Global solutions to the initial value problem for the equations of one-dimensional motion of viscous polytropic gases

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§ 1. Introduction.

The one-dimensional motion of compressible, viscous and heat-conductive fluids is described by the equations in the Lagrangian mass coordinate (the suffix denotes the partial differentiation with respect to the variable $t$ or $x$):

$$
\begin{aligned}
\rho_t + \rho u_x &= 0 \\
\rho u_t + \rho u_x &= (\mu p u_x)_x \\
\left(e + \frac{u^2}{2}\right)_t + (p u)_x &= (\kappa \rho \theta_x)_x + (\mu p u u_x)_x,
\end{aligned}
$$

where $\rho$ is the density, $u$ is the velocity, $\theta$ is the absolute temperature, $\mu$ is the viscosity coefficient and $\kappa$ is the coefficient of heat conduction, and the pressure $p$ and the internal energy $e$ are related to $\rho$, $\theta$ by the equation of state of the fluids. Here we assume that the equation of state is one for the ideal polytropic gas:

(i) $p = R \rho \theta, \quad e = \frac{R}{\gamma - 1} \theta + \text{constant},$

where $R$ is the gas constant and the constant $\gamma \geq 1$ is the adiabatic exponent, and also assume that

(ii) $\mu, \kappa$ are positive constants.

If we take $\rho$ and $S = \text{entropy}/R$ as the basic variables, the equation of state (i) can be written in the form (cf. [1])

$$
\begin{aligned}
e &= \frac{R \bar{\theta}}{\gamma - 1} \left((\rho/\bar{\rho})^{\gamma - 1} \exp(\gamma - 1) (S - \bar{S}) - 1\right) \\
p &= \rho e_s = R \bar{\rho} \bar{\theta} \left((\rho/\bar{\rho})^{\gamma - 1} \exp(\gamma - 1) (S - \bar{S}) \right) \\
\theta &= e_s / R = \bar{\theta} \left((\rho/\bar{\rho})^{\gamma - 1} \exp(\gamma - 1) (S - \bar{S}) \right),
\end{aligned}
$$

(1.2)
where $\bar{\rho} > 0$, $\bar{\theta} > 0$ and $\bar{S}$ are typical states of $\rho$, $\theta$ and $S$ respectively. In particular we consider these states to be the constants $\bar{\rho}$, $\bar{\theta}$ and $\bar{S}$ in (1.5) (1.7). Then the system of equations of motion (1.1) is transformed to

\[
\begin{cases}
\rho_t + \rho^2 u_x = 0 \\
u_t + R\bar{\rho} ((\rho/\bar{\rho})^\gamma (S - \bar{S}))_x = \mu (\rho u_x)_x \\
S_t = \frac{\kappa (\gamma - 1)}{R} (\rho S_x + \rho x)_x + \frac{\kappa (\gamma - 1)^2}{\rho} (\rho S_x + \rho x)^2 + \frac{\mu}{\rho} \theta u_x^2 ,
\end{cases}
\]

where $\theta$ is the function of $\rho$, $S$ given by (1.2).

In the present paper we prove an existence theorem of the unique global solution for the initial value problem (1.1) under (i) (ii) provided $(\gamma - 1) \cdot \|H^1\text{-norm of the initial data}\) is appropriately small. In order to describe the arguments precisely we introduce some function spaces.

**Definition.** Let $l$ be a nonnegative integer and $0 < \sigma < 1$. $B^{1+\varepsilon}$ denotes the Hölder space of continuous functions $f(x)$, $x \in \mathbb{R}$, which have the $l$-th order derivatives of Hölder continuity with exponent $\sigma$. $\|_{1+l+\varepsilon}$ means its norm. $H^1$ denotes the Sobolev space of $L_2$-functions $f(x)$ which have the $l$-th order derivatives of $L_2$-functions. $\|_1$ is its norm. $B^{\sigma/2, \varepsilon} (Q_T)$ denotes the Hölder space of Hölder continuous functions $u(t, x)$ with the Hölder exponents $\sigma/2$ and $\varepsilon$ with respect to $t \in [0, T]$ and $x \in \mathbb{R}$ respectively. $\|_{\sigma/2, \varepsilon (Q_T)}$ is its norm. $B^{1+\varepsilon/2, 1+\varepsilon} (Q_T) = \{u \in B^{\sigma/2, \varepsilon} (Q_T); u_t, u_x, u_{xx} \in B^{\sigma/2, \varepsilon} (Q_T)\}$. $\|_{1+\varepsilon/2, 2+\varepsilon (Q_T)}$ is its norm. Let $X$ be a Banach space and $0 \leq t_1 < t_2 \leq \infty$. $C([t_1, t_2]; X)$ denotes the Banach space of continuous functions $u(t)$ on $[t_1, t_2]$ with the values in $X$. $L_q ([t_1, t_2]; X)$ denotes the Banach space of square summable functions $u(t)$ on $[t_1, t_2]$ with the values in $X$.

We assume that the initial data

\[(\rho, u, \theta) (0) = (\rho, u, \theta) (0, x) \in B^{1+\varepsilon}\]

for some $\varepsilon \in (0, 1)$, and satisfy the following conditions:

(iii) There are two constants $\bar{\rho} > 0$ and $\bar{S}$ such that for $1 \leq \gamma \leq 2$

\[(1.5) \quad E_0 = \|\rho(0) - \bar{\rho}, u(0), \sqrt{\gamma - 1} (S(0) - \bar{S})\|_1 < \infty.\]

(iv) The constant

\[(1.6) \quad c_0 = \inf_{x} \rho(0, x) / \bar{\rho} \leq 1\]

is positive and is independent of $\gamma \in [1, 2]$. The initial data $\theta(0)$ for the absolute temperature is determined by $\rho(0), S(0)$ using (1.2):

\[(1.7) \quad \theta(0, \cdot) = \tilde{\theta} \left( \rho(0, \cdot) / \bar{\rho} \right)^{-1} \exp (\gamma - 1) (S(0, \cdot) - \bar{S}) > 0,\]
where $\bar{\theta} > 0$ is any fixed constant.

Now we solve the problem (1.1) (1.4) globally in time as follows:

**Theorem.** Let us assume the conditions (i) (ii) for the system (1.1) and the conditions (iii) (iv) for the initial data (1.4). Then there exists a constant $\gamma_i$ $(1 < \gamma_i \leq 2)$ depending only on $c_0$ and $E_0$ such that the initial value problem (1.1) (1.4) has a unique solution $$(p, u, \theta) (t)$$ globally in time for any $\gamma \in [1, \gamma_i]$. The solution belongs to $B^{1+\gamma_i, 2+\gamma} (Q_T)$ for any $T \geq 0$ and satisfies the following:

$$\left\{ \begin{array}{l} \rho (t) - \bar{\rho}, u (t), (\theta (t) - \bar{\theta}) / \sqrt{\gamma - 1} \in C^0 (0, \infty ; H^1), \\ \rho_i (t), u_s (t), \theta_s (t) \in L^2 (0, \infty ; H^s) \text{ for any } \gamma \in [1, \gamma_i]. \end{array} \right.$$ (1.8)

The solution decays to the constant state in the maximum norm:

$$\lim_{t \to \infty} ||\rho (t) - \bar{\rho}, |u (t)|, \theta (t) - \bar{\theta}| / (\gamma - 1)^{1/4}|| = 0$$

for any $\gamma \in [1, \gamma_i]$.

The result and argument is an improved one of nice paper [4] of Kanel' for the barotropic model gas and the proof is based on the local existence theorem and on the a priori estimates of the energy in $H^1$. The initial boundary value problem (1.1) under the conditions (i) (ii) is solved globally in time by [5], but their arguments are restricted to the case of the finite interval with respect to $x$, and the solution is bounded by the estimate of exponential growth with respect to $t$. Our theorem for $1 \leq \gamma \leq \gamma_i$ is also valid for the initial boundary value problem and gives a uniform bound for the solution with respect to $t \geq 0$ and the exponential decay of the solution to the constant state as $t \to \infty$. In our theorem the constant $\gamma_i$ can be larger than 2 provided the initial data are small. This is a special case of that proved by [6]. The isothermal case (1.2) of $\gamma = 1$ is solved globally in time for general initial data by [3].

**§ 2. A priori estimates in $H^1$.**

In order to obtain a priori estimates let us assume that there is a solution $(\rho, u, \theta) (t), 0 \leq t \leq T,$ for any fixed $T > 0$ of the initial value problem (1.1), (1.4) which satisfies the following:

$$\left\{ \begin{array}{l} \rho (t) - \bar{\rho}, u (t), \theta (t) - \bar{\theta} \in C^0 (0, T ; H^1) \cap B^{1+\gamma_i, 2+\gamma} (Q_T), \\ \inf_{Q_T} \{ \rho (t, x), \theta (t, x) \} > 0. \end{array} \right.$$ (2.1)

Then we remind ourselves by (1.2) (1.7) that

(2.2) $$S(t) - \bar{S} \in C^0 (0, T ; H^1) \cap B^{1+\gamma_i, 2+\gamma} (Q_T)$$ and $(\rho, u, S) (t)$ satisfies the system (1.3).
**Proposition 2.1.** Let us assume the conditions (i)–(iv) for the initial value problem (1.1) (1.4) and let the solution \((\rho, u, \theta)(t)\) be given as (2.1) (2.2). If the solution also satisfies the inequalities in \(Q_T\)

\[
\begin{aligned}
\left\{ \frac{5}{4} \frac{r-1}{\gamma} \leq \rho(t, x)/\bar{\rho}, \\
M^{-1} \leq \exp(\gamma - 1) (S(t, x) - \bar{S}), \quad (\rho(t, x)/\bar{\rho})^{r-1} \leq M,
\end{aligned}
\]

where \(M = \min\left\{ \frac{5}{4}, 1 + \frac{2}{5} \frac{R_0^2}{\kappa} \right\} > 1\), then it has the following a priori estimates:

\[
\|\rho(t) - \bar{\rho}, u(t), (\theta(t) - \bar{\theta})/\sqrt{\gamma - 1}\|^2
+ \int_0^t \|\rho_x(s)\|^2 + \|(u_x, \theta_x)(s)\|^2 ds \leq C_1E_0^2,
\]

for \(0 \leq t \leq T\) and \(1 \leq \gamma \leq 2\), where the constant \(C_1 = C_1(c_0, E_0)\) does not depend on \(t, T\) and \(\gamma\).

At the same time we have the a priori estimates:

\[
\begin{aligned}
c_i^1 \leq \rho(t, x)/\bar{\rho} \leq c_i, \\
|u(t, x)| \leq c_i, \\
|S(t, x) - \bar{S}| \leq c_i/\sqrt{\gamma - 1}, \\
\exp(-\sqrt{\gamma - 1} c_i) \leq \theta(t, x)/\bar{\theta} \leq \exp(\sqrt{\gamma - 1} c_i),
\end{aligned}
\]

for any \((t, x) \in Q_T\), where the constant \(c_i = c_i(c_0, E_0) \geq 1, c_2 = c_2(C_1, E_0) \geq 0\) and \(c_3 = c_3(c_1, c_2) \geq 0\) do not depend on \(t, T\) and \(\gamma\).

The proof is given by lemmas 2.2–2.6. We use the function \(F(\rho, u, S)\) to get the a priori estimates of \(L^r\)-norm:

\[
F(\rho, u, S) = \frac{u^2}{2} + R\bar{\theta} f(\rho/\bar{\rho}) \exp(\gamma - 1) (S - \bar{S})
+ 2R\bar{\theta} (1 - \bar{\rho}/\rho) \exp(\gamma - 1) (S - \bar{S}) \text{sh} \frac{r-1}{2} (S - \bar{S}) + \frac{2R\bar{\theta}}{\gamma - 1} \text{sh}^2 \frac{r-1}{2} (S - \bar{S}),
\]

where

\[
f(y) = \int_0^y \frac{\xi^r - 1}{\xi^{r-1}} d\xi.
\]

The function \(F\) is equivalent to the quadratic function in a definite sense as follows:

**Lemma 2.2.** If \(\rho, S\) satisfy the condition

\[
\begin{aligned}
\left\{ \frac{5}{4} \frac{r-1}{\gamma} \leq \min\{\rho/\bar{\rho}, 1\} \\
\exp(\gamma - 1) |S - \bar{S}| \leq \frac{5}{4},
\end{aligned}
\]

then...
then it holds

\begin{equation}
\frac{u^2}{2} + R\tilde{\theta} f(\rho/\bar{\rho}) + \frac{R\tilde{\theta}}{20} (\gamma - 1) (S - \bar{S})^2 \leq F(\rho, u, S).
\end{equation}

In addition if $\rho/\bar{\rho} \geq c_0 > 0$ for some $c_0 \leq 1$, then it holds

\begin{equation}
F(\rho, u, S) \leq \frac{u^2}{2} + \frac{5R\tilde{\theta}}{2c_0^2} (\rho/\bar{\rho} - 1)^2 + R\tilde{\theta} (\gamma - 1) (S - \bar{S})^2.
\end{equation}

It is easy to see that

\begin{equation}
2(1 - \bar{\rho}/\rho) \exp \frac{-1}{2} (S - \bar{S}) \sinh \frac{-1}{2} (S - \bar{S})
\end{equation}

\begin{equation}
\leq \frac{5}{9} (\gamma - 1) (1 - \bar{\rho}/\rho)^2 \exp (\gamma - 1) (S - \bar{S}) + \frac{9}{5} \frac{1}{\gamma - 1} \sinh \frac{-1}{2} (S - \bar{S}).
\end{equation}

The mean value theorem gives the inequalities:

\begin{equation}
\left\{ \begin{array}{ll}
\frac{\gamma - 1}{2y} (y - 1)^2 \leq f(y) \leq \frac{\gamma - 1}{2} (y - 1)^2 & \text{for} \ 0 < y < 1, \\
\frac{\gamma - 1}{2y} (y - 1)^2 \leq f(y) \leq \frac{\gamma - 1}{2} (y - 1)^2 & \text{for} \ 1 < y < \infty.
\end{array} \right.
\end{equation}

Therefore the inequality (2.8) follows from (2.10) (2.11) and the condition (2.7).

\textbf{Lemma 2.3.} Under the hypotheses of Proposition 2.1 the solution has the following a priori estimate:

\begin{equation}
\int u^2 + f(\rho/\bar{\rho}) + (\gamma - 1) (S - \bar{S})^2 + (\rho z/\rho)^4 dx
\end{equation}

\begin{equation}
+ \int_0^t \int \rho u_t^2 + \frac{\rho}{\theta} \theta_t^2 + \frac{\theta}{\rho} \rho_t^2 dx ds \leq C(c_0) E_0 \quad \text{for} \quad 0 \leq t \leq T,
\end{equation}

where $c_0$ and $E_0$ are constants in (iii) (iv), and the constant $C(c_0)$ depending on $c_0$ does not depend on $t$, $T$ and $\gamma$.

First let us note the identity for the function $F(\rho, u, S)$ of the solution of (1.3).

\begin{equation}
F(\rho, u, S) + \mu \rho u_t^2 \frac{\delta}{\theta} \text{ch}(\gamma - 1) (S - \bar{S})
\end{equation}

\begin{equation}
+ \kappa \frac{\rho}{\theta} \theta_t^2 \frac{\delta}{\theta} \{ \text{ch}(\gamma - 1) (S - \bar{S}) - \text{sh}(\gamma - 1) (S - \bar{S}) \} = 0.
\end{equation}
\[
= \left\{ \mu u u_z - R \bar{\theta} ((\rho/\bar{\theta})^\prime \exp(\gamma - 1)(S - \bar{S}) - 1) u \\
+ \kappa \rho \theta_x \left( 1 - \frac{\bar{\theta}}{\theta} \text{ch}(\gamma - 1)(S - \bar{S}) \right) \right\} \theta_x - \kappa(\gamma - 1) \rho \theta_x \frac{\bar{\theta}}{\theta} \text{sh}(\gamma - 1)(S - \bar{S}),
\]

where the equality \((\gamma - 1) S_z = \theta_z/\theta - (\gamma - 1) \rho_z/\rho\) is used. Next the first two equations of (1.1) give the following identity.

\[
(2.14) \left\{ \frac{\mu}{2} \left( \frac{\rho_z^2}{\rho} \right) + \mu u \frac{\rho_z}{\rho} \right\} + R \mu \frac{\theta}{\rho} \rho_z^2 = \mu \rho u_z^2 - R \mu \rho \theta_x - \mu (\rho u u_z)_z.
\]

Add (2.13) to \(\beta\) times (2.14) for a constant \(\beta > 0\).

\[
(2.15) \left\{ F(\rho, u, S) + \beta \left( \frac{\mu}{2} \left( \frac{\rho_z^2}{\rho} \right) + \mu u \frac{\rho_z}{\rho} \right) \right\} t \\
+ \frac{\bar{\theta}}{\theta} \left\{ \text{ch}(\gamma - 1)(S - \bar{S}) - \beta \frac{\theta}{\bar{\theta}} \right\} \mu \rho u_z^2 + \frac{\bar{\theta}}{\theta} \text{exp} \left\{ - (\gamma - 1)(S - \bar{S}) \right\} \kappa \frac{\theta}{\bar{\theta}} \theta_x^2 \\
+ \beta R \mu \frac{\theta}{\rho} \rho_z^2 + \{ \} x \\
= - \left\{ \beta R \mu + \kappa(\gamma - 1) \frac{\theta}{\bar{\theta}} \text{sh}(\gamma - 1)(S - \bar{S}) \right\} \rho_x \theta_x.
\]

If we note the inequality as a result of (2.3)

\[
(2.16) M^{-1} \leq \theta(t, x)/\bar{\theta} \leq M^2,
\]

we have the estimate for the right hand side of (2.15).

\[
(2.17) \left| \beta R \mu + \kappa(\gamma - 1) \frac{\theta}{\bar{\theta}} \text{sh}(\gamma - 1)(S - \bar{S}) \right| \rho_x \theta_x \\
\leq \left| \beta R \mu + \frac{\kappa}{2} M (M^2 - 1) \right| \rho_x \theta_x = 2 \beta R \mu |\rho_x \theta_x| \\
\leq \frac{9}{10} \beta R \mu \frac{\theta}{\rho} \rho_z^2 + \frac{10}{9} \beta R \mu \kappa \frac{\theta}{\rho} \theta_x^2,
\]

where we take

\[
(2.18) \beta = \frac{\kappa}{2 R \mu} M (M^2 - 1) \leq \frac{9}{16}.
\]

By virtue of (2.3) (2.16) and (2.18) we have the estimates

\[
\frac{\bar{\theta}}{\theta} \left\{ \text{ch}(\gamma - 1)(S - \bar{S}) - \beta \frac{\theta}{\bar{\theta}} \right\} \geq M^{-1} \left( 1 - \frac{\kappa}{2 R \mu} M (M^2 - 1) \right) \geq \frac{1}{10 M^2},
\]

\[
\frac{\bar{\theta}}{\theta} \exp \left\{ - (\gamma - 1)(S - \bar{S}) \right\} - \frac{10}{9} \beta R \mu \kappa \geq M^{-1} \left( 1 - \frac{5}{9} M (M^2 - 1) \right) \geq \frac{1}{9 M^2}.
\]
from which we have the following by (2.15) (2.17).

\[
(2.19) \quad \left\{ F(\rho, u, S) + \beta \left( \frac{\mu}{2} \left( \frac{\theta z}{\rho} \right) + \mu u \frac{\theta z}{\rho} \right) \right\}_t + \left\{ \right\}_x + \frac{\mu}{10M^2} u_i^2 + \frac{c}{9M^3} \frac{\rho \theta^2}{\theta} + \frac{\beta R \mu}{10} \rho \theta^2 \leq 0.
\]

Therefore using lemma 2.2 and (2.18), we can conclude the estimate (2.12) by integration of (2.19) in \([0, t] \times \mathbb{R} \).

**Lemma 2.4.** Under the same assumptions of Proposition 2.1 there exists a constant \(c_i \geq 1\) depending only on \(c_0\) and \(E_0\) such that the density satisfies the a priori estimate:

\[
(2.20) \quad c_i \rho(t, x)/\rho \leq c_i \quad \text{in} \quad Q_T.
\]

The proof of (2.20) is the same as that of [4] because we have obtained the estimate (2.12). In fact it suffices to note the property of the function \(\phi\):

\[
\phi(\rho) = \int_\rho^\infty (\sqrt{f(\rho/\varrho)} d\rho/\varrho) \to -\infty,
\]

as \(\rho \to 0\) and \(\rho \to \infty\) respectively, and the estimate given by (2.12)

\[
|\phi(\rho(t, x))| \leq \left\{ \int f(\rho/\varrho) dx \int (\rho/\varrho)^{1/2} dx \right\}^{1/2} \leq C(c_0) E_0^3.
\]

**Lemma 2.5.** Under the same assumptions we have the a priori estimates of \(L_2\)-norm for the derivatives of \((u, \theta)(t)\) for \(0 \leq t \leq T \).

\[
(2.21) \quad \int u_i^2 dx + \int_0^t \int \rho u_i^2 dx ds \leq C E_i^3,
\]

\[
(2.22) \quad \int \theta_i^2 dx + (\gamma - 1) \int_0^t \int \rho \theta_i^2 dx ds \leq (\gamma - 1) C E_i^3,
\]

where the constant \(C = C(c_0, E_0)\) does not depend on \(t, T\) and \(\gamma\).

It follows from the equation (1.1) that

\[
(2.23) \quad \frac{1}{2} \int u_i^2 dx + \frac{\mu}{2} \int_0^t \int \rho u_i^2 dx ds \leq \frac{1}{2} \int u_i^2(0) dx + \frac{R^2}{\mu} \int_0^t \int \frac{1}{\rho} (\rho \theta_x + \theta_x)^2 dx ds + \mu \int_0^t \int \frac{1}{\rho} \rho \theta_x^2 dx ds,
\]

\[
(2.24) \quad \frac{1}{2} \int \theta_i^2 dx + \frac{\kappa (\gamma - 1)}{2R} \int_0^t \int \rho \theta_x^2 dx ds \leq \frac{1}{2} \int \theta_i^2(0) dx
\]
\[
+\frac{3R(\gamma-1)}{2\kappa} \int_0^t \int \rho^\gamma u_2^2 \, dx \, ds + \frac{3\kappa(\gamma-1)}{2R} \int_0^t \int \frac{1}{\rho} \theta_2^2 \, dx \, ds \\
+\frac{3\mu(\gamma-1)}{2R\kappa} \int_0^t \int \rho u_2^2 \, dx \, ds.
\]

If we use the estimate (2.12) and the condition (2.16), and note the inequalities for any \(\varepsilon > 0\)

\[
\begin{align*}
\rho u_2^2 & \leq \varepsilon \int \rho u_2^2 \, dx + \left( \frac{4}{\varepsilon} + C(c_0) E_0^3 \right) \int \rho u_2^2 \, dx \\
\rho \theta_2^2 & \leq \varepsilon \int \rho \theta_2^2 \, dx + \left( \frac{4}{\varepsilon} + C(c_0) E_0^3 \right) \int \rho \theta_2^2 \, dx,
\end{align*}
\]

the estimates (2.21) (2.22) are consequences of (2.23) (2.24) where we also use the relation

\[\| \theta_\ast (0) \| = \| (\gamma-1) (S_\ast + \rho_\ast / \rho) (0) \| \leq C \sqrt{\gamma-1} E_0.\]

At last we show the a priori estimates for \(S\) and \(\theta\).

**Lemma 2.6.** Under the same hypotheses we have the a priori estimates for \(S\) and \(\theta\) in \(Q_T\).

\[
|S(t, x) - \overline{S}| \leq c_5 / \sqrt{\gamma-1},
\]

\[
\exp \left( - \sqrt{\gamma-1} c_5 \right) \leq \theta(t, x) / \overline{\theta} \leq \exp \sqrt{\gamma-1} c_5,
\]

where \(c_5 = c_5(C_1, E_0)\) and \(c_5 = c_5(c_1, c_2)\) are constants independent of \(t, T\) and \(\gamma\).

We have the estimate by (2.25)

\[
\int S_2^2 \, dx \leq \frac{2}{(\gamma-1)^2} \int (\theta / \overline{\theta})^2 \, dx + 2 \int (\rho / \overline{\rho})^2 \, dx \leq CE_0^2 / (\gamma-1),
\]

where we use the condition (2.16) and the inequalities (2.12) and (2.22). The estimate (2.26) is an easy consequence of the inequalities (2.12) and (2.28). The estimate (2.27) follows from (1.2), (1.7), (2.20) and (2.26). Thus lemmas 2.2–2.6 give the main a priori estimates in (2.4), (2.5) of Proposition 2.1 from which follow the remaining estimates in (2.4) (2.5).

This completes the proof of Proposition 2.1.

**§ 3. A priori estimates of Hölder continuity.**

It is not difficult to get the estimate of Hölder continuity by use of Proposition 2.1 and the equation (1.1).
**Proposition 3.1.** Under the same assumptions as Proposition 2.1 we have the a priori estimates of the Hölder norm for the solution in $Q_T$:

\[(3.1) \quad \| \rho, u, \theta \|_{1+\alpha/2,1+\alpha/2(Q_T)} \leq K(T), \]

where $K(T)$ is a constant depending only on $T$, $c_0$, $E_0$ and $\mathcal{B}^{+\alpha}$-norm of the initial data.

**Proof.** First we show using (2.4) that

\[(3.2) \quad \| \rho, u, \theta \|_{1+\alpha/2,1+\alpha/2(Q_T)} \leq C(c_0, E_0). \]

In fact we have the following for the density.

\[
|\rho(t, x) - \rho(t', x')| \leq |\rho(t, x) - \rho(t', x)| + |\rho(t', x) - \rho(t', x')| \\
\leq \left\{ 2 \int_{\infty}^t (\rho(t, x) - \rho(t', x))(\rho_x(t, x) - \rho_x(t', x)) dx \right\}^{1/2} + \int_{t'}^t |\rho_x(t', \xi)| d\xi \\
\leq \left\{ 2 \int_{\infty}^t |t-t'|^{1/2} \left( \int_{t'}^t \rho_{xx}(s, x) ds \right)^{1/2} (|\rho_x(t, x)| + |\rho_x(t', x)|) dx \right\}^{1/2} \\
+ |x-x'|^{1/2} \left( \int_{t'}^t \rho_x(t', \xi) d\xi \right)^{1/2} \\
\leq 2^{1/4} |t-t'|^{1/4} \left\{ \int_{t'}^t \int_{t'}^t \rho_x d\xi ds \cdot \left( \int \rho_x^2(t) + \rho_x^2(t') dx \right) \right\}^{1/4} \\
+ |x-x'|^{1/2} \left( \int \rho_x^2(t') dx \right)^{1/2} \\
\leq C(|t-t'|^{1/4} + |x-x'|^{1/2}) C_{1/2}^4 E_0.
\]

The Hölder continuity of $u$, $\theta$ is proved in the same way. Next we have the estimate

\[(3.3) \quad \| \rho_x \|_{1+\alpha/2(Q_T)} \leq K(T) \]

in the following way. It follows from the equation (1.1) that

\[(3.4) \quad \frac{\rho_x(t, x)}{\rho} + R \int_0^t \rho \theta \frac{\partial \rho_x(s, x)}{\rho} ds = \frac{\rho_x(0, x)}{\rho} - R \int_0^t \rho \theta \rho_x(s, x) ds - u(t, x) + u(0, x) \]

Using (3.2) (3.4) we know for a constant $K = K(T, c_0, E_0)$

\[
\| \rho_x \|_{1+\alpha/2(Q_T)} \leq K \{ |\rho_x(0)|_{1+\alpha/2} + \| \rho, u, \theta \|_{1+\alpha/2(Q_T)} + 1 \}
\]

if we note that

\[(3.5) \quad \left| \int_0^t \theta_x(s, x) ds - \int_0^t \theta_x(s, x) ds \right| \]
\[ \leq C |t-t'|^{3/4} \sup_{t \leq t \leq t'} \| \theta_x(t) \|^{1/2} \left( \int_0^T \int \theta_x^2 \ dx \ ds \right)^{1/2} \]

\[ \left| \int_0^t \theta_x(s,x) - \theta_x(s,x') \ ds \right| \leq \sqrt{T} |x-x'|^{1/4} \left( \int_0^T \int \theta_x^2 \ dx \ ds \right)^{1/2} \]

Now we can consider the equations for \( u \) and \( \theta \) as the linear parabolic equations with the Hölder continuous coefficients \( \rho, \rho_x \) by (3.2) and (3.3)

\[ (3.6) \quad u_t = \mu \rho u_{xx} + \mu \rho_x u_x - R \rho \theta_x - R \theta \rho_x, \]

\[ (3.7) \quad \theta_t = \frac{\kappa (\gamma - 1)}{R} \rho \theta_{xx} + \frac{\kappa (\gamma - 1)}{R} \rho_x \theta_x - (\gamma - 1) \rho \theta u_x + \frac{\mu (\gamma - 1)}{R} \rho u_x^2. \]

The classical results \([2]\) give the Hölder estimate in \( Q_T \) for the equation (3.7)

\[ (3.8) \quad \| \theta \|_{1+\sigma/2 + \sigma/\ell (Q_T)} \leq K + K \{ \| u_x \|_{\sigma/2} \| u_{xx} \|_{\sigma/2} + \| u_x \|_{\sigma/2} \}, \]

and for the equation (3.6)

\[ (3.9) \quad \| u \|_{1+\sigma/2 + \sigma/\ell (Q_T)} \leq K + K \| \theta_x \|_{\sigma/2}, \]

where \( K = K(T) \) is a constant and \( 0 < \sigma \leq 1/2 \) is assumed. It follows from the interpolation theorem \([7]\) that

\[ \| u_x \| \leq C \| u \|_{1+\sigma/2} \| u_x \|_{\sigma/\ell + \sigma}, \]

\[ \| u_x \|_{\sigma/\ell} \leq C \| u \|_{1+\sigma/\ell} \| u_x \|_{\sigma/\ell + \sigma}, \]

where \( a = 1/(3+2\sigma) \), \( b = (1+2\sigma)/(3+2\sigma) \). Thus the right hand side of (3.9) has the estimate

\[ \| u \|_{1+\sigma/\ell + \sigma} \leq K + K \{ \| u \|_{1+\sigma/\ell} \| u_x \|_{\sigma/\ell + \sigma} + \| u_x \|_{\sigma/\ell} \} \leq K + K \| u \|_{1+\sigma/\ell + \sigma} \text{ in } Q_T, \]

where (3.2) is used. If we note that \( 0 < a + b < 1 \), we obtain by Young's inequality

\[ (3.10) \quad \| u \|_{1+\sigma/2 + \sigma/\ell (Q_T)} \leq K. \]

Therefore the same is true for \( \theta \) from (3.8).

\[ (3.11) \quad \| \theta \|_{1+\sigma/2 + \sigma/\ell (Q_T)} \leq K. \]

By use of (3.4) and (3.10) (3.11) we also know

\[ (3.12) \quad \| \rho \|_{1+\sigma/2 + \sigma/\ell (Q_T)} \leq K(T). \]

When \( 1/2 < \sigma < 1 \), we can repeat the above arguments to get the desired
estimate (3.1) using the estimates (3.10) (3.11) (3.12) with $\sigma=1/2$.
This completes the proof of Proposition 3.1.

§ 4. Proof of Theorem.

First we make a summary of the local existence theorem for the initial value problem (1.1) (1.4) under the conditions (i) (ii).

Theorem 4.1 (local existence).
Let the initial data (1.4) satisfy

$$
\begin{align*}
0 &< c_i^1 \leq \rho(0, \cdot)/\bar{\rho} \leq c_i < \infty \\
0 &< c_i^1 \leq \theta(0, \cdot)/\bar{\theta} \leq c_i < \infty
\end{align*}
$$

for some constants $c_i$, $c_i^1 \geq 1$. If we assume the conditions (i) (ii) on the system for $1 \leq \gamma \leq 2$ then there exist $M_i>1$ ($M_i^2 < M$) and $T_i>0$ such that the initial value problem (1.1) (1.4) has a unique solution $(\rho, u, \theta) \in \mathbb{B}^{1+\alpha+\gamma}(Q_{T_i})$ for $1 \leq \gamma \leq 2$ which satisfies the following

$$
\begin{align*}
1/M_i c_i &\leq \rho(t, x)/\bar{\rho} \leq M_i c_i \\
1/M_i^{-1} c_i &\leq \theta(t, x)/\bar{\theta} \leq M_i^{-1} c_i
\end{align*}
$$

in $Q_{T_i}$.

Now let us solve the initial value problem (1.1) (1.4) under the conditions (i) (iv). Remembering Proposition 2.1 let us fix a constant $\gamma_i > 1$ so small that for any $\gamma \in [1, \gamma_i]$ we have the inequalities

$$
\begin{align*}
\frac{5}{4} \gamma &- 1 \leq 1/M_i c_i \\
(M_i^2 c_i)^{-1} \exp \sqrt{\gamma - 1} c_i &\leq M_i
\end{align*}
$$

where the constants $c_i$ and $c_i^1$ are given by (2.5) of Proposition 2.1. If we take $c_1 = c_i$ and $c_2 = \exp \sqrt{\gamma - 1} c_i$, the local solution of (1.1) for any $\gamma \in [1, \gamma_i]$ can be continued globally in time by Proposition 2.1. In fact since $c_i \geq \max \{1/c_0, \sup \rho(0, \cdot)/\bar{\rho}\}$ and $\exp \sqrt{\gamma - 1} c_i \geq \max \{\sup \theta/\bar{\theta}(0, \cdot), \sup \theta(0, \cdot)/\bar{\theta}\}$, it follows from Theorem 4.1 that there exist a constant $T_i = T_i(c_i, c_2)$ and a solution $(\rho, u, \theta)(t, \cdot)$ in $0 \leq t \leq T_i$ which satisfies the inequalities for any $(t, x) \in Q_{T_i}$

$$
\begin{align*}
1/M_i c_i &\leq \rho(t, x)/\bar{\rho} \leq M_i c_i \\
1/M_i^{1-\gamma} \exp \sqrt{\gamma - 1} c_i &\leq \theta(t, x)/\bar{\theta} \leq M_i^{1-\gamma} \exp \sqrt{\gamma - 1} c_i \\
\exp (\gamma - 1) |S(t, x) - \bar{S}| &\leq (M_i^2 c_i)^{\gamma - 1} \exp \sqrt{\gamma - 1} c_i,
\end{align*}
$$

where (1.2) and (1.7) are used for the last inequality. On the other hand by (4.4) and the definition of $T_i$ in (4.3) the solution satisfies the assumption (2.3) for $(t, x) \in Q_{T_i}$. Therefore it has the a priori estimates (2.5) and
(3.1) for $0 \leq t \leq T_1$, in particular at $t = T_1$. Then Theorem 4.1 again gives a local solution $(\rho, u, \theta)(t)$ in $T_1 \leq t \leq T_1 + T_2$ which satisfies (4.4) for $T_1 \leq t \leq T_1 + T_2$, where $T_2$ is independent of $T_1$. Thus the solution $(\rho, u, \theta)(t)$ satisfies the assumption (2.3) for $0 \leq t \leq T_1 + T_2$ and consequently has the a priori estimates (2.5) and (3.1) for $0 \leq t \leq T_1 + T_2$. Therefore we can continue the solution for $T_1 + T_2 \leq t$ in the same way and get a global solution.

Last we show the asymptotic decay of solutions to the constant state $(\bar{\rho}, 0, \bar{\theta})$ as $t \to \infty$. Set $P(t) = \int \rho_2(t, x) \, dx$. It follows from the $H^1$ energy estimate (2.4) and the equation (1.1) that

$$
\int_0^\infty P(s) \, ds, \quad \int_0^\infty \left| \frac{dP(s)}{ds} \right| \, ds \leq CE_0^2.
$$

Therefore it holds $\lim_{t \to \infty} P(t) = 0$, from which $\lim_{t \to \infty} |\rho(t, \cdot) - \bar{\rho}| = 0$ by (2.4).

Set $U(t) = \int u_2^2(t, x) \, dx$ and note

$$
\frac{U(t + \Delta t) - U(t)}{\Delta t} = \int (u_2(t + \Delta t) - u_2(t)) \left( \frac{u(t + \Delta t) - u(t)}{\Delta t} \right) \, dx
= - \int (u_{xx}(t + \Delta t) + u_{xx}(t)) \left( \frac{u(t + \Delta t) - u(t)}{\Delta t} \right) \, dx.
$$

Then we have by (2.4) (1.1)

$$
\int_0^\infty U(s) \, ds, \quad \int_0^\infty \left| \frac{dU(s)}{ds} \right| \, ds \leq CE_0^2,
$$

which gives $\lim_{t \to \infty} U(t) = 0$ and consequently $\lim_{t \to \infty} u(t) = 0$. In the same way we know $\theta(t) = \int \theta_2^2(t, x) \, dx \to 0$ as $t \to \infty$. Then we have

$$
(\theta(t, x) - \bar{\theta})^{1/\gamma} \leq 2 \left\{ \int (\theta - \bar{\theta})^2 \, dx / (\gamma - 1) \right\}^{1/\gamma} \theta(t)^{1/\gamma},
$$

which gives (1.9) for $\theta(t)$.

This completes the proof of Theorem.

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References