Simple transcendental extensions of valued fields

Dedicated to A. Seidenberg on his 65th birthday

By

Jack OGM

(Communicated by Prof. M. Nagata, Jan. 21, 1981)

Let $K_0 \subset K = K_0(x)$ be fields with $x$ transcendental over $K_0$; let $v_0$ be a valuation of $K_0$ and $v$ be an extension of $v_0$ to $K$; and let $V_0 \subset V$, $k_0 \subset k$, and $G_0 \subset G$ be the respective valuation rings, residue fields, and value groups.

If $k$ is not algebraic over $k_0$, then there exists $y \in V$ such that $y$ specializes to a transcendental $y^*$ over $k_0$ under the canonical homomorphism $V \rightarrow k$; if this $y$ should happen to be a generator of $K/K_0$, then it is easily seen that $k = k_0(y^*)$ and $G = G_0$.

Our main theorem asserts that, under the assumption that $\text{char } k_0 = 0$, if $v_0$ is henselian, then the converse holds: if $k/k_0$ is simple transcendental and $G = G_0$, then there exists a generator of $K/K_0$ which specializes to a transcendental over $k_0$. We also prove that "$v_0$ is henselian" can be replaced by "$v_0$ is rk 1" and that for arbitrary finite rk $v_0$ one must assume, in addition, that for every valuation ring $W \supseteq V$ of $K$, the residue field of $W$ is simple transcendental over the residue field of $W \cap K_0$.

It requires no new considerations to prove this theorem under the a priori weaker hypothesis that $k_0$ is algebraically closed in $k$ and $\cong k$ and $K/K_0$ is generically of index 1 (i.e. every generator of $K/K_0$ has value in $G_0$), and in this form the theorem yields as a corollary the char 0 case of the following conjecture of Nagata:

**Ruled Residue Conjecture.** $k$ is either algebraic or ruled over $k_0$.

("Ruled" means that there should be a field $k_1$ with $k_0 \subset k_1 \subset k$ and $k$ simple transcendental over $k_1$; in the present setting such a $k_1$ is necessarily finite algebraic over $k_0$.) Nagata [7] has proved, without assumption on the characteristic, that this conjecture holds for discrete $v_0$ and that $k$ is always either algebraic over $k_0$ or contained in a finite algebraic extension of $k_0$ followed by a simple transcendental extension.

The paper divides into two parts. Part I, consisting of §§ 1–5, is devoted to proving the above theorem for henselian $v_0$ (3.7) and to deriving the above conjecture in char 0 from it (4.6). In Part II (§§ 6–8) the corresponding theorem for $v_0$ of finite rk is proved.
The main portion of this work was done (1978–79) while the author was on sab-
batical leave from LSU, during which time he enjoyed the hospitality of the
University of Wisconsin-Milwaukee.

Notation and terminology.

We fix fields $K_0 < K$ with $K$ a simple extension of $K_0$, i.e. there exists $x \in K$, $x \notin K_0$
such that $K = K_0(x)$. Usually $x$ will be transcendental (abbreviated tr.) over $K_0$, but we do not a priori assume this. We also fix a valuation $v$ of $K$ and its restriction $v_0$ to $K_0$. Moreover, we shall consistently use $x$ to denote a generator of $K/K_0$
of value 0; there always exists such a generator since one of $x, 1+x, \text{ or } 1+(1/x)$ must have value 0.

The valuation ring, residue field, and value group of $v$ will be denoted $\mathcal{V}, k_v, \text{ and } \mathcal{G}$ respectively; a subscript 0 will indicate the corresponding objects for $v_0$: and $K^v, v^v$ will denote the henselization (cf. [4] or [9]) of $K, v$. By the index of $K/K_0$ we shall mean $[G: G_0]$; and we shall say that $K/K_0$ ($or v/v_0$) is generically of index 1 if for every generator $z$ of $K/K_0, v(z) \in G_0$. For example, $K/K_0$ is generically of index 1 if $[G: G_0] = 1$. This condition will be used in § 3 and will be discussed in § 4.

The notation $(\cdot)$ will be reserved for image under the canonical homomorphism $V \rightarrow V/m_v = k$; thus, if $a \in V$, $a^*$ denotes the image of $a$ under $V \rightarrow k$. To enlarge on this notation, $K \rightarrow k$ will signify in our diagrams that $k$ is the residue field of $v$; and for $a \in K$, $a \rightarrow a^*$ (read “$a$ specializes to $a^*$ under $v$”) will mean $a \in V$ and $a^*$ is the image of $a$ under $V \rightarrow k$. The reference to $v$ will be omitted when the valuation involved is clear. Similarly, if $f(X) \in V[X], f(X)^*$ will denote the image of $f(X)$ under the homomorphism $V[X] \rightarrow k[X]$ obtained by specializing coefficients.

In addition, we shall use $\mathbb{Z}$ to denote the integers, $\mathbb{Q}$ the rationals, $\mathbb{C}$ the complex numbers, and $X$ an indeterminate.

Part I: The theorem for henselian $v_0$, and the Ruled Residue Conjecture.

1. Preliminaries.

As specified above, $K = K_0(x), x \notin K_0$, with $x$ either transcendental or algebraic
over $K_0$ and $v(x) = 0$.

In a few special cases it is easy to describe a generating set for $k/k_0$. To begin with, note that we always have $k_0(x^*) \subseteq k$ since $k_0 \subseteq k$ and $x^* \in k$.

1.1. Inf extensions (See also 4.3).

For any $z \in K$, $v$ will be called the inf extension (to $K_0(z)$) of $v_0$ w.r.t. $v(z)$ if for every
$\zeta = a_0 + a_1 z + \cdots + a_n z^n, a_i \in K_0, v(\zeta) = \inf \{ v_0(a_i) + iv(z) | i = 0, \ldots, n \}$. If $z$ is tr. over $K_0$, then it is easily verified that an extension of $v_0$ to $K_0(z)$ may be so defined (cf. [2, p. 160, Lemma 1]). We are mainly interested in the inf extension of $v_0$ w.r.t. $v(x) = 0$, for which the following simple fact is basic: $x^*$ is tr. over $k_0 \leftrightarrow x$ is
tr. over $K_0$ and $v$ is the inf extension of $v_0$ w.r.t. $v(x) = 0$; and when this is the case,
then $k=k_0(x^*)$ and $G=G_0$ (cf. [2, p. 161, Prop. 2]).

1.2. Suppose $x$ is algebraic over $K_0$. Then $K$ is algebraic over $K_0$, and therefore also $k$ is algebraic over $k_0$. Moreover, it is a classical result that $[K : K_0] \geq [k : k_0] \times [G : G_0]$ (cf. [2, p. 138, Lemma 2]). Therefore if $[K_0(x) : K_0] = [k_0(x^*) : k_0]$, then $k=k_0(x^*)$ and $G=G_0$; or if $[K_0(x) : K_0] = [G : G_0]$, then $k=k_0$. (A strong form of the above inequality [2, p. 143, Theorem 1] shows that in these two cases $v$ is the only extension of $v_0$, up to equivalence.) Note also that the inequality implies that $x$ is tr. over $K_0$ whenever $k$ is not finite algebraic over $k_0$.

1.3. Suppose $k$ is not algebraic over $k_0$. Then there exists $z \in k$ such that $z$ is tr. over $k_0$. Let $y$ be a preimage in $V$ for $z$. By 1.1, $y$ is tr. over $K_0$ and the restriction of $v$ to $K_0(y)$ has residue field $k_0(z)$ and value group $G_0$. Since $x$ is algebraic over $K_0(y)$, by the inequality of 1.2 we have $[G : G_0] < \infty$ and $[k : k_0(z)] < \infty$. In particular, $k$ is then a finitely generated extension of $k_0$ of tr. degree 1; so if $k/k_0$ is an algebraic extension followed by a simple tr. extension, then it is automatically a finite algebraic extension followed by a simple tr. extension. (If $k/k_0$ is algebraic, then it can happen that $[k : k_0] = \infty$; see 5.1.)

1.4. The residue field and value group for the henselization $v^\wedge$, $K^\wedge$ of $v$, $K$ are again $k$ and $G$ (cf. [4, p. 136]). If $a \in k$ is separably algebraic of deg $n$ over $k_0$ then by Hensel’s lemma (cf. [4, p. 118, Cor. 16.6]) there exists a preimage $a \in K^\wedge$ for $a$ such that $a$ is separably algebraic over $K_0$ of deg $n$. It follows from 1.2 that $G_0$ is the value group and $k_0(a)$ the residue field of $v^\wedge$ restricted to $K_0(a)$.

A consequence is that if $K^\wedge_0$ is the separable algebraic closure of $K_0$ in $K^\wedge$, then the restriction $v^\wedge_0$ of $v^\wedge$ to $K^\wedge_0$ has a residue field $k_1$ which contains the separable algebraic closure of $k_0$ in $k$, and hence which is itself separably algebraically closed in $k$, and a value group $G_1$ such that $G_0 \subseteq G_1 \subseteq G$, and the restriction of $v^\wedge$ to $K^\wedge_0(x)$ has residue field $k$ and value group $G$.

Moreover, $v^\wedge_0$, $K^\wedge_0$ is henselian [4, p. 130, Theorem 17.9]. Thus, in considering the Ruled Residue Conjecture, we may assume $k_0$ is separably algebraically closed in $k$ and $K_0$ is henselian.

A word of caution is in order, however. In passing from $K_0$ to $K^\wedge_0$, the notion of “generator” changes: for if $r \in K^\wedge_0|K_0$, then $x-r$ is a generator of $K^\wedge_0(x)$ over $K^\wedge_0$ but is not a generator of $K_0(x)$ over $K_0$, since it is not even in $K_0(x)$.

2. Generating pairs.

Throughout § 2 $x, y$ will be elements of $k$ of value 0, with $x$ tr. over $K_0$. 
2.1. Definition. \( x \) will be called a generator for \( y \) if \( y \in K[X] \), or, equivalently if \( y = af(x) \) for some \( a \neq 0 \in K \) and some primitive \( f(X) \in V_0[X] \). \((f(X) \in V_0[X] \) is called primitive if some coefficient has value 0.) The pair \( x, f(X) \) will be called a generating pair for \( y \). The \( a \) and \( f(X) \) are unique up to unit multiples from \( V_0 \); to be precise, if \( y = a_i f_i(x) \) for some \( a_i \neq 0 \in K \) and some primitive \( f_i(X) \in V_0[X] \), then there exists a unit \( u \in V_0 \) such that \( a = ua_i \) and \( f(X) = (1/u)f_i(X) \). Note also that \( u(y) = 0 \) implies \( u(f(x)) = -u(a) \geq 0 \).

2.2. Multiplicity. The generator for \( y \) (or the generating pair \( x, f(X) \)) will be said to have multiplicity \( n \geq 0 \) if \( x^a \) is a root of multiplicity \( n \) for \( f(X)^* \), i.e. if \( f(X)^* = (X - x^*)^n h(X) \), with \( h(X) \in k_0[X] \) and \( h(x^*) \neq 0 \).

Suppose \( r \in V \) is such that \( r^* = x^a \). We may write \( f(X) = a_0 + a_1(X-r) + \cdots + a_d(X-r)^d \), where the \( a_i \) are uniquely determined elements of \( V_0[r] \); in fact, \( a_i = f^{(i)}(r) \), where \( f^{(i)}(r) \) shall denote the \( i \)th derivative of \( f(X) \) with the coefficients formally divided by \( i! \), evaluated at \( r \). Then \( x^* \) is a root of multiplicity \( n \) for \( f(X)^* \), \( \Rightarrow a_0 = \cdots = a_{n-1} = 0 \) and \( a_n \neq 0 \). For further reference, note also that if \( f(X) = b_0 + b_1 X + \cdots + b_n X^n \), then \( x^* \) is a root of multiplicity \( \leq n \) for \( f(X)^* \) if \( v(b_0) = 0 \) and \( v(b_j) > 0 \) for \( j > n \); for then \( f^{(n)}(x^*) = b_n + \text{(terms of value > 0)} \), so \( f^{(n)}(x^*) = b_n^* \neq 0 \).

2.3. Multiplicity 0. \( x \) is a generator for \( y \) of multiplicity 0 if \( y \in V_0[x] \). For, suppose \( x, f(X) \) is a generating pair for \( y \). Since \( y = af(X) \), \( a \in K \), and \( v(y) = 0 \), \( v(a) = 0 \) if \( f(X)^* \) is a generator of \( y \), then \( a \in V_0 \) and hence \( y \in V_0[X] \). Conversely, if \( y \in V_0[X] \), then there exists \( a \in V_0 \) and a primitive \( f(X) \in V_0[X] \) such that \( y = af(X) \). Since \( v(a) \geq 0 \), it follows from \( u(y) = 0 \) that \( u(a) = 0 \); so \( x, f(X) \) has multiplicity 0.

2.4. Existence of generating pairs.

Proposition. Assume \([G : G_0] = n < \infty \), let \( x \) be a tr. generator of \( K \) over \( K_0 \) of value 0, and let \( l \) be any field such that \( k_0 \subset l \subset k \). If there exists \( x \in k \) such that \( x^* \not\in l \), then there exists \( y \in K_0[x] \) of value 0 such that \( y^* \not\in l \).

Proof. Choose a preimage \( a \in K \) for \( x \). Since \( K = K_0(x) \), \( a = f_1(x)/f_2(x) \), \( f_1(X) \in K_0[X] \). Let \( b = a^n = f_1(x)^n/f_2(x)^n \). The hypothesis \([G : G_0] = n \) implies \( v(f_i(x)^n) \in G_0 \), \( i = 1, 2 \). Therefore there exist \( c_i \in K_0 \) such that \( v(c_i f_i(x)^n) = 0 \); and then \( b = (c_2/c_1)(c_1 f_1(x)^n/c_2 f_2(x)^n) \), where \( c_2/c_1, c_1 f_1(x)^n \), \( i = 1, 2 \), all have value 0. But then \( b^* = (c_2/c_1)^*[(c_1 f_1(x)^n)^*(c_2 f_2(x)^n)^*] \) implies either \( c_1 f_1(x)^n \) or \( c_2 f_2(x)^n \) is not in \( l \) since \( b^* = x^* \not\in l \). Thus, for \( i = 1 \) or 2, \( y = c_i f_i(x)^n \) is the required element.

Corollary. Let \( x \) be a generator of \( K \) over \( K_0 \) of value 0, and suppose \( k \) is not algebraic over \( k_0 \). Then there exists \( y \in K_0[X] \) of value 0 such that \( y^* \) is tr. over \( k_0 \) if, moreover, \( x \) is a generator of multiplicity 0 for this \( y \), then \( x^* \) is tr. over \( k_0 \) and \( k = k_0(x^*) \).

Proof. For the first assertion, note that \([G : G_0] < \infty \) by 1.3, and then apply the above proposition with \( l = \text{algebraic closure of } k_0 \) in \( k \). For the second assertion,
apply 2.3 to conclude \( y \in V_0[x] \). It follows that \( y^* \in k_0[x^*] \) and hence that \( x^* \) is tr. over \( k_0 \). Then by 1.1, \( k = k_0(x^*) \).

2.5. Nagata’s proof [7, p. 91, Thm. 5] that \( k/k_0 \) is either algebraic or \( k \) is contained in a finite algebraic extension of \( k_0 \) followed by a simple tr. extension:

Suppose \( K_0 \) is algebraically closed and \( k/k_0 \) is not algebraic. By 2.4 there exists \( y \in K_0[x] \) such that \( y^* \) is tr. over \( k_0 \). Factor: \( y = a(x - r_1) \cdots (x - r_m) \), \( a, r_i \in K_0 \). Since \([G: G_0] < \infty \) (1.3) and \( G_0 \) is now divisible, we have \( G = G_0 \). Therefore there exist \( b_1, \ldots, b_m \in K_0 \) such that \( v(x - r_i) = b_i \). Then \( y^* = (ab_1 \cdots b_m)((x - r_1)/b_1)^* \cdots ((x - r_m)/b_m)^* \), so \( y^* \) is tr. over \( k_0 \) implies \( (x - r_i)/b_i \) is tr. over \( k_0 \) for some \( i \). Thus, we have found a generator \( x_1 = (x - r_i)/b_i \) of \( K/K_0 \) such that \( x_1^* \) is tr. over \( k_0 \).

By 1.1, \( k = k_0(x_1^*) \).

If \( K_0 \) is not algebraically closed, pass to the algebraic extension \( K_0' = K_0(r_i, b_i) \). The residue field of \( K_0'(x) \) is \( k_0(x_1^*) \), where \( k_0 \) is the residue field of \( K_0' \) and hence is finite algebraic over \( k_0 \). Thus, \( k_0 < k < k_0(x_1^*) \).

3. Proof of the theorem.

We remind the reader that \( x \) always denotes a generator of \( K \) over \( K_0 \) of value 0 \( (x \not\in K_0) \). In addition, throughout §3 \( x \) will be assumed tr. over \( K_0 \) and \( y \) will be an element of \( K \) of value 0 having a fixed generating pair \( x, f(x) \) of multiplicity \( n > 0 \).

3.1. Definition. We shall call \( x \) rational if \( x^* \in k_0 \), or equivalently, if there exists \( r \in K_0 \) such that \( v(x - r) > 0 \). For such an \( r \), \( v(r) = 0 \) and \( r^* = x^* \).

Let \( \mathfrak{Z}(x) = \{ x \in K \mid \text{there exist } r, 0 \neq b \in K_0 \text{ such that } x_1 = (x - r)/b \text{ and } v(x - r) = v(b) > 0 \} \). Whenever we write \( x_1 = (x - r)/b \in \mathfrak{Z}(x) \), we shall tacitly assuming that \( r, 0 \neq b \in K_0 \) and \( v(x - r) = v(b) > 0 \). Note that \( \mathfrak{Z}(x) \neq \emptyset \) if \( x \) is rational and \( K \) is generically of index 1 over \( K_0 \). (Reminder: generically index 1 means every generator of \( K \) over \( K_0 \) has value in \( G_0 \).) If \( x_1 \in \mathfrak{Z}(x) \) and there exist \( r_i, 0 \neq b_i \in K_0 \) such that \( v(x_1 - r_i) = v(b_i) > 0 \), then \( x_2 = (x_1 - r_i)/b_i \in \mathfrak{Z}(x) \) too. Thus, every \( x_1 \in \mathfrak{Z}(x) \) is a generator of \( K \) over \( K_0 \) of value 0, and \( \mathfrak{Z}(x_1) \subseteq \mathfrak{Z}(x) \).

The next lemma is crucial to the proof of the main theorem.

3.2. Lemma. Suppose there exists \( x_1 = (x - r)/b \in \mathfrak{Z}(x) \) such that \( x_1 \) is not a generator for \( y \) of multiplicity \( n \), and write \( f(X) = a_0 + a_1(X - r) + \cdots + a_n(X - r)^n + \cdots + a_m(X - r)^m \), \( a_i \in V_0[r] (\subset V_0) \). Then

i) \( v(a_i(x - r)^i) \geq v((x - r)^n) \) for \( i = 0, \ldots, n - 1 \);

ii) \( x_1 \) is a generator for \( y \) of multiplicity \( n \); and

iii) if \( char k \neq n \), then \( v(a_{n-1}) = v(x - r) \).

Remark. Since we are assuming throughout §3 that \( x \) is a generator for \( y \) of multiplicity \( n > 0 \), ii) may be rephrased: if \( x \) is a generator for \( y \) of multiplicity \( n > 0 \), then every element of \( \mathfrak{Z}(x) \) is a generator for \( y \) of multiplicity \( \leq n \). Also, iii) implies \( a_{n-1} \neq 0 \) because \( x \not\in K_0 \) implies \( x - r \neq 0 \).

Proof. Note to begin with that \( v(a_n) = 0 \) since \( r^* \) is a root of multiplicity \( n \) of \( f(X)^* \).
i): Suppose there exists \( i < n \) such that \( v(a_i(x-r)^j) < v((x-r)^n) \). Choose \( q \) to be the largest integer in \( \{0, \ldots, n-1\} \) such that \( v(a_i(x-r)^q) = \min \{ v(a_j(x-r)^j) \mid j = 0, \ldots, n-1 \} \), i.e. choose \( q \in \{0, \ldots, n-1\} \) such that

\[
\begin{align*}
v(a_q(x-r)^q) &< v(a_j(x-r)^j), \quad j = q + 1, \ldots, n, \\
\text{and } v(a_q(x-r)^q) &\leq v(a_j(x-r)^j), \quad j = 0, \ldots, q.
\end{align*}
\]

It follows that \( v(a_q(x-r)^q) < v((x-r)^q) \), since \( v(a_q(x-r)^q) = v((x-r)^q) < v(a_j(x-r)^j), \quad j > n \).

Now consider \( (1/a_q b^n) f(x) = b_0 + b_1 x + \cdots + b_n x^n \), where \( b_j = a_j / a_q b^{q-j} \), \( j = 0, \ldots, m \). By (i), \( v(b_j) \geq 0 \), \( j = 0, \ldots, q \),

\[
\begin{align*}
v(b_j) &\geq 0, \quad j = 0, \ldots, q, \\
b_q &\geq 1, \\
v(b_j) &> 0, \quad j = q + 1, \ldots, m.
\end{align*}
\]

Let \( f_1(X) = b_0 + b_1 X + \cdots + b_m X^m \). Then \( y = af(x) = a b^n f_1(x) \), so \( x_1, f_1(X) \) is a generating pair for \( y \). Moreover, by 2.2 the multiplicity of \( x_1, f_1(X) \) is \( \leq q < n \).

Thus, we have a contradiction to the hypothesis that \( x_1 \) is not a generator for \( y \) of multiplicity \( < n \).

ii): Consider \( (1/a_q b^n) f(x) = b_0 + b_1 X + \cdots + b_m X^m \), where now \( b_j = a_j / a_q b^{q-j} \), \( j = 0, \ldots, m \); and again let \( f_1(X) = b_0 + b_1 X + \cdots + b_m X^m \). By i), \( v(b_j) \geq 0 \) for \( j = 0, \ldots, n \); and also \( v(b_j) > 0 \) for \( j = n + 1, \ldots, m \) since \( v(b) > 0 \). By 2.2 we again see that \( x_1, f_1(X) \) is a generating pair for \( y \) of multiplicity \( \leq n \); and the hypothesis that \( x_1 \) is not a generator for \( y \) of multiplicity \( < n \) yields the equality.

iii): Let \( f_1(X) \) be as in ii). Then \( f_1^{n-1}(x_1) = b_{n-1} + nx_1 + c_2 b_{n+1} x_1^2 + \cdots + c_{m-n+1} b_m x_1^{m-n+1} \), where the \( c_i \) are natural numbers. Therefore \( f_1^{n-1}(x_1)^* = b_{n-1}^* + nx_1^* \) since \( v(b_j) > 0 \), \( j = n + 1, \ldots, m \). But \( nx_1^* \neq 0 \) because char \( k \neq n \); so we must have \( b_{n-1}^* \neq 0 \) too, for otherwise \( x_1 \) would be a generator for \( y \) of multiplicity \( < n \), contrary to hypothesis. But \( b_{n-1}^* \neq 0 \) implies \( v(b_{n-1}^*) = 0 \), so \( v(a_{n-1}) = v(a_n b) = v(b) = v(x-r) \).

3.3 Corollary. Suppose \( \text{char } k \neq n \) and \( K/K_0 \) is generically of index 1. If \( x \) is rational and \( \deg f(X) = n \), then there exists \( x_1 \in \mathfrak{X}(x) \) which is a generator for \( y \) of multiplicity \( < n \).

Proof. Since \( x \) is rational, there exists \( r \in K_0 \) such that \( v(x-r) > 0 \). Then \( r \in V_0, \) and \( f(X) = a_0 + a_1(X-r) + \cdots + a_n(X-r)^n \), \( a_i \in V_0[r] = V_0 \). By our initial assumption, \( x, f(X) \) is a generating pair for \( y \) of multiplicity \( n \), so \( a_n^* \neq 0 \) and \( a_{n-1}^* = 0 \). Let \( t = -a_{n-1} / a_n \). Then \( t \in K_0 \) and \( v(t) > 0 \). Now let \( r_1 = r + t \), and rewrite \( f(X) = b_0 + b_1(X-r_1) + \cdots + b_n(X-r_1)^n \), where \( b_i = a_i, b_{n-1} = nb_n, b_{n-2} = a_{n-1}, \ldots \). Since \( K/K_0 \) is generically of index 1, there exists \( b \neq 0 \in K_0 \) such that \( v(x-r_1) = v(b) > 0 \), and hence \( x_1 = (x-r_1)/b \in \mathfrak{X}(x) \). But \( t \) was chosen so that \( b_{n-1} = 0 \). Thus, the failure of 3.2—iii) yields the conclusion that \( x_1 \) must be a generator for \( y \) of multiplicity \( < n \).
3.4. Lemma. Suppose \( x \) is rational and \( K_0 \) is henselian. Then there exists \( s \in V_0[x] \) of value 0 such that \( y/s \) has a generating pair \( x, g(X) \) of multiplicity \( n \) and

\[ \frac{g(x) + X}{11(X)}, \text{char } \overline{\mathbf{V}} \]

over, by 3.3.2-Corollary, there exists \( x, f(x) \) which is a generator for \( y/s \) of multiplicity \( < n \). This means there exists \( a \in K_0 \) and a primitive \( f_1(X) \in V_0[X] \) such that \( y/s = af_1(x_1) \) and \( x_1^* \) is a root of multiplicity \( < n \) for \( f_1(X)^* \). If we write \( s = s(x) \in V_0[x] \), and if \( x_1 = (x - r)/b \), then \( s(x_1, b + r) = s_1(x_1) \in V_0[x_1] \). Moreover, \( s_1(x_1^*) = s^* \neq 0 \), so \( x_1^* \) is a root of multiplicity \( 0 \) of \( s_1(X)^* \).\] Thus, \( y = s_1(x_1)f_1(x_1) \), and it follows that \( x_1, s_1(X)f_1(X) \) is a generating pair for \( y/s \) of multiplicity \( < n \).

3.6 Corollary. Suppose \( K_0 \) is henselian, \( K/K_0 \) is generically of index 1, and \( \text{char } k \neq n \). If \( x \) is rational, then there exists \( x_1 \in \mathfrak{M}(x) \) such that \( x_1 \) is a generator for \( y \) of multiplicity \( 0 \).

Proof. Since \( x \) is rational and \( K/K_0 \) is generically index 1, \( \mathfrak{M}(x) \neq \emptyset \). Moreover, by 3.2 every element of \( \mathfrak{M}(x) \) is a generator for \( y \) of multiplicity \( \leq n \). Choose \( x_1 \in \mathfrak{M}(x) \) of multiplicity \( \mu \) and such that no element of \( \mathfrak{M}(x) \) has multiplicity \( < \mu \). If \( \mu = 0 \), we are done; if not, by 3.5 there exists \( x_2 \in \mathfrak{M}(x) \ subset \mathfrak{M}(x) \) such that \( x_2 \) is a generator for \( y \) of multiplicity \( < \mu \), a contradiction to the choice of \( x_1 \).

3.7 Theorem. Assume \( K = K_0(x) \), where \( x \) is tr. over \( K_0 \) and \( v(x) = 0 \); \( \text{char } k = 0 \); and \( K_0 \) is henselian. If \( K/K_0 \) is generically of index 1 and \( k_0 \) is algebraically closed in \( k \) and \( \neq k \), then there exists \( x \in \mathfrak{M}(x) \cup \{x\} \) such that \( x^* \) is tr. over \( k_0 \).

Proof. If there exists \( x_1 \in \mathfrak{M}(x) \) such that \( x_1^* \neq k_0 \), then by hypothesis \( x_1^* \) is tr. over \( k_0 \) and we are done. Thus we may assume every element of \( \mathfrak{M}(x) \cup \{x\} \) is rational.

By 2.4-Corollary, there exists \( y_1 \in K \) of value 0 such that \( x \) is a generator for \( y_1 \) and \( y_1^* \) is tr./\( k_0 \); and also by 2.4-Corollary, we may further assume that \( x \) is a generator for \( y_1 \) of multiplicity \( n > 0 \). But then by 3.6 there exists \( x_1 \in \mathfrak{M}(x) \) such that \( x_1 \) is a generator for \( y_1 \) of multiplicity \( 0 \), which means \( y_1 \in V_0[x_1] \). Therefore \( y_1^* \in k_0[x_1^*] \), and hence \( x_1^* \) is tr./\( k_0 \). Q. E. D.
In view of the reduction of 1.4 whereby \( k_0 \) may be assumed separably algebraically closed in \( k \) and \( K_0 \) henselian, 3.7 yields the Ruled Residue Conjecture (char 0) in the case that \([G:G_0]=1\). For by 1.1 if a generator of \( K/K_0 \) specializes to a tr., then \( k/k_0 \) is simple transcendental.

4. Extensions generically of index 1.

We assume throughout \( \S \, 4 \) that \( K = K_0(x) \), \( x \) tr. over \( K_0 \) and \( v(x) = 0 \).

Before proceeding to the final ingredient in the proof of the Ruled Residue Conjecture (char 0), we shall make a couple of comments on the notion of “generically index 1.” Recall that \( K/K_0 \) is of index 1 means \( v(x) \in G_0 \) for every \( x \in K \) and that \( K/K_0 \) is generically of index 1 was defined to mean \( v(x) \in G_0 \) for every generator \( x \) of \( K/K_0 \).

4.1 Proposition. The following are equivalent:

i) \( K/K_0 \) is generically of index 1.

ii) If \( r \in K_0 \) and \( v(x - r) > 0 \), then \( v(x - r) \in G_0 \).

iii) Either \( \{ v(x - r) | r \in K_0 \text{ and } v(x - r) > 0 \} \) has no maximal element, or its maximal element is in \( G_0 \).

Proof. Since \( x - r \) is a generator of \( K/K_0 \) for all \( r \in K_0 \), the implications i) \( \Rightarrow \) ii) \( \Rightarrow \) iii) are immediate. ii) \( \Rightarrow \) i): Every generator of \( K_0(x)/K_0 \) is of the form \( \xi = (ax + b)/(cx + d) \); \( a, b, c, d \in K_0 \), \( ad - bc \neq 0 \) (cf. [10, p. 198]). Therefore it suffices to show \( v(ax + b) \in G_0 \) whenever \( a \neq 0 \), \( b \in K_0 \), or equivalently, to show \( v(x + (b/a)) \in G_0 \). Since \( v(a) = 0 \), either \( v(b/a) < 0 \) and \( v(x + (b/a)) = v(a) \in G_0 \), or \( v(b/a) \geq 0 \), in which case \( v(x + (b/a)) \geq 0 \) and ii) applies. iii) \( \Rightarrow \) ii): If there exist \( r, r' \in K_0 \) such that \( 0 < v(x - r) < v(x - r') \), then \( v(x - r) = v((x - r) - (x - r')) = v(r' - r) \in G_0 \). Thus, if \( v(x - r) \) is not a maximal element of the set, then it is automatically in \( G_0 \).

Q. E. D.

4.2 Example of \( K/K_0 \) which is generically of index 1 but not of index 1 and which has \( x \) rational, i.e. \( x \in k_0 \).

Let \( v \) be the \( X \)-adic valuation of \( Q(\sqrt{2}, \pi)(X) \), i.e. \( v \) is the inf extension of the 0-valuation of \( Q(\sqrt{2}, \pi)(X) \) w.r.t. \( v(X) = 1 \); let \( K_0 = Q(X^2) \); and let \( K = K_0(x) \), where \( x = 1 + \sqrt{2}X^2 + \pi X^3 \). In view of 4.1 to prove \( K/K_0 \) is generically of index 1 it suffices to show \( v(x - r) > 0 \) for \( r \in K_0 \), implies \( v(x - r) = 2 \). Note first that \( v(x - r) > 0 \) implies \( 1 = x^* = r^* \), so \( r = 1 - a \), \( a \in K_0 \) and \( v(a) > 0 \). Therefore \( x - r = a + \sqrt{2}X^2 + \pi X^3 \) and \( x - r)/X^2 = (a/X^2) + \sqrt{2}/X^2 \); so it remains to show \( v(a/X^2) = 2 \). But \( a \in K_0 \) and \( v(a) > 0 \) implies \( v(a) \geq 2 \). Then \( (a/X^2) + \sqrt{2}/X^2 \), \( (a/X^2)* + \sqrt{2}/X^2 \); and since \( a/X^2 \in K_0 \), \( (a/X^2)* \in K_0 = Q \). Since \( \sqrt{2} \notin Q \), it follows that \( (a/X^2)* + \sqrt{2}/X^2 \). Hence \( v(a/X^2) + \sqrt{2}/X^2 \).

Finally, to see that \( K/K_0 \) is not of index 1, note that \( [(x - 1)/X^2] - 2 = 2\sqrt{2} \pi X^2 + \pi^2 X^2 \) has value \( 1 \in G_0 \). Thus, \( G_0 = 2Z \) and \( G = Z \). Q. E. D.

Exactly when generically index 1 does imply index 1 for fields \( K/K_0 \) is not clear. For example, a consequence of 6.2 is that this implication holds if \( rk v = 1 \),
$k_0$ is algebraically closed in $k$ and $\neq k$, and either char $k=0$ or $v$ is discrete.

The following proposition relates arbitrary inf extensions to those defined with respect to value 0.

**4.3 Proposition (continuation of 1.1).** Let $z$ be a (tr.) generator of $K/K_0$, let $v(z)=g$, and suppose $g+G_0$ is of finite order $n \geq 1$ in $G/G_0$. Let $v_1 = v|_{K_1}$, where $K_1=K_0(z^n)$, and let $k_1$ be the residue field of $v_1$. Then the following are equivalent:

i) $v$ is the inf extension of $v_0$ w.r.t. $v(z)=g$.

ii) $v_1$ is the inf extension of $v_0$ w.r.t. $v_1(z^n)=ng$.

iii) There exists $b \neq 0 \in K_0$ such that $v_1$ is the inf extension of $v_0$ w.r.t. $v_1(z^n/b)=0$.

iv) There exists $b \neq 0 \in K_0$ such that $z^n/b \overset{v_0}{\sim} \text{tr. over } k_0$.

Moreover, when these hold, then $k=k_1=k_0(z)$ and $G/G_0$ is cyclic, generated by $g+G_0$.

**Proof.** i)$\Rightarrow$ii)$\Rightarrow$iii) are immediate from the definitions, and iii)$\Rightarrow$iv) by 1.1. It remains to show iii)$\Rightarrow$i). The value group of $v_1$ is $G_0$ and the residue field is $k_0(z)$ by 1.1. Since $[K:K_1]=n$ and $[G:G_0] \geq n$, it follows from 1.2 that $[G:G_0]=n$, $[k:k_1]=1$, and $v_1$ extends uniquely, up to equivalence, to $K$. In particular, then $G=G_0+Zg$ and $k=k_1$. But the inf extension $w$ of $v_0$ w.r.t. $w(z)=g$ is an extension of $v_1$ to $K$ (cf 1.1), so $w$ is equivalent to $v$. Since $G=G_0+Zg$ and $w(z)=g=v(z)$, we must actually have $w=v$.

Q. E. D.

We are now ready for the technical device (4.4 and 4.5) needed to complete the proof of the Ruled Residue Conjecture (char 0).

**4.4 Lemma.** Let $\xi \in K$, $\not\in K_0$ and $v(\xi)=g$, where $g+G_0$ is of finite order $n \geq 1$ in $G/G_0$; let $t$ be tr. over $K$, and let $v_i$ denote the inf extension of $v$ (to $K(t)$) w.r.t. $v_i(t)=g$; and let $v^\wedge_i$ be the henselization of $v_i$, $K(t)$.

If char $k$ $\nmid n$, $k_0$ is algebraically closed in $k$, and $v$ is not the inf extension of $v_0$ (to $K_0(\xi)$) w.r.t. $v(\xi)=g$, then there exists $b \in K(t)^\wedge$ algebraic over $K_0(t)$ with the following properties:

i) $b \overset{v}{\sim} b^* \text{ tr. over } k$.

ii) The residue fields of $K'=K(t, b)$ and $K_0'=K_0(t, b)$ are $k(b^*)$ and $k_0(b^*)$, respectively.

iii) The value groups of $K'$ and $K_0'$ are $G$ and $G_0+Zg$, respectively.

**Proof.** Since $v_i(t^n)=ng \in G_0$, there exists $d \in K_0$ such that $v_i(t^n)=v_i(d)$; and by 4.3, $t^n/d \overset{\alpha}{\sim} \text{tr. over } k_0$ and the residue field of $K_0(t)$ is $k_0(x)$. Also, by 1.1, $t/\xi \sim b^* \text{ tr. over } k$ and the residue field of $K(t)$ is $k(b)$. But $v(d)=v(\xi^n)$ implies there exists $u \in K$ of value 0 such that $\xi^n=ud$; and therefore $(t/\xi)^n=(1/u)(t^n/d)$, and consequently $\beta^n=(1/u^*)\alpha$.

Claim: $u^* \in k_0$. For otherwise $u^*$ is tr. over $k_0$ by hypothesis. But then $u=\xi^*d \overset{\alpha}{\sim} u^* \text{ tr. over } k_0$ implies by 4.3 that $v$ is the inf extension of $v_0$ w.r.t. $v(\xi)=g$, a contradiction to our hypotheses.

Thus, $\beta$ is separably algebraic of deg $n$ over $k_0(\alpha)$; so by Hensel's lemma [4, p.
118, Cor. (16.6)] there exists \( b \in K(t)^\wedge \) algebraic of deg \( n \) over \( K_0(t) \) such that \( b \to \beta. \) Then the residue field and value group for \( K(t, b) \) are \( k(\beta) \) and \( G \) since \( K(t) \subseteq K(t, b) \subseteq K(t)^\wedge. \) By 1.2 and 4.3 the residue field and value group for \( K_0(t, b) \) are \( k_0(\alpha, \beta) = k_0(\beta) \) and \( G_0 + \mathbb{Z}g = \text{value group of } K_0(t) \). Q.E.D.

Note that if \( \xi \) is a generator of \( K/K_0, \) then by 4.3 \( k/k_0 \) is not simple transcendental implies \( v \) is not the inf extension of \( v_0 \) w.r.t. \( v(\xi). \) This is how we shall fulfill the above hypothesis in the following corollary.

**4.5 Corollary.** If there exist (valued) fields \( K \supseteq K_0 \) such that

i) \( K/K_0 \) is simple tr. and \( \text{char } k = 0, \)

ii) \( K_0 \) is henselian,

iii) \( k_0 \) is algebraically closed in \( k \) and \( k \not\subseteq k_0, \)

iv) \( k/k_0 \) is not simple tr.,

then there exist such fields with the additional property that \( K/K_0 \) is generically of index 1.

**Proof.** Suppose there exists a generator \( z \) of \( K/K_0 \) such that \( v(z) = g \not\subseteq G_0. \) By 4.4 there exist fields \( K_0' \supseteq K' = K_0(z) \) having residue fields \( k'_0 = k_0(\beta), \) \( k' = k(\beta), \) respectively, \( \beta \) tr. over \( k, \) and value groups \( G_0, G, \) respectively, with \( [G: G_0] < [G: G_0]. \) It follows from [11, p. 167, Lem. 2] that \( k'/k_0' \) satisfies iii) and from the generalized Lüroth theorem [8, p. 137, Thm. 4.12.2] that \( k'/k_0' \) satisfies iv). Now replace \( K' \) by its henselization \( (K_0')^\wedge \) (inside \( (K')^\wedge \)) and \( K' \) by \( (K_0')^\wedge(z); \) this does not alter the residue fields or value groups (cf. [4, p. 136, Thm. 17.19] or [8, p. 193, Thm. 5.11.11]). Thus, under the assumption that \( K/K_0 \) is not generically index 1 we have found fields \( (K_0')^\wedge \supseteq (K_0')^\wedge(z) \) satisfying i)–iv) and the additional condition that \( [G: G_0] < [G: G_0]. \) The corollary now follows by induction on \( [G: G_0]. \)

**4.6 Ruled Residue Theorem** (char 0). Let \( K_0 \) and \( K = K_0(x) \) be fields with \( x \) tr. over \( K_0, \) let \( v \) be a valuation of \( K \) with residue field \( k, \) and let \( k_0 \) be the residue field of \( v|K_0. \) Suppose \( \text{char } k = 0 \) and \( k \) is not algebraic over \( k_0. \) Then there exists a finite algebraic extension \( k_1 \) of \( k_0 \) and an \( x \) tr. over \( k_1 \) such that \( k = k_1(x). \)

**Proof.** By 1.3 it suffices to show \( k \) is of the form \( k_1(x), k_1 \text{ algebraic over } k_0 \) and \( x \text{ tr. over } k_1. \) By 1.4 we may assume \( K_0 \) is henselian and \( k_0 \) is algebraically closed in \( k, \) and by 4.5 we may additionally assume \( K/K_0 \) is generically of index 1. The theorem now follows from 3.7. Q.E.D.

**4.7 Remarks.**

1. It is only in the reduction step of 4.5 that field extensions of \( K \) lying outside \( v^\wedge, K^\wedge \) are used. If one wants to think in terms of working inside a fixed valued field, he can proceed as follows: If order of \( G/G_0 = s, \) choose preimages \( g_1, \ldots, g_s \in G \) for the elements of \( G/G_0. \) Then let \( t_1, \ldots, t_s \) be indeterminates, and extend \( v \) to \( K(t_1, \ldots, t_s) \) by infs w.r.t. \( v(t_i) = g_i. \) Now the construction of 4.5 can be carried out inside the henselization \( K(t_1, \ldots, t_s)^\wedge. \)

2. On the char \( k = 0 \) assumption: It is not at all clear how to adapt our
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methods to the non-zero characteristic case. As noted in the introduction, Nagata has proved without restriction on the characteristic that the statement of 4.6 remains valid a) if \(v\) is discrete, \(\text{rk} n\), i.e. if \(G\) is a lexicographic direct sum of \(n\) copies of \(\mathbb{Z}\), or b) if the conclusion is weakened to \(k \subset k_1(x)\) (cf. [7, Thms. 1 and 5], [8, p. 198, Thm. 5.12.1]). When \(K_0 = \mathbb{Q}\), it seems that the discrete, \(\text{rk} 1\) case of a) (from which a) follows by induction) is implicit in the early paper [5] of Mac Lane, although the terminology of that paper obscures this conclusion (See [5, Thms. 8.1, 12.1, and 14.1]). As for further progress in removing the characteristic 0 assumption from 4.6, in generalizing from Nagata’s result a) above there are two extreme cases to take into account: one is the case of discrete, infinite \(\text{rk} v\), i.e. \(G\) is the lexicographic direct sum of infinitely many copies of \(\mathbb{Z}\); and the other (probably the more difficult) is the case of non-discrete, \(\text{rk} 1\) \(v\), e.g. \(G = \mathbb{Q}\).

3. Addendum (Oct., 1980). W. Heinzer, after reading a preprint of this paper, has pointed out that the Ruled Residue Conjecture for \(k_0\) perfect can be proved as follows: Let \(D = K_0[x] \cap V\); and note that \(V = D_S\), where \(S = \{ \text{units of } V \} \cap D\). For, if \(\zeta \in V\), write \(\zeta = f_1f_2, f_i \in K_0[x]\): since \([G : G_0] < \infty\), there exist \(a \in K_0\) and an integer \(n > 0\) such that \(v(f_i) = v(a)\); and therefore \((f_i/a)\zeta \in D\) and \(\zeta \in D_S\). It follows that \(k\) is the quotient field of \(D^*\), where \(D \rightarrow D^*\). Next, Nagata’s argument (cf. 2.5) shows there exists a finite algebraic extension \(K_0\) of \(K_0\) and an \(x_i = (x - r)/b \in K_0[x] = K_0[x_i]\) such that \(x_i^*\) is tr. over \(k_0\). By 1.1, then \(K_0'(x_i), \nu'\) is the inf extension of \(K_0, \nu_0\) w.r.t. \(\nu'(x_i) = 0\), from which it follows that \(D' \rightarrow k_0'[x_i]\), where \(D' = K_0'[x_i] \cap V'\). Thus, we have \(k_0 \subset D* \subset k_0'[x_i]\); so by [1, p. 322, (2.9)] the integral closure of \(D^*\) is of the form \(k_0'[z]\), \(k_0'\) algebraic over \(k_0\) and \(z\) tr. over \(k_0'\). But then \(k = k_0'(z)\).

Q. E. D.

The theorem of [1] on which Heinzer’s proof rests requires two non-elementary facts about 1-dim function fields: i) genus does not decrease under a finite separable extension of the base field and ii) genus 0 plus the existence of a rational place implies simple tr. Thus, while his proof yields the more general case of a perfect \(k_0\), it is not nearly as simple-minded as our proof of 4.6. In any case, both approaches should be of interest in further efforts to remove the restrictive hypothesis involving the characteristic.

5. Complements.

We begin with a class of examples to illustrate that all of the possibilities for \(k/k_0\) suggested by theorem 4.6 can occur.

5.1. Let \(k_0\) be a subfield of \(C\) = complex numbers, let \(C((t))\) be the field of formal Laurent series in the indeterminate \(t\) with coefficients in \(C\), and let \(v\) be the \(t\)-adic valuation of \(C((t))\). Let \(x = a_0 + a_1t + a_2t^2 + \cdots \in C[[t]]\), and consider the residue fields given by

\[
\begin{align*}
K_0 = k_0(t) & \quad K = k_0(t) (x) \quad C((t)) \\
\downarrow & \quad \downarrow \quad \downarrow \\
k_0 & \quad k \quad C
\end{align*}
\]
What is a generating set for \( k \) over \( k_0 \)?

**Lemma.** If \( a_0, a_1, \ldots, a_i \) (\( i \geq 0 \)) are algebraic over \( k_0 \), then \( a_0, a_1, \ldots, a_i, a_{i+1} \in k \).

**Proof.** Note that \( x \rightarrow a_0 \) implies \( a_0 \in k \). Let \( f(X) \in k_0[X] \) be the irreducible polynomial for \( a_0 \) over \( k_0 \), and let \( y_1 = f(x)/f'(a_0)((x-a_0)/t) + (f''(a_0)/2)((x-a_0)/t)^2 + \cdots \). Since \( (x-a_0)/t = a_1 + a_2 t + \cdots \), we can write \( y_1 = f'(a_0)a_1 + f''(a_0)a_2 + f'''(a_0)a_3 + \cdots \). But \( y_1 \rightarrow y_1^* = f'(a_0)a_1 \) and \( f'(a_0) \neq 0 \), so \( a_1 \in k \) since \( y_1^* \) and \( f'(a_0) \) are in \( k \).

Now let \( f_0(X) \in k_0[X] \) be the irreducible polynomial for \( y_1^* \) over \( k_0 \), and let \( y_2 = f_0(y_1)/f'(y_1^*)((y_1-y_1^*)/t) + (f''(y_1^*)/2)((y_1-y_1^*)/t)^2 + \cdots \). Since \( (y_1-y_1^*)/t = (c_1^1)2 + (c_1^2)3 + (c_1^3)4 + \cdots \), where \( c_1^1 = f'(a_0) \neq 0 \in k_0(a_0) \) and \( b_1^j \in k_0(a_0, a_1) \), we can write \( y_2 = (c_2^1)2 + (c_2^2)3 + (c_2^3)4 + \cdots \), with \( c_2^1 \neq 0 \in k_0(a_0, a_1) \) and \( b_2^j \in k_0(a_0, a_1) \). Then \( y_2 \rightarrow y_2^* = c_2^12 + b_2^23 \) implies \( a_2 \in k_0(a_0, a_1, y_2^*) \subset k \).

We have thus demonstrated the lemma for \( i = 0, 1 \); the general case is by induction on \( i \) and is identical to the \( i = 1 \) case.

**Corollary.** If \( a_0, \ldots, a_{n-1} \) (\( n \geq 1 \)) are algebraic over \( k_0 \) and \( a_n \) is tr. over \( k \), then \( k = k_0(a_0, a_1, a_2). \) If \( a_0, a_1, \ldots \) are all algebraic over \( k_0 \), then \( k = k_0(a_0, a_1, \ldots) \).

**Proof.** The inclusion \( \supset \) is by the lemma. Suppose \( a_n \) is tr. over \( k_0 \), and consider the finite algebraic extension of \( K = k_0(t), L = k_0(t, a_0, \ldots, a_{n-1}) \). Then \( L(x) = L(x_n) \), where \( x_n = a_n + a_{n+1} t + \cdots \). The residue field of \( L \) is \( k_0(a_0, \ldots, a_{n-1}) \). Moreover, since \( x_n \rightarrow a_n \) tr. over \( k_0(a_0, \ldots, a_{n-1}) \), by \( 1.1 \) the residue field of \( L(x) \) must be \( k_0(a_0, \ldots, a_{n-1})/(a_n) \). But \( K \subset L(x) \) implies \( k \subset \) the residue field \( k_0(a_0, \ldots, a_{n-1}, a_n) \) of \( L \). Thus, we have proved the first assertion of the corollary. For the second, observe that \( K \subset k_0(a_0, a_1, \ldots) ((t)) \) implies \( k = k_0(a_0, a_1, \ldots) \).

Q. E. D.

Note that \( x \) is necessarily tr. over \( k_0(t) \) whenever \( k/k_0 \) is not finite algebraic, by \( 1.2 \). In conclusion, the corollary shows that it is possible to get the residue field \( k \) to be an arbitrary finite algebraic extension of \( k_0 \) followed by a simple tr. extension (actually, it is only necessary to take \( n = 1 \) in the corollary since any finite algebraic extension of \( k_0 \) can be realized as a simple extension), or to be an arbitrary countably generated algebraic extension of \( k_0 \). See also [2, p. 173, Exercise 1] and [12, p. 104, Example 4] for examples of this latter type. (Incidentally, the Remark on p. 162 of [2] seems to ignore examples of the former type.)

It is interesting to pursue this example a bit further and inquire about the completion \( v^c, K^c \) of \( v, K \) in \( C((t)) \) when, say, \( a_0 \) is algebraic over \( k_0 \) and \( a_1 \) tr. over \( k_0 \). First observe that \( V = k_0(y_1)[x]_{(f(x))} \), where \( f(X) \) is the irreducible polynomial for \( a_0 \) over \( k_0 \). For, we have seen that \( y_1 \) specializes to \( a \) tr. over \( k_0 \), which implies \( k_0(y_1) \subset V \); and since \( f(X) \) is irreducible over \( k_0 \) and therefore also over \( k_0(y_1) \), \( k_0(y_1)[x]_{(f(x))} \) is a DVR contained in \( V \) and having the same quotient field \( k_0(t, x) \) as \( V \), and hence must be \( V \). We have also seen that the residue field \( k \) of \( V \) is \( k_0(a_0, a_1) \), so by Hensel’s lemma (cf. [4, p. 120, 16.7]) there exists a preimage for \( a_0 \).
in $V^c$ which is algebraic over $k_0$. But the only such preimage in $C[[t]]$ is $a_0$ itself, so $a_0 \in V^c$. Thus, $k_0(y_1)[a_0] = k_0(y_1, a_0) \subset V^c$ is a coefficient field for $V^c$, and $V^c$ is the $t$-adic topological closure of $k_0(a_0, y_1)[t]_{(t)}$ in $C[[t]]$; so $V^c$ may be thought of as being the subset of $C[[t]]$ obtained by taking power series in $t$ with coefficients in $k_0(a_0, y_1)$ and rewriting them as power series with coefficients in $C$.

5.2. As mentioned in the introduction, Nagata [7, p. 91, Thm. 5] has proved that if $k/k_0$ is not algebraic, then $k$ is contained in a (finite) algebraic extension of $k_0$ followed by a simple tr. extension. Does this result in itself imply 4.6? That is, given fields $k_0 \subset k \subset k_1(t)$ with $k_1$ finite algebraic over $k_0$, $t$ tr. over $k_1$, and $k/k_0$ not algebraic, is $k$ necessarily a finite algebraic extension of $k_0$ followed by a simple tr. extension? The following example (cf. [3, p. 23] and [8, p. 144, 2]) shows that the answer is "no".

Let $k_0 = \mathbb{R}$; $k = k_0(x, y)$, where $x^2 + y^2 + 1 = 0$; and $k_1 = \mathbb{C}$ = complexes. Then $k_0 \subset k \subset C(x + iy)$. For $x - iy = -1/(x + iy)$ implies $x - iy, x + iy \in C(x + iy)$, and hence $x, y \in C(x + iy)$.

Next observe that $k_0$ is algebraically closed in $k$, which amounts to verifying $i \notin k$. For, if $i \in k$, then $k_0(x, y) = k_0(x, y, i)$; and hence $[k_0(x, y, i): k_0(x)] = 2$. But $[k_0(x, i): k_0(x)] = 2$, and it follows from Gauss’s lemma that $Y^2 + x^2 + 1$ is irreducible over $k_0(x, i) = C(x)$; so $[k_0(x, y, i): k_0(x)] = 4$.

Now suppose $k/k_0$ is simple tr.. Then there exists a valuation $v$ of $k/k_0$ having residue field $k_0$. If $v(x) \geq 0$, then $y^2 + x^2 + 1 = 0$ implies $v(y) \geq 0$ too; and therefore in the residue field $k_0$, $y^2 + x^2 + 1 = 0$, which is impossible because $k_0 = \mathbb{R}$. If $v(x) < 0$, then the same argument applied to $(y/x)^2 + (1/x)^2 + 1 = 0$ works. Thus, $k$ is not a simple tr. extension of $k_0$.

The function field $k/k_0$ is known to have genus 0, but the additional fact needed to be able to conclude that $k$ is a simple tr. extension of $k_0$ is the existence of a $k_0$-rational place. See [3, p. 23].

5.3. An application of the Ruled Residue Theorem (inspired by the applications of Nagata in [7]. See also [8, p. 199, Thm. 5.12.2]).

Let $k_0 < k$ be fields of char. 0 and $G$ be any torsion-free abelian group (written additively). Let $k[G]$ be the group ring of $G$ with coefficients in $k$, i.e. $k[G] = \bigoplus \{kX^g \mid g \in G\}$, with multiplication defined linearly by $X^gX^h = X^{g+h}$. Let $k(G)$ denote the quotient field of $k[G]$. Then $k_0(G) \subset k(G)$.

Cancellation theorem. If $k(G)$ is a simple tr. extension of $k_0(G)$, then $k$ is a simple tr. extension of $k_0$.

Proof. Since $G$ is torsion-free, $G$ can be totally ordered. Then any $\zeta \in k[G]$ may be written $\zeta = a_1X^g + \cdots + a_tX^{g_t}$, $a_i \neq 0 \in k$, $g_1 < \cdots < g_t \in G$. Define $v: k[G] \to G$ by $v(\zeta) = \inf \{g_i \mid i = 1, \ldots, t\}$; and extend to a valuation $v$ of $k(G)$ having value group $G$ and residue field $k$. The restriction $v_0$ of $v$ to $k_0(G)$ is similarly a valuation with residue field $k_0$.

Claim: $k_0$ is algebraically closed in $k(G)$, and hence a fortiori in $k$. Since $k_0(G)$ is algebraically closed in $k(G)$ by hypothesis, it suffices to show $k_0$ is
algebraically closed in \( k_0(G) \). If \( x \in k_0(G) \) is algebraic over \( k_0 \), then \( k_0[x] = k_0(x) \subset V_0 \), and hence \( k_0(x) \) would map isomorphically under the residue map \( V_0 \to k_0 \), thereby yielding \( x \in k_0 \).

Thus, by theorem 4.6 and the fact that \( k_0 \) is algebraically closed in \( k \) and \( \neq k \), we conclude that \( k \) is a simple tr. extension of \( k_0 \).

Q. E. D.

In the statement of the cancellation theorem, we can replace the hypothesis that \( k(G) \) is a simple tr. extension of \( k_0(G) \) by the weaker hypothesis that \( k(G) \) is a simple tr. extension of \( k_0 \), for by Lüroth’s theorem the former hypothesis is a consequence of the latter. Finally, the cancellation theorem may be rephrased in terms of quotient fields of group rings as follows: If \( G \) is identified with \( 0 \oplus G \) in \( Z \oplus G \), then \( k_0(Z \oplus G) = k(G) \) implies \( k \equiv K_0(Z) \).

5.4. The set \( J(x) \cup \{ x \} \). The statements of 3.6 and 3.7 concerning elements of \( J(x) \cup \{ x \} \) imply comparable statements for arbitrary generators of value 0, as we shall now show. Assume \( K = K_0(x) \), where \( x \) is tr. over \( K_0 \) of value 0.

**Proposition.** Suppose \( K/K_0 \) is generically of index 1, and let \( l \) be a field such that \( k_0 \subset l \subset k \). If there exists a generator \( y \) of \( K/K_0 \) of value 0 such that \( y^* \notin l \), then there exists \( x_1 \in J(x) \cup \{ x \} \) such that \( x_1 \notin l \).

**Proof.** By [10, p. 198], \( y = (ax + b)/(cx + d) \), \( a, b, c, d \in K_0 \), \( ad - bc \neq 0 \). Since \( K/K_0 \) is generically of index 1, there exists \( e \notin K_0 \) such that \( \nu(ax + b) = \nu(cx + d) = \nu(e) \). Then \( y=((ax + b)/e)/(cx + d)/e) \) implies one of \( ((ax + b)/e)^* \) or \( ((cx + d)/e)^* \notin l \). Therefore we may assume \( y = (ax + b)/e \). Dividing \( a, b, e \) by the element of least value from among \( a, b, e \), we may further assume \( a, b, e \) have value \( > 0 \) and one of them has value 0. If \( \nu(e) = 0 \), then \( y^* = (a*/e^*)x^* + (b*/e^*) \) implies \( x^* \notin l \), so \( x_1 = x \) works; if \( \nu(e) > 0 \) but \( \nu(a) = 0 \), then \( x_1 = y = (x + (b/a))/(e/a) \in J(x) \); and if \( \nu(e) > 0 \) and \( \nu(b) = 0 \), then \( \nu(ax + b) = \nu(e) > 0 \) implies \( \nu(a) = 0 \) and we are in the previous case.

Q. E. D.

By taking \( l = k_0 \) (resp., \( l \) algebraic closure of \( k_0 \) in \( k \)), we have

**Corollary.** Suppose \( K/K_0 \) is generically of index 1. If there exists a generator \( y \) of \( K/K_0 \) such that \( y^* \notin k_0 \) (resp., \( y^* \) is tr. over \( k_0 \)), then there exists \( x_1 \in J(x) \cup \{ x \} \) such that \( x_1 \notin k_0 \) (resp., \( x_1^* \) is tr. over \( k_0 \)).

To carry this a bit further, let us define \( K \) to be generically rational over \( K_0 \) if for every generator \( y \) of \( K/K_0 \) of value 0, \( y^* \notin k_0 \). Then under the assumption that \( K/K_0 \) is generically of index 1, the condition of 3.6 “every element of \( J(x) \cup \{ x \} \) is rational” is equivalent to “\( K \) is generically rational over \( K_0 \)”.

**Part II: The theorem for \( \nu \) of finite rk.**

We retain the notation established in the introduction; in particular, \( K = K_0(x) \), where \( \nu(x) = 0 \). In addition, we assume throughout II that \( x \) is tr. over \( K_0 \).
6. Theorem 3.7 revisited.

Theorem 3.7 is false without the assumption that $K_0$ is henselian if $\text{rk} \; v > 1$, as example 7.2 will show; indeed, the henselian hypothesis was employed precisely to deal with valuations of infinite $\text{rk}$, and if we restrict attention to valuations of finite $\text{rk}$, a sharper result, which in the $\text{rk} = 1$ case amounts to deleting the henselian hypothesis and in the discrete, $\text{rk} = 1$ case amounts to deleting both the henselian and char 0 hypotheses, can be obtained. Since we are ignorant of the status of this result in the cases of infinite $\text{rk}$ or of non-zero characteristic and arbitrary value group, we shall first phrase it as a conjecture.

6.1 Conjecture. For every valuation overring $W$ of $V (W \subset K)$, the residue field $l_0$ of $W \cap K_0$ is algebraically closed in the residue field $l$ of $W$, $k_0 \neq K$, and $K/K_0$ is generically of index 1 $\Rightarrow$ there exists a generator $x$ of $K/K_0$ such that $v$ is the inf extension of $v_0$ w.r.t. $\tau(x) = 0$; or, equivalently, there exists a generator $x$ of $K/K_0$ which specializes to a tr. over $k_0$.

What we know about this conjecture, aside from the henselian case of 3.7, is summed up in the following theorem.1)

6.2 Theorem. The implication $\Rightarrow$ of 6.1 is true if either a) $\text{rk} \; v$ is finite and char $k = 0$, or b) $v$ is discrete.

The converse implication $\Leftarrow$ to 6.1 is always true. For, if $x \rightarrow x*$ tr. over $k_0$ and $w$ is the valuation of $K$ whose ring is $W$, then there exists a valuation $u$ of the residue field $l$ of $w$ such that $x \rightarrow x' \rightarrow x*$. (See §7). But $x*$ is tr. over $k_0$, so $x'$ is tr. over $l_0$, and therefore 1.1 yields $l/l_0$ is simple tr., and hence $l_0$ is algebraically closed in $l$.

In b) $\text{rk} \; v$ is necessarily finite, since by definition of discrete, $G$ is a lexicographic direct sum of finitely many copies of $Z$; but char $k$ may be arbitrary. In both a) and b) the crux of the proof lies in the $\text{rk} = 1$ case, from which the finite $\text{rk}$ case follows by induction.

The remainder of §6 will be devoted to establishing a) and b) for $\text{rk} = 1 \; v$. Just as theorem 3.7 follows from 3.6, this will follow from

6.3 Proposition. Suppose $v$ is $\text{rk} = 1$ and either a) char $k = 0$ or b) $v$ is discrete, and suppose $K/K_0$ is generically of index 1 and every element of $\mathcal{Y}(x) \cup \{x\}$ is rational. If $y$ is an element of $K$ of value 0 and $x$ is a generator for $y$ of multiplicity $> 0$, then there exists $x_1 \in \mathcal{Y}(x)$ such that $x_1$ is a generator for $y$ of multiplicity 0.

Proof. We first need a lemma.

Lemma. Suppose $y \in K$ has a generating pair $x, f(X)$ of multiplicity $n > 0$, where char $k \nmid n$. If $x = (x-r)/b \in \mathcal{Y}(x)$, then either $x_1$ is a generator for $y$ of multiplicity $< n$ or there exists a generating pair $x_1, f_1(X)$ for $y$ of multiplicity $n$

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1) Added August, 1981: I now have an example (to appear in a sequel) in char. $p$ for which $G_1 = G = Q$, $k/k_0$ is simple tr., and yet no generator of $K/K_0$ specializes to a tr. over $k_1$. Thus, the remaining undecided case of 6.1 is char $k = 0$ and $\text{rk} \; v$ infinite.
and an \( r_1 \in \mathbb{K}_0 \) such that \( f(x) = b^n f_1(x_1), \) \( v(x_1 - r_1) > 0, \) and \( v(f_1^{(n-1)}(r_1)) \geq 2v(f^{(n-1)}(r_1)) = 2v(b) \).

**Proof of lemma.** Suppose \( x_1 \) is not a generator for \( y \) of multiplicity \(< n \). We may write \( f(X) = a_0^* + a_1(X - r) + \cdots + a_n(X - r)^n \) where the \( a_i \) are in \( V_0 \). Then \( a_0^* = a_0 \neq 0 \) and \( a_{n-1} = f_1^{(n-1)}(r) \) (cf. 2.2). By 3.2, \( v(a_0(x - r)^i) \geq v((x - r)^n) \) for \( i = 0, \ldots, n - 1 \), and \( v(a_n - 1) = v(x - r) = v(b) \). Therefore if we write \( b^n f(x) = b^0 + b_1((x - r)/b) + \cdots \), where \( b_i = a_i/b^{n-i} \), then \( v(b_i) \geq 0 \), \( i = 0, \ldots, n - 1 \), and \( v(b_{n-1}) = 0 \); moreover, the \( b_i \) for \( i \geq n \) are of the form \( b_n = a_n, b_{n+1} = a_{n+1} b, \ldots \), and hence also are in \( V_0 \). Let \( f_1(x) = b_0 + b_1 x + \cdots + b_n X^n \). Then \( x_1, f_1(X) \) is a generator pair for \( y \), and \( f(x) = b^n f_1(x_1) \). Moreover, computing \( f_1^n(X_1) = a_n + b(\cdots) \), we see that \( f_1^n(x_1)^* = a_n^* \neq 0 \). Therefore \( x_1, f_1(X) \) is a generating pair for \( y \) of multiplicity \( \leq n \), and hence by our initial assumption of multiplicity \( n \).

It remains to show there exists \( r_1 \in \mathbb{K}_0 \) with the specified properties. We have \( f_1^{n-1}(x_1) = (a_{n-1}/b) + a_n x_1 + b(\cdots) \), and \( f_1^{n-1}(x_1)^* = 0 \) since \( x_1, f_1(X) \) has multiplicity \( n \); so \( 0 = (a_{n-1}/b)^* + a_n x_1^* \) and \( x_1^* = -(a_{n-1}/b)^*/a_n \). Let \( x = -(a_{n-1}/b)/(a_n) \). Now, as far as the requirement \( v(x_1 - r_1) > 0 \) is concerned, we are free to choose \( r_1 \) to be any element of the form \( r_1 = z + t, t \in \mathbb{K}_0 \) and \( v(t) > 0 \). For any such \( r_1 \), \( f_1^{n-1}(r_1) = (a_{n-1}/b) + a_n r_1 + ((n + 1)n/2)a_{n+1} b_1^2 + b^2(\cdots) = a_n t + ((n + 1)n/2)a_{n+1} \). Therefore if we choose \( t = -(n + 1)/2a_n \times (a_{n+1} b_1^2) \) (Note: If char \( K = 2 \), our hypotheses imply \( n + 1 \) is even.), then \( f_1^{n-1}(r_1) = (\text{terms involving } b_1, t^2, \text{ and } b^2) \). It follows that \( v(t) \geq v(b) > 0 \) and \( v(f_1^{n-1}(r_1)) = 2v(b) = 2v(a_{n-1}) \).

Q. E. D.

We shall only use the inequality of the lemma in the weak form \( v(f_1^{n-1}(r_1)) \geq v(f^{(n-1)}(r_1)) \). We now continue the proof of 6.3.

Choose \( x_1 \in \mathbb{K}(x) \cup \{x \} \) such that \( x_1 \) is a generator for \( y \) of multiplicity \( n \) and no element of \( \mathbb{K}(x) \cup \{x \} \) is a generator for \( y \) of multiplicity \(< n \). If \( n = 0 \), we are done, so assume \( n > 0 \). Every element of \( \mathbb{K}(x_1) = \mathbb{K}(x) \) is rational by hypothesis, and by 3.2 every element of \( \mathbb{K}(x_1) \) is a generator for \( y \) of multiplicity \( n \). Thus, by replacing \( x \) by \( x_1 \) in the formulation of proposition 6.3, we may additionally assume that every element of \( \mathbb{K}(x) \) is a generator for \( y \) of multiplicity \( < n \).

**Proof of 6.3–a):** Assume \( \text{char } k = 0 \). Suppose we have a generating pair \( x_r, f_r(X) \) of multiplicity \( n \) for \( y, x_r \in \mathbb{K}(x) \), and \( r_1 \in \mathbb{K}_0 \) such that \( v(x_1 - r_1) > 0 \). Since \( \mathbb{K}/\mathbb{K}_0 \) is generically of index 1, there exists \( b_1 \in \mathbb{K}_0 \) such that \( (x_1 - r_1)/b_1 = x_{i+1} \in \mathbb{K}(x) \). By the above lemma, there exists a generating pair \( x_{i+1}, f_{i+1}(X) \) for \( y \) of multiplicity \( n \) and \( r_{i+1} \in \mathbb{K}_0 \) such that \( f_{i+1}(x_1) = b_1 f_{i+1}(x_{i+1}), v(x_i - r_1) > 0 \), and \( v(f_{i+1}^{(n-1)}(r_{i+1})) \geq v(f_1^{(n-1)}(r_1)) = v(b) \). We thus define inductively a sequence \( x_i, f_i(X), i = 1, 2, \ldots \), of generating pairs for \( y \) and elements \( b_i \in \mathbb{K}_0 \) such that \( f_i(x_1) = b_1^i f_i(x_{i+1}) \) and \( v(b_{i+1}) \geq v(b_i) \). Then \( y = a f_i(x_1) = a b_1^i f_{i+1}(x_{i+1}) = a b_1^i b_2^i \cdots f_3(x_3) = \cdots \), where \( 0 < v(b_{i+1}) \leq v(b_i) \leq \cdots \). Since \( v \) is \( \text{rk } 1 \), for sufficiently large \( i \) \( v(a b_1^i b_2^i \cdots b_{i+1}) > 0 \). But then \( v(y) > 0 \), a contradiction.

**Proof of 6.3–b):** Assume \( v \) is discrete. To every generating pair \( x_1, f_1(X) \) for \( y \)
with $x_i \in \mathfrak{J}(x)$ there is associated a coefficient $a = y/f_i(x_i) \in K_0$. Since $v(f_i(x_i)) > 0$ because $x_i$ is assumed to be a generator for $y'$ of multiplicity $> 0$, we have $v(a) < 0$. Choose a generating pair $x_i, f_i(X)$ of this type (i.e. for $y$ with $x_i \in \mathfrak{J}(x)$) for which $-v(a)$ is minimal. (This uses $v$ is discrete, rk 1.) Since $x_i$ is rational and $K/K_0$ is generically of index 1, there exists $x_2 = (x_i - r_i)/b_i \in \mathfrak{J}(x) \subset \mathfrak{J}(x)$. Expand: $f_i(X) = a_0 + a_1(X - r_i) + \cdots + a_m(X - r_i)^m, a_i \in V_0$. Then $f_i(x_i) = b_i^n [c_0 + c_1((x_i - r_i)/b_i) + \cdots + c_m((x_i - r_i)/b_i)^m]$, where $c_i = a_i b_i^{-1}$. By 3.2-l, $c_0, \ldots, c_{n-1} \in V_0$; and $c_n = a_n, c_{n+1} = a_{n+1} b_1, \ldots, c_m = a_m b_m^{-1}$ are also in $V_0$. Therefore if $f_2(X) = c_0 + c_1 X + \cdots + c_m X^m$ and $x_2 = (x_i - r_i)/b_1$, it follows that $x_2, f_2(X)$ is a generating pair for $y$. But $y = a f_i(x_i) = ab_i f_2(x_i)$, and $v(b_i) > 0$ (since $v(b_i) = v(x_i - r_i) > 0$); so $-v(a b_i^n) < -v(a)$, a contradiction to our choice of $x_i, f_i(X)$.


Recall (cf. [12, pp. 43.53]) that a valuation $v$ of $K$ is called composite with valuations $w$ of $K$ and $u$ of $l$ if $V \subseteq W, l$ is the residue field of $w$, and the image $V'$ of $V$ under $W \to W/m_v = l$ is the valuation ring $U$ of $u$. The canonical homomorphism $V \to V/m_v = k$ may then be factored: $V \to V' = U \to k$. In terms of specialization maps (or “places”: cf. [12, p. 3]), one should keep in mind the following diagram:

[Diagram not provided]

7.1. We shall now finish the proof of 6.2 by induction on rk $v$, the rk 1 case having been established in § 6. If rk $v > 1$ and finite, then $v$ is composite with valuations $w$ and $u$ of strictly smaller rk.

First observe that $w/v_0$ is generically of index 1. For, $u/v_0$ is generically of index 1 implies for any generator $z$ of $K/K_0$ there exists $a \in K_0$ such that $z/a$ is a unit of $V$. But $V \subseteq W$, so $z/a$ is also a unit of $W$, and therefore $w(z) = w(a)$ and $w/v_0$ is generically of index 1.

By induction hypothesis applied to $w$, there exists a generator $z$ of $K/K_0$ such that $z \not\to z'$ tr. over $l_0$. Replacing $z$ by either $1 + z$ or $1 + (1/z)$ if necessary, we may further assume $v(z) = 0$ and hence also $u(z') = 0$. Now let $l_1 = l_0(z') \subseteq l$, and let $u_1 = u|l_1$. We want to check next that the hypotheses of 6.1 hold for $u_1/l_0$.

Claim: $u_1/l_0$ is generically of index 1. First observe that for any element $\beta \not\in 0$ of $l$ which has a $w$-preimage $b \in K$ which is a generator of $K/K_0, u(\beta) \in u(l_0)$. For $u/v_0$ is generically of index 1 implies there exists $a \not\in 0 \in K_0$ such that $b/a$ is a unit of $V \subseteq W$. Then $w(a) = w(b) = 0$, $a \not\to x \not\in l_0$, and $b/a \not\to \beta/x$. But $b/a$ is a unit of $V$ implies $\beta/x$ is a unit of $V = U$, so $u(\beta) = u(x) \in u(l_0)$. Next observe that to check $u_1/l_0$ is generically of index 1, it suffices by 4.1 to show that for any $r' \in l_0$ such that
u(z' - r') > 0, u(z' - r') \in u(l_0). But z' - r' has a w-preimage z - r, r \in K_0, in K which is a generator of K/K_0; so the previous observation applies.

Claim: Given any valuation overring R_1 of U_1 in l_1, the residue field of \( R_1 \cap l_0 = R_0 \) is algebraically closed in the residue field of R_1. To see this, first note that there exists a valuation overring R of U in l such that \( R \cap l_1 = R_1 \) (cf. [12, p. 53, Lemma 4]). The inverse image of R under W \to l is a valuation ring T lying between V and W; so by the hypothesis on V, the residue field \( \mathcal{O}_0 \) of T \cap K_0 is algebraically closed in the residue field \( \mathcal{O} \) of T. But \( \mathcal{O}, \mathcal{O}_0 \) are also the residue fields of R, R_0, respectively, and the residue field of R_1 lies between \( \mathcal{O}_0 \) and \( \mathcal{O} \), thereby establishing our assertion.

Claim: The residue field of \( u_0, l_0(=k_0) \neq \) residue field of \( u_1, l_1 \). For, \( l \) is algebraic over \( l_1 \) implies k is algebraic over the residue field of \( u_1 \). Since \( k/k_0 \) is not algebraic by hypothesis, \( k_0 \neq \) residue field of \( u_1 \).

Thus, we may apply the induction hypothesis to \( u_1/u_0 \) to conclude there exists a generator of \( l_1/l_0 \) which specializes under \( u \) to a tr. over \( k_0 \). By 5.4-Corollary this generator may be assumed to be of the form \( (z' - r')/s' \), for some \( r', 0 \neq s' \in l_0 \). But then if \( r, s \) are w-preimages in \( K_0 \) for \( r', s', (z - r)/s - (z' - r')/s' \); and therefore \( (z - r)/s \) is the desired generator of \( K/K_0 \) which specializes under \( v \) to a tr. over \( k_0 \).

Q. E. D.

7.2. We give next an example to show "\( K_0 \) is henselian" cannot be omitted from 3.7 and the condition on the residue fields in 6.1 cannot be weakened to "\( k_0 \) is algebraically closed in k". The example will have the following properties: \( v, v_0 \) are discrete, rk 2; index of \( v/v_0 = 1 \); \( k/k_0 \) is simple tr.; \( k_0 = Q \). The idea is to construct discrete, rk 1 valuations \( w, u \) such that \( v \) is composite with \( w \) and \( u \) and such that (in the initial notation of § 7) \( l/l_0 \) is not simple tr. Then no generator of \( K/K_0 \) can specialize under \( v \) to a tr. over \( k_0 \); for if it did, it would also specialize under \( w \) to a tr. over \( l_0 \), and by 1.1 this would imply \( l/l_0 \) is simple tr.

Let \( s, z \) be complex numbers algebraically independent over \( Q \), and let \( t \) be an indeterminate over \( C \). Let \( K_0 = Q(s, t) \) and \( K = K_0(x) \), where \( x = (1 + s)^{1/2} + xt \), and let \( w \) be the restriction of the \( t \)-adic valuation of \( C(t) \) to \( K \). Then \( l_0 = Q(s) \) and \( l = l_0((1 + s)^{1/2}, z) \), as we have seen in 5.1. Now let \( u_0 \) be the \( s \)-adic valuation of \( l_0 \); extend first to a valuation \( u_1 \) of \( l_0((1 + s)^{1/2}) \) and then to a valuation \( u \) of \( l \) by infs w.r.t. \( u(z) = 0 \).

The residue field \( k_0 \) of \( u_0 \) is \( Q \); and the residue field \( k_1 \) of \( u_1 \) remains \( Q \), since \( u_0 \) extends in two ways to \( l_0((1 + s)^{1/2}) \) (because if \( \xi = (1 + s)^{1/2} \), then \( s = \xi^2 - 1 = (\xi - 1) \cdot (\xi + 1) \) implies \( u_0 \) extends to \( l_0(\xi) = Q(\xi) \) either by \( u_1(\xi - 1) = 1, u_1(\xi + 1) = 0 \), or the reverse). Therefore by 1.1 the residue field \( k \) of \( l \) is \( Q(z^\ast) \), where \( z = z^\ast \).

Finally, \( v/v_0 \) is of index 1 because \( w/w_0 \) and \( u/u_0 \) are of index 1. (To see this, let \( a \neq 0 \in K \). Then \( w/w_0 \) is of index 1 implies there exists \( a_0 \neq 0 \in K_0 \) such that \( a/a_0 \not\sim b_0 \neq 0 \). Similarly, \( u/u_0 \) is of index 1 implies there exists \( b_0 \neq 0 \in l_0 \) such that \( b_0 \not\sim b_0 \neq 0 \). Let \( b_0 \) be a w-preimage for \( b_0 \) in \( K_0 \). Then \( a/a_0 b_0 \not\sim b_0 \neq 0 \), so \( v(a) = v(a_0 b_0) \in v(K_0) \).
7.3. We conclude § 7 with a proposition on composite valuations needed in § 8.

**Proposition.** Let $z$ be a generator of $K/K_0$, and suppose $[G: G_0]<\infty$ and $v$ is composite with a valuation $w$ of $K$. If $v$ is the inf extension of $v_0$ w.r.t. $v(z)$, then $w$ is the inf extension of $w_0$ w.r.t. $w(z)$ (and $w/w_0$ is of finite index).

**Proof.** Let $H$ be the value group of $w$. If the coset $v(z)+G_0$ has order $n$ in $G/G_0$, then $w(z)+H_0$ has order $n_1$ dividing $n$ in $H/H_0$. For, if there exists $b \notin K_0$ such that $v(z^n/b)=0$, then $z^n/b$ is a unit of $V$ and a fortiori a unit of $W$; and therefore $w(z^n)=w(b)\in H_0$. Thus, $n=n_1m$ for some integer $m \geq 1$.

By 4.3, there exists $b \notin K_0$ such that $z^n/b \mapsto \eta$ tr. over $k_0$, which implies $z^n/b \mapsto \eta'$ tr. over $l_0$. Also, there exists $c \notin K_0$ such that $w(z^n)=w(c)$. Then $(z^n/c)^m=z^n/c^m=z^n/db$, a unit of $W_0$. Hence $(z^n/c)^m \mapsto \eta'/d'$, $d' \in l_0$. But $\eta'$ is tr. over $l_0$, so we must have $z^n/c$ also specializes under $w$ to a tr. over $l_0$. Therefore by 4.3 $w$ is the inf extension of $w_0$ w.r.t. $w(z)$.

8. **Conjecture 6.1 for arbitrary inf extensions.**

What is the appropriate generalization of conjecture 6.1 to arbitrary inf extensions? It is a somewhat surprising fact that the obvious reformulation is not quite correct; one needs an extra condition, “every generator of $K_0(z^n)/K_0$ has value in $G_0$” below, as we shall show in example 8.2.

8.1. **Conjecture.**

For every valuation overring $W \subset K$ of $V$ the residue field $l_0$ of $W \cap K_0$ is algebraically closed in the residue field $l$ of $W$; $k_0 \neq k$; and there exists a generator $z$ of $K/K_0$ with $v(z)+G_0$ of order $n \geq 1$ in $G/G_0$ such that every generator of $K/K_0$ has value in $\{iv(z)+G_0 \mid i=0, \ldots, n-1\}$ and every generator of $K_0(z^n)/K_0$ has value in $G_0(\iff) v$ is the inf extension of $v_0$ w.r.t. $v(z)$ for some generator $z_1$ of $K/K_0$ such that $v(z_1)+G_0$ has order $n$ in $G/G_0$.

Note that the converse ($\iff$) to the conjecture is true: if $v$ is the inf extension of $v_0$ w.r.t. $v(z)$, then the value group of $K_0(z^n)/K_0$ is $G_0$ by 4.3; the group $G/G_0$ is cyclic generated by $v(z)+G_0$ by the definition of inf extension w.r.t. $v(z)$; and $l/l_0$ is simple tr., by 7.3 and 4.3, and a fortiori satisfies the hypothesis of the conjecture.

8.2. **Examples.** If $\Gamma$ is any totally ordered abelian group and $L$ a field, then the group ring $L[\Gamma]=\oplus \{LX^\gamma \mid \gamma \in \Gamma\}$, with multiplication defined by $X^\gamma X^\delta=X^{\gamma+\delta}$, may be given a valuation $w$ by defining $w(a_0X^{\gamma_0}+\cdots+a_tX^{\gamma_t})=\inf \{|\gamma_i| \mid i=0, \ldots, t\}$; and, as usual, this valuation extends to the quotient field $L(\Gamma)$ of $L[\Gamma]$. Moreover, the value group of $w$ is $\Gamma$, and one verifies easily that the residue field is $L$.

Let $Q(t)$ be a simple tr. extension of $Q$, let $\Gamma$ be the additive subgroup of the reals consisting of $\{x+\beta \pi \mid x, \beta \in \mathbb{Z}\}$, and let $w$ be the (rk 1) valuation of $Q(t)(\Gamma)$ described above. Let $z=X^t+tx^n$, let $K=K_0(z)$, where $K_0 \subset Q(t)(\Gamma)$ will be described presently, and let $v_0, v$ be the restrictions of $w$ to $K_0, K$ respectively. a) Example where $k$ is simple tr. over $k_0$ but $G/G_0$ is not cyclic (and hence $v$ cannot be an inf extension of $v_0$ w.r.t. any choice of generator of $K/K_0$). Take $K_0=Q(G_0)$, where
\( G_0 \) is the subgroup of \( \Gamma \) consisting of \( \{x + \beta \pi \mid x, \beta \in 2\mathbb{Z}\} \). Then the value group of \( v_0 \) is \( G_0 \) and the residue field \( k_0 \) is \( Q \). Since \( z^2 = X^2 + 2tX^{1+\pi} + t^2X^{2\pi} \) and \( X^2 = r \in K_0, z^2 - r \in K \), and therefore \( v(z^2 - r) = 1 + \pi \) is in the value group \( G \) of \( v \). Since \( v(z) = 1 \), it follows that \( 1, \pi \in G \); so \( G = \Gamma \). Then \( G/G_0 \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \).

Now let us compute \( k \). (Incidentally, we know \( Q = k_0 \subset k \subset Q(t) \) is residue field of \( w \), so without further ado we already know by Lüroth’s theorem that \( k/k_0 \) is simple tr.) We have \( (z^2 - r)^2 = 4t^2X^{2+2\pi} + 4t^3X^{1+3\pi} + t^4X^{4\pi} \). Let \( s = 4X^{2+2\pi} \in K_0 \). Then \( \xi = (z^2 - r)^2/s \cdot \xi^2 \). Since \( t^2 \) is tr. over \( k_0 \), it follows that the residue field of \( K_0(\xi) \) is \( Q(t^2) \) (cf. 1.1); and the value group of \( K_0(\xi) \) is \( G_0 \). But then \( [G : G_0] = 4 \) and \([K : K_0(\xi)] \leq 4\) imply (by 1.2) that the residue field \( k \) of \( K \) is also \( Q(t^2) \).

**Remark.** In light of this example, it would be interesting to know just what finite groups \( G/G_0 \) can occur when \( k \) is simple tr. over \( k_0 \) (and, of course, also \( K \) is simple tr. over \( K_0 \)).

b) Example to show that the hypothesis “every generator of \( K_0(z^n)/K_0 \) has value in \( G_0 \)” is needed in 8.1. Take \( K = \mathbb{Z}/2\mathbb{Z} \), and \( v(z) = 1 \). Therefore \( 1/\xi \) has value in \( K \). Since \( y \) is the inf extension of \( (z - s)^2/d \) specializes to a tr. over \( k_0 \).

**Claim:** \( v \) cannot be the inf extension of \( v_0 \) w.r.t. any generator of \( K/K_0 \). Note first that for any \( s \in K_0 \), \( v(z) = 1 \neq v(s) \). If \( v(s) < v(z) \), then \( v(z - s) = v(s) \) and \((z - s)^2/s^2 \to -1 \). If, on the other hand, \( v(z) < v(s) \), then \( v(z - s) = v(z) = 1 \) and \((z - s)^2/X^2 \to 1 \). The claim now follows from the Proposition below, which asserts that if \( v \) is the inf extension of \( v_0 \) w.r.t. some generator of \( K/K_0 \), then there exists \( s \in K_0 \) such that for any \( d \neq 0 \in K_0 \) with \( v(d) = v((z-s)^2), (z-s)^2/d \) specializes to a tr. over \( k_0 \).

**Lemma.** Let \( \xi \in K \). If \( \xi/b \to \text{tr. over } k_0 \) for some \( b \neq 0 \in K_0 \), then \( \xi/b \to \text{tr. over } k_0 \) for every \( b' \in K_0 \) such that \( v(b') = v(\xi) \).

**Proof.** \( v(b') = v(\xi/b) = v(b) \) implies there exists a unit \( u \) of \( V_0 \) such that \( b' = ub \). Therefore \( \xi/b' = (1/u)(\xi/b) \to (1/u^*)(\xi/b)* \). But \( 1/u^* \in k_0 \).

**Proposition (4.3 continued).** Suppose \( z_1 \) is a (tr.) generator of \( K/K_0 \) such that \( v(z_1) + G_0 \) has finite order \( n \geq 1 \) in \( G/G_0 \). If \( v \) is the inf extension of \( v_0 \) w.r.t. \( v(z_1) \), then for any generator \( z \) of \( K/K_0 \), there exists \( s \in K_0 \) such that for any \( d \in K_0 \) with \( v(d) = n v(z - s), (z - s)^n/d \to \text{tr. over } k_0 \).

**Proof.** By 4.3, there exists \( b \neq 0 \in K_0 \) such that \( z_1^n/b \to \text{tr. over } k_0 \). We may write \( z_1 = (a_1 + c_1)(a_2 + c_2), a_1, c_1 \in K_0, a_1c_2 - a_2c_1 \neq 0 \). Since \( [G : G_0] = n \), there

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2) Added August, 1981: W. Heinzer has now proved that \( G/G_0 \) may be any finite abelian group.
exist \( d_i \in K_0 \) such that \( \nu(a_iz - c_i) = v(d_i), \ i = 1, 2. \) Therefore \( z_i^{*i}(d_i/d_2) = N_i/N_2, \) where \( N_i = (a_iz - c_i)^\nu/d_i \) has value 0. By the lemma, \( z_i^{*i}(d_i/d_2) \to \text{tr. over } k_0, \) so either \( N_i \) or \( N_2 \) specializes to a tr. over \( k_0, \) say \( N_1 \) does. Then \( a_1 \neq 0 \) and \( N_1 = (z - (c_1/a_1))^{*i}/(d_i/a_i). \) In view of the above lemma, we are done. Q. E. D.

In order to apply this example to 8.1, it remains to verify \( k_0 \) is algebraically closed in \( k. \) (As in a) we know a priori by Lüroth’s theorem that \( k/k_0 \) is simple tr., but it is also easy to compute \( k \) directly.) We have seen in a) that the residue field of \( K_0(\zeta) \) is \( Q(t^2) \) and the value group is \( G_0. \) Since \( [K_1: K_0(\zeta)] \leq 2 \) and \( [G_1: G_0] = 2, \) it follows that the residue field of \( K_1 \) must remain \( Q(t^2). \) But \( [K: K_1] \leq 2 \) and \( (z^2 - r)/2X^*z - t \in k, \) so we must have \( k = Q(t). \)

LOUISIANA STATE UNIVERSITY
BATON ROUGE, LOUISIANA 70803

References