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Simply connected compact simple Lie group $E_{8(-248)}$ of type E_8

By

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It is known that there exist three simple Lie groups of type E_8 up to local isomorphism, one of them is compact and the others are non-compact. In this paper, we shall consider the compact case. (As for one of the non-compact cases, see [6]). Our results are as follows. The group

$$E_8 = \{\alpha \in \mathrm{Aut}(\mathfrak{e}_8^C) \mid \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}$$

is a simply connected compact simple Lie group of type E_8, where $\mathfrak{e}_8^C$ is the complex Lie algebra of type E_8 and $\langle R_1, R_2 \rangle$ a positive definite Hermitian inner product in $\mathfrak{e}_8^C$. This group E_8 contains a subgroup

$$E_7 = \{\alpha \in E_8 \mid \alpha \underline{1} = \underline{1}\}$$

which is a simply connected compact simple Lie group of type E_7.

Thus we have been able to construct all simply connected compact simple Lie groups of exceptional type explicitly [1], [4], [5], [9], [10]:

$$\begin{aligned}
G_2 &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}, \mathfrak{C}) \mid \alpha(xy) = (\alpha x)(\alpha y)\}, \\
F_4 &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\} \\
&= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J}) \mid \alpha(X \times Y) = \alpha X \times \alpha Y\}, \\
E_6 &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{J}^C, \mathfrak{J}^C) \mid \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\} \\
&= \{\alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{J}^C, \mathfrak{J}^C) \mid \tau\alpha\tau(X \times Y) = \alpha X \times \alpha Y, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}, \\
E_7 &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^C, \mathfrak{P}^C) \mid \alpha\mathfrak{M}^C = \mathfrak{M}^C, \{\alpha P, \alpha Q\} = \{P, Q\}, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\} \\
&= \{\alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{P}^C, \mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\}
\end{aligned}$$

and

$$E_8 = \{\alpha \in \mathrm{Iso}_{\boldsymbol{C}}(\mathfrak{e}_8^C, \mathfrak{e}_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2], \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}.$$

In the last section, we calculate the Killing form of the Lie algebra $\mathfrak{e}_8^C$.

Throughout this paper, we refer many results of [1], [2] with their proofs. We pay our height tribute to Freudenthal's excellent works of the exceptonal Lie algebras.

§ 1. Inner products in Lie algebras $\mathfrak{f}_4^C$, $\mathfrak{e}_6^C$ and $\mathfrak{e}_7^C$.

1.1. Exceptional Jordan algebra $\mathfrak{J}^C$

Let $\mathfrak{C}^C$ denote the split Cayley algebra over the field of complex numbers $\boldsymbol{C}$ and $\mathfrak{J}^C=\mathfrak{J}(3,\mathfrak{C}^C)$ the split exceptional Jordan algebra over $\boldsymbol{C}$. This $\mathfrak{J}^C$ is the Jordan algebra consisting of all 3×3 Hermitian matrices with entries in $\mathfrak{C}^C$

$$X=\begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \qquad \xi_i\in \boldsymbol{C},\ x_i\in\mathfrak{C}^C$$

($\overline{x}$ is the conjugate of x in the Cayley algebra) with respect to the multiplication

$$X\circ Y=\frac{1}{2}(XY+YX).$$

In $\mathfrak{J}^C$, the symmetric inner product (X,Y), the positive definite Hermitian inner product $\langle X,Y\rangle$, the crossed product $X\times Y$, the cubic form (X,Y,Z) and the determinant $\det X$ are defined respectively by

$$\begin{aligned}
&(X,Y)=\mathrm{tr}(X\circ Y),\\
&\langle X,Y\rangle=(\tau X,Y)=(\overline{X},Y),\\
&X\times Y=\frac{1}{2}(2X\circ Y-\mathrm{tr}(X)Y-\mathrm{tr}(Y)X+(\mathrm{tr}(X)\mathrm{tr}(Y)-(X,Y))E),\\
&(X,Y,Z)=(X\times Y,Z),\\
&\det X=\frac{1}{3}(X,X,X)
\end{aligned}$$

where $\tau:\mathfrak{J}^C\to\mathfrak{J}^C$ is the complex conjugation with respect to the basic field $\boldsymbol{C}$ (τX is also denoted by $\overline{X}$) and E the 3×3 unit matrix.

1.2. Lie algebra $\mathfrak{f}_4^C$.

For later use, we review some properties of the exceptional simple Lie algebras $\mathfrak{e}_6^C$ and $\mathfrak{f}_4^C$ over $\boldsymbol{C}$ [1]:

$$\begin{aligned}
\mathfrak{e}_6^C&=\{\phi\in\mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{J}^C,\mathfrak{J}^C)\mid(\phi X,X,X)=0\}\\
&=\{\phi\in\mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{J}^C,\mathfrak{J}^C)\mid(\phi X,Y,Z)+(X,\phi Y,Z)+(X,Y,\phi Z)=0\},\\
\mathfrak{f}_4^C&=\{\delta\in\mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{J}^C,\mathfrak{J}^C)\mid\delta(X\circ Y)=\delta X\circ Y+X\circ\delta Y\}\\
&=\{\delta\in\mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{J}^C,\mathfrak{J}^C)\mid\delta(X\times Y)=\delta X\times Y+X\times\delta Y\}
\end{aligned}$$

$$= \{\delta \in \mathfrak{e}_6^C \mid (\delta X, Y) + (X, \delta Y) = 0\}$$
$$= \{\delta \in \mathfrak{e}_6^C \mid \delta E = 0\}.$$

For $A \in \mathfrak{J}^C$, we define a linear transformation $\tilde{A}$ of $\mathfrak{J}^C$ by

$$\tilde{A}X = A \circ X, \qquad X \in \mathfrak{J}^C.$$

If $A \in \mathfrak{J}_0^C = \{A \in \mathfrak{J}^C \mid \mathrm{tr}(A) = 0\}$, then $\tilde{A} \in \mathfrak{e}_6^C$. In fact,

$$(\tilde{A}X, X, X) = (A \circ X, X \times X) = (A, X \circ (X \times X)) = (A, (\det X) E)$$
$$= (\det X)(A, E) = (\det X)\,\mathrm{tr}(A) = 0.$$

And, for $A, B \in \mathfrak{J}^C$, we have $[\tilde{A}, \tilde{B}] = \tilde{A}\tilde{B} - \tilde{B}\tilde{A} \in \mathfrak{f}_4^C$. In fact,

$$[\tilde{A}, \tilde{B}] = \left[\widetilde{\left(A - \frac{1}{3}\,\mathrm{tr}(A) E\right)},\ \widetilde{\left(B - \frac{1}{3}\,\mathrm{tr}(B) E\right)}\right] \in \mathfrak{e}_6^C \text{ and } [\tilde{A}, \tilde{B}]E = 0.$$

Proposition 1. *For $\delta \in \mathfrak{f}_4^C$ and $A, B \in \mathfrak{J}^C$, we have*

$$[\delta, [\tilde{A}, \tilde{B}]] = [\widetilde{\delta A}, \tilde{B}] + [\tilde{A}, \widetilde{\delta B}].$$

In particular, $\{[\tilde{A}, \tilde{B}] \mid A, B \in \mathfrak{J}^C\}$ generates $\mathfrak{f}_4^C$ additively.

Proof. $[\delta, [\tilde{A}, \tilde{B}]]X = \delta[\tilde{A}, \tilde{B}]X - [\tilde{A}, \tilde{B}]\delta X$

$$= \delta(A \circ (B \circ X) - B \circ (A \circ X)) - A \circ (B \circ \delta X) + B \circ (A \circ \delta X)$$
$$= \delta A \circ (B \circ X) + A \circ (\delta B \circ X) - \delta B \circ (A \circ X) - B \circ (\delta A \circ X)$$
$$= [\widetilde{\delta A}, \tilde{B}]X + [\tilde{A}, \widetilde{\delta B}]X, \qquad \text{for any } X \in \mathfrak{J}^C.$$

This shows that $\mathfrak{a} = \{\sum_i [\tilde{A}_i, \tilde{B}_i] \mid A_i, B_i \in \mathfrak{J}^C\}$ is an ideal of $\mathfrak{f}_4^C$. From the simplicity of $\mathfrak{f}_4^C$, we have $\mathfrak{a} = \mathfrak{f}_4^C$.

In $\mathfrak{f}_4^C$, we define an inner product $\langle \delta_1, \delta_2 \rangle$ by

$$\langle \delta, [\tilde{A}, \tilde{B}] \rangle = \langle \delta \bar{B}, A \rangle, \qquad \delta \in \mathfrak{f}_4^C,\ A, B \in \mathfrak{J}^C.$$

More precisely, for $\delta_1 = \sum_i [\tilde{A}_i, \tilde{B}_i]$, $\delta_2 = \sum_j [\tilde{C}_j, \tilde{D}_j]$, $A_i, B_i, C_j, D_j \in \mathfrak{J}^C$, we define

$$\langle \delta_1, \delta_2 \rangle = \sum_{i,j} \langle [\tilde{A}_i, \tilde{B}_i] \bar{D}_j, C_j \rangle.$$

Proposition 2. *The inner product $\langle \delta_1, \delta_2 \rangle$ in $\mathfrak{f}_4^C$ is Hermitian and positive definite.*

Proof. The inner product $\langle \delta_1, \delta_2 \rangle$ is Hermitian, since

$$\langle [\tilde{A}, \tilde{B}], [\tilde{C}, \tilde{D}] \rangle = \langle [\tilde{A}, \tilde{B}]\bar{D}, C \rangle = \langle A \circ (B \circ \bar{D}) - B \circ (A \circ \bar{D}), C \rangle$$

$$=\langle B\circ\overline{D}, \overline{A}\circ C\rangle-\langle A\circ\overline{D}, \overline{B}\circ C\rangle$$
$$=\langle\overline{C}\circ(\overline{D}\circ B), \overline{A}\rangle-(\overline{D}\circ(\overline{C}\circ B), \overline{A}\rangle$$
$$=\langle[\tilde{\overline{C}}, \tilde{\overline{D}}]B, \overline{A}\rangle=\overline{\langle[\tilde{C}, \tilde{D}]\overline{B}, A\rangle}=\overline{\langle[\tilde{C}, \tilde{D}], [\tilde{A}, \tilde{B}]\rangle}.$$

From this, for $\delta_2=\sum_j[\tilde{C}_j, \tilde{D}_j]$, we have

$$\langle[\tilde{A}, \tilde{B}], \delta_2\rangle=\langle[\tilde{A}, \tilde{B}], \sum_j[\tilde{C}_j, \tilde{D}_j]\rangle=\overline{\langle\sum_j[\tilde{C}_j, \tilde{D}_j], [\tilde{A}, \tilde{B}]\rangle}$$
$$=\overline{\langle\sum_j[\tilde{C}_j, \tilde{D}_j]\overline{B}, A\rangle}=\overline{\langle\delta_2\overline{B}, A\rangle}.$$

This shows that the definition of $\langle\delta_1, \delta_2\rangle$ is independent of expressions of δ_2 and hence of δ_1. Finally, under the following notations in $\mathfrak{J}^C$

$$E_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad E_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad E_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$
$$E_1(x)=\begin{pmatrix}0&0&0\\0&0&x\\0&\overline{x}&0\end{pmatrix},\quad F_2(x)=\begin{pmatrix}0&0&\overline{x}\\0&0&0\\x&0&0\end{pmatrix},\quad F_3(x)=\begin{pmatrix}0&x&0\\\overline{x}&0&0\\0&0&0\end{pmatrix},$$

we can easily verify that

$$\sqrt{2}[\tilde{E}_1, \tilde{F}_2(e_i)],\quad \sqrt{2}[\tilde{E}_1, \tilde{F}_3(e_i)],\quad \sqrt{2}[\tilde{E}_3, \tilde{F}_1(e_i)],\quad i=0,1,2,\cdots,7,$$
$$\frac{1}{\sqrt{2}}[\tilde{F}_1(e_i), \tilde{F}_1(e_j)],\quad 0\leqq i<j\leqq 7$$

(where $\{e_0, e_1, e_2, \cdots, e_7\}$ is an orthonormal basis in $\mathfrak{C}^C$) is an orthonormal basis in $\mathfrak{f}_4^C$. Hence this inner product $\langle\delta_1, \delta_2\rangle$ is positive definite.

1.3. Lie algebra $\mathfrak{e}_6^C$.

For later use, we continue to consider the Lie algebra $\mathfrak{e}_6^C$.

Proposition 3. *Any $\phi\in\mathfrak{e}_6^C$ can be represented uniquely by*

$$\phi=\delta+\tilde{A},\quad \delta\in\mathfrak{f}_4^C,\ A\in\mathfrak{J}_0^C.$$

Proof. Put $A=\phi E$, then $\mathrm{tr}(A)=(\phi E, E, E)=0$, so $\tilde{A}\in\mathfrak{e}_6^C$ and $(\phi-\tilde{A})E=0$, hence $\delta=\phi-\tilde{A}\in\mathfrak{f}_4^C$.

For $\phi\in\mathfrak{e}_6^C$, we denote the skew-transpose of ϕ by ϕ' with respect to the inner product (X, Y) in $\mathfrak{J}^C$:

$$(\phi X, Y)+(X, \phi' Y)=0.$$

Proposition 4. *For $\phi\in\mathfrak{e}_6^C$, we have*

(1) *If $\phi=\delta+\tilde{A}$, $\delta\in\mathfrak{f}_4^C$, $A\in\mathfrak{J}_0^C$, then $\phi'=\delta-\tilde{A}$. In particular, $\phi'\in\mathfrak{e}_6^C$.*

(2) $\phi(X\times Y)=\phi'X\times Y+X\times\phi'Y$, $\quad X, Y\in\mathfrak{J}^C$.

Proof. (1) is easy. (2) is also easy, since

$$\begin{aligned}(\phi(X\times Y), Z) &= -(X\times Y, \phi'Z) = -(X, Y, \phi'Z)\\ &= (\phi'X, Y, Z)+(X, \phi'Y, Z)\\ &= (\phi'X\times Y+X\times\phi'Y, Z), \qquad \text{for any } Z\in\mathfrak{J}^C,\end{aligned}$$

For $A, B\in\mathfrak{J}^C$ we define $A^\vee B\in\mathfrak{e}_6^C$ by

$$A^\vee B=[\tilde{A}, \tilde{B}]+\widetilde{\left(A\circ B-\frac{1}{3}(A, B)E\right)}.$$

Proposition 5. *For $A, B\in\mathfrak{J}^C$, we have*

(1) $$(A^\vee B)'=-B^\vee A.$$

(2) $$(A^\vee B)X=\frac{1}{2}(B, X)A+\frac{1}{6}(A, B)X-2B\times(A\times X), \quad X\in\mathfrak{J}^C.$$

Proof. (1) is easy. (2) It suffices to show that (in the case of $A=X$)

$$(X^\vee B)X=\frac{1}{2}(B, X)X+\frac{1}{6}(X, B)X-2B\times(X\times X), \quad X\in\mathfrak{J}^C,$$

that is,

$$2B\times(X\times X)=B\circ(X\circ X)-2(B\circ X)\circ X+(B, X)X$$

and furthermore for $X\in\mathfrak{J}=\{X\in\mathfrak{J}^C\,|\,\overline{X}=X\}$. Since any $X\in\mathfrak{J}$ can be transformed in a diagonal form by the group

$$\begin{aligned}F_4 &= \{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J})\,|\,\alpha(X\circ Y)=\alpha X\circ\alpha Y\}\\ &= \{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J})\,|\,\alpha(X\times Y)=\alpha X\times\alpha Y\}\\ &= \{\alpha\in\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}, \mathfrak{J})\,|\,\alpha(X\times Y)=\alpha X\times\alpha Y,\ (\alpha X, \alpha Y)=(X, Y)\}\end{aligned}$$

[1], it suffices to show it for $X=\begin{pmatrix}\xi_1 & 0 & 0\\ 0 & \xi_2 & 0\\ 0 & 0 & \xi_3\end{pmatrix}$ (and $B=\begin{pmatrix}\beta_1 & b_3 & \overline{b}_2\\ \overline{b}_3 & \beta_2 & b_1\\ b_2 & \overline{b}_1 & \beta_3\end{pmatrix}$). Now, by the direct calculations, we see that both sides of the above are

$$\begin{pmatrix}\beta_2\xi_2\xi_1+\beta_3\xi_3\xi_1 & -\xi_1\xi_2b_3 & *\\ * & \beta_3\xi_3\xi_2+\beta_1\xi_1\xi_2 & -\xi_2\xi_3b_1\\ -\xi_3\xi_1b_2 & * & \beta_1\xi_1\xi_3+\beta_2\xi_2\xi_3\end{pmatrix}.$$

Proposition 6. *For $\phi \in \mathfrak{e}_6^C$ and $A, B \in \mathfrak{J}^C$, we have*

$$[\phi, A^\vee B] = \phi A^\vee B + A^\vee \phi' B\,.$$

In particular, $\{A^\vee B \mid A, B \in \mathfrak{J}^C\}$ generates $\mathfrak{e}_6^C$ additively.

Proof. (Propositions 4, 5). $[\phi, A^\vee B]X = \phi((A^\vee B)X) - (A^\vee B)\phi X$

$$= \phi\Big(\frac{1}{2}(B, X)A + \frac{1}{6}(A, B)X - 2B\times(A\times X)\Big) - (A^\vee B)\phi X$$

$$= \frac{1}{2}(B, X)\phi A + \frac{1}{6}(A, B)\phi X - 2\phi' B\times(A\times X) - 2B\times(\phi A\times X)$$

$$-2B\times(A\times\phi X) - \frac{1}{2}(B, \phi X)A - \frac{1}{6}(A, B)\phi X + 2B\times(A\times\phi X)$$

$$= \frac{1}{2}(B, X)\phi A + \frac{1}{6}(\phi A, B)X - 2B\times(\phi A\times X) \quad + \frac{1}{2}(\phi' B, X)A$$

$$+\frac{1}{6}(A, \phi' B)X - 2\phi' B\times(A\times X)$$

$$= (\phi A^\vee B)X + (A^\vee \phi' B)X\,, \qquad \text{for any } X \in \mathfrak{J}^C.$$

This shows that $\mathfrak{a} = \{\sum_i (A_i{}^\vee B_i) \mid A_i, B_i \in \mathfrak{J}^C\}$ is an ideal of $\mathfrak{e}_6^C$. From the simplicity of $\mathfrak{e}_6^C$, we have $\mathfrak{a} = \mathfrak{e}_6^C$.

For $\phi \in \mathfrak{e}_6^C$, we denote the skew-transpose of ϕ by ${}'\phi$ with respect to the inner product $\langle X, Y\rangle$ in $\mathfrak{J}^C$:

$$\langle \phi X, Y\rangle + \langle X, {}'\phi Y\rangle = 0\,.$$

Then obviously we have

Proposition 7. *For $\phi \in \mathfrak{e}_6^C$, we have ${}'\phi = \tau\phi'\tau$. In particular, ${}'\phi \in \mathfrak{e}_6^C$.*

Now, in $\mathfrak{e}_6^C$, we define a positive definite Hermitian inner product $\langle \phi_1, \phi_2\rangle$ by

$$\langle \phi_1, \phi_2\rangle = \langle \delta_1, \delta_2\rangle + \langle A_1, A_2\rangle$$

where $\phi_i = \delta_i + \tilde{A}_i$, $\delta_i \in \mathfrak{f}_4^C$, $A_i \in \mathfrak{J}_0^C$, $i = 1, 2$.

Proposition 8. *For $\phi \in \mathfrak{e}_6^C$ and $A, B \in \mathfrak{J}^C$, we have*

$$\langle \phi, A^\vee B\rangle = \langle \phi \overline{B}, A\rangle\,.$$

Proof. If $\phi = \delta + \tilde{C}$, $\delta \in \mathfrak{f}_4^C$, $C \in \mathfrak{J}_0^C$, then

$$\langle\phi, A^{\vee}B\rangle=\langle\delta+\tilde{C}, [\tilde{A}, \tilde{B}]+\widetilde{\left(A\circ B-\frac{1}{3}(A, B)E\right)}\rangle$$

$$=\langle\delta, [\tilde{A}, \tilde{B}]\rangle+\langle C, A\circ B-\frac{1}{3}(A, B)E\rangle$$

$$=\langle\delta\overline{B}, A\rangle+\langle C\circ\overline{B}, A\rangle-\frac{1}{3}(A, B)\overline{\mathrm{tr}(C)}$$

$$=\langle(\delta+\tilde{C})\overline{B}, A\rangle=\langle\phi\overline{B}, A\rangle\,.$$

Proposition 9. *For $A, B, C\in\mathfrak{J}^C$, we have*

$$A^{\vee}(B\times C)+B^{\vee}(C\times A)+C^{\vee}(A\times B)=0\,.$$

Proof. (Propositions 4, 7, 8). For any $\phi\in\mathfrak{e}_6^C$,

$$\langle\phi, A^{\vee}(A\times A)\rangle=\langle\phi(\bar{A}\times\bar{A}), A\rangle=\langle 2\phi'\bar{A}\times\bar{A}, A\rangle$$
$$=2\langle\phi'\bar{A}, A\times A\rangle=2\langle\tau'\phi A, A\times A\rangle=\overline{2\langle'\phi A, \bar{A}\times\bar{A}\rangle}$$
$$=-\overline{2\langle A, \phi(\bar{A}\times\bar{A})\rangle}=-2\langle\phi(\bar{A}\times\bar{A}), A\rangle=-2\langle\phi, A^{\vee}(A\times A)\rangle\,.$$

Therefore $\langle\phi, A^{\vee}(A\times A)\rangle=0$ and hence $A^{\vee}(A\times A)=0$. Polarize this, we have the required result.

1.4. Lie algebra $\mathfrak{e}_7^C$.

Let $\mathfrak{P}^C$ be a 56 dimensional vector space defined by

$$\mathfrak{P}^C=\mathfrak{J}^C\oplus\mathfrak{J}^C\oplus\boldsymbol{C}\oplus\boldsymbol{C}\,.$$

For $\phi\in\mathfrak{e}_6^C$, $A, B\in\mathfrak{J}^C$ and $\rho\in\boldsymbol{C}$, we define a linear transformation $\Phi(\phi, A, B, \rho)$ of $\mathfrak{P}^C$ by

$$\Phi(\phi, A, B, \rho)\begin{pmatrix}X\\ Y\\ \xi\\ \eta\end{pmatrix}=\begin{pmatrix}\phi-\frac{1}{3}\rho 1 & 2B & 0 & A\\ 2A & \phi'+\frac{1}{3}\rho 1 & B & 0\\ 0 & A & \rho & 0\\ B & 0 & 0 & -\rho\end{pmatrix}\begin{pmatrix}X\\ Y\\ \xi\\ \eta\end{pmatrix}$$

$$=\begin{pmatrix}\phi X-\frac{1}{3}\rho X+2B\times Y+\eta A\\ 2A\times X+\phi'Y+\frac{1}{3}\rho Y+\xi B\\ (A, Y)+\rho\xi\\ (B, X)-\rho\eta\end{pmatrix}$$

Then Freudenthal has shown in [2] that

$$\mathfrak{e}_7^C=\{\Phi=\Phi(\phi,A,B,\rho)\in\mathrm{Hom}_C(\mathfrak{P}^C,\mathfrak{P}^C)\,|\,\phi\in\mathfrak{e}_6^C,\ A,B\in\mathfrak{J}^C,\ \rho\in\boldsymbol{C}\}$$

is a simple Lie algebra over $\boldsymbol{C}$ of type E_7. The Lie bracket $[\Phi_1,\Phi_2]$ in $\mathfrak{e}_7^C$ is given by

$$[\Phi(\phi_1,A_1,B_1,\rho_1),\Phi(\phi_2,A_2,B_2,\rho_2)]=\Phi(\phi,A,B,\rho)$$

where

$$\begin{cases}\phi=[\phi_1,\phi_2]+2A_1{}^\vee B_2-2A_2{}^\vee B_1\,,\\ A=\Big(\phi_1+\dfrac{2}{3}\rho_1 1\Big)A_2-\Big(\phi_2+\dfrac{2}{3}\rho_2 1\Big)A_1\,,\\ B=\Big(\phi_1'-\dfrac{2}{3}\rho_1 1\Big)B_2-\Big(\phi_2'-\dfrac{2}{3}\rho_2 1\Big)B_1\,,\\ \rho=(A_1,B_2)-(B_1,A_2)\,.\end{cases}$$

For $P=(X,Y,\xi,\eta)$, $Q=(Z,W,\zeta,\omega)\in\mathfrak{P}^C$, we define $P\times Q\in\mathfrak{e}_7^C$ by

$$P\times Q=\Phi(\phi,A,B,\rho),\quad\begin{cases}\phi=-\dfrac{1}{2}(X^\vee W+Z^\vee Y),\\ A=-\dfrac{1}{4}(2Y\times W-\xi Z-\zeta X),\\ B=\dfrac{1}{4}(2X\times Z-\eta W-\omega Y),\\ \rho=\dfrac{1}{8}((X,W)+(Z,Y)-3(\xi\omega+\zeta\eta)).\end{cases}$$

Proposition 10. *For $\Phi\in\mathfrak{e}_7^C$ and P, $Q\in\mathfrak{P}^C$, we have*

$$[\Phi,P\times Q]=\Phi P\times Q+P\times\Phi Q\,.$$

In particular, $\{P\times Q\,|\,P,Q\in\mathfrak{P}^C\}$ generates $\mathfrak{e}_7^C$ additively.

Proof. (Propositions 4, 5, 6, 9). It suffices to show $[\Phi,P\times P]=2\Phi P\times P$. For $\Phi=\Phi(\phi,A,B,\rho)\in\mathfrak{e}_7^C$, $P=(X,Y,\xi,\eta)\in\mathfrak{P}^C$,

$$\begin{aligned}&[\Phi,P\times P]\\ &=\Big[\Phi(\phi,A,B,\rho),\Phi\Big(-X^\vee Y,\ -\frac{1}{2}(Y\times Y-\xi X),\\ &\qquad\qquad\frac{1}{2}(X\times X-\eta Y),\frac{1}{4}((X,Y)-3\xi\eta)\Big)\Big]\\ &=\Phi\Big([\phi,\ -X^\vee Y]+2A^\vee\frac{1}{2}(X\times X-\eta Y)-2\Big(-\frac{1}{2}(Y\times Y-\xi X)\Big)^\vee B,\end{aligned}$$

$$\Big(\phi+\frac{2}{3}\rho 1\Big)\Big(-\frac{1}{2}(Y\times Y-\xi X)\Big)-\Big(-X^{\vee}Y+\frac{2}{3}\frac{1}{4}((X,Y)-3\xi\eta)1\Big)A,$$

$$\Big(\phi'-\frac{2}{3}\rho 1\Big)\Big(\frac{1}{2}(X\times X-\eta Y)\Big)-\Big(-(X^{\vee}Y)'-\frac{2}{3}\frac{1}{4}((X,Y)-3\xi\eta)1\Big)B,$$

$$\Big(A,\frac{1}{2}(X\times X-\eta Y)\Big)-\Big(-\frac{1}{2}(Y\times Y-\xi X),B\Big)\Big)$$

$$=2\Phi\Big(-\frac{1}{2}\Big(\phi X-\frac{1}{3}\rho X+2B\times Y+\eta A\Big)^{\vee}Y$$

$$-\frac{1}{2}X^{\vee}\Big(2A\times X+\phi'Y+\frac{1}{3}\rho Y+\xi B\Big),$$

$$-\frac{1}{4}\Big(2\Big(2A\times X+\phi'Y+\frac{1}{3}\rho Y+\xi B\Big)\times Y-((A,Y)+\rho\xi)X$$

$$-\xi\Big(\phi X-\frac{1}{3}\rho X+2B\times Y+\eta A\Big)\Big),$$

$$\frac{1}{4}\Big(2\Big(\phi X-\frac{1}{3}\rho X+2B\times Y+\eta A\Big)\times X-((B,X)-\rho\eta)Y$$

$$-\eta\Big(2A\times X+\phi'Y+\frac{1}{3}\rho Y+\xi B\Big)\Big),$$

$$\frac{1}{8}\Big(\Big(\phi X-\frac{1}{3}\rho X+2B\times Y-2\eta A,Y\Big)+\Big(X,2A\times X+\phi'Y+\frac{1}{3}\rho Y-2\xi B\Big)\Big)\Big)$$

$$=2\Phi P\times P.$$

This shows that $\mathfrak{a}=\{\sum_i (P_i\times Q_i)\,|\,P_i,Q_i\in\mathfrak{P}^C\}$ is an ideal of $\mathfrak{e}_7^C$. From the smplicity of $\mathfrak{e}_7^C$, we have $\mathfrak{a}=\mathfrak{e}_7^C$.

In $\mathfrak{P}^C$, we define a positive definite Hermitian inner product $\langle P,Q\rangle$ by

$$\langle P,Q\rangle=\langle X,Z\rangle+\langle Y,W\rangle+\bar{\xi}\zeta+\bar{\eta}\omega$$

where $P=(X,Y,\xi,\eta)$, $Q=(Z,W,\zeta,\omega)\in\mathfrak{P}^C$. For $\Phi\in\mathfrak{e}_7^C$, we denote the skew-transpose of Φ by ${}'\Phi$ with respect to this inner product $\langle P,Q\rangle$:

$$\langle\Phi P,Q\rangle+\langle P,{}'\Phi Q\rangle=0,$$

Proposition 11. *For $\Phi=\Phi(\phi,A,B,\rho)\in\mathfrak{e}_7^C$, we have*

$${}'\Phi=\Phi({}'\phi,-\bar{B},-\bar{A},-\bar{\rho}).$$

In particular, ${}'\Phi\in\mathfrak{e}_7^C$.

Proof. For $P=(X,Y,\xi,\eta)$, $Q=(Z,W,\zeta,\omega)\in\mathfrak{P}^C$,

$$\langle \Phi P, Q\rangle = \langle \phi X - \frac{1}{3}\rho X + 2B\times Y + \eta A, Z\rangle + \langle 2A\times X + \phi' Y + \frac{1}{3}\rho Y + \xi B, W\rangle$$
$$+ \overline{((A, Y) + \rho\xi)}\zeta + \overline{((B, X) - \rho\eta)}\omega$$
$$= -\langle X, {}'\phi Z + \frac{1}{3}\bar{\rho} Z - 2\bar{A}\times W - \omega\bar{B}\rangle - \langle Y, -2\bar{B}\times Z + ({}'\phi)' W$$
$$- \frac{1}{3}\bar{\rho} W - \zeta\bar{A}\rangle - \bar{\xi}(-(\bar{B}, W) - \bar{\rho}\zeta) - \bar{\eta}(-(\bar{A}, Z) + \bar{\rho}\omega)$$
$$= -\langle P, {}'\Phi Q\rangle .$$

Now, in $\mathfrak{e}_7^C$, we define a positive definite Hermitian inner product $\langle \Phi_1, \Phi_2\rangle$ by

$$\langle \Phi_1, \Phi_2\rangle = 2\langle \phi_1, \phi_2\rangle + 4\langle A_1, A_2\rangle + 4\langle B_1, B_2\rangle + \frac{8}{3}\bar{\rho}_1\rho_2$$

where $\Phi_i = \Phi(\phi_i, A_i, B_i, \rho_i) \in \mathfrak{e}_7^C$, $i = 1, 2$.

Proposition 12. *For $\Phi \in \mathfrak{e}_7^C$ and $P, Q \in \mathfrak{P}^C$, we have*

$$\langle \Phi, P\times Q\rangle = \langle \Phi\hat{P}, Q\rangle$$

where $\hat{P} = (-\bar{Y}, \bar{X}, -\bar{\eta}, \bar{\xi})$ for $P = (X, Y, \xi, \eta)$.

Proof. (Proposition 8). It suffices to show $\langle \Phi, P\times P\rangle = \langle \Phi\hat{P}, P\rangle$. For $\Phi = \Phi(\phi, A, B, \rho) \in \mathfrak{e}_7^C$, $P = (X, Y, \xi, \eta) \in \mathfrak{P}^C$,

$$\langle \Phi, P\times P\rangle = \langle \Phi(\phi, A, B, \rho), \Phi\Big(-X^\vee Y, -\frac{1}{2}(Y\times Y - \xi X),$$
$$\frac{1}{2}(X\times X - \eta Y), \frac{1}{4}((X, Y) - 3\xi\eta)\Big)\rangle$$
$$= 2\langle \phi, -X^\vee Y\rangle + 4\langle A, -\frac{1}{2}(Y\times Y - \xi X)\rangle + 4\langle B, \frac{1}{2}(X\times X - \eta Y)\rangle$$
$$+ \frac{8}{3}\frac{1}{4}\bar{\rho}((X, Y) - 3\xi\eta)$$
$$= \langle -\phi\bar{Y} + \frac{1}{3}\rho\bar{Y} + 2B\times\bar{X} + \bar{\xi}A, X\rangle + \langle -2A\times\bar{Y} + \phi'\bar{X} + \frac{1}{3}\rho\bar{X} - \bar{\eta}B, Y\rangle$$
$$+ \overline{((A, \bar{X}) - \rho\bar{\eta})}\xi + \overline{(-(B, \bar{Y}) - \rho\bar{\xi})}\eta$$
$$= \langle \Phi\hat{P}, P\rangle .$$

Proposition 13. *For $\Phi \in \mathfrak{e}_7^C$ and $P \in \mathfrak{P}^C$, we have*

$$\widehat{\Phi P} = {}'\Phi\hat{P} .$$

Proof. (Proposition 12). For any $Q\in\mathfrak{P}^C$,

$$\langle{}'\Phi\hat{P}, Q\rangle=\langle{}'\Phi, P\times Q\rangle=\langle{}'\Phi\hat{Q}, P\rangle=-\langle\hat{Q}, \Phi P\rangle=\langle\widehat{\Phi P}, Q\rangle .$$

Hence $\widehat{\Phi P}={}'\Phi\hat{P}$.

Proposition 14. *For $\Phi, \Phi_1, \Phi_2\in\mathfrak{e}_7^C$, we have*

$$\langle[\Phi, \Phi_1], \Phi_2\rangle+\langle\Phi_1, [{}'\Phi, \Phi_2]\rangle=0 .$$

Proof. (Propositions 10, 12, 13). It suffices to show it for $\Phi_1=P\times Q$, $P, Q\in\mathfrak{P}^C$.

$$\begin{aligned}\langle[\Phi, P\times Q], \Phi_2\rangle&=\langle\Phi P\times Q+P\times\Phi Q, \Phi_2\rangle\\&=\langle\Phi P, \Phi_2\hat{Q}\rangle+\langle P, \Phi_2\widehat{\Phi Q}\rangle=-\langle P, {}'\Phi\Phi_2\hat{Q}\rangle+\langle P, \Phi_2{}'\Phi\hat{Q}\rangle\\&=-\langle P, [{}'\Phi, \Phi_2]\hat{Q}\rangle=-\langle P\times Q, [{}'\Phi, \Phi_2]\rangle .\end{aligned}$$

Finally, we define a skew-symmetric inner product $\{P, Q\}$ in $\mathfrak{P}^C$ by

$$\{P, Q\}=(X, W)-(Z, Y)+\xi\omega-\zeta\eta$$

where $P=(X, Y, \xi, \eta)$, $Q=(Z, W, \zeta, \omega)\in\mathfrak{P}^C$.

Proposition 15. *For $P, Q\in\mathfrak{P}^C$, we have*

$$(P\times Q)P=(P\times P)Q-\frac{3}{8}\{P, Q\}P .$$

Proof. (Proposition 5). For $P=(X, Y, \xi, \eta)$, $Q=(Z, W, \zeta, \omega)\in\mathfrak{P}^C$,

$$\begin{aligned}&(P\times Q)P\\&=\Phi\Big(-\frac{1}{2}(X^\vee W+Z^\vee Y), -\frac{1}{4}(2Y\times W-\xi Z-\zeta X), \frac{1}{4}(2X\times Z-\eta W-\omega Y),\\&\qquad\frac{1}{8}((X, W)+(Z, Y)-3(\xi\omega+\zeta\eta))\Big)(X, Y, \xi, \eta)\end{aligned}$$

$$=\left(\begin{array}{l}-\frac{1}{2}(X^\vee W)X-\frac{1}{2}(Z^\vee Y)X-\frac{1}{24}((X, W)+(Z, Y)-3(\xi\omega+\zeta\eta))X\\ \quad+\frac{1}{2}(2X\times Z-\eta W-\omega Y)\times Y-\frac{1}{4}\eta(2Y\times W-\xi Z-\zeta X)\\ -\frac{1}{2}(2Y\times W-\xi Z-\zeta X)\times X+\frac{1}{2}(W^\vee X)Y+\frac{1}{2}(Y^\vee Z)Y\\ \quad+\frac{1}{24}((X, W)+(Z, Y)-3(\xi\omega+\zeta\eta))Y+\frac{1}{4}\xi(2X\times Z-\eta W-\omega Y)\end{array}\right.$$

$$
\left.\begin{matrix}
-\dfrac{1}{4}(2Y\times W-\xi Z-\zeta X, Y)+\dfrac{1}{8}((X, W)+(Z, Y)-3(\xi\omega+\zeta\eta))\xi \\[2ex]
\dfrac{1}{4}(2X\times Z-\eta W-\omega Y, X)-\dfrac{1}{8}((X, W)+(Z, Y)-3(\xi\omega+\zeta\eta))\eta
\end{matrix}\right)
$$

$$
=\begin{pmatrix}
-\dfrac{1}{8}(Y, Z)X-\dfrac{1}{4}(X, Y)Z-\dfrac{3}{8}(X, W)X+2Y\times(X\times Z)+(X\times X) \\
\quad\times W-\eta W\times Y-\dfrac{1}{2}\omega Y\times Y+\dfrac{1}{4}\xi\eta Z+\dfrac{1}{8}\xi\omega X+\dfrac{3}{8}\zeta\eta X \\[2ex]
\dfrac{1}{8}(X, W)X+\dfrac{1}{4}(X, Y)W+\dfrac{3}{8}(Z, Y)Y-2X\times(Y\times W)-(Y\times Y) \\
\quad\times Z+\xi Z\times X+\dfrac{1}{2}\zeta X\times X-\dfrac{1}{4}\xi\eta W-\dfrac{1}{8}\zeta\eta Y-\dfrac{3}{8}\xi\omega Y \\[2ex]
-\dfrac{1}{2}(Y\times Y, W)+\dfrac{3}{8}\xi(Z, Y)+\dfrac{1}{4}\zeta(X, Y)+\dfrac{1}{8}\xi(X, W)-\dfrac{3}{8}\xi^2\omega-\dfrac{3}{8}\xi\zeta\eta \\[2ex]
\dfrac{1}{2}(X\times X, Z)-\dfrac{3}{8}\eta(W, X)-\dfrac{1}{4}\omega(Y, X)-\dfrac{1}{8}\eta(Z, Y)+\dfrac{3}{8}\xi\omega\eta+\dfrac{3}{8}\zeta\eta^2
\end{pmatrix}
$$

$$
=\begin{pmatrix}
-(X^\vee Y)Z-\dfrac{1}{12}((X, Y)-3\xi\eta)Z+(X\times X-\eta Y)\times W \\
\quad-\dfrac{1}{2}\omega(Y\times Y-\xi X)-\dfrac{3}{8}((X, W)-(Z, Y)+\xi\omega-\zeta\eta)X \\[2ex]
-(Y\times Y-\xi X)\times Z+(Y^\vee X)W+\dfrac{1}{12}((X, Y)-3\xi\eta)W \\
\quad+\dfrac{1}{2}\zeta(X\times X-\eta Y)-\dfrac{3}{8}((X, W)-(Z, Y)+\xi\omega-\zeta\eta)Y \\[2ex]
-\dfrac{1}{2}(Y\times Y-\xi X, W)+\dfrac{1}{4}((X, Y)-3\xi\eta)\zeta-\dfrac{3}{8}((X, W)-(Z, Y) \\
\quad+\xi\omega-\zeta\eta)\xi \\[2ex]
\dfrac{1}{2}(X\times X-\eta Y, Z)-\dfrac{1}{4}((X, Y)-3\xi\eta)\omega-\dfrac{3}{8}((X, W)-(Z, Y) \\
\quad+\xi\omega-\zeta\eta)\eta
\end{pmatrix}
$$

$$
=\varPhi\Big(-X^\vee Y, -\frac{1}{2}(Y\times Y-\xi X), \frac{1}{2}(X\times X-\eta Y), \frac{1}{4}((X, Y-3)\xi\eta)\Big)(Z, W, \zeta, \omega)
$$

$$
-\frac{3}{8}((X, W)-(Z, Y)+\xi\omega-\zeta\eta)\,(X, Y, \xi, \eta)
$$

$$
=(P\times P)Q-\frac{3}{8}\{P, Q\}P\,.
$$

2. Lie algebra $\mathfrak{e}_8^C$.

We consider the simple Lie algebra $\mathfrak{e}_8^C$ over $\boldsymbol{C}$ of type E_8 constructed by Freudenthal in [2]. Let $\mathfrak{e}_8^C$ be a 248 dimensional vector space defined by

$$\mathfrak{e}_8^C=\mathfrak{e}_7^C\oplus\mathfrak{P}^C\oplus\mathfrak{P}^C\oplus\boldsymbol{C}\oplus\boldsymbol{C}\oplus\boldsymbol{C}\,.$$

The Lie bracket $[R_1, R_2]$ is defined by

$$[(\Phi_1, P_1, Q_1, r_1, s_1, t_1),\ (\Phi_2, P_2, Q_2, r_2, s_2, t_2)]=(\Phi, P, Q, r, s, t)$$

where

$$\begin{cases} \Phi=[\Phi_1, \Phi_2]+P_1\times Q_2-P_2\times Q_1\,, \\ P=\Phi_1P_2-\Phi_2P_1+r_1P_2-r_2P_1+s_1Q_2-s_2Q_1\,, \\ Q=\Phi_1Q_2-\Phi_2Q_1-r_1Q_2+r_2Q_1+t_1P_2-t_2P_1\,, \\ r=-\dfrac{1}{8}\{P_1, Q_2\}+\dfrac{1}{8}\{P_2, Q_1\}+s_1t_2-s_2t_1\,, \\ s=\dfrac{1}{4}\{P_1, P_2\}+2r_1s_2-2r_2s_1\,, \\ t=-\dfrac{1}{4}\{Q_1, Q_2\}-2r_1t_2+2r_2t_1\,. \end{cases}$$

By the straight-forward calculations, we see that $\mathfrak{e}_8^C$ is a Lie algebra.

Remark. For $(\Phi, P, Q, r, s, t)\in\mathfrak{e}_8^C$, the notation used by Freudenthal in [2] is $\ulcorner\begin{pmatrix}\Phi+r & s \\ t & \Phi-r\end{pmatrix}, \begin{pmatrix}P \\ Q\end{pmatrix}\urcorner$.

Theorem 16. *$\mathfrak{e}_8^C$ is a simple Lie algebra of type E_8.*

Proof. We use the following notations in $\mathfrak{e}_8^C=\mathfrak{e}_7^C\oplus\mathfrak{P}^C\oplus\mathfrak{P}^C\oplus\boldsymbol{C}\oplus\boldsymbol{C}\oplus\boldsymbol{C}$ $=\mathfrak{e}_7^C\oplus\mathfrak{K}$ briefly:

$$\begin{array}{ll} (\Phi, 0, 0, 0, 0, 0)=\Phi\,, & (0, P, 0, 0, 0, 0)=\bar{P}\,, \\ (0, 0, Q, 0, 0, 0)=\underline{Q}\,, & (0, 0, 0, r, 0, 0)=r\,, \\ (0, 0, 0, 0, s, 0)=\bar{s}\,, & (0, 0, 0, 0, 0, t)=\underline{t}\,. \end{array}$$

Let $\mathfrak{a}$ be a non-trivial ideal of $\mathfrak{e}_8^C$.

Case (1) $\mathfrak{a}\cap\mathfrak{e}_7^C=\{0\}$ and $\mathfrak{a}\cap\mathfrak{K}=\{0\}$ (this case does not occur). Let $p\colon\mathfrak{e}_8^C\to\mathfrak{e}_7^C$, $q\colon\mathfrak{e}_8^C\to\mathfrak{K}$ denote the projections. Now, in this case, $p|\mathfrak{a}\colon\mathfrak{a}\to\mathfrak{e}_7^C$ is a monomorphism. Then, since $p(\mathfrak{a})$ is a non-trivial ideal of $\mathfrak{e}_7^C$, we have $p(\mathfrak{a})=\mathfrak{e}_7^C$ from the simplicity of $\mathfrak{e}_7^C$. Therefore $p|\mathfrak{a}\colon\mathfrak{a}\to\mathfrak{e}_7^C$ gives an isomorphism between $\mathfrak{a}$ and $\mathfrak{e}_7^C$, so $\dim\mathfrak{a}=\dim\mathfrak{e}_7^C=133$. On the other hand, $q|\mathfrak{a}\colon\mathfrak{a}$

$\to\mathfrak{K}$ is a also a monomorphism, so $\dim\mathfrak{a}\leqq\dim\mathfrak{K}=115$. This is a contradiction.

Case (2) $\mathfrak{a}\cap\mathfrak{e}_7^C\neq\{0\}$. From the simplicity of $\mathfrak{e}_7^C$, we have $\mathfrak{e}_7^C\subset\mathfrak{a}$. And

$$[\Phi(0,0,0,1),\ (0,0,1,0)^-]=(0,0,1,0)^-\in\mathfrak{a}\,,$$
$$[\Phi(0,0,0,1),\ (0,0,1,0)_-]=(0,0,1,0)_-\in\mathfrak{a}\,,$$
$$[(0,0,1,0)^-,\ (0,0,0,4)^-]=\bar{1}\in\mathfrak{a}\,,$$
$$[(0,0,1,0)_-,\ (0,0,0,-4)_-]=\underline{1}\in\mathfrak{a}\,,$$
$$[\bar{1},\underline{1}]=1\in\mathfrak{a}\,,$$
$$[\bar{1}+\underline{1},\bar{Q}+\underline{P}]=\bar{P}+\underline{Q}\in\mathfrak{a}\,,\qquad\text{for}\quad P,Q\in\mathfrak{P}^C.$$

Therefore $\mathfrak{a}=\mathfrak{e}^C$.

Case (3) $\mathfrak{a}\cap\mathfrak{K}\neq\{0\}$. Let R be a non-zero element of $\mathfrak{a}\cap\mathfrak{K}$.

(i) $R=(\Phi,P,Q,r,s,t)$, $P\neq 0$. In this case we have

$$[[[R,1],\underline{1}],1]=\underline{P}\in\mathfrak{a}\,.$$

Choose $P_1\in\mathfrak{P}^C$ such that $P\times P_1\neq 0$. (Such P_1 exists. In fact, if contrary, $P\times P=0$, i.e., $P\in\mathfrak{M}^C=\{P\in\mathfrak{P}^C\,|\,P\times P=0,P\neq 0\}$, so there exists $\alpha\in E_{7(-133)}$ (see § 5) such that $P=c\alpha(0,0,1,0)$ for some $c\in\boldsymbol{R}$ ([4] Theorem 9). However, for $(0,0,1,0)\in\mathfrak{P}^C$, we can find $P_2\in\mathfrak{P}^C$ such that $(0,0,1,0)\times P_2\neq 0$, so $P\times\alpha P_2\neq 0$. This is a contradiction). Next choose $\Phi\in\mathfrak{e}_7^C$ such that $[P\times P_1,\Phi]\neq 0$. (Such Φ exists because $\mathfrak{e}_7^C$ is simple). Then we have

$$[[\underline{P},\bar{P}_1],\Phi]=-[P\times P_1,\Phi]\in\mathfrak{a}\,.$$

So we can reduce to the case (2).

(ii) $R=(\Phi,0,Q,r,s,t)$, $Q\neq 0$. This case is similar to (i).

(iii) $R=(\Phi,0,0,r,s,t)$, $r\neq 0$. In this case we have

$$[[[R,\underline{1}],\bar{1}],\bar{P}]=2r\bar{P}\in\mathfrak{a}\,,\qquad\text{for}\quad 0\neq P\in\mathfrak{P}^C.$$

So we can reduce to (ii).

(iv) $R=(\Phi,0,0,0,s,t)$, $s\neq 0$. In this case we have

$$[R,\underline{1}]=s\in\mathfrak{a}\,.$$

So we can reduce to (iii).

(v) $R=(\Phi,0,0,0,0,t)$, $t\neq 0$. This case is similar to (iv).

Therefore in any case we have $\mathfrak{a}=\mathfrak{e}_8^C$. Thus we see that $\mathfrak{e}_8^C$ is simple. Since the dimension of $\mathfrak{e}_8^C$ is 248, it must be of type E_8.

For $R=(\Phi,P,Q,r,s,t)\in\mathfrak{e}_8^C$, we denote the adjoint transformation ad R

of $\mathfrak{e}_8^C$ by $\Theta(\Phi, P, Q, r, s, t)$:

$$\Theta(\Phi, P, Q, r, s, t)\begin{pmatrix}\Phi_1\\ P_1\\ Q_1\\ r_1\\ s_1\\ t_1\end{pmatrix}=\begin{pmatrix}\operatorname{ad}\Phi & -Q & P & 0 & 0 & 0\\ -P & \Phi+r1 & s & -P & -Q & 0\\ -Q & t & \Phi-r1 & Q & 0 & -P\\ 0 & -\frac{1}{8}Q & -\frac{1}{8}P & 0 & -t & s\\ 0 & \frac{1}{4}P & 0 & -2s & 2r & 0\\ 0 & 0 & -\frac{1}{4}Q & 2t & 0 & -2r\end{pmatrix}\begin{pmatrix}\Phi_1\\ P_1\\ Q_1\\ r_1\\ s_1\\ t_1\end{pmatrix}$$

$$=\begin{pmatrix}[\Phi, \Phi_1]-Q\times P_1+P\times Q_1\\ -\Phi_1P+\Phi P_1+rP_1+sQ_1-r_1P-s_1Q\\ -\Phi_1Q+tP_1+\Phi Q_1-rQ_1+r_1Q-t_1P\\ -\frac{1}{8}\{Q, P_1\}-\frac{1}{8}\{P, Q_1\}-ts_1+st_1\\ \frac{1}{4}\{P, P_1\}-2sr_1+2rs_1\\ -\frac{1}{4}\{Q, Q_1\}+2tr_1-2rt_1\end{pmatrix}=[R, R_1]=(\operatorname{ad}R)R_1 .$$

Since $\mathfrak{e}_8^C$ is simple, the Lie algebra $\operatorname{Der}(\mathfrak{e}_8^C)$ of all derivations of $\mathfrak{e}_8^C$ consists of $\operatorname{ad}R$, $R\in\mathfrak{e}_8^C$:

$$\operatorname{Der}(\mathfrak{e}_8^C)=\{\Theta(\Phi, P, Q, r, s, t)\mid\Phi\in\mathfrak{e}_7^C, P, Q\in\mathfrak{P}^C, r, s, t\in\boldsymbol{C}\}$$

and it is also isomorphic to the Lie algebra $\mathfrak{e}_8^C$. We denote $\operatorname{Der}(\mathfrak{e}_8^C)$ sometimes by the same notation $\mathfrak{e}_8^C$.

Now, in $\mathfrak{e}_8^C$, we define a positive definite Hermitian inner Product $\langle R_1, R_2\rangle$ by

$$\langle R_1, R_2\rangle=\langle\Phi_1, \Phi_2\rangle+\langle P_1, P_2\rangle+\langle Q_1, Q_2\rangle+8\bar{r}_1r_2+4\bar{s}_1s_2+4\bar{t}_1t_2$$

where $R_i=(\Phi_i, P_i, Q_i, r_i, s_i, t_i)\in\mathfrak{e}_8^C$, $i=1, 2$. For $\Theta\in\mathfrak{e}_8^C$, we denote the skew-transpose of Θ by ${}'\Theta$ with respect to this inner product $\langle R_1, R_2\rangle$:

$$\langle\Theta R_1, R_2\rangle+\langle R_1, {}'\Theta R_2\rangle=0 .$$

Proposition 17. *For $\Theta=\Theta(\Phi, P, Q, r, s, t)\in\mathfrak{e}_8^C$, we have*

$${}'\Theta=({}'\Phi, -\widehat{Q}, \widehat{P}, -\bar{r}, -\bar{t}, -\bar{s}).$$

In particular, ${}'\Theta\in\mathfrak{e}_8^C$.

Proof. (Propositions 12, 14). For $R_i=(\Phi_i, P_i, Q_i, r_i, s_i, t_i)\in\mathfrak{e}_8^C$, $i=1, 2$,

$$
\begin{aligned}
\langle \Theta R_1, R_2\rangle &= \langle [\Phi, \Phi_1] - Q\times P_1 + P\times Q_1, \Phi_2\rangle + \langle -\Phi_1 P + \Phi P_1 + rP_1 + sQ_1 \\
&\qquad - r_1P - s_1Q, P_2\rangle + \langle -\Phi_1 Q - tP_1 + \Phi Q_1 - rQ_1 + r_1Q - t_1P, Q_2\rangle \\
&\qquad + 8\overline{\Big(-\frac{1}{8}\{Q, P_1\} - \frac{1}{8}\{P, Q_1\} - ts_1 + st_1\Big)}r_2 + 4\overline{\Big(\frac{1}{4}\{P, P_1\} - 2sr_1 + 2rs_1\Big)}s_2 \\
&\qquad + 4\overline{\Big(-\frac{1}{4}\{Q, Q_1\} + 2tr_1 - 2rt_1\Big)}t_2 \\
&= -\langle \Phi_1, [{}'\Phi, \Phi_2] - \widehat{P}\times P_2 - \widehat{Q}\times Q_2\rangle - \langle P_1, \Phi_2\widehat{Q} + {}'\Phi P_2 - \bar{r}P_2 - \bar{t}Q_2 \\
&\qquad + r_2\widehat{Q} - s_2\widehat{P}\rangle - \langle Q_1, -\Phi_2\widehat{P} - \bar{s}P_2 + {}'\Phi Q_2 + \bar{r}Q_2 + r_2\widehat{P} + t_2\widehat{Q}\rangle \\
&\qquad - 8\bar{r}_1\Big(-\frac{1}{8}\{\widehat{P}, P_2\} + \frac{1}{8}\{\widehat{Q}, Q_2\} + \bar{s}s_2 - \bar{t}t_2) - 4s_1\Big(-\frac{1}{4}\{\widehat{Q}, P_2\} + 2\bar{t}r_2 \\
&\qquad - 2\bar{r}s_2\Big) - 4t_1\Big(-\frac{1}{4}\{P, Q_2\} - 2\bar{s}r_2 + 2\bar{r}t_2\Big) \\
&= -\langle R_1, {}'\Theta R_2\rangle .
\end{aligned}
$$

3. Complex Lie group E_8^C.

The group E_8^C is defined to be the automorphism group of the Lie algebra $\mathfrak{e}_8^C$:

$$E_8^C = \mathrm{Aut}(\mathfrak{e}_8^C) = \{\alpha \in \mathrm{Iso}_C(\mathfrak{e}_8^C, \mathfrak{e}_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}.$$

Theorem 18. *The group E_8^C is a connected complex Lie group of type E_8.*

Proof. The type of the group E_8^C is obviously E_8, because its Lie algebra is $\mathrm{Der}(\mathfrak{e}_8^C)$ which is isomrphic to $\mathfrak{e}_8^C$. The group $E_8^C = \mathrm{Aut}(\mathfrak{e}_8^C)$ coincides with the inner automorphism group $\mathrm{Innaut}(\mathfrak{e}_8^C)$ which is the group generated by $\{\exp(\mathrm{ad}\, R) \mid R \in \mathfrak{e}_8^C\}$, since, as is well known (e.g. [7]),

$$\mathrm{Aut}(\mathfrak{e}_8^C)/\mathrm{Innaut}(\mathfrak{e}_8^C) = \begin{pmatrix}\text{the group of the symmetries of} \\ \text{the Dynkin diagram of } \mathfrak{e}_8^C\end{pmatrix} = \{1\}.$$

Hence $E_8^C = \mathrm{Innaut}(\mathfrak{e}_8^C)$ which is connected.

4. Compact Lie group E_8.

The group E_8 is defined to be the subgroup of E_8^C which leaves the inner product $\langle R_1, R_2\rangle$ in $\mathfrak{e}_8^C$ invariant:

$$
\begin{aligned}
E_8 &= \{\alpha \in \mathrm{Aut}(\mathfrak{e}_8^C) \mid \langle \alpha R_1, \alpha R_2\rangle = \langle R_1, R_2\rangle\} \\
&= \{\alpha \in \mathrm{Iso}_C(\mathfrak{e}_8^C, \mathfrak{e}_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2], \langle \alpha R_1, \alpha R_2\rangle = \langle R_1, R_2\rangle\}.
\end{aligned}
$$

Theorem 19. *E_8 is a compact simple Lie group of type E_8.*

Proof. The group E_8 is compact as a closed subgroup of the unitary group

$$U(248)=U(\mathfrak{e}_8^C)=\{\alpha\in \mathrm{Iso}_C(\mathfrak{e}_8^C,\mathfrak{e}_8^C)\,|\,\langle\alpha R_1,\alpha R_2\rangle=\langle R_1,R_2\rangle\}.$$

The Lie algebra $\mathfrak{e}_8$ of the group E_8 is

$$\begin{aligned}\mathfrak{e}_8&=\{\Theta\in \mathrm{Der}(\mathfrak{e}_8^C)\,|\,\langle\Theta R_1,R_2\rangle+\langle R_1,\Theta R_2\rangle=0\}\\&=\{\Theta\in \mathrm{Der}(\mathfrak{e}_8^C)\,|\,{}'\Theta=\Theta\}\\&=\{\Theta(\Phi,P,\hat{P},r,s,-\bar{s})\,|\,{}'\Phi=\Phi\in\mathfrak{e}_7^C,P\in\mathfrak{P}^C,r,s\in\boldsymbol{C},r+\bar{r}=0\}\end{aligned}$$

from Proposition 17. Therefore the complexification of $\mathfrak{e}_8$ is $\mathfrak{e}_8^C$, so the type of the group E_8 is E_8.

In order to prove that the group E_8 is connected, we shall give a polar decomposition of the group E_8^C.

Lemma 20 ([3] p. 345). *Let G be an algebraic subgroup of the general linear group $GL(n,\boldsymbol{C})$ such that the condition $A\in G$ implies $A^*\in G$. Then G is homeomorphic to the topological product of $G\cap U(n)$ and a Euclidean space $\boldsymbol{R}^d$:*

$$G\simeq(G\cap U(n))\times\boldsymbol{R}^d$$

where $U(n)$ is the unitary subgroup of $GL(n,\boldsymbol{C})$.

To use the above Lemma, we show the following

Proposition 21. *E_8^C is an algebraic subgroup of the general linear group $GL(248,\boldsymbol{C})=\mathrm{Iso}_C(\mathfrak{e}_8^C,\mathfrak{e}_8^C)$ and satisfies the condition $\alpha\in E_8^C$ implies $\alpha^*\in E_8^C$, where α^* is the transpose of α with respect to the inner product $\langle R_1,R_2\rangle$: $\langle\alpha R_1,R_2\rangle=\langle R_1,\alpha^*R_2\rangle$.*

Proof. As mentioned in Theorem 18, the group $E_8^C=\mathrm{Innaut}(\mathfrak{e}_8^C)$ is generated by $\{\exp\Theta\,|\,\Theta=\Theta(\Phi,P,Q,r,s,t)\in\mathfrak{e}_8^C\}$. From Proposition 17, ${}'\Theta\in\mathfrak{e}_8^C$ for $\Theta\in\mathfrak{e}_8^C$, so $(\exp\Theta)^*=\exp(-{}'\Theta)\in E_8^C$, hence $\alpha\in E_8^C$ implies $\alpha^*\in E_8^C$. It is obvious that E_8^C is algebraic, because E_8^C is defined by the algebraic relation $\alpha[R_1,R_2]=[\alpha R_1,\alpha R_2]$, $R_1,R_2\in\mathfrak{e}_8^C$.

From the definition of the group E_8 obviously

$$E_8^C\cap U(\mathfrak{e}_8^C)=E_8$$

and the dimension of the Euclidean part of E_8^C is

$$\dim E_8^C - \dim E_8 = 2 \times 248 - 248 = 248 .$$

Hence we have

Theorem 22. *The group E_8^C is homeomorphic to the topological product of the group E_8 and a 248 dimensional Euclidian space $\boldsymbol{R}^{248}$:*

$$E_8^C \simeq E_8 \times \boldsymbol{R}^{248} .$$

In particular, the group E_8 is connected (from Theorem 18).

From the general theory of the compact Lie groups, it is known that the center $z(E_{8(-248)})$ of the simply connected compact simple Lie group $E_{8(-248)}$ of type E_8 is trivial [8]: $z(E_{8(-248)}) = \{1\}$. Therefore the connectedness of E_8 implies the simply connectedness of E_8. Thus we have the following Theorem which was our purpose.

Theorem 23. *The group $E_8 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{e}_8^C, \mathfrak{e}_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_8], \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}$ is a simply connected compact simple Lie group of type E_8.*

5. Subgroup E_7 of E_8.

We have proved in [5] that the group

$$E_{7(-133)} = \{\beta \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \beta(P \times Q)\beta^{-1} = \beta P \times \beta Q, \langle \beta P, \beta Q \rangle = \langle P, Q \rangle\}$$

is a simply connected compact simple Lie group of type E_7. We shall find a subgroup in E_8 which is isomorphic to $E_{7(-133)}$.

Proposition 24. (1) *For $\beta \in E_{7(-133)}$ and $P, Q \in \mathfrak{P}^C$, we have*

$$\{\beta P, \beta Q\} = \{P, Q\} .$$

(2) *For $\beta \in E_{7(-133)}$ and $\Phi_1, \Phi_2 \in \mathfrak{e}_7^C$ we have*

$$\langle \beta \Phi_1 \beta^{-1}, \beta \Phi_2 \beta^{-1} \rangle = \langle \Phi_1, \Phi_2 \rangle .$$

Proof. (1) (Proposition 15). If $P = 0$, then (1) is obviously valid. If $P \neq 0$,

$$\begin{aligned}\frac{3}{8}\{\beta P, \beta Q\}\beta P &= (\beta P \times \beta Q)\beta Q - (\beta P \times \beta P)\beta P \\ &= \beta(P \times P)\beta^{-1}\beta Q - \beta(P \times Q)\beta^{-1}\beta P \\ &= \beta((P \times P)Q - (P \times Q)P) = \frac{3}{8}\{P, Q\}\beta P .\end{aligned}$$

Hence we have $\{\beta P, \beta Q\} = \{P, Q\}$, since $\beta P \neq 0$.

(2) (Proposition 13). First we shall show that, for $\beta \in E_{7(-133)}$ and $P \in \mathfrak{P}^C$, we have

$$\widehat{\beta P} = \beta \hat{P} .$$

The Lie algebra of the group $E_{7(-133)}$ is $\mathfrak{e}_7 = \{\Phi \in \mathfrak{e}_7^C \mid \Phi = '\Phi\}$. Since the group $E_{7(-133)}$ is connected and compact, any $\beta \in E_{7(133)}$ can be written by $\beta = \exp \Phi$ for some $\Phi \in \mathfrak{e}_7$. So

$$\widehat{\beta P} = (\widehat{(\exp \Phi) P}) = \sum_{k=0}^{\infty} \frac{1}{k!} (\widehat{\Phi^k P}) = \sum_{k=0}^{\infty} \frac{1}{k!} '\Phi^k \hat{P} \quad \text{(Proposision 13)}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \Phi^k \hat{P} = (\exp \Phi) \hat{P} = \beta \hat{P} .$$

Now, to prove (2), it suffices to show it for $\Phi_2 = P \times Q$.

$$\langle \beta\Phi_1\beta^{-1}, \beta(P \times Q)\beta^{-1}\rangle = \langle \beta\Phi_1\beta^{-1}, \beta P \times \beta Q\rangle = \langle \beta\Phi_1\beta^{-1}\widehat{\beta P}, \beta Q\rangle$$
$$= \langle \beta\Phi_1\beta^{-1}\beta\hat{P}, \beta Q\rangle = \langle \beta\Phi_1\hat{P}, \beta Q\rangle = \langle \Phi_1\hat{P}, Q\rangle = \langle \Phi_1, P \times Q\rangle .$$

In the followings, we use the notations $\Phi, \bar{P}, \underline{Q}, 1, \bar{1}, \underline{1}$ of Theorem 16.

Proposition 25. *If $\alpha \in E_8$ satisfies $\alpha\underline{1} = \underline{1}$ then $\alpha 1 = 1$, $\alpha\bar{1} = \bar{1}$.*

Proof. Put $\alpha 1 = (\Phi, P, Q, r, s, t)$, then

$$-2\underline{1} = \alpha(-2\underline{1}) = \alpha[1, \underline{1}] = [\alpha 1, \underline{1}] = (0, 0, -P, s, 0, -2r),$$

hence $P = 0$, $s = 0$, $r = 1$. And the condition $\langle \alpha 1, \alpha 1\rangle = \langle 1, 1\rangle = 8$, that is, $\langle \Phi, \Phi\rangle + \langle Q, Q\rangle + 8 + 4\bar{t}t = 8$ implies $\Phi = 0$, $Q = 0$, $t = 0$. Therefore $\alpha 1 = 1$. Simlarly $\alpha\bar{1} = \bar{1}$.

Theorem 26. *The group E_8 contains a subgroup*

$$E_7 = \{\alpha \in E_8 \mid \alpha\underline{1} = \underline{1}\}$$

which is a simply connected compact simple Lie group of type E_7.

Proof. We shall show that the group E_7 is isomorphic to the group $E_{7(-133)}$. Making use of Proposition 24, it is easy to verify that, for $\beta \in E_{7(-133)}$, the linear mapping $\alpha : \mathfrak{e}_8^C \to \mathfrak{e}_8^C$,

$$\alpha = \begin{pmatrix} \mathrm{Ad}\,\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(where $\mathrm{Ad}\,\beta : \mathfrak{e}_7^C \to \mathfrak{e}_7^C$ is defined by $(\mathrm{Ad}\,\beta)\Phi = \beta\Phi\beta^{-1}$) belongs to E_8. Conversely, suppose $\alpha \in E_8$ satisfies $\alpha 1 = 1$, $\alpha \bar{1} = \bar{1}$ and $\alpha \underline{1} = \underline{1}$ (Proposition 25). Since α leaves the orthogonal complement $\mathfrak{e}_7^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C$ of $\boldsymbol{C} \oplus \boldsymbol{C} \oplus \boldsymbol{C}$ invariant, α has the from

$$\alpha = \begin{pmatrix} \beta_1 & \beta_{12} & \beta_{13} & 0 & 0 & 0 \\ \beta_{21} & \beta_2 & \beta_{23} & 0 & 0 & 0 \\ \beta_{31} & \beta_{32} & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

where $\beta_1 : \mathfrak{e}_7^C \to \mathfrak{e}_7^C$, $\beta_2, \beta_3 : \mathfrak{P}^C \to \mathfrak{P}^C$, $\beta_{21}, \beta_{31} : \mathfrak{e}_7^C \to \mathfrak{P}^C$, $\beta_{12}, \beta_{13} : \mathfrak{P}^C \to \mathfrak{e}_7^C$, $\beta_{23}, \beta_{32} : \mathfrak{P}^C \to \mathfrak{P}^C$ are linear mappings respectively. From the condition $[\alpha\Phi, 1] = \alpha[\Phi, 1] = 0$, that is,

$$\begin{aligned} 0 &= [(\beta_1\Phi, \beta_{21}\Phi, \beta_{31}\Phi, 0, 0, 0), (0, 0, 0, 1, 0, 0)] \\ &= (0, -\beta_{21}\Phi, \beta_{31}\Phi, 0, 0, 0) \qquad \text{for any} \quad \Phi \in \mathfrak{e}_7^C, \end{aligned}$$

we have $\beta_{21} = \beta_{31} = 0$. Similarly, from $[\alpha\bar{P}, 1] = -\alpha\bar{P}$, $[\alpha\underline{Q}, 1] = \alpha\underline{Q}$, we have $\beta_{12} = \beta_{32} = 0$, $\beta_{13} = \beta_{23} = 0$ respectively. Thus

$$B = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_2 \end{pmatrix}.$$

Furthermore the relation $[\bar{P}, \underline{Q}] = \left(P \times Q, 0, 0, -\frac{1}{8}\{P, Q\}, 0, 0\right)$ implies

$$\beta_1(P \times Q) = \beta_2 P \times \beta_3 Q\,, \qquad \{\beta_2 P, \beta_3 Q\} = \{P, Q\}, \tag{i}$$

and $[\bar{P}, \bar{Q}] = \frac{1}{4}\{P, Q\}\bar{1}$, $[\Phi, \bar{P}] = (\Phi P)^{-}$ imply

$$\{\beta_2 P, \beta_2 Q\} = \{P, Q\}, \qquad \beta_2(\Phi P) = \beta_1\Phi\beta_2 P \tag{ii}$$

respectively. From the above we have $\beta_2 = \beta_3$ (put β), and from (ii) we have $\beta\Phi\beta^{-1} = \beta_1\Phi$. Therefore from (i) β satisfies $\beta(P \times Q)\beta^{-1} = \beta P \times \beta Q$, and obviously $\langle \beta P, \beta Q \rangle = \langle \alpha\bar{P}, \alpha\bar{Q} \rangle = \langle \bar{P}, \bar{Q} \rangle = \langle P, Q \rangle$. Hence $\beta \in E_{7(-133)}$ and $\beta_1 = \mathrm{Ad}\,\beta$. Thus Theorem 26 is proved.

6. Killing from of $\mathfrak{e}_8^C$.

In this section, we calculate the Killing form of the Lie algebra $\mathfrak{e}_8^C$ according to the preceding notations. For this purpose, we first describe the

Killing forms of the other exceptional Lie algebras with several representations.

Proposition 27. *The Killing forms $B=B_{\mathfrak{g}}$ of the Lie algebras $\mathfrak{g}=\mathfrak{f}_4^C$, $\mathfrak{e}_6^C$ and $\mathfrak{e}_7^C$ are given by*

F_4: (1) $B(\delta_1, \delta_2)=3\mathrm{tr}(\delta_1\delta_2), \quad \delta_i \in \mathfrak{f}_4^C$.

(2) $B(\delta, [\tilde{A}, \tilde{B}])=-9(\delta A, B), \quad \delta\in\mathfrak{f}_4^C, A, B\in\mathfrak{J}^C$.

E_6: (1) $B(\phi_1, \phi_2)=4\mathrm{tr}(\phi_1\phi_2), \quad \phi_i\in\mathfrak{e}_6^C$

(2) $B(\phi, A^\vee B)=-12(\phi A, B), \quad \phi\in\mathfrak{e}_6^C, A, B\in\mathfrak{J}^C$

(3) $B(\delta_1+\tilde{T}_1, \delta_2+\tilde{T}_2)=\dfrac{4}{3}B_{\mathfrak{f}_4 C}(\delta_1, \delta_2)+12(T_1, T_1), \delta_i\in\mathfrak{f}_4^C, T\in\mathfrak{J}_0^C$.

E_7: (1) $B(\Phi_1, \Phi_2)=3\,\mathrm{tr}(\Phi_1\Phi_2), \quad \Phi_i\in\mathfrak{e}_7^C$.

(2) $B(\Phi, P\times Q)=-9\{\Phi P, Q\}, \quad \Phi\in\mathfrak{e}_7^C, P, Q\in\mathfrak{P}^C$.

(3) $B(\Phi(\phi_1, A_1, B_1, \rho_1),\ \Phi(\phi_2, A_2, B_2, \rho_2))=\dfrac{3}{2}B_{\mathfrak{e}_6^C}(\phi_1, \phi_2)$

$+36(A_1, B_2)+36(B_1, A_2)+24\rho_1\rho_2, \quad \phi_i\in\mathfrak{e}_6^C, A_i, B_i\in\mathfrak{J}^C, \rho_i\in \boldsymbol{C}$.

Theorem 28. *The Killing form of the Lie algebra $\mathfrak{e}_8^C$ is given by*

$$B((\Phi_1, P_1, Q_1, r_1, s_1, t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2))$$
$$=\frac{5}{3}B_{\mathfrak{e}_7^C}(\Phi_1, \Phi_2)+15\{Q_1, P_2\}-15\{P_1, Q_2\}+120r_1r_2+60t_1s_2+60s_1t_2.$$

Proof. For $Q=(\Phi, P, Q, r, s, t)\in\mathfrak{e}_8^C$, we define $'R\in\mathfrak{e}_8^C$ by $'R=('\Phi, -Q, P, -\bar{r}, -\bar{t}, -\bar{s})$. Then we have

$$'[R_1, R_2]=['R_1, 'R_2].$$

In fact, if we identify $R\in\mathfrak{e}_8^C$ with $\mathrm{ad}R\in\mathrm{ad}\,\mathfrak{e}_8^C$, then we have $\langle '[R_1, R_2]R_3, R_4\rangle=-\langle R_3, [R_1, R_2]R_4\rangle=-\langle R_3, R_1R_2R_4-R_2R_1R_3-'R_1'R_2R_3, R_4\rangle=\langle ['R_1, 'R_2]R_3, R_4\rangle$ (Proposition 17) for $R_3, R_4\in\mathfrak{e}_8^C$. Now, we define a linear form B_1 of $\mathfrak{e}_8^C$ by

$$B_1(R_1, R_2)=\langle 'R_1, R_2\rangle.$$

Then this B_1 is an invariant form of $\mathfrak{e}_8^C$, since $B_1([R, R_1], R_2)=\langle '[R, R_1], R_2\rangle=\langle ['R, 'R_1], R_2\rangle=\langle 'R'R_1, R_2\rangle=-\langle 'R_1, RR_2\rangle=-B(R_1, [R, R_2])$ (Proposition 17) for $R\in\mathfrak{e}_8^C$. Therefore the Killing form B of $\mathfrak{e}_8^C$ is equal to B_1 up to a constant k: $B=kB_1$. Considering special elements of $\mathfrak{e}_8^C$, for example $R_1=R_2=(0, 0, 0, 0, 0, 1)$, we can easily obtain $k=-15$ and we see that $-15B_1$ has the form stated in Theorem 28.

Remarks. The Killing form of the Lie algebra $\mathfrak{g}_2^C = \mathrm{Der}(\mathfrak{C}^C) = \{D \in \mathrm{Hom}_C(\mathfrak{C}^C, \mathfrak{C}^C) \mid D(xy) = (Dx)y + x(Dy)\}$ is given by

(1) $B(D_1, D_2) = 4\,\mathrm{tr}(D_1 D_2), \quad D_i \in \mathfrak{g}_2^C$.

(2) $B(D, D_{a,b}) = -28(Da, b), \quad D, D_{a,b} \in \mathfrak{g}_2^C$ (where $D_{a,b}$ $(a, b \in \mathfrak{C}^C, \bar{a} = -a, \bar{b} = -b)$ is defined by $D_{a,b}x = a(bx) - b(ax) + a(xb) - (ax)b + (xb)a - (xa)b, x \in \mathfrak{C}^C)$.

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