\exp\{-\sqrt{-1}(\pi/4)(X'_\alpha+X'_{-\alpha})\}$ of $Sp(2,\boldsymbol{C})$, and similarly for $\nu_{\alpha'}$. Then we get by a simple calculation

$$\nu_F H'_\alpha=\nu_\alpha H'_\alpha=\begin{pmatrix}0 & -iT\\ iT & 0\end{pmatrix},\quad \nu_F H'_{\alpha'}=\nu_{\alpha'}H'_{\alpha'}=\begin{pmatrix}0 & -iU\\ iU & 0\end{pmatrix}.$$

Remark that $\nu_F^{-1}H'_\alpha=-\nu_F H'_\alpha$, $\nu_F^{-1}H'_{\alpha'}=-\nu_F H'_{\alpha'}$, that is, $\nu_F^{-1}|\mathfrak{h}_c=-\nu_F|\mathfrak{h}_c$. Moreover $\nu_F H'_\gamma$, $\nu_F H'_{\gamma'}$ are given easily and we get two root vectors Z_ε ($\varepsilon=\pm1$) for $\varepsilon\nu_F\gamma$ as follows:

$$\nu_F H'_\gamma=\begin{pmatrix}0 & -D\\ D & 0\end{pmatrix},\quad \nu_F H'_{\gamma'}=\begin{pmatrix}0 & -iE\\ iE & 0\end{pmatrix},\quad Z_\varepsilon=\begin{pmatrix}A & \varepsilon B\\ -\varepsilon B & A\end{pmatrix},$$

where

$$D=\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix},\quad E=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad A=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad B=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}.$$

We see that Z_ε belong to $\mathfrak{k}_c=\{X\in\mathfrak{g}_c\,;\,\theta X=X\}$ for $\theta X=-{}^tX$, whence $\nu_F\gamma$ is compact.

Now consider for $\mathfrak{g}=\mathfrak{sp}(2,\boldsymbol{R})$ another choice of $F^*=\{\alpha^*,\alpha^{*\prime}\}$, $X^*_{\pm\alpha^*}$, $X^*_{\pm\alpha^{*\prime}}$ satisfying Condition 5.1. Let $\gamma^*=2^{-1}(\alpha^{*\prime}-\alpha^*)$, $\gamma^{*\prime}=2^{-1}(\alpha^{*\prime}+\alpha^*)$, then by assumption, $X^*_{\alpha^{*\prime}}=[Y^*_{\gamma^*},Y^*_{\gamma^{*\prime}}]$, $X^*_{\alpha^*}=[Y^*_{-\gamma^*},Y^*_{\gamma^{*\prime}}]$ for some root vectors $Y^*_{\pm\gamma^*}$, $Y^*_{\gamma^{*\prime}}$. Since all the roots of $\mathfrak{h}$ are real, there exists an element $g\in Sp(2,\boldsymbol{R})$ such that

$$\mathrm{Ad}(g)\gamma=\gamma^*,\ \mathrm{Ad}(g)\gamma'=\gamma^{*\prime}\,;\ \mathrm{Ad}(g)Y_\gamma=\varepsilon Y^*_{\gamma^*},\ \mathrm{Ad}(g)Y_{\gamma'}=\varepsilon' Y^*_{\gamma^{*\prime}},$$

where ε, $\varepsilon'=1$ or -1. Then we have $\mathrm{Ad}(g)X_{\pm\alpha}=\varepsilon\varepsilon' X^*_{\pm\alpha^*}$, $\mathrm{Ad}(g)X_{\pm\alpha'}=\varepsilon\varepsilon' X^*_{\pm\alpha^{*\prime}}$, and hence

$$\mathrm{Ad}(g)\circ\nu_F\circ\mathrm{Ad}(g)^{-1}=(\nu_{F^*})^{\varepsilon\varepsilon'}.$$

Note that $\mathrm{Ad}(g)^{-1}\gamma^*=\gamma$, $(\nu_F)^{\varepsilon\varepsilon'}\gamma=(\varepsilon\varepsilon')\nu_F\gamma$, then we get $\nu_{F^*}\gamma^*=(\varepsilon\varepsilon')\mathrm{Ad}(g)\nu_F\gamma$. Since $\mathrm{Ad}(g)$ maps the compact roots of $\mathfrak{h}^F$ to the compact roots of $\mathfrak{h}^{F^*}$, we see that $\nu_{F^*}\gamma^*$ is a compact root of $\mathfrak{h}^{F^*}$. Q. E. D.

Applying this lemma to our situation, we see that $\nu_{F_0}(e_{n-2j-1}-e_{n-2j})$ is compact for $0\leqq j\leqq N-1$, as desired.

5. For $\alpha=2e_i$ or $e_i\pm e_j\in\Sigma_R(A)$, we have

$$\text{(A9)}\qquad (s\Lambda,\nu_{F_0}\alpha)/|\alpha|^2=\begin{cases}(s\Lambda)_i/2 & \text{for } \alpha=2e_i\,,\\ ((s\Lambda)_i\pm(s\Lambda)_j)/2 & \text{for } \alpha=e_i\pm e_j\,,\end{cases}$$

where $(s\Lambda)_i$ denotes the i-th component of the co-ordinates of $s\Lambda$: $(s\Lambda)_i=\varepsilon_i\lambda'_{s(i)}$. Note that $\varepsilon_1\varepsilon_2\cdots\varepsilon_l=1$ and $\mathrm{sgn}(p_i^2-p_j^2)=\mathrm{sgn}(j-i)$, we obtain under the condition (A7),

$$\text{(A10)}\qquad \mathrm{sgn}_{P(E_k)}(s\Lambda)=\nu^l\prod_{1\leq i\leq m}\mathrm{sgn}(s(2i)-s(2i-1))\prod_{1\leq j\leq k}\mathrm{sgn}(s(l+2j)-s(l+2j-1))\,.$$

For an element E in (A1), there exists a $u\in W(\Sigma_R(A))$ such that $E=uE_k$. Then by (3.7), we have for $X=(t_1,t_2,\cdots,t_n)\in\mathfrak{h}$,

$$Y(\exp X\,;E_k,u,s\Lambda)=\nu^l\prod_{1\leq i\leq m}\mathrm{sgn}(s(2i)-s(2i-1))$$
$$\times\prod_{1\leq j\leq k}\mathrm{sgn}(s(l+2j)-s(l+2j-1))\cdot\exp(-f(t,p,\sigma,s))\,,$$

where

$$f(t, p, \sigma, s)=\sum_{1\leq j\leq l} p_{s(j)}t_{\sigma(l)}$$

$$+\sum_{1\leq i\leq k} 2^{-1}\{|p_{s(l+2i-1)}+p_{s(l+2i)}|(t_{\sigma(l+2i-1)}+t_{\sigma(l+2i)})$$

$$+|p_{s(l+2i-1)}-p_{s(l+2i)}|(t_{\sigma(l+2i-1)}-t_{\sigma(l+2i)})\}.$$

Here we used the fact that $\varepsilon_{l+2i-1}=\varepsilon_{l+2i}$. Note that $\nu^l=\nu^n$ and

(A11)
$$\begin{cases} 2^{-1}(|p_i+p_j|+|p_i-p_j|)=p_{\min(i,j)}, \\ 2^{-1}(|p_i+p_j|-|p_i-p_j|)=p_{\max(i,j)}. \end{cases}$$

Consider the following condition on $s\in\mathfrak{S}_n$:

(A12k) $$s(l+2i-1)<s(l+2i) \qquad (1\leqq i\leqq k).$$

Then if $s\in\mathfrak{S}_n$ satisfies (A12k), we have

$$f(t, p, \sigma, s)=\sum_{1\leq j\leq n} p_{s(j)}t_{\sigma(j)}.$$

Taking into account (A11) and the factor $\text{sgn}(s)\prod_{1\leq j\leq k}\text{sgn}(s(l+2j)-s(l+2j-1))$, we get

$$\sum_{s\in\mathfrak{S}_n}\text{sgn}(s)Y(\exp X; E_k, u, s\Lambda)=\nu^n \sum_{(s,\sigma):(A2k)(A12k)}\text{sgn}(s)$$

$$\times\prod_{1\leq i\leq m}\text{sgn}(s(2i)-s(2i-1))2^k\exp(-\sum_{1\leq j\leq n} p_{s(j)}t_{\sigma(j)}),$$

where the sum runs over all (s, σ) satisfying (A2k)=(A2) for k, and (A12k).

Put $j'=\sigma(j)$, $s'=\sigma^{-1}s$, then $s(j)=s(\sigma^{-1}(j'))=s'(j')$, $\text{sgn}(s)=\text{sgn}(s')\text{sgn}(\sigma)$,

$$\sum_{1\leq j\leq n} p_{s(j)}t_{\sigma(j)}=\sum_{1\leq j\leq n} p_{s'(j)}t_j,$$

and the condition (A12k) takes the form

(A13$s'k$) $$s'(\sigma(l+2j-1))<s'(\sigma(l+2j)) \qquad (1\leqq j\leqq k).$$

For a fixed $s'\in\mathfrak{S}_n$, (A13$s'k$) is considered as a condition on σ. Thus

$$Z(\exp X; E_k, \Lambda, P_R(\Lambda))=\nu^n\sum_{s'\in\mathfrak{S}_n}\ \sum_{\sigma:(A2k)(A13s'k)}\text{sgn}(s')$$

$$\times 2^k\text{sgn}(\sigma)\prod_{1\leq i\leq m}\text{sgn}(s'(\sigma(2i))-s'(\sigma(2i-1))\exp(-\sum_{1\leq j\leq n} p_{s'(j)}t_j).$$

Since $\varepsilon(E_k)=(-1)^{n-N+k}$, we have

$$\sum_{0\leq k\leq N}\varepsilon(E_k)Z(\exp X; E_k, \Lambda, P_R(\Lambda))=(-\nu)^n\sum_{s'\in\mathfrak{S}_n}\text{sgn}(s')J(s')\exp(-\sum_{1\leq j\leq n} p_{s'(j)}t_j),$$

where

(A14) $$J(s')=\sum_{0\leq k\leq N}\ \sum_{\sigma:(A2k)(A13s'k)}(-1)^{N+k}2^k\text{sgn}(\sigma)$$

$$\times\prod_{1\leq i\leq m}\text{sgn}(s'(\sigma(2i))-s'(\sigma(2i-1))).$$

Thus our task is to prove $J(s')=1$ for any $s'\in\mathfrak{S}_n$.

6. Now fix $s'\in\mathfrak{S}_n$. Denote by $C_k(s')$ the set of all $\sigma\in\mathfrak{S}_n$ satisfying (A2k) and (A13$s'k$). For $k=0$, (A13$s'k$) does not exist and $C_0(s')=C_0$ is independent of s'. For $0\leqq k\leqq N$ and $\sigma\in\mathfrak{S}_n$, denote by $P_k(\sigma)$ the set of ordered pairs $(\sigma(2i-1), \sigma(2i))$ $(1\leqq i\leqq m)$ and $(\sigma(l+2j-1), \sigma(l+2j))$ $(1\leqq j\leqq k)$. For $\tau\in C_0$, let $C_k(s', \tau)$ be the subset of $C_k(s')$ consisting of $C_k(s')$ such that $P_k(\sigma)=P_0(\tau)$. Put

$$P_0(\tau, s')=\{(i, j)\in P_0(\tau)\,;\ s'(i)<s'(j)\},$$

and $M=M_{s'}=\# P_0(\tau, s')$. Then $\# C_k(s', \tau)=\binom{M}{k}$. In fact, put

$$D_\sigma=\{(\sigma(l+2j-1), \sigma(l+2j))\,;\quad 1\leqq j\leqq k\},$$

then $D_\sigma\in P_0(\tau, s')$ for $\sigma\in C_k(s', \tau)$, and conversely, for any subset D of k-elements of $P_0(\tau, s')$, there exists exactly one element $\sigma\in C_k(s', \tau)$ such that $D_\sigma=D$. Moreover note that for any $\sigma\in C_k(s', \tau)$,

$$\operatorname{sgn}(\sigma)=\operatorname{sgn}(\tau),\qquad \prod_{1\leq i\leq m}\operatorname{sgn}(s'(\sigma(2i))-s'(\sigma(2i-1)))=(-1)^{N-M}.$$

Then we get for a fixed $s'\in\mathfrak{S}_n$ and $\tau\in C_0$,

$$\sum_{0\leq k\leq N}\ \sum_{\sigma\in C_k(s',\tau)}(-1)^{N+k}2^k\operatorname{sgn}(\sigma)\prod_{1\leq i\leq m}\operatorname{sgn}(s'(\sigma(2i))-s'(\sigma(2i-1))$$

$$=\operatorname{sgn}(\tau)(-1)^{N-M}\sum_{0\leq k\leq M}(-1)^{N+k}2^k\binom{M}{k}=\operatorname{sgn}(\tau)(-1)^{N-M}(-1)^N(1-2)^M$$

$$=\operatorname{sgn}(\tau).$$

Note that $C_k(s')$ is the disjoint union of $C_k(s', \tau)$ over $\tau\in C_0$. Then we get

$$J(s')=\sum_{\tau\in C_0}\operatorname{sgn}(\tau).$$

On the other hand, we have

(A15) $$\sum_{\tau\in C_0}\operatorname{sgn}(\tau)=1, \text{ or equivalently, } \sum_{\tau:\,(\mathrm{A16})}\operatorname{sgn}(\tau)=N!,$$

where the last sum runs over all $\tau\in\mathfrak{S}_n$ satisfying

(A16) $$\tau(2i-1)<\tau(2i)\qquad (1\leqq i\leqq N).$$

Hence we have $J(s')=1$ as desired.

Department of Mathematics
Kyoto University

Symbols

References

[1] N. Bourbaki, Groupes et algèbres de Lie, Chap. 4, 5 et 6, Hermann, Paris, 1968.

[2] Harish-Chandra,

(a) Fourier transforms on a semi-simple Lie algebra I, Amer. J. Math., **79** (1957), 193-257.

(b) Invariant eigendistributions on a semi-simple Lie group, Trans. Amer. Math. Soc., **119** (1965), 457-508.

(c) Discrete series for semi-simple Lie groups I, Acta Math., **113** (1965), 241-318.

(d) Discrete series for semi-simple Lie groups II, Acta Math., **116** (1966), 1-111.

[3] H. Hecht, Characters of some representations of Harish-Chandra, Math. Ann., **219** (1976), 213-226.

[4] S. Helgason, Differential geometry and symmetric spaces, Academic Press, New York-London, 1962.

[5] T. Hirai,

(a) Invariant eigendistributions of Laplace operators on real simple Lie groups I, Case os $SU(p, q)$. Japan. J. Math., **39** (1970), 1-68.

(b) Explicit form of the characters of discrete series representations of semi-simple Lie groups, In: Proceedings of Symposia in Pure Mathematics **26**, pp. 281-287, Amer. Math. Soc., 1973.

(c) Invariant eigendistributions of Laplace operators on real simple Lie groups II, General theory for semisimple Lie groups, Japan. J. Math. New Series **2** (1976), 27-89.

(d) Invariant eigendistributions of Laplace operators on real simple Lie groups III, Methods of construction for semisimple Lie groups, Ibid. **2** (1976), 269-

341.
(e) Invariant eigendistributions of Laplace operators on real simple Lie groups IV, Explicit forms of the characters of discrete series representations for $Sp(n, \boldsymbol{R})$, Ibid. **3** (1977), 1-48.

[6] S. Martens, The characters of the holomorphic discrete series, Proc. Nat. Acad. Sci. USA, **72** (1975), 3275-3276.

[7] H. Midorikawa,
(a) On a relation between characters of discrete series and non-unitary principal series representations, Proc, Japan Acad., **50** (1974), 29-32,
(b) On the explicit formulae of characters for discrete series represetations, Japan. J. Math., New Series **3** (1977), 313-368.

[8] W. Schmid,
(a) On the characters of the discrete series, Invent. math., **30** (1975), 47-144.
(b) Some remarks about the discrete series characters of $Sp(n, \boldsymbol{R})$. In: Proceedings of the Colloquium on Noncommutative Harmonic Analysis, Marseille, 1974. Lecture Notes in Mathematics, Springer, Berlin-Heidelberg-New York, 1975.

[9] M. Sugiura, Conjugacy classes of Cartan subalgebras in real semisimple Lie algebras, J. Math. Soc. Japan, **11** (1959), 474-534.

[10] J. Arthur, The characters of discrete series as orbital integrals. Invent. math., **32** (1976), 205-261.

Added for Appendix:

[11] S. Mikami, A simple expression of the characters of certain discrete series representations, J. Math. Kyoto Univ., **19** (1979), 435-470.

[12] J. A. Vargas, A character formula for the discrete series of a semisimple Lie group, preprint.